The Split Closure of a Strictly Convex Body

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Abstract

The Chvátal-Gomory closure and the split closure of a rational polyhedron are rational polyhedra. It was recently shown that the Chvátal-Gomory closure of a strictly convex body is also a rational polytope. In this note, we show that the split closure of a strictly convex body is defined by a finite number of split disjunctions, but is not necessarily polyhedral. We also give a closed form expression in the original variable space of a split cut for full dimensional ellipsoids.

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1. Introduction

Cutting planes are inequalities that separate fractional points from the convex hull of the integer feasible solutions of an Integer Programming (IP) problem. Together with branch and bound techniques, cutting planes drive the engine of state-of-the-art integer programming solvers [8, 9].

One of the most successful class of cutting planes for linear IP problems are obtained via split disjunctions [3, 5]. Split disjunctions can be applied to both linear and nonlinear IP problems as follows. Given a convex set $C \subseteq \mathbb{R}^n$, we are interested in obtaining convex relaxations of $C \cap \mathbb{Z}^n$. If $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$, then $C \cap \mathbb{Z}^n \subseteq (C \cap \{x \in \mathbb{R}^n : \langle \pi, x\rangle \leq \pi_0\}) \cup (C \cap \{x \in \mathbb{R}^n : \langle \pi, x\rangle \geq \pi_0 + 1\})$ where $\langle u, v\rangle$ is the inner product between u and v. Therefore, a convex relaxation of $C \cap \mathbb{Z}^n$ that is potentially tighter than C is given by $C^{\pi,\pi_0} := \operatorname{conv}((C \cap \{x \in \mathbb{R}^n : \langle \pi, x\rangle \leq \pi_0\}) \cup (C \cap \{x \in \mathbb{R}^n : \langle \pi, x\rangle \geq \pi_0 + 1\}))$. Observe that if C is a polyhedron, then C^{π,π_0} is a polyhedron. In this case, the nontrivial linear inequalities defining C^{π,π_0} (i.e. inequalities not valid for C) are called split cuts. A usual way to study split cuts for linear IP is to consider properties of the object obtained by adding all split cuts with the original linear inequalities. This object is called the split closure and for both linear and nonlinear IP, it can be formally defined as follows.

Definition 1.1. *Let C be a closed convex set. Let $SC_D(C) = \cap_{(\pi,\pi_0)\in D} C^{\pi,\pi_0}$, $D \subseteq \mathbb{Z}^n \times \mathbb{Z}$. The split closure of C is the convex set $SC_{\mathbb{Z}^n\times\mathbb{Z}}(C)$. For simplicity, we refer to $SC_{\mathbb{Z}^n\times\mathbb{Z}}(C)$ as $SC(C)$.*

If C is bounded or a rational polyhedron, then it is known that $SC(C)$ is a closed set. In Section 2 we give a short proof of this fact for any closed convex set C. If C is a rational polyhedron, then C^{π,π_0} is a polyhedron. However, because the number of disjunctions considered in the construction of the split closure is not finite, $SC(C)$ may not necessarily be a polyhedron. The first proof of the polyhedrality of the split closure of a rational polyhedron was introduced by Cook, Kannan and Schrijver in 1990 [5] and other proofs were subsequently presented [2, 7, 15] using different techniques. The approach of all these proofs is to use different properties of rational polyhedra, split disjunctions and their interactions to show that a finite number of split disjunctions is sufficient to describe the split closure.

In the case where C is not a polyhedron, C^{π,π_0} is not always a polyhedron and therefore C^{π,π_0} may not be describable by a finite number of linear inequalities. Thus, given a general convex set, a reasonable generalization of the polyhedrality result of split closure for rational polyhedra, is to show that a finite number of split disjunctions is sufficient to describe the split closure. In this note, we verify this for a wide range of strictly convex sets. As a direct corollary we obtain that the split closure preserves conic quadratic representability [4] of strictly convex sets. These results can be stated formally as follows.

Definition 1.2. *We say a set C is strictly convex if for all $u, v \in C$, $u \neq v$ we have that $\lambda u + (1-\lambda)v \in$ rel.int(C) for all $0 < \lambda < 1$. We say C is a strictly convex body if C is a full dimensional, strictly convex and compact set.*

Theorem 1.3. *Let $C \subseteq \mathbb{R}^n$ be a closed bounded strictly convex set such that the affine hull of C is a rational affine subspace of $\mathbb{R}^n$. Then the split closure of C is finitely defined, that is, there exists a finite set $D \subseteq \mathbb{Z}^n \times \mathbb{Z}$ such that $SC(C) = SC_D(C)$.*

Definition 1.4. *A conic quadratic representable set is a set of the form $\{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^p \; Ax+Dy-b \in K\}$ for $A \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times p}$, $b \in \mathbb{R}^m$ and K is the product of* Lorentz cones *of the form $\{u \in \mathbb{R}^l : \|(u_1, ..., u_{l-1})\| \leq u_l\}$ where $\|\cdot\|$ is the Euclidean norm.*

Corollary 1.5. *If C is a bounded conic quadratic representable strictly convex set, then $SC(C)$ is conic quadratic representable.*

Corollary 1.5 is a direct consequence of Theorem 1.3 by using the fact that the convex hull of the union of a finite number of bounded conic quadratic representable sets is also conic quadratic representable [4].

We note here that, while we show that $SC(C)$ is described by a finite number of disjunctions, verifying that $SC(C)$ is not always a polyhedron is also interesting; especially since the Chvátal-Gomory closure of a strictly convex set is a rational polyhedron [6]. For this reason we present an example that illustrates how the split closure of strictly convex sets can indeed be non-polyhedral. To achieve this we will give a closed form expression in the original variable space of a split cut for a full dimensional ellipsoid. Although it is straightforward to obtain a closed form expression using auxiliary variables, there is a theoretical and practical interest in getting an expression without auxiliary variables. In general, obtaining expressions without auxiliary variables can result in more efficient cutting plane methods (e.g. [12, 13]). Also, for example, the lifting approach for conic programming presented in [1] does not introduce any new auxiliary variables.

2. Proof of Theorem 1.3

Let C be a closed convex set, $\mathrm{bd}(C)$ be its boundary and $\sigma_C(a) := \sup\{\langle a, x\rangle \,:\, x \in C\}$ be its support function. Given $a \in \mathbb{R}$, $\lfloor a \rfloor$ represents the largest integer smaller than or equal to a. Let I^p represent the p-by-p identity matrix and $0^{a\times b}$ be the a-by-b matrix with all zero entries. Given a matrix P and a set C, let $PC = \{Px \,:\, x \in C\}$.

To prove Theorem 1.3, we will use the fact that the Chvátal-Gomory closure of a strictly convex body is a rational polyhedron.

Definition 2.1. *Let $C \subseteq \mathbb{R}^n$ be a closed convex set. For any set $S \subseteq \mathbb{Z}^n$ let*

$$\mathrm{CGC}_S(C) = \bigcap_{a\in S} \{x \in \mathbb{R}^n \,:\, \langle a, x\rangle \le \lfloor \sigma_C(a)\rfloor\}\,.$$

The Chvátal-Gomory closure of C is $\mathrm{CGC}(C) := \mathrm{CGC}_{\mathbb{Z}^n}(C)$.

Theorem 2.2 ([6])**.** *Let C be a strictly convex body. Then there exists a finite set $S \subseteq \mathbb{Z}^n$, such that $\mathrm{CGC}(C) = \mathrm{CGC}_S(C)$. Moreover $\mathrm{bd}(C) \cap \mathrm{CGC}(C) \subseteq \mathrm{bd}(C) \cap \mathbb{Z}^n$.*

We first present a few basic properties of split closures.

Lemma 2.3. *Let $C \subset \mathbb{R}^n$ be a closed convex set. Then*

1. *For any $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ we have that C^{π,π_0} is a closed convex set.*
2. *$SC(C)$ is a closed convex set.*

Proof. Let C_∞ be the recession cone of C and for a fixed $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ let $C^1 := \{x \in C \,:\, \langle \pi, x\rangle \le \pi_0\}$, $C^2 := \{x \in C \,:\, \langle \pi, x\rangle \ge \pi_0 + 1\}$, $C^1_+ := C^1 + C_\infty$ and $C^2_+ := C^2 + C_\infty$. If C^1 or C^2 is empty, then the first part of the lemma is direct. For the case in which both sets are non-empty we first claim that

$$C^{\pi,\pi_0} = \mathrm{conv}\left(C^1_+ \cup C^2_+\right). \tag{1}$$

For this it suffices to show that $C^{\pi,\pi_0} \supseteq \mathrm{conv}\left(C^1_+ \cup C^2_+\right)$. Let $x^i \in C^i$, $r^i \in C_\infty$ and $\lambda_i \ge 0$ for $i \in \{1, 2\}$ such that $\lambda_1 + \lambda_2 = 1$. To show that $\sum_{i=1}^2 \lambda_i(x^i + r^i) \in C^{\pi,\pi_0}$, we show that $x^i + r^i \in C^{\pi,\pi_0}$ for $i \in \{1, 2\}$. We only do this for $i = 1$ as the other case is analogous. If $\langle \pi, r^1\rangle \le 0$, then $x^1 + r^1 \in C^1$ so the result it direct. If not, then $\langle \pi, r^1\rangle > 0$ and there exists $t \ge 1$ such that $x^1 + tr^1 \in C^2$. Because $x^1 + r^1 \in \mathrm{conv}\left(\{x^1, x^1 + tr^1\}\right)$ we conclude that $x^1 + r^1 \in C^{\pi,\pi_0}$.

Now, C^1_+ and C^2_+ are non-empty closed convex sets with the same recession cone and hence by Corollary 9.8.1 of [11], we obtain that $\mathrm{conv}\left(C^1_+ \cup C^2_+\right)$ is a closed convex set. The first part of the lemma then follows from (1) and the second part is a direct consequence of the first part. □

Lemma 2.4. *Let $C \subseteq \mathbb{R}^n$ be a compact convex set. Then*

1. *$SC(C) \subseteq CGC(C) \subseteq C$.*
2. *If $C^1 \subseteq C^2$, then $SC(C^1) \subseteq SC(C^2)$.*

Proof. 1. Note that the inequality $\langle a, x\rangle \le \lfloor \sigma_C(a)\rfloor$ is valid for $C^{a,\lfloor \sigma_C(a)\rfloor}$. Therefore, $SC_{\mathbb{Z}^n\times\mathbb{Z}}(C) \subseteq CGC_{\mathbb{Z}^n}(C) \subseteq C$, where the last inclusion is proven in [6].

2. This follows from the fact that $(C^1)^{\pi,\pi_0} \subseteq (C^2)^{\pi,\pi_0}$ for all $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$.

$\square$

Lemma 2.5. *Let $B(u, \varepsilon) \subseteq \mathbb{R}^n$ be the closed ball with the center u and radius ε and let $\pi \in \mathbb{Z}^n$ and $\|\pi\| > \frac{1}{\varepsilon}$. Then $u \in B(u, \varepsilon)^{\pi,\pi_0}$ for all $\pi_0 \in \mathbb{Z}$.*

Proof. If $\langle \pi, u\rangle \leq \pi_0$ or $\langle \pi, u\rangle \geq \pi_0 + 1$, then $u \in B(u, \varepsilon)^{\pi,\pi_0}$. Next consider the case where $\pi_0 < \langle \pi, u\rangle < \pi_0 + 1$. Since the distance between the sets $\{x \in \mathbb{R}^n : \langle \pi, x\rangle = \pi_0\}$ and $\{x \in \mathbb{R}^n : \langle \pi, x\rangle = \pi_0 + 1\}$ is less than ε, there exists two points of the form $u + \delta_1 \frac{\pi}{\|\pi\|}$ and $u - \delta_2 \frac{\pi}{\|\pi\|}$ such that

1. $u + \delta_1 \frac{\pi}{\|\pi\|} \in \{x \in \mathbb{R}^n : \langle \pi, x\rangle \geq \pi_0 + 1\}$ and $u - \delta_2 \frac{\pi}{\|\pi\|} \in \{x \in \mathbb{R}^n : \langle \pi, x\rangle \leq \pi_0\}$,
2. $\delta_1 \geq 0$, $\delta_2 \geq 0$ and $\delta_1 + \delta_2 = \frac{1}{\|\pi\|} < \varepsilon$.

Therefore, $u + \delta_1 \frac{\pi}{\|\pi\|}, u - \delta_2 \frac{\pi}{\|\pi\|} \in B(u, \varepsilon)$ and u is a convex combination of $u + \delta_1 \frac{\pi}{\|\pi\|}, u - \delta_2 \frac{\pi}{\|\pi\|}$. Thus $u \in B(u, \varepsilon)^{\pi,\pi_0}$. $\square$

Before presenting the proof of Theorem 1.3, we present two key related results. See [14] for a proof.

Lemma 2.6 (Hermite Normal Form)**.** *Let $A \in \mathbb{Z}^{p\times n}$ be a matrix with full row rank. Then there exists an unimodular matrix $U \in \mathbb{Z}^{n\times n}$ such that $AU = [B\ 0^{p\times n-p}]$ and $B \in \mathbb{Z}^{p\times p}$ is an invertible matrix.*

Theorem 2.7 (Integer Farkas's Lemma)**.** *Let A be a rational matrix and b be a rational vector. Then the system $Ax = b$ has integral solutions if and only if $\langle y, b\rangle$ is integer whenever y is a rational vector and $A^T y$ is an integer vector.*

Proof of Theorem 1.3. Let L be the affine hull of C. By assumption, L is a rational affine subspace. If L contains no integer points, then by Theorem 2.7 we have that there exists $\pi \in \mathbb{Z}^n$ such that $\langle \pi, x\rangle = b \notin \mathbb{Z}$ $\forall x \in C$. Thus, $C^{\pi,\lfloor b\rfloor} = \emptyset$. Therefore if L contains no integer points, then the proof of Theorem 1.3 is complete. We now assume that L contains an integer point. Let $u \in L \cap \mathbb{Z}^n$. Note that the split closure of C is finitely defined if and only if the split closure of $C - \{u\}$ is finitely defined. Therefore we may assume $u = 0$ and $L = \{x \in \mathbb{R}^n : Ax = 0\}$ where $A \in \mathbb{Z}^{n-k\times n}$ with full row rank. Let U be the unimodular matrix given by Lemma 2.6. Let $P = [0^{k\times n-k}\ I^k]U^{-1}$ and $Q = U \begin{bmatrix} 0^{n-k\times k} \\ I^k \end{bmatrix}$. Then observe that

1. If $x \in L$, then $QPx = x$. Also for $y \in \mathbb{R}^k$, $PQy = y$.
2. PC is strictly convex body.
3. If $\pi \in \mathbb{Z}^n$, then $Q^T\pi \in \mathbb{Z}^k$. Therefore $C^{\pi,\pi_0} = Q\left((PC)^{Q^T\pi,\pi_0}\right)$.
4. If $\eta \in \mathbb{Z}^k$, then $P^T\eta \in \mathbb{Z}^n$. Therefore $(PC)^{\eta,\eta_0} = P\left((C)^{P^T\eta,\eta_0}\right)$.

Then the split closure of C is finitely defined if and only if the split closure of PC is finitely defined. Hence it is sufficient to verify Theorem 1.3 for full-dimensional sets.

Let $T := \mathrm{CGC}(C)$. Since $SC(C) \subseteq T$, it is sufficient to verify that for all but a finite number of vectors π and scalars π_0, the relationship $T \subseteq C^{\pi,\pi_0}$ holds.

By Theorem 2.2, $T \cap \mathrm{bd}(C) \subseteq \mathbb{Z}^n$. Therefore, if $x \in T \cap \mathrm{bd}(C)$, then $x \in SC(C)$. Let $\mathrm{ext}(T)$ be the set of vertices of the polytope T. Because $\mathrm{ext}(T) \setminus \mathrm{bd}(C) \subseteq C \setminus \mathrm{bd}(C) = \mathrm{int}(C)$ and $|\mathrm{ext}(\mathrm{CGC}_S(C))| < \infty$ we have that there exists $\varepsilon > 0$ such that $B(v, \varepsilon) \subseteq C \quad \forall v \in \mathrm{ext}(T) \setminus \mathrm{bd}(C)$.

Now if $\|\pi\| > \frac{1}{\varepsilon}$, then by Lemma 2.4 and Lemma 2.5 we obtain that $v \in C^{\pi,\pi_0}$ $\forall v \in \mathrm{ext}(T)$ $\forall \pi_0 \in \mathbb{Z}$. Finally note that since C is bounded, given $\pi \in \mathbb{Z}^n$, there exists only a finite possibilities of π_0 such that $C \neq C^{\pi,\pi_0}$. Therefore, $T \not\subseteq C^{\pi,\pi_0}$ holds for only a finite number of split disjunctions, completing the proof. $\square$

We note here that the above proof can be modified to prove that whenever $\mathrm{CGC}(C)$ is a polyhedron and $\mathrm{CGC}(C) \cap \mathrm{bd}(C) = SC(C) \cap \mathrm{bd}(C)$, the split closure is finitely defined.

3. Example of Non-polyhedral Split Closure

We now give an example of a conic quadratic representable strictly convex body whose split closure is non-polyhedral. Using a simple lemma we have as a direct corollary of Theorem 1.3 that the split closure of all strictly convex bodies in $\mathbb{R}^2$ is a (not necessarily rational) polyhedron, so the example will be in $\mathbb{R}^3$.

Lemma 3.1. *In $\mathbb{R}^2$, all nontrivial inequalities from a split disjunction are linear inequalities. Moreover every split disjunction yields at most two linear inequalities.*

Proof. Note that

$$C^{\pi,\pi_0} = \{x \in C : \langle \pi, x\rangle \le \pi_0\} \cup \{x \in C : \langle \pi, x\rangle \ge \pi_0 + 1\}$$
$$\cup \operatorname{conv}\left(\{x \in \operatorname{bd}(C) : \langle \pi, x\rangle = \pi_0\} \cup \{x \in \operatorname{bd}C : \langle \pi, x\rangle = \pi_0 + 1\}\right)$$

and if $C \subset \mathbb{R}^2$, then the last term in the union is a polytope with at most four facets. A simple case analysis shows that at most two of this facets are faces of C^{π,π_0}. The linear inequalities inducing these facets are the only nontrivial split cuts for C^{π,π_0}. □

Corollary 3.2. *The split closure of a strictly convex set in $\mathbb{R}^2$ is polyhedral.*

Proof. Direct from Theorem 1.3 and Lemma 3.1 by noting that $SC(C)$ is obtained by adding split cuts to $\mathrm{CGC}(C)$, which is a polyhedron. □

To construct the example in $\mathbb{R}^3$ we will need an explicit formula for split cuts for ellipsoids. For polyhedral sets we can readily talk about split cuts, as C^{π,π_0} is always defined by a finite number of linear inequalities. For general convex sets the following straightforward lemma gives us at least one case in which we can also talk about *linear* split cuts.

Lemma 3.3. *Let $C \subseteq \mathbb{R}^n$ be a closed convex set and $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$.*

- *If $C \cap \{x \in \mathbb{R}^n : \langle \pi, x\rangle \ge \pi_0 + 1\} = \emptyset$, then $C^{\pi,\pi_0} = \{x \in C : \langle \pi, x\rangle \le \pi_0\}$.*
- *If $C \cap \{x \in \mathbb{R}^n : \langle \pi, x\rangle \le \pi_0\} = \emptyset$, then $C^{\pi,\pi_0} = \{x \in C : \langle \pi, x\rangle \ge \pi_0 + 1\}$.*

In the case depicted by Lemma 3.3 the obtained split cuts are simply Chvátal-Gomory cuts, so an interesting question is if there are other cases in which we can talk about, possibly nonlinear, split cuts in closed form. For conic quadratic representable sets Corollary 1.5 almost gives one such case.

Let $C \subseteq \mathbb{R}^n$ be a conic quadratic representable strictly convex body. By Corollary 1.5 we know that $SC(C)$ is conic quadratic representable and in particular so is C^{π,π_0} for any $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. However, this result does not tell us the structure of split cuts for conic quadratic representable sets in the original space. The issue is that, for any $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, we only know that there exists $A \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times p}$ and $b \in \mathbb{R}^m$ such that $C^{\pi,\pi_0} = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^p\ Ax + Dy - b \in K\}$ where K a product of Lorentz cones. Unfortunately, we do not know if this representation is possible without matrix D and auxiliary variables y. In fact [4] contains many sets that require these auxiliary variables for their conic quadratic representation. However, because of its very particular structure we might expect C^{π,π_0} to have a conic quadratic representation without auxiliary variables. We show that this is true at least when C is a full dimensional bounded ellipsoid. blueFor this we will need the following facts about positive (semi-) definite matrices.

Definition 3.4. *We say that $A \in \mathbb{R}^{n\times n}$ is positive (semi-)definite if A is symmetric and*

$$\langle x, Ax\rangle \quad >(\ge)\ 0 \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}.$$

We write $A \succ (\succeq)\ 0$ to denote that A is positive (semi-)definite. The relation $\succ (\succeq)$ defines a natural partial order where $A \succ (\succeq) B \Leftrightarrow A - B \succ (\succeq) 0$. We recall some basic facts about positive semi-definite matrices.

Fact 3.5.

1. $A \succ 0$ *iff* A *is non-singular and* $A^{-1} \succ 0$.
2. $A \succeq 0$ *iff* $\exists\ B$ *s.t.* $B^t B = A$. *Furthermore* $\exists$ *unique* $B \succeq 0$ *such that* $B^2 = A$ *which we denote* $A^{1/2}$.
3. $A \succeq 0$. *Define* $\|x\|_A = \sqrt{\langle x, Ax\rangle}$ *for* $x \in \mathbb{R}^n$. *Then we have that*
   (a) $\|x+y\|_A \leq \|x\|_A + \|y\|_A \quad \forall\ x, y \in \mathbb{R}^n$.
   (b) $\|ax\|_A = |a|\|x\|_A \quad \forall\ x \in \mathbb{R}^n, a \in \mathbb{R}$.

We define the ellipsoid $E(A,c)$ as

$$E(A,c) = \{x \in \mathbb{R}^n : \langle x-c, A(x-c)\rangle \leq 1\} = \{x \in \mathbb{R}^n : \|x-c\|_A \leq 1\}.$$

For convenience, we denote $E(A,0) \equiv E(A)$. Take $A \in \mathbb{R}^{n\times n}$, $A \succ 0$. For $\pi \in \mathbb{R}^n \setminus \{0\}$ define

$$A_\pi^\perp = A - \frac{\pi\pi^T}{\langle \pi, A^{-1}\pi\rangle}.$$

For $r \in \mathbb{R}$, $-\|\pi\|_{A^{-1}} \leq r \leq \|\pi\|_{A^{-1}}$, define

$$R_A^\pi(r) = \sqrt{1 - \frac{r^2}{\langle \pi, A^{-1}\pi\rangle}}.$$

Lemma 3.6. *Take* $A \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$ *where* $A \succ 0$. *For* $\pi \in \mathbb{R}^n$, $\pi_0 \in \mathbb{R}$, $-\|\pi\|_{A^{-1}} \leq \pi_0 \leq \pi_0+1 \leq \|\pi\|_{A^{-1}}$, *we have that*

$$\begin{aligned} E(A,c)^{\pi,\pi_0} = \{x \in \mathbb{R}^n : \|x-c\|_A \ &\leq 1, \\ \|x-c\|_{A_\pi^\perp} &\leq (\pi_0 + 1 - \langle \pi, x\rangle) R_A^\pi(\pi_0 - \langle \pi, c\rangle) \\ &\quad + (\langle \pi, x\rangle - \pi_0) R_A^\pi(\pi_0 - \langle \pi, c\rangle + 1)\}. \end{aligned} \tag{2}$$

Since the proof of this lemma is technical, we include it in the Appendix.

Lemma 3.6 and Lemma 3.3 tell us that if C is a full dimensional bounded ellipsoid, then for each $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ such that $C^{\pi,\pi_0} \subsetneq C$ there exist exactly one split cut associated to (π, π_0), which is given by $\langle \pi, x\rangle \leq \pi_0$, $\langle \pi, x\rangle \geq \pi_0 + 1$ or conic quadratic inequality (2). Together with Theorem 1.3 we have the following direct corollary.

Corollary 3.7. *If* $C \subseteq \mathbb{R}^n$ *is a bounded full dimensional ellipsoid, then* $SC(C)$ *is described by a finite number of linear inequalities and conic quadratic inequalities.*

It is not clear from Corollary 3.7 if in this case $SC(C)$ is a polyhedral set or not. In particular, it would be possible for the linear inequalities describing $SC(C)$ to dominate all the conic quadratic inequalities (e.g. when $\mathrm{CGC}(C) = \mathrm{conv}(C \cap \mathbb{Z}^n)$). However, the following example shows that $SC(C)$ can, in fact, be non-polyhedral.

Example 3.8. *Let* $C = \{x \in \mathbb{R}^3 : \|x-c\|_A \leq 1\}$ *for* $A = \frac{1}{33/64}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/10000 \end{bmatrix}$ *and* $c = (1/2, 1/2, 1/2)^T$. *Using Lemma 3.6 we have that the split cuts for* C *associated to* $x_1 \leq 0 \vee x_1 \geq 1$ *and* $x_2 \leq 0 \vee x_2 \geq 1$ *are*

$$\sqrt{\frac{64}{33}\left(x_2 - \frac{1}{2}\right)^2 + \frac{4}{20625}\left(x_3 - \frac{1}{2}\right)^2} \leq \sqrt{\frac{17}{33}}$$

and

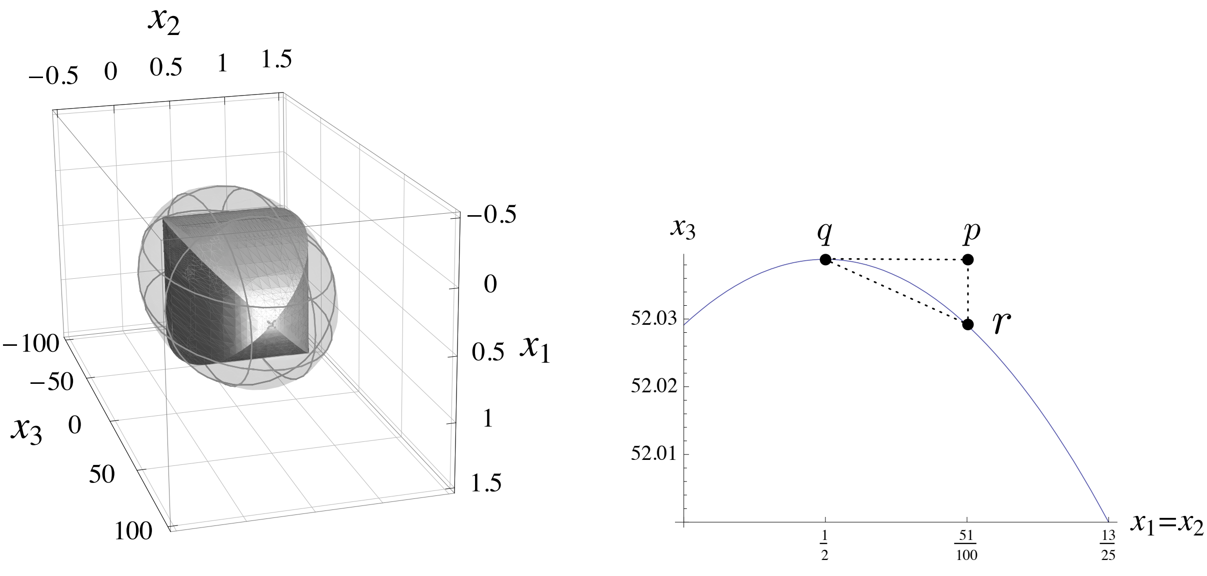
$$\sqrt{\frac{64}{33}\left(x_1-\frac{1}{2}\right)^2+\frac{4}{20625}\left(x_3-\frac{1}{2}\right)^2}\le\sqrt{\frac{17}{33}}$$

respectively.

Let R *be the convex set obtained by adding these two split cuts to* C*. As illustrated in Figure 1(a),* R *is a non-polyhedral 'rugby ball like' convex set contained in* C*. To show that the split closure of* C *is not a polyhedron we will show that part of the surface of* R *remains in the surface of the split closure of* C*. The part of the surface of* R *we consider is a portion 'of the seams of the rugby ball' starting at the point in* R *with highest* x_3 *value and with increasing* x_1 *and* x_2*. Specifically, we will show that the curve* $\gamma := \left\{\left(t, t, \frac{1+25\sqrt{17-64(t-1/2)^2}}{2}\right) : \frac{1}{2} \le t \le \frac{1}{2}+\frac{1}{100}\right\}$ *belongs to the boundary of the split closure of* C*.*

Because γ *is in the boundary of* R *it suffices to show that it is not cut by any split cut. To achieve this we show that points* $q := (1/2, 1/2, \frac{1+25\sqrt{17}}{2})^T$ *and* $r := \left(1/2+1/100, 1/2+1/100, \frac{1+25\sqrt{17-64(1/100-1/2)^2}}{2}\right)^T$ *are not separated by any split cut for* C *and that point* $p := (1/2+1/100, 1/2+1/100, \frac{1+25\sqrt{17}}{2})^T$ *is only separated by the two split cuts defining* R*. The result will then follow because, as illustrated in Figure 1(b),* $\gamma \subseteq \mathrm{conv}(p, q, r)$*.*

To show that p*,* q *and* r *are not separated by any split cut besides the ones defining* R *we first note that* $B(v, 0.34) \subseteq \{x \in \mathbb{R}^3 : \max_{i=1}^3 |x_i - v_i| \le 0.34\} \subseteq C$ *for all* $v \in \{p, q, r\}$*. Then, similar to the proof of Theorem 1.3, we have that by Lemma 2.4 and Lemma 2.5 the only split cuts that can separate* p*,* q *or* r *are those associated to* $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ *with* $\|\pi\| \le 1/0.34$*. Aided by Lemma 3.6 and Lemma 3.3 we tested this finite list of split cuts using a simple Mathematica [10] program to show that none of these cuts separates* p*,* q *or* r*.*



(a) Rugby ball given by two split cuts.

(b) Non-polyhedral curve contained in convex hull of three points.

Figure 1: Illustration of Example 3.8

4. Observation and Open Questions

Theorem 2.2 for the Chvátal-Gomory closure is extended in [6] to include the intersection of strictly convex sets with rational polyhedra. Although this extension is relatively simple for the Chvátal-Gomory closure, an analogous extension for the split closure seems much harder. Besides the polyhedrality of the Chvátal-Gomory closure, the proof of Theorem 1.3 relies on separating every point in $\mathrm{bd}(C) \setminus \mathbb{Z}^n$ with a Chvátal-Gomory cut and this is no longer possible when C is the intersection of a strictly convex set with a rational polyhedron. One way to deal with this issue and extend the proof of Theorem 1.3 is to prove the existence of a convex set T such that

- $T \supseteq SC(S)$
- $T \cap \mathrm{bd}(S) = SC(S) \cap \mathrm{bd}(S)$
- There exists $\delta > 0$ such that $v \in \mathrm{ext}(T) \setminus \mathrm{bd}(S)$ implies that v is at least a distance δ from $\mathrm{bd}(S)$.

Unfortunately, the existence of such set remains an open question.

Another interesting observation concerns the split closure of polyhedral approximations of a strictly convex body. Example 3.8 shows that some strictly convex sets lack a polyhedral approximation whose split closure is the same as that of the strictly convex set. In contrast, every strictly convex set has a polyhedral (outer) approximation with exactly the same Chvátal-Gomory closure.

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References

[1] A. Atamtürk and V. Narayanan, *Lifting for conic mixed-integer programming*, Mathematical Programming **To appear. DOI:10.1007/s10107-009-0282-9** (2009).
[2] K. Andersen, G. Cornuéjols, and Y. Li, *Split closure and intersection cuts*, Mathematical Programming **102** (2005), 457–493.
[3] E. Balas, *Disjunctive programming*, Annals of Discrete Mathematics **5** (1979), 3–51.
[4] A. Ben-Tal and A. Nemirovski, *Lectures on modern convex optimization: analysis, algorithms, and engineering applications*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2001.
[5] W. J. Cook, R. Kannan, and A. Schrijver, *Chvátal closures for mixed integer programming problems*, Mathematical Programming **58** (1990), 155–174.
[6] D. Dadush, S. S. Dey, and J. P. Vielma, *The Chvátal-Gomory Closure of Strictly Convex Body*, http://www.optimization-online.org/DB_HTML/2010/05/2608.html.
[7] S. Dash, O. Günlük, and A. Lodi, *MIR closures of polyhedral sets*, Mathematical Programming **121** (2010), 33–60.
[8] M. Jünger, T. Liebling, D. Naddef, G. Nemhauser, W. Pulleyblank, G. Reinelt, G. Rinaldi, and L. Wolsey (eds.), *50 years of integer programming 1958-2008: From the early years to the state-of-the-art*, Springer-Verlag, New York, 2010.
[9] A. Lodi, *Mixed integer programming computation*, [8], pp. 619–645.
[10] Wolfram Research, *Wolfram Mathematica 6*, `http://www.wolfram.com/products/mathematica/index.html`.
[11] R. T. Rockafellar, *Convex analysis*, Princeton University Press, 1996.
[12] A. Saxena, P. Bonami, and Jon Lee, *Convex relaxations of non-convex mixed integer quadratically constrained programs: extended formulations*, Mathematical Programming, Series B **124** (2010), 383–411.
[13] ———, *Convex relaxations of non-convex mixed integer quadratically constrained programs: projected formulations*, Mathematical Programming, Series B **To appear. DOI:10.1007/s10107-010-0340-3** (2010).
[14] A. Schrijver, *Theory of linear and integer programming*, John Wiley & Sons, Inc., New York, NY, 1986.
[15] J. P. Vielma, *A constructive characterization of the split closure of a mixed integer linear program*, Operations Research Letters **35** (2007), 29–35.

6. Appendix A: Ellipsoid Split Cut

To prove Lemma 3.6, we will need the following lemma.

Lemma 6.1. *Take $A \in \mathbb{R}^{n\times n}$ where $A \succ 0$. For $\pi \in \mathbb{R}^n \setminus \{0\}$, the following holds:*

1. $A \succeq A_\pi^\perp \succeq 0$.
2. $\forall\, x \in \mathbb{R}^n,\ t \in \mathbb{R},\ \|x + tA^{-1}\pi\|_{A_\pi^\perp} = \|x\|_{A_\pi^\perp}$.
3. $E(A) = \{x \in \mathbb{R}^n : \|x\|_{A_\pi^\perp} \le R_A^\pi(\langle \pi, x\rangle),\ \ -\|\pi\|_{A^{-1}} \le \langle \pi, x\rangle \le \|\pi\|_{A^{-1}}\}$.

Proof.

Proof of 1. For all $y \in \mathbb{R}^n$, since $A \succ 0 \Leftrightarrow A^{-1} \succ 0$, we have that

$$\left\langle y, \frac{\pi\pi^T}{\langle \pi, A^{-1}\pi\rangle} y\right\rangle = \frac{\langle \pi, y\rangle^2}{\langle \pi, A^{-1}\pi\rangle} \ge 0.$$

Therefore $\frac{\pi\pi^T}{\langle \pi, A^{-1}\pi\rangle} = A - A_\pi^\perp \succeq 0 \Rightarrow A \succeq A_\pi^\perp$ as needed. Since A is positive definite by Fact 3.5 we know that both $A^{1/2}$ and $A^{-1/2}$ exist. Now for $x \in \mathbb{R}^n$, we have that

$$\begin{aligned}\langle x, Ax\rangle \left\langle \pi, A^{-1}\pi\right\rangle &= \left\langle A^{1/2}x, A^{1/2}x\right\rangle \left\langle A^{-1/2}\pi, A^{-1/2}\pi\right\rangle = \|A^{1/2}x\|^2\|A^{-1/2}\pi\|^2\\ &\ge \left\langle A^{-1/2}x, A^{1/2}\pi\right\rangle^2 = \langle x, \pi\rangle^2 = \langle x, \pi\pi^T x\rangle \qquad \text{(by Cauchy-Schwarz)}.\end{aligned}$$

Hence we have that

$$\langle x, Ax\rangle \ge \left\langle x, \frac{\pi\pi^T}{\langle \pi, A^{-1}\pi\rangle} x\right\rangle \quad \forall\, x \in \mathbb{R}^n \quad \Rightarrow \quad A - \frac{\pi\pi^T}{\langle \pi, A^{-1}\pi\rangle} \succeq 0 \quad \Rightarrow \quad A_\pi^\perp \succeq 0$$

as needed.

Proof of 2. We first note that

$$A_\pi^\perp A^{-1}\pi = \left(A - \frac{\pi\pi^T}{\langle \pi, A^{-1}\pi\rangle}\right) A^{-1}\pi = AA^{-1}\pi - \pi\frac{\langle \pi, A^{-1}\pi\rangle}{\langle \pi, A^{-1}\pi\rangle} = \pi - \pi = 0.$$

For $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, using the above, we have that

$$\begin{aligned}\|x + tA^{-1}\pi\|_{A_\pi^\perp}^2 &= \left\langle x + tA^{-1}\pi, A_\pi^\perp(x + tA^{-1}\pi)\right\rangle\\ &= \left\langle x, A_\pi^\perp x\right\rangle + 2t\left\langle x, A_\pi^\perp A^{-1}\pi\right\rangle + t^2\left\langle \pi, A^{-1}A_\pi^\perp A^{-1}\pi\right\rangle\\ &= \left\langle x, A_\pi^\perp x\right\rangle + 2t\langle x, 0\rangle + t^2\left\langle \pi, A^{-1}0\right\rangle = \left\langle x, A_\pi^\perp x\right\rangle = \|x\|_{A_\pi^\perp}^2,\end{aligned}$$

as needed.

Proof of 3. Take $x \in E(A)$. Then $\langle x, Ax\rangle \le 1$. Remembering $A = A_\pi^\perp + \frac{\pi\pi^T}{\langle \pi, A^{-1}\pi\rangle}$, we get that

$$\langle x, Ax\rangle \ \le\ 1 \quad \Leftrightarrow \quad \left\langle x, \left(A_\pi^\perp + \frac{\pi\pi^T}{\langle \pi, A^{-1}\pi\rangle}\right) x\right\rangle \ \le\ 1 \quad \Leftrightarrow \quad \|x\|_{A_\pi^\perp}^2 \ \le\ 1 - \frac{\langle \pi, x\rangle^2}{\langle \pi, A^{-1}\pi\rangle}. \tag{3}$$

From Part 1, we know that $A_\pi^\perp \succeq 0$. Moreover, $\langle x, Ax\rangle \le 1$ for all $x \in E(A)$. Therefore we have that

$$\left\langle x, A_\pi^\perp x\right\rangle \ \ge\ 0 \quad \Rightarrow \quad 1 - \frac{\langle \pi, x\rangle^2}{\langle \pi, A^{-1}\pi\rangle} \ \ge\ 0 \quad \Rightarrow \quad -\|\pi\|_{A^{-1}} \le \langle \pi, x\rangle \le \|\pi\|_{A^{-1}}. \tag{4}$$

Now for $-\|\pi\|_{A^{-1}} \le \langle \pi, x\rangle \le \|\pi\|_{A^{-1}}$, we have that $R_A^\pi(\langle \pi, x\rangle)$ is defined and $R_A^\pi(\langle \pi, x\rangle)^2 = 1 - \frac{\langle \pi, x\rangle^2}{\langle \pi, A^{-1}\pi\rangle}$. Combining (4) and (3) (taking square roots) yields the result.

□

Proof of Lemma 3.6. Let

$$S_0 = \{x \in \mathbb{R}^n : \|x-c\|_A \le 1,\ \langle \pi, x-c\rangle \le \pi_0\} \quad \text{and} \quad S_1 = \{x \in \mathbb{R}^n : \|x-c\|_A \le 1,\ \langle \pi, x-c\rangle \ge \pi_0+1\}$$

and let

$$C = \left\{x \in \mathbb{R}^n : \|x-c\|_A \le 1,\ \|x-c\|_{A_\pi^\perp} \le (\pi_0+1-\langle\pi,x-c\rangle)R_A^\pi(\pi_0) + (\langle\pi,x-c\rangle-\pi_0)R_A^\pi(\pi_0+1)\right\}.$$

Note that by definition $E(A,c)^{\pi,\pi_0+\langle\pi,c\rangle} = \text{conv}\{S_0,S_1\}$. Our goal is to show that $\text{conv}\{S_0,S_1\} = C$, which is equivalent to (2). Since the above relationships are all preserved under shifts, we may assume that $c=0$ and hence focus our attention to the ellipsoid $E(A,0)=E(A)$. Letting $w_r = \frac{r}{\langle\pi,A^{-1}\pi\rangle}A^{-1}\pi$, for $r\in\mathbb{R}$, we note that

$$\langle \pi, w_r\rangle = \left\langle \pi, \frac{r}{\langle\pi,A^{-1}\pi\rangle}A^{-1}\pi\right\rangle = r\frac{\langle\pi,A^{-1}\pi\rangle}{\langle\pi,A^{-1}\pi\rangle} = r \tag{5}$$

By Lemma 6.1 Part 2, for $r\in\mathbb{R}$, $-\|\pi\|_{A^{-1}} \le r \le \|\pi\|_{A^{-1}}$, we see that $\|w_r\|_{A_\pi^\perp} = 0 \le R_A^\pi(\langle\pi,w_r\rangle) = R_A^\pi(r)$. Therefore by Lemma 6.1 Part 3, $w_r \in E(A)$. From the above and our assumption on π_0, we get that $w_{\pi_0}\in S_0$ and $w_{\pi_0+1}\in S_1$, and therefore $S_0,S_1 \ne \emptyset$.

Take $x\in\text{conv}\{S_0,S_1\}$. Clearly $x\in E(A)$, so we need only check whether x satisfies the additional conic inequality. Since S_0 and S_1 are convex, we can write $x=\alpha x_0+(1-\alpha)x_1$, $0\le\alpha\le1$, where $x_0\in S_0, x_1\in S_1$. By Lemma 6.1 Part 1, we know that $A_\pi^\perp \succeq 0$ and hence $\|\cdot\|_{A_\pi^\perp}$ is convex by Fact 3.5. By the convexity of $\|\cdot\|_{A_\pi^\perp}$, and that $x_0,x_1\in E(A)$ together with Lemma 6.1 Part 3, we get

$$\|x\|_{A_\pi^\perp} = \|\alpha x_0+(1-\alpha)x_1\|_{A_\pi^\perp} \le \alpha\|x_0\|_{A_\pi^\perp} + (1-\alpha)\|x_1\|_{A_\pi^\perp} \le \alpha R_A^\pi(\langle\pi,x_0\rangle) + (1-\alpha)R_A^\pi(\langle\pi,x_1\rangle)$$

Since $\alpha\langle\pi,x_0\rangle+(1-\alpha)\langle\pi,x_1\rangle = \langle\pi,x\rangle$, and $\langle\pi,x_0\rangle\le\pi_0<\pi_0+1\le\langle\pi,x_1\rangle$ by concavity of the function R_A^π (since $\sqrt{1-y^2}$ is concave) we get that

$$\alpha R_A^\pi(\langle\pi,x_0\rangle) + (1-\alpha)R_A^\pi(\langle\pi,x_1\rangle) \le (\pi_0+1-\langle\pi,x\rangle)R_A^\pi(\pi_0) + (\langle\pi,x\rangle-\pi_0)R_A^\pi(\pi_0+1),$$

as needed.

Now take $x\in C$. We will verify that if x satisfies $\|x\|_{A_\pi^\perp} \le (\pi_0+1-\langle\pi,x\rangle)R_A^\pi(\pi_0)+(\langle\pi,x\rangle-\pi_0)R_A^\pi(\pi_0+1)$, then $x\in\text{conv}\{S_0,S_1\}$. If $\langle\pi,x\rangle\le\pi_0$, then $x\in S_0$ and if $\langle\pi,x\rangle\ge\pi_0+1$, then $x\in S_1$, so we may assume that $\pi_0<\langle\pi,x\rangle<\pi_0+1$. Let $\alpha=(\pi_0+1-\langle\pi,x\rangle)$ and $1-\alpha=\langle\pi,x\rangle-\pi_0$, where by the previous sentence we get that $0<\alpha<1$ and that $\alpha\pi_0+(1-\alpha)(\pi_0+1)=\langle\pi,x\rangle$.

We may write $x=z+w_{\langle\pi,x\rangle}$ for some $z\in\mathbb{R}^n$. By construction and (5), we have that $\langle\pi,z\rangle=0$. Since $x\in C$ and $\pi_0<\langle\pi,x\rangle<\pi_0+1$, by Lemma 6.1 Part 2 and 3 and concavity of R_A^π, we have that

$$\|z\|_{A_\pi^\perp} = \|z+w_{\langle\pi,x\rangle}\|_{A_\pi^\perp} = \|x\|_{A_\pi^\perp} \le R_A^\pi(\langle\pi,x\rangle) \le \alpha R_A^\pi(\pi_0)+(1-\alpha)R_A^\pi(\pi_0+1). \tag{6}$$

Let

$$z_0 = z\left(\frac{R_A^\pi(\pi_0)}{\alpha R_A^\pi(\pi_0)+(1-\alpha)R_A^\pi(\pi_0+1)}\right) \quad \text{and} \quad z_1 = z\left(\frac{R_A^\pi(\pi_0+1)}{\alpha R_A^\pi(\pi_0)+(1-\alpha)R_A^\pi(\pi_0+1)}\right).$$

Now note that

$$\begin{aligned}\alpha z_0+(1-\alpha)z_1 = z \quad \text{and} \quad & \alpha w_{\pi_0}+(1-\alpha)w_{\pi_0+1} = w_{\langle\pi,x\rangle}\\ &\Rightarrow \alpha(z_0+w_{\pi_0})+(1-\alpha)(z_1+w_{\pi_0+1}) = z+w_{\langle\pi,x\rangle} = x.\end{aligned}$$

We claim that $z_0+w_{\pi_0}\in S_0$ and $z_1+w_{\pi_0+1}\in S_1$. Assuming this, the above equation then gives us that $x\in\text{conv}\{S_0,\ S_1\}$ and so we are done. Since $\langle\pi,z\rangle=0\Rightarrow\langle\pi,z_0\rangle=0$, we get that $\langle\pi,z_0+w_{\pi_0}\rangle=\langle\pi,w_{\pi_0}\rangle=\pi_0$. Then, by definition of z_0, Lemma 6.1 Part 2 and (6) we have that

$$\|z_0+w_{\pi_0}\|_{A_\pi^\perp} = \|z_0\|_{A_\pi^\perp} = R_A^\pi(\pi_0)\frac{\|z\|_{A_\pi^\perp}}{\alpha R_A^\pi(\pi_0)+(1-\alpha)R_A^\pi(\pi_0+1)} \le R_A^\pi(\pi_0) = R_A^\pi(\langle\pi,z_0+w_{\pi_0}\rangle) \tag{7}$$

From (7) and Lemma 6.1 Part 3, we see that $z_0+w_{\pi_0}\in E(A_0)$ and $\langle\pi,z_0+w_{\pi_0}\rangle\le\pi_0$. Therefore $z_0+w_{\pi_0}\in S_0$ as needed. The argument for $z_1+w_{\pi_0+1}$ is symmetric. □