The Chvátal-Gomory Closure of an Ellipsoid is a Polyhedron

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Abstract. It is well-know that the Chvátal-Gomory (CG) closure of a rational polyhedron is a rational polyhedron. In this paper, we show that the CG closure of a bounded full-dimensional ellipsoid, described by rational data, is a rational polytope. To the best of our knowledge, this is the first extension of the polyhedrality of the CG closure to a non-polyhedral set. A key feature of the proof is to verify that all non-integral points on the boundary of ellipsoids can be separated by some CG cut. Given a point u on the boundary of an ellipsoid that cannot be trivially separated using the CG cut parallel to its supporting hyperplane, the proof constructs a sequence of CG cuts that eventually separates u. The polyhedrality of the CG closure is established using this separation result and a compactness argument. The proof also establishes some sufficient conditions for the polyhedrality result for general compact convex sets.

1 Introduction

Nonlinear Integer Programming has received significant attention from the Integer Programming (IP) community in recent time. Although, some special classes are efficiently solvable [32], even simple nonlinear IP problems can be NP-Hard or undecidable [33]. However, there has been considerable progress in the development of practical algorithms that can be effective for many important applications (e.g. [1, 8–10, 32, 36, 37]). Building on work for linear IP, practical algorithms for nonlinear IP have benefited from the development of several classes of cutting planes or valid inequalities (e.g. [3–6, 13, 14, 25, 29–31, 35, 28, 39, 40, 43]). Many of these inequalities are based on the generalization of ideas used in linear IP. For example, [4, 5, 39, 14] exploit the interaction between superadditive functions and nonlinear constraints to develop techniques that can yield several strong valid inequalities.

Following the success of such approaches we study some theoretical properties of this interaction when the superadditive function is the integer round down operation $\lfloor \cdot \rfloor$ and the nonlinear constraints are convex. Specifically we study the polyhedrality of the (first) Chvátal-Gomory (CG) closure [15, 26, 27, 41] of a non-polyhedral convex set. The study of properties of the CG closure of a rational polyhedron has yielded many well known results for linear IP. In this case, the closure is a rational polyhedron [41] for which the associated optimization, separation and membership problems are NP-hard even for restricted cases [11, 12, 21, 34]. However, optimization over the CG closure of a polyhedron has been successfully used to show its strength computationally [22, 23]. Similar results have also been obtained for closures associated to other valid inequalities such as split cuts [2, 7, 12, 17, 19, 20, 44].

CG cuts for non-polyhedral sets are considered implicitly in [15, 41] and explicitly in [14], but only [41] deals with the polyhedrality of the CG closure. Although [41] shows that for rational polyhedra the closure is a rational polyhedron, the result does not automatically extend to non-polyhedral sets. Furthermore, neither of the known proofs of the result for rational polyhedra [16, 41, 42] can be easily adapted to consider other convex sets. In fact, as noted in [41] even the polyhedrality of the CG closure of non-rational polytopes remains unknown. Because of this, we study the polyhedrality of the CG closure of an ellipsoid as the first natural step towards understanding the closure of other non-polyhedral convex sets.

Let a *rational ellipsoid* be the image of an Euclidean ball under a rational affine transformation. Our main result is to show that the CG closure of a full-dimensional bounded rational ellipsoid is a rational polytope. To the best of our knowledge, this is the first extension to a non-polyhedral set of the well known result for rational polyhedra. Additionally, the proof of our main result reveals some general sufficient conditions for the polyhedrality of the CG closure and other interesting properties. For example, we show that every non-integral point on the boundary of an ellipsoid can be separated by a CG cut. We recently verified [18] that this geometrically natural property holds for some other classes of convex sets.

The rest of the paper is organized as follows. In Section 2, we give some background on CG cuts, formally state the main result of the paper and present an outline of its proof. In Section 3, we present notation and review some standard results from convex analysis. In Section 4, we consider two separation results that are needed for the proof of the main theorem, which we present in Section 5. We end with some remarks in Section 6.

2 Background, Main Result and Proof Outline

For a polyhedron $P \subset \mathbb{R}^n$, the CG cutting plane procedure [15, 26, 27] can be described as follows. For an integer vector $a \in \mathbb{Z}^n$, let $d \in \mathbb{R}$ be such that $\{x \in \mathbb{R}^n : \langle a, x \rangle \leq d\} \supset P$ where $\langle u, v \rangle$ is the inner product between u and v. We then have that $P_I := P \cap \mathbb{Z}^n \subset \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \lfloor d \rfloor\}$ and hence the CG cut $\langle a, x \rangle \leq \lfloor d \rfloor$ is a valid inequality for $conv(P_I)$. The first CG closure P^1 of P is defined as the convex set obtained by adding all possible CG cuts to P. If P is a rational polyhedron, then P^1 is also a polyhedron [41] and hence we can recursively define the k-th CG closure P^k of P as the first CG closure of P^{k-1}. Furthermore, for any rational polyhedron P we have that there exists $k \in \mathbb{Z}_+$ such that $P^k = conv(P_I)$ [15, 41]. Non-rational polytopes are also considered in [15] and the CG procedure is extended to the feasible region of Conic Programming (CP) problems in [14]. In fact, the CG procedure can be extended to, at least, any compact convex set as follows.

Let $C \subset \mathbb{R}^n$ be a compact convex set and let $\sigma_C(a) := \sup_{x \in C} \langle a, x \rangle$ be its support function so that $C = \bigcap_{a \in \mathbb{R}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \sigma_C(a)\}$. Because $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ and $\sigma_C(a)$ is positively homogeneous and continuous, it can be verified that $C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \sigma_C(a)\}$.

Definition 1. *For any $S \subset \mathbb{Z}^n$, let $CC(S, C) := \bigcap_{a \in S} \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}$. We recursively define the k-th CG closure C^k of C as $C^1 := CC(\mathbb{Z}^n, C)$ and $C^{k+1} := CC(\mathbb{Z}^n, C^k)$ for all $k > 1$.*

The definition is consistent because C^1 is a closed convex set contained in C and when C is a polyhedron it coincides with the traditional definition. Furthermore, $C_I := C \cap \mathbb{Z} \subset C^k$ for all k and, as noted in [41], the following theorem follows from [15, 41].

Theorem 1 ([15, 41]). *There exist k such that $C^k = conv(C_I)$.*

Theorem 1 is also shown in [14] for CP problems with bounded feasible regions. However, the result neither implies nor requires the polyhedrality of C^1. In fact, the original proof of Theorem 1 in [15] does not use the polyhedrality of either P or P^1. Although surprising, it could be entirely possible for Theorem 1 to hold and for C^k to be the only polyhedron in the hierarchy $\{C^l\}_{l=1}^k$. Our main result is the first proof of the polyhedrality of C^1 for a non-polyhedral set C.

Theorem 2 (Main Theorem). *Let T be a full-dimensional bounded rational ellipsoid. Then $CC(\mathbb{Z}^n, T)$ is a rational polytope.*

Before presenting an outline of our proof of Theorem 2, we discuss why some of the well-known polyhedrality proofs and results do not easily extend to ellipsoids. We begin by noting that it is not clear how to extend the polyhedrality proofs in [16, 41, 42] beyond rational polyhedra because they rely on properties that are characteristic of these sets such as TDI systems and finite integral generating sets. However, we could try to prove Theorem 2 by using the polyhedrality of the first CG closure of polyhedral approximations of T. One natural scheme could be to attempt constructing a sequence of rational polytope pairs $\{P_i, Q_i\}_{i \in \mathbb{N}}$ such that (i) $P_i \cap \mathbb{Z}^n = Q_i \cap \mathbb{Z}^n = T \cap \mathbb{Z}^n$, (ii) $P_i \subset T \subset Q_i$ and (iii) $Vol(Q_i \setminus P_i) \leq 1/i$. We then would have that

$$P_i^k \subset T^k \subset Q_i^k, \tag{1}$$

for all $i, k \geq 1$. As noted in [41], using this approach Theorem 1 in general follows directly from Theorem 1 for rational polytopes. Unfortunately, it is not clear how to show that there exists i such that (1) holds as equality for $k = 1$ without knowing a priori that T^1 is a polyhedron. Finally, we note that cut domination arguments commonly used in polyhedrality proofs of closures do not seem to adapt well to the proof of Theorem 2.

Due of the reasons stated above, to prove Theorem 2 we resort to a different approach that relies on being able to separate with a CG cut every non-integral point on the boundary of T. Specifically, we show that $CC(\mathbb{Z}^n, T)$ can be generated with the procedure described in Figure 1.

Fig. 1. A procedure to generate the first CG closure for ellipsoid

Step 1 Construct a polytope Q defined by a finite number of CG cuts such that:
- $Q \subset T$.
- $Q \cap bd(T) \subset \mathbb{Z}^n$.

Step 2 Update Q with a CG cut that separates a point of $Q \setminus CC(\mathbb{Z}^n, T)$ until no such cut exists.

To show that Step 1 can be accomplished, we first show that every non-integral point on the boundary of T can be separated by a CG cut. If there are no integral points on the boundary of T, then this separation result allows us to cover the boundary of T with a possibly infinite number of open sets that are associated to the CG cuts. We then use compactness of the boundary of T to obtain a finite sub-cover that yields a finite number of CG cuts that separate every point on the boundary of T. If there are integer points on the boundary, then we use a stronger separation result and a similar argument to show that there is a finite set of CG cuts that separate every non-integral point on the boundary of T.

To show that Step 2 terminates finitely, we simply show that the set of CG cuts that separate at least one point in $Q \setminus CC(\mathbb{Z}^n, T)$ is finite.

We note that the separation of non-integral points using CG cuts on the boundary of T, required in Step 1 of Figure 1, is not straightforward. A natural first approach to separate a non-integral point u on the boundary of T is to take an inequality $\langle a, x\rangle \le \sigma_T(a)$ that is supporting at u, scale it so that $a \in \mathbb{Z}^n$, and then generate the CG cut $\langle a, x\rangle \le \lfloor \sigma_T(a)\rfloor$. If $\sigma_T(a) \notin \mathbb{Z}$, then the CG cut will separate u because a was selected such that $\langle a, u\rangle = \sigma_T(a)$. Unfortunately, as the following examples show, this approach can fail either because a cannot be scaled to be integral or because $\sigma_T(a) \in \mathbb{Z}$ for any scaling that yields $a \in \mathbb{Z}^n$.

Example 1. Let $T := \{x \in \mathbb{R}^2 \mid \sqrt{x_1^2 + x_2^2} \le 1\}$ and $u = (1/2, \sqrt{3}/2)^T \in bd(T)$. We have that the supporting inequality for u is $a_1x_1 + a_2x_2 \le \sigma_T(a)$ where $a = u$. Since u is irrational in only one component, observe that a cannot be scaled to be integral.

For Example 1, it is easy to see that selecting an alternative integer left-hand-side vector a' resolves the issue. We can use $a' = (1,1)$ which has $\sigma_T(a') = \sqrt{2}$ to obtain the CG cut $x_1 + x_2 \leq 1$. In Example 1 this CG cut separates every non-negative non-integral point on the boundary of T. In Section 4, we will show that given any non-integral point u on the boundary of T such that the left-hand-side of its supporting hyperplane cannot be scaled to be integral, there exists an alternative left-hand-side integer vector a' such that the CG cut $\langle a', x\rangle \leq \lfloor \sigma_T(a')\rfloor$ separates u. This vector a' will be systematically obtained using simultaneous diophantine approximation of the left-hand-side of an inequality describing the supporting hyperplane at u.

Example 2. Let $T := \{x \in \mathbb{R}^2 \,|\, \sqrt{x_1^2 + x_2^2} \leq 5\}$ and $u = (25/13, 60/13)^T \in bd(T)$. We have that the supporting inequality for u can be scaled to $a_1x_1 + a_2x_2 \leq \sigma_T(a)$ for $a = (5,12)^T$ which has $\sigma_T(a) = 65$. Because 5 and 12 are coprime and $\sigma_T(\cdot)$ is positively homogeneous, a cannot be scaled so that $a \in \mathbb{Z}^2$ and $\sigma_T(a) \notin \mathbb{Z}$.

Observe that Example 2 is not an isolated case. In fact, these cases are closely related to primitive Pythagorean triples. For $T := \{x \in \mathbb{R}^2 \,|\, \sqrt{x_1^2 + x_2^2} \leq r\}$, select any primitive Pythagorean triple $v_1^2 + v_2^2 = v_3^2$, and consider the point $r(\frac{v_1}{v_3}, \frac{v_2}{v_3})$ (such that $r(\frac{v_1}{v_3}, \frac{v_2}{v_3}) \notin \mathbb{Z}^2$). Then since v_1 and v_2 are coprimes, the behavior in Example 2 will be observed. Also note that these examples are not restricted only to Euclidean balls in $\mathbb{R}^2$, since it is easy to construct integers $a_1, ..., a_n, a_{n+1}$ such that $\sum_{i=1}^n a_i^2 = a_{n+1}^2$ (e.g. $3^2 + 4^2 + 12^2 = 13^2$). For the class of points $u \in bd(T)$ where the left-hand-side of an inequality describing the supporting hyperplane is scalable to an integer vector a, we will show in Section 4 that there exists a systematic method to obtain $a' \in \mathbb{Z}^n$ such that $\langle a', x\rangle \leq \lfloor \sigma_T(a')\rfloor$ separates u.

3 Notation and Standard Results from Convex Analysis

In this paper we consider an ellipsoid given by a non-singular and surjective rational linear transformation of an Euclidean ball followed by a rational translation. Without loss of generality, we may assume that that this ellipsoid is described as $T := \{x \in \mathbb{R}^n : \gamma_B(x - c) \leq 1\}$ where $c \in \mathbb{Q}^n$, and $\gamma_B(x) := \|Ax\|$ is the gauge of $B := \{x \in \mathbb{R}^n : \|Ax\| \leq 1\}$ such that $A \in \mathbb{Q}^{n\times n}$ is a symmetric positive definite matrix. Then T is the translation by c of B. The set B is a full dimensional compact convex set with the zero vector in its interior and hence has the following properties.

- The support function of B is $\sigma_B(a) = \|A^{-1}a\|$.
- The polar of B given by $B^\circ := \{a \in \mathbb{R}^n \,|\, \langle a, x\rangle \leq 1 \quad \forall x \in B\} = \{a \in \mathbb{R}^n \,|\, \sigma_B(a) \leq 1\}$ is a full-dimensional and compact convex set.
- For any $u \in bd(B)$ we have that $s_B(u) := A^T A(u)$ is such that $\langle s_B(u), u\rangle = \sigma_B(s_B(u)) = 1$ and hence $\langle s_B(u), x\rangle \leq 1 = \sigma_B(s_B(u))$ is a valid inequality for B that is supporting at u.

– $\langle a, x\rangle \le \sigma_B(a)\gamma_B(x)$.
– The boundary of B is $bd(B) := \{x \in \mathbb{R}^n : \gamma_B(x) = 1\}$.

Because $T = B + c$ we also have the following properties of T.

– The support function of T is $\sigma_T(a) = \sigma_{B+c}(a) = \sigma_B(a) + \langle a, c\rangle = \|A^{-1}a\| + \langle a, c\rangle$.
– For any $u \in bd(T)$ we have that $s_T(u) := s_B(u - c) = A^T A(u - c)$ is such that $\langle s_T(u), u - c\rangle = \langle s_B(u - c), u - c\rangle = \sigma_B(s_B(u - c)) = \sigma_B(s(u)) = 1$ and hence $\langle s(u), x\rangle \le 1 + \langle s(u), c\rangle = \sigma_T(s(u))$ is a valid inequality for T that is supporting at u.
– The boundary of T is $bd(T) := \{x \in \mathbb{R}^n : \gamma_B(x - c) = 1\}$.

To simplify the notation, we regularly drop the T from $\sigma_T(\cdot)$, $s_T(\cdot)$ and $CC(\cdot, T)$ so that $\sigma(\cdot) := \sigma_T(\cdot)$, $s(\cdot) := s_T(\cdot)$ and $CC(\cdot) := CC(\cdot, T)$. In addition, for $u \in \mathbb{R}$ we denote its fractional part by $F(u) := u - \lfloor u\rfloor$.

4 Separation

To prove Theorem 2 we need two separation results. The first one simply states that every non-integral point on the boundary of T can be separated by a CG cut.

Proposition 1. *If $u \in bd(T) \setminus \mathbb{Z}^n$, then there exists a CG cut that separates point u.*

An integer point $u \in bd(T) \cap \mathbb{Z}^n$ cannot be separated by a CG cut, but Proposition 1 states that every point in $bd(T)$ that is close enough to u will be separated by a CG cut. However, for the compactness argument to work we need a stronger separation result for points on the boundary that are close to integral boundary points. This second result states that all points in $bd(T)$ that are sufficiently close to an integral boundary point can be separated by a finite number of CG cuts.

Proposition 2. *Let $u \in bd(T) \cap \mathbb{Z}^n$. Then there exists $\varepsilon_u > 0$ and a finite set $W_u \subset \mathbb{Z}^n$ such that*

$$\langle w, u\rangle = \lfloor\sigma(w)\rfloor \quad \forall w \in W_u, \tag{2}$$

$$\forall v \in bd(T) \cap \{x \in \mathbb{R}^n : \|x - u\| < \varepsilon_u\} \setminus \{u\} \quad \exists w \in W_u \ s.t\ \langle w, v\rangle > \lfloor\sigma(w)\rfloor, \tag{3}$$

and

$$\forall v \in int(T) \quad \exists w \in W_u \ s.t.\ \langle w, v\rangle < \lfloor\sigma(w)\rfloor. \tag{4}$$

The main ideas used in the proof of Proposition 2 are as follows. First, it is verified that for any nonzero integer vector q, there exists a finite $i \in \mathbb{Z}_+$ such that the CG cut of the form $\langle q + i\lambda s(u), x\rangle \le \lfloor\sigma(q + i\lambda s(u))\rfloor$ satisfies (2) (here $\lambda s(u) \in \mathbb{Z}^n$ for some scalar $\lambda \neq 0$). Second, it is verified that by carefully selecting a finite number of integer vectors and applying the above construction, all points in a sufficiently small neighborhood of u can be separated. Finally, (4) is established by adding the supporting hyperplane at u which is trivially a CG cut.

Although this proof of Proposition 2 is similar to the proof of Proposition 1, it is significantly more technical. We therefore refer the readers to [18] where a more general version of Proposition 2 is proven and confine our discussion to an outline of the proof of Proposition 1 here.

4.1 Outline of Proof of Proposition 1

To prove Proposition 1 we construct a separating CG cut for $u \in bd(T) \setminus \mathbb{Z}^n$ by modifying the supporting inequality for T at u. In the simplest case, we scale $\langle s(u), x\rangle \le \sigma(s(u))$ by $\lambda > 0$ so that $\lambda s(u) \in \mathbb{Z}^n$, to obtain a CG cut $\langle \lambda s(u), x\rangle \le \lfloor \sigma(\lambda s(u))\rfloor$ that separates u. If this is not successful, then we approximate the direction $s(u)$ by a sequence $\{s^i\}_{i\in\mathbb{N}} \subset \mathbb{Z}^n$ such that $s^i/\|s^i\| \to s(u)/\|s(u)\|$ and for which $\langle s^i, x\rangle \le \lfloor \sigma(s^i)\rfloor$ separates u for sufficiently large i. For this approach to work we will need a sequence that complies with the following two properties.

C1 $\lim_{i\to+\infty} \langle s^i, u\rangle - \sigma(s^i) = 0$
C2 $\lim_{i\to+\infty} F(\sigma(s^i)) = \delta > 0$. (A weaker condition like $\limsup_{i\to+\infty} F(\sigma(s^i)) > 0$ is sufficient, but we will verify the stronger condition).

Neither condition holds for every sequence such that $s^i/\|s^i\| \to s(u)/\|s(u)\|$. For instance, for $s(u) = (0,1)^T$ the sequence $s^i = (k, k^2)$ does not comply with condition C1. For these reason we need the following proposition.

Proposition 3. *Let $u \in bd(T) \setminus \mathbb{Z}^n$ and let e^l be the l-th unit vector for some $l \in \{1,\ldots,n\}$ such that $u_l \notin \mathbb{Z}$.*

(a) If there exists $\lambda > 0$ such that $p := \lambda s(u) \in \mathbb{Z}^n$ and $\sigma(\lambda s(u)) \in \mathbb{Z}$, then $s^i := e^l + ip$ complies with conditions C1 and C2.
(b) If $\lambda s(u) \notin \mathbb{Z}^n$ for all $\lambda > 0$, then let $\{(p^i, q_i)\}_{i\in\mathbb{N}} \subset \mathbb{Z}^n \times (\mathbb{Z}_+ \setminus \{0\})$ be the coefficients obtained using Dirichlet's Theorem to approximate $s(u)$. That is $\{(p^i, q_i)\}_{i\in\mathbb{N}}$ is such that

$$|q_i s(u)_j - p^i_j| < \frac{1}{i} \ \forall j \in \{1,...,n\}.$$

For $M \in \mathbb{Z}_+$ such that $Mc \in \mathbb{Z}^n$ we have that $s^i := e^l + Mp^i$ complies with conditions C1 and C2.

With this proposition we can proceed to the proof of Proposition 1

Proof (Proof of Proposition 1). Let $u \in bd(T) \setminus \mathbb{Z}^n$. There are three possible cases:

1. There exists $\lambda > 0$ such that $\lambda s(u) \in \mathbb{Z}^n$ and $\sigma(\lambda s(u)) \notin \mathbb{Z}$.
2. There exists $\lambda > 0$ such that $\lambda s(u) \in \mathbb{Z}^n$ and $\sigma(\lambda s(u)) \in \mathbb{Z}$.
3. $\lambda s(u) \notin \mathbb{Z}^n$ for all $\lambda > 0$.

Case 1: $\langle \lambda s(u), x\rangle \leq \lfloor \sigma(\lambda s(u))\rfloor$ is a CG cut that separates u.
Cases 2 and 3: From Proposition 3, we have that in both cases there exists a sequence $\{s^i\}_{i\in\mathbb{N}} \subset \mathbb{Z}^n$ satisfying conditions C1 and C2. Together with

$$\langle s^i, u\rangle - \lfloor \sigma(s^i)\rfloor = \langle s^i, u\rangle - \sigma(s^i) + F(\sigma(s^i)), \tag{5}$$

conditions C1 and C2 yields that for sufficiently large i we have $\langle s^i, u\rangle - \lfloor \sigma(s^i)\rfloor > 0$ and hence $\langle s^i, x\rangle \leq \lfloor \sigma(s^i)\rfloor$ separates u.

We next discuss the proof of Proposition 3 in the next two subsections.

Condition C1 in Proposition 3 Condition C1 is not difficult to satisfy. In fact, it is satisfied by any sequence for which the angle between s^i and s converges fast enough (e.g. if $\|s^i\| \to +\infty$, then C1 is satisfied if we have that $\left\|(s^i/\|s^i\|) - (s(u)/\|s(u)\|)\right\| \in o(1/\|s^i\|)$). For the specific sequences in Proposition 3 (a) and 3 (b) condition C1 can be verified using properties from Section 3 and the following lemma which we do not prove here.

Lemma 1. *Let $w \in \mathbb{R}^n$ and $\{v^i\}_{i\in\mathbb{N}} \subset \mathbb{R}^n$ be any sequence such that there exists $\mathcal{N} > 0$ for which*

$$|v^i_j w_k - v^i_k w_j| < \mathcal{N} \quad \forall i \in \mathbb{N},\, j,k \in \{1,...,n\},\, j \neq k \tag{6}$$

and $\lim_{i\to+\infty}\langle v^i, w\rangle = +\infty$. *Then*

$$\lim_{i\to+\infty} \langle v^i, w\rangle - \|v^i\|\|w\| = 0.$$

Condition C2 in Proposition 3 Condition C2 is much more interesting and showing that it holds for our specific sequences is the crux of the proof of Proposition 3. The intuition behind the proof is the following: For the sequence in Proposition 3 (a) we have $s^i = e^l + ip$. For large enough i, $\sigma(s^i) \approx \langle e^l + ip, u\rangle = u_l + i\langle \lambda s(u), u\rangle = u_l + i\sigma(\lambda s(u))$. Now since $\sigma(\lambda s(u))$ is integral, the fractional part of $\sigma(s^i)$ is therefore approximately equal to u_l. The formal proof is presented next. We first present a simple lemma.

Lemma 2. *Let $\alpha \in \mathbb{R}$, $t \in \mathbb{R}_+$ and $\{\beta_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ be such that* $\lim_{i\to\infty}\beta_i = \infty$. *Then, for every $\varepsilon > 0$ there exists N_ε such that*

$$\alpha + \beta_i \leq \sqrt{(\alpha+\beta_i)^2 + t} \leq \alpha + \beta_i + \varepsilon \quad \forall i \geq N_\varepsilon$$

Lemma 3. *The sequence in Proposition 3 (a) satisfies Condition C2.*

Proof. Let $\alpha = \langle A^{-1}e^l, A^{-1}p\rangle/\|A^{-1}p\|$, $\beta_i = i\|A^{-1}p\|$ and $t = \|A^{-1}e^l\|^2 - (\langle A^{-1}r^l, A^{-1}p\rangle/\|A^{-1}p\|)^2$. We have that $\lim_{i\to\infty}\beta_i = \infty$ because $\|A^{-1}p\| > 0$ and $t \geq 0$ by Cauchy-Schwarz inequality. Observe that,

$$\begin{aligned}
\|A^{-1}s^i\| &= \sqrt{i^2\|A^{-1}p\|^2 + 2i\langle A^{-1}e^l, A^{-1}p\rangle + \|A^{-1}e^l\|^2} \\
&= \sqrt{\left(\frac{\langle A^{-1}e^l, A^{-1}p\rangle}{\|A^{-1}p\|} + i\|A^{-1}p\|\right)^2 + \|A^{-1}e^l\|^2 - \left(\frac{\langle A^{-1}e^l, A^{-1}p\rangle}{\|A^{-1}p\|}\right)^2} \\
&= \sqrt{(\alpha+\beta_i)^2 + t}.
\end{aligned}$$

Then, by Lemma 2, we have that

$$\begin{aligned}
\sigma(s^i) &= \sqrt{(\alpha+\beta_i)^2+t} + \langle c, s^i\rangle \\
&\geq \frac{\langle A^{-1}e^l, A^{-1}p\rangle}{\|A^{-1}p\|} + i\|A^{-1}p\| + \langle c, e^l + ip\rangle \\
&= i\sigma(p) + \left\lfloor \frac{\langle A^{-1}e^l, A^{-1}p\rangle}{\|A^{-1}p\|} + \langle c, e^l\rangle \right\rfloor + F\left(\frac{\langle A^{-1}e^l, A^{-1}p\rangle}{\|A^{-1}p\|} + \langle c, e^l\rangle\right)
\end{aligned}$$

and, similarly, for $i \geq N_\varepsilon$ we have

$$\sigma(s^i) \leq i\sigma(p) + \left\lfloor \frac{\langle A^{-1}e^l, A^{-1}p\rangle}{\|A^{-1}p\|} + \langle c, e^l\rangle \right\rfloor + F\left(\frac{\langle A^{-1}e^l, A^{-1}p\rangle}{\|A^{-1}p\|} + \langle c, e^l\rangle\right) + \varepsilon.$$

Hence, by setting $k := i\sigma(p) + \lfloor\langle A^{-1}e^l, A^{-1}p\rangle/\|A^{-1}p\| + \langle c, e^l\rangle\rfloor$ and $\delta := F(\langle A^{-1}e^l, A^{-1}p\rangle/\|A^{-1}p\| + \langle c, e^l\rangle)$ we have that

$$k + \delta \leq \sigma(s^i) \leq k + \delta + \varepsilon \quad \forall i \geq N_\varepsilon. \tag{7}$$

Now, $k \in \mathbb{Z}$ and

$$\begin{aligned}
\delta &= F(\langle A^{-1}e^l, A^{-1}p\rangle/\|A^{-1}p\| + \langle c, e^l\rangle) \\
&= F(\langle A^{-1}e^l, \lambda A(u-c)\rangle/\lambda + \langle c, e^l\rangle) \\
&= F(\langle e^l, A^{-1}\lambda A(u-c)\rangle/\lambda + \langle c, e^l\rangle) \\
&= F(\langle e^l, u\rangle) = F(u_l) \in (0,1),
\end{aligned}$$

because $u_l \notin \mathbb{Z}$ and because $p = \lambda s(u)$ implies $\|A^{-1}p\| = \lambda\|A^{-1}s(u)\| = \lambda$. Thus $\lim_{i\to+\infty} F(\sigma(s^i)) = \delta > 0$.

Lemma 4. *The sequence in Proposition 3 (b) satisfies Condition C2.*

Proof. Let $\bar{\epsilon}^i := p^i - q_i s(u)$, so that $\|\bar{\epsilon}^i\| \leq \frac{\sqrt{n}}{i}$. We then have that

$$\lim_{i\to+\infty} \|A^{-1}\bar{\epsilon}^i\| = \lim_{i\to+\infty} \|A^{-1}(-\bar{\epsilon}^i)\| = 0. \tag{8}$$

Now observe that

$$\begin{aligned}
\|A^{-1}s^i\| &= \|A^{-1}(Mp^i + e^l)\| \\
&= \|Mq_iA^{-1}s(u) + A^{-1}e^l + MA^{-1}\bar{\epsilon}^i\| \\
&\leq \|Mq_iA^{-1}s(u) + A^{-1}e^l\| + M\|A^{-1}\bar{\epsilon}^i\| \\
&= \sqrt{\left(\frac{\langle A^{-1}s(u), A^{-1}e^l\rangle}{\|A^{-1}s(u)\|} + Mq_i\right)^2 + t} + M\|A^{-1}\bar{\epsilon}^i\|.
\end{aligned}$$

where $t := \|A^{-1}e^l\|^2 - \left(\frac{\langle A^{-1}s(u), A^{-1}e^l\rangle}{\|A^{-1}s(u)\|}\right)^2$. Since $\|A^{-1}s(u)\| = \|A(u-c)\| = 1$, $t = \|A^{-1}e^l\|^2 - \langle A(u-c), A^{-1}e^l\rangle^2$ which is non-negative by the Chauchy-Schwartz inequality. Therefore by setting $\alpha := \frac{\langle A^{-1}s(u), A^{-1}e^l\rangle}{\|A^{-1}s(u)\|}$, $\beta_i := Mq_i$ we can use Lemma 2 and the fact that $\|A^{-1}s(u)\| = \|A(u-c)\| = 1$ to obtain that for every $\varepsilon > 0$ there exists N_ε such that

$$\|A^{-1}s^i\| \leq Mq_i + \langle A(u-c), A^{-1}e^l\rangle + \varepsilon + M\|A^{-1}\bar{\epsilon}^i\| \quad \forall i \geq N_\varepsilon. \tag{9}$$

Similarly, we also have that

$$\begin{aligned}
\|A^{-1}s^i\| &= \|A^{-1}(Mp^i + e^l)\| \\
&\geq \|MA^{-1}p^i + A^{-1}e^l - MA^{-1}\bar{\epsilon}^i\| - M\|A^{-1}(-\bar{\epsilon}^i)\| \\
&= \|Mq_iA^{-1}s(u) + A^{-1}e^l\| - M\|A^{-1}(-\bar{\epsilon}^i)\| \\
&= \sqrt{\left(\frac{\langle A^{-1}s(u), A^{-1}e^l\rangle}{\|A^{-1}s(u)\|} + Mq_i\right)^2 + t} - M\|A^{-1}(-\bar{\epsilon}^i)\| \\
&\geq Mq_i + \langle A(u-c), A^{-1}e^l\rangle - M\|A^{-1}(-\bar{\epsilon}^i)\| \\
&= Mq_i + \langle u-c, e^l\rangle - \|A^{-1}(-\bar{\epsilon}^i)\|.
\end{aligned} \tag{10}$$

Combining (9) and (10) and using (8) and the definition of $\sigma(\cdot)$ we obtain that for every $\tilde{\varepsilon} > 0$ there exists $N_{\tilde{\varepsilon}}$ such that

$$Mq_i + \langle p^i, Mc\rangle + \langle u, e^l\rangle - \tilde{\varepsilon} \leq \sigma(s^i) \leq Mq_i + \langle p^i, Mc\rangle + \langle u, e^l\rangle + \tilde{\varepsilon} \tag{11}$$

holds for all $i \geq N_{\tilde{\varepsilon}}$. Noting that $Mq_i + \langle p^i, Mc\rangle \in \mathbb{Z}$ for all i we obtain that $\lim_{i\to+\infty} F(\sigma(s^i)) = \langle u, e^l\rangle > 0$.

5 Proof of Main Theorem

To prove Theorem 2, we first verify that Step 2 in Figure 1 can be accomplished using a finite number of CG cuts.

Proposition 4. *If there exists a finite set $S \subset \mathbb{Z}^n$ such that*

$$CC(S,T) \subset T \tag{12a}$$

$$CC(S,T) \cap bd(T) \subset \mathbb{Z}^n, \tag{12b}$$

then $CC(\mathbb{Z}^n, T)$ is a rational polytope.

Proof. Let V be the set of vertices of $CC(S)$. By (12) we have that $bd(T) \cap V \subset \mathbb{Z}^n \cap T \subset CC(\mathbb{Z}^n)$. Hence any CG cut that separates $u \in CC(S) \setminus CC(\mathbb{Z}^n)$ must also separate a point in $V \setminus bd(T)$. It is then sufficient to show that the set of CG cuts that separates some point in $V \setminus bd(T)$ is finite. To achieve this we will use the fact that, because $V \setminus bd(T) \subset T \setminus bd(T)$ and $|V| < \infty$, there exists $1 > \varepsilon > 0$ such that

$$\gamma_B(v-c) \leq 1 - \varepsilon \quad \forall v \in V \setminus bd(T). \tag{13}$$

Now, if a CG cut $\langle a, x\rangle \le \lfloor \sigma(a)\rfloor$ for $a \in \mathbb{Z}^n$ separates $v \in V \setminus bd(T)$, then

$$\begin{aligned}
& \langle a, v\rangle > \lfloor \sigma(a)\rfloor && (14)\\
\Rightarrow\quad & \langle a, v\rangle > \sigma(a) - 1 && (15)\\
\Rightarrow\quad & \langle a, v\rangle > \sigma_B(a) + \langle a, c\rangle - 1 && (16)\\
\Rightarrow\quad & \sigma_B(a)\gamma_B(v-c) \ge \langle a, v-c\rangle > \sigma_B(a) - 1 && (17)\\
\Rightarrow\quad & \sigma_B(a) < \frac{1}{1-\gamma_B(v-c)} \le 1/\varepsilon && (18)\\
\Rightarrow\quad & a \in (1/\varepsilon)B^\circ. && (19)
\end{aligned}$$

The result follows from the fact that $(1/\varepsilon)B^\circ$ is a bounded set.

The separation results from Section 4 allows the construction of the set required in Proposition 4, which proves our main result.

Proof (Proof of Theorem 2). Let $\mathcal{I} := bd(T) \cap \mathbb{Z}^n$ be the finite (and possibly empty) set of integer points on the boundary of T. We divide the proof into the following cases

1. $CC(\mathbb{Z}^n) = \emptyset$.
2. $CC(\mathbb{Z}^n) \neq \emptyset$ and $CC(\mathbb{Z}^n) \cap int(T) = \emptyset$.
3. $CC(\mathbb{Z}^n) \cap int(T) \neq \emptyset$.

For the first case, the result follows directly. For the second case, by Proposition 1 and the strict convexity of T, we have that $|\mathcal{I}| = 1$ and $CC(\mathbb{Z}^n) = \mathcal{I}$ so the result again follows directly. For the third case we show the existence of a set S complying with conditions (12) presented in Proposition 4.

For each $u \in \mathcal{I}$, let $\varepsilon_u > 0$ be the value from Proposition 2. Let $\mathcal{D} := bd(T) \setminus \bigcup_{u\in\mathcal{I}}\{x \in \mathbb{R}^n : \|x-u\| < \varepsilon_u\}$. Observe that $\mathcal{D} \cap \mathbb{Z}^n = \emptyset$ by construction and that $\mathcal{D}$ is compact because it is obtained from compact set $bd(T)$ by removing a finite number of open sets. Now, for any $a \in \mathbb{Z}^n$ let $O(a) := \{x \in bd(T) \mid \langle a, x\rangle > \lfloor \sigma(a)\rfloor\}$ be the set of points of $bd(T)$ that are separated by the CG cut $\langle a, x\rangle \le \lfloor \sigma(a)\rfloor$. This set is open with respect to $\mathcal{D}$. Furthermore, by Proposition 1 and the construction of $\mathcal{D}$, we have that $\mathcal{D} \subset \bigcup_{a\in\mathcal{A}} O(a)$ for a possibly infinite set $\mathcal{A} \subset \mathbb{Z}^n$. However, since $\mathcal{D}$ is a compact set we have that there exists a finite subset $\mathcal{A}_0 \subset \mathcal{A}$ such that

$$\mathcal{D} \subset \bigcup_{a\in\mathcal{A}_0} O(a). \tag{20}$$

Let $S := \mathcal{A}_0 \cup \bigcup_{u\in\mathcal{I}} W_u$ where, for each $u \in \mathcal{I}$, W_u is the set from Proposition 2. Then by (20) and Proposition 2 we have that S is a finite set that complies with condition (12b).

To show that S complies with condition (12a) we will show that if $p \notin T$, then $p \notin CC(S,T)$. To achieve this, we use the fact that $CC(\mathbb{Z}^n) \cap int(T) \neq \emptyset$. Let $\tilde{c} \in CC(\mathbb{Z}^n) \cap int(T)$, $\tilde{B} = B + c - \tilde{c}$ and $\gamma_{\tilde{B}}(x) = \inf\{\lambda > 0 : x \in \lambda\tilde{B}\}$ be the gauge of $\tilde{B}$. Then $\tilde{B}$ is a convex body with $0 \in int(\tilde{B})$, $T = \{x \in \mathbb{R}^n : \gamma_{\tilde{B}}(x-\tilde{c}) \le 1\}$ and $bd(T) = \{x \in \mathbb{R}^n : \gamma_{\tilde{B}}(x-\tilde{c}) = 1\}$. Now, for $p \notin T$, let $\bar{p} := \tilde{c} + (p - \tilde{c})/\gamma_{\tilde{B}}(p - \tilde{c})$ so that $\bar{p} \in \{\mu\tilde{c} + (1-\mu)p \,:\, \mu \in (0,1)\}$ and $\bar{p} \in bd(T)$. If $\bar{p} \notin \mathbb{Z}^n$, then by the definitions of S and $\tilde{c}$ we have that there exists $a \in S$ such that $\langle a, \tilde{c}\rangle \le \lfloor\sigma(a)\rfloor$ and $\langle a, \bar{p}\rangle > \lfloor\sigma(a)\rfloor$. Then $\langle a, p\rangle > \lfloor\sigma(a)\rfloor$ and hence $p \notin CC(S,T)$. If $\bar{p} \in \mathbb{Z}^n$ let $w \in W_{\bar{p}}$ be such that $\langle w, \tilde{c}\rangle < \lfloor\sigma(w)\rfloor$ and $\langle w, \bar{p}\rangle = \lfloor w\rfloor$. Then $\langle w, p\rangle > \lfloor\sigma(w)\rfloor$ and hence $p \notin CC(S,T)$.

6 Remarks

We note that the proof of Proposition 4 only uses the fact that T is a convex body and Theorem 2 uses the fact that T is additionally an ellipsoid only through Proposition 1 and Proposition 2. Therefore, we have the following general sufficient conditions for the polyhedrality of the first CG closure of a compact convex set.

Corollary 1. *Let T be any compact convex set. $CC(\mathbb{Z}^n, T)$ is a rational polyhedron if any of the following conditions hold*

Property 1 *There exists a finite $S \subset \mathbb{Z}^n$ such that* (12) *holds.*
Property 2 *For any $u \in bd(T) \setminus \mathbb{Z}^n$ there exists a CG cut that separates u and for any $u \in bd(T) \cap \mathbb{Z}^n$ there exist $\varepsilon_u > 0$ and a finite set $W_u \subset \mathbb{Z}^n$ such that* (2)–(4) *hold.*

A condition similar to (12) was considered in [41] for polytopes that are not necessarily rational. Specifically the author stated that if P is a polytope in real space such that $CC(\mathbb{Z}^n, P) \cap bd(P) = \emptyset$, then $CC(\mathbb{Z}^n, P)$ is a rational polytope. We imagine that the proof he had in mind could have been something along the lines of Proposition 4.

We also note that Step 2 of the procedure described in Section 2 can be directly turned into a finitely terminating algorithm by simple enumeration. However, it is not clear how to obtain a finitely terminating algorithmic version of Step 1 because it requires obtaining a finite subcover of the boundary of T from a quite complicated infinite cover.

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