The Chvátal-Gomory Closure of a Strictly Convex Body

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In this paper, we prove that the Chvátal-Gomory closure of a set obtained as an intersection of a strictly convex body and a rational polyhedron is a polyhedron. Thus, we generalize a result of Schrijver[29] which shows that the Chvátal-Gomory closure of a rational polyhedron is a polyhedron.

1. Introduction   A cutting plane, also known as a cut, is typically a linear inequality that separates fractional points from the convex hull of integer feasible solutions of an Integer Programming (IP) problem. Cutting planes have proven to be crucial in the development of successful IP solver technology. See [21, 25, 26, 27] for general expositions on cutting plane methods.

Chvátal-Gomory (CG) cuts are one of the first classes of cutting planes presented in the literature [14]. They have been at the heart of various fundamental theoretical and computational breakthroughs in IP. For example, Gomory [14] introduced CG cuts to present the first finite cutting plane algorithm for bounded IP problems. CG cuts can be used to obtain the convex hull of integer feasible solutions of some sets such as the Matching Polytope, as shown by Edmonds [12], which is a pioneering result in the area of polyhedral combinatorics.

For a rational polyhedron $P \subseteq \mathbb{R}^n$, the CG cutting plane procedure [9, 14, 15] can be described as follows. For $a \in \mathbb{Z}^n$, let $d \in \mathbb{R}$ be such that $\{x \in \mathbb{R}^n : \langle a, x \rangle \leq d\} \supset P$ where $\langle u, v \rangle$ is the inner product between u and v. We then have that $P_I := P \cap \mathbb{Z}^n \subseteq \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \lfloor d \rfloor\}$ and hence the CG cut $\langle a, x \rangle \leq \lfloor d \rfloor$ is a valid inequality for $\mathrm{conv}(P_I)$. The first CG closure of P is defined as the convex set obtained by adding all CG cuts. Because the number of CG cuts is infinite we have that the CG closure is not automatically a polyhedron. The first proof of the polyhedrality of the CG closure was introduced by Schrijver in 1979.

THEOREM 1.1 ([29]) *The CG closure of a rational polyhedron is a rational polyhedron.*

Convex Nonlinear Integer Programming, i.e. problems where the continuous relaxation of the feasible region is a non-polyhedral convex set, has received considerable attention from the IP community recently. There has been significant progress in the development of practical algorithms that can be effective for many important applications (e.g. [1, 5, 6, 19, 24]). Building on work for linear IP, practical algorithms for convex nonlinear IP have benefited from the development of several classes of cutting planes or valid inequalities (e.g. [2, 3, 4, 7, 8, 13, 17, 18, 23, 16, 28, 31]). Many of these inequalities are based on the generalization of ideas used in linear IP.

One particular idea for generating cutting planes for convex nonlinear IP that has been motivated by the linear case is that of CG cuts. CG cuts for general convex IP were discussed implicitly in [29] and described explicitly in [8] for convex IP problems where the continuous relaxation of the feasible region is conic representable. CG cuts can be extended to the case of a general convex set $C \subseteq \mathbb{R}^n$ using its support function $\sigma_C(a) := \sup_{x \in C} \langle a, x \rangle$. A valid cut for $\text{conv}(C \cap \mathbb{Z}^n)$ is $\langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor$ where a is an integral vector. Similar to the case of rational polyhedra, the CG closure for C is the convex set obtained by adding all CG cuts.

To the best of our knowledge, the only version of Theorem 1.1 for the case where C is not a rational polyhedron was shown in [11]. It was shown in [11] that when C is a full-dimensional bounded ellipsoid described by rational data the CG closure is a polytope. A set C is called strictly convex when the strict convex combination of any two points belonging to C lies in the relative interior of C. In this paper, we will verify Theorem 1.1 for the case where C is either a strictly convex body (full dimensional compact strictly convex set) or an intersection of a strictly convex body and a rational polyhedron. The first result generalizes the result in [11] and the second result effectively generalizes Theorem 1.1.

We observe here that while various proofs of Theorem 1.1 have been presented, unfortunately none of them seem to extend to the case of non-polyhedral convex sets. For example, it not clear how to extend the proofs in [10, 29, 30] beyond rational polyhedra because they rely on properties that are characteristic to these sets such as totally dual integral systems and finite integral generating sets. Cut domination arguments, commonly used in the polyhedrality proofs, also do not seem to adapt well to the non-polyhedral setting. Note that one key property of the CG closure for rational polyhedra is that the CG closure of a facet of a rational polyhedron is equal to the intersection of the facet with the CG closure of the polyhedron. This property together with an induction on the dimension of the polyhedron can be used to prove the polyhedrality of the CG closure of a rational polyhedron. While we will prove a similar statement regarding the zero dimensional faces of a strictly convex set, we will need to develop completely new techniques to do so. Moreover in the case of strictly convex sets there are an infinite number of facets, thus requiring a different approach than used in the case of rational polyhedron. Finally we note that our polyhedrality proof for the case of a strictly convex body intersected with a rational polyhedron will in parts use the ideas developed for rational polyhedra.

Instead of attempting to use the proof techniques for rational polyhedra, another possibility for proving the polyhedrality of the CG closure of a strictly convex body C is to directly use the polyhedrality of the first CG closure of rational polyhedral approximations of C. One natural scheme would be to try and construct a sequence of rational polytope pairs $\{P_i, Q_i\}_{i \in \mathbb{N}}$ such that (i) $P_i \cap \mathbb{Z}^n = Q_i \cap \mathbb{Z}^n = C \cap \mathbb{Z}^n$, (ii) $P_i \subseteq C \subseteq Q_i$ and (iii) $\text{vol}(Q_i \setminus P_i) \leq 1/i$. We would then have that that CG closure of P_i is a subset of CG closure of C which in turn is in the CG closure of Q_i for any i. Unfortunately, it is not clear how to show that there exists i such that CG closure of P_i is equal to the CG closure of Q_i and hence equal to the CG closure of C.

We note that strictly convex sets are completely 'rounded' without any flat faces of dimension greater than 0, which is completely in contrast with polyhedra. Thus, in a sense, the polyhedrality of the CG closure of strictly convex sets represents a result which is on the other end of the spectrum with respect to the polyhedrality result for the CG closure of rational polytopes. While the result on the intersection of strictly convex sets and rational polyhedra does fill the void slightly, we believe that a whole new set of methodologies and insights need to be developed to understand the structure of the CG closure of a general convex set.

The rest of the paper is organized as follows. In Section 2 we give some background material, formally state our main results and give an overview of their proofs. Then in Section 3 we study the separation of points in the boundary of strictly convex sets using CG cuts. The separation results in this section are

key to the proofs of our main results. Section 4 contains the proof of the polyhedrality of the CG closure for strictly convex bodies and Section 5 does the same for the intersection of a strictly convex body and a rational polyhedron.

We follow most of the notational conventions of [20], which we will use as a reference for all convex analysis results. However, for completeness, the appendix includes proofs of any result that does not explicitly appear in [20].

2. Background and Proof Outline   Before stating our main result we review some definitions and known results.

DEFINITION 2.1   (*i*) *For $C \subseteq \mathbb{R}^n$, we define the following:*

- *We denote the closure of C by $\mathrm{cl}(C)$, its boundary by C by $\mathrm{bd}(C)$, its interior by $\mathrm{int}(C)$, its affine hull by $\mathrm{aff}(C)$, relative interior by $\mathrm{rel.int}(C)$, and its convex hull by $\mathrm{conv}(C)$.*
- *We say that C is a convex body if it is convex, full dimensional and compact. C is strictly convex if for all $u, v \in C$, $u \neq v$ we have that $\lambda u + (1 - \lambda)v \in \mathrm{rel.int}(C)$ for all $0 < \lambda < 1$. Similarly C is a strictly convex body if C is strictly convex and a convex body.*
- *For C convex, $u \in C$, we denote the tangent and normal cones to C at u by $T_C(u) = \mathrm{cl}\left(\{v \in \mathbb{R}^n : \exists\ \varepsilon > 0 \text{ s.t. } u + \varepsilon v \in C\}\right)$ and $N_C(u) = \{v \in \mathbb{R}^n : \langle v, u \rangle = \sigma_C(v)\}$ respectively.*

(*ii*) *We denote the set $\{x \in \mathbb{R}^n : \|x\| = 1\}$ as S^{n-1}, where $\|\cdot\|$ is the Euclidean norm.*

(*iii*) *Given a vector $v \in \mathbb{R}^n \setminus \{0\}$, we denote $\frac{v}{\|v\|}$ as $\overline{v}$.*

(*iv*) *For notational convenience, for an invertible matrix $U \in \mathbb{R}^{n \times n}$ we write $U^{-T} := U^{-1^T}$.*

To formally define the CG closure of a closed convex set it is useful to use the following characterization.

PROPOSITION 2.1 *Let C be a compact convex set and $\sigma_C(\cdot)$ be its support function. Then*

$$C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n\ :\ \langle a, x \rangle \leq \sigma_C(a)\}. \tag{1}$$

This is the standard outer description of a closed convex set (e.g. Theorem C.2.2.2 in [20]) with the exception that we take an intersection over $a \in \mathbb{Z}^n$ instead to the usual $a \in \mathbb{R}^n$ or $a \in S^{n-1}$. The validity of this alternative representation for compact convex sets is straightforward, but for completeness Proposition 2.1 is proven in the Appendix.

DEFINITION 2.2 *Let C be a compact convex set. For any set $S \subseteq \mathbb{Z}^n$ let*

$$\mathrm{CGC}_S(C) = \bigcap_{a \in S} \{x \in \mathbb{R}^n\ :\ \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\} \tag{2}$$

Let $\mathrm{CGC}(C) = \mathrm{CGC}_{\mathbb{Z}^n}(C)$, we recursively define the k-th CG closure C^k of C as $C^0 := C$ and $C^k := \mathrm{CGC}(C^{k-1})$ for $k \geq 1$.

$\mathrm{CGC}_S(C)$ is a closed convex set containing $C_I := C \cap \mathbb{Z}^n$ for any $S \subseteq \mathbb{Z}^n$ and by Proposition 2.1 we also have $C^1 \subseteq C$. Then $\mathrm{conv}(C_I) \subseteq C^l \subseteq C^k \subseteq C^0 = C$ for all $l > k > 0$. The last two containments are strict unless $C = \mathrm{conv}(C_I)$ or $C^k = \mathrm{conv}(C_I)$ and, as noted in [29], the following theorem follows from [9, 29].

THEOREM 2.1 ([9, 29]) *For every convex body C there exist $r \in \mathbb{N}$ such that $C^r = \mathrm{conv}(C_I)$.*

Theorem 2.1 is also shown in [8] for Conic Quadratic Programming problems with bounded feasible regions. However, the result neither implies nor requires the polyhedrality of C^1. In fact, the original proof of Theorem 2.1 in [9] does not use the polyhedrality of either C or C^1. Although surprising, it

could be entirely possible for Theorem 2.1 to hold and for C^r to be the only polyhedron in the hierarchy $\{C^k\}_{l=1}^r$.

Our first main result is the following.

THEOREM 2.2 *Let C be a strictly convex body. Then* $\mathrm{CGC}(C)$ *is a rational polytope.*

Together with Theorem 1.1 this characterizes the polyhedrality of the CG closure of convex sets at two extremes of the curvature spectrum: strictly convex bodies sets and rational polyhedra. Unfortunately, it is not clear how to use these results or adapt their proofs to generalize the result to every convex body. However, our second main result is to prove that the CG closure of the intersection of a strictly convex body with a rational polyhedron is a rational polytope.

THEOREM 2.3 *Let C be a strictly convex body and P be a rational polyhedron. Then* $\mathrm{CGC}(C \cap P)$ *is a rational polytope.*

Theorem 2.3 allows us to relax the full dimensional requirement of Theorem 2.2, but only if $\mathrm{aff}(C)$ is a rational affine subspace.

2.1 Proof Outline of Theorem 2.2   The general proof strategy for Theorem 2.2 is the same one used in [11] to show the result for rational ellipsoids. The main difference is the generalization of some separation results from rational ellipsoids to arbitrary strictly convex bodies. One of these separation results essentially states that any non-integral point in the surface of a strictly convex body can be separated by a CG cut. Using this property we can show that the CG closure of a strictly convex body can be generated using the procedure described in Figure 1.

Figure 1: A procedure to generate the first CG closure for a strictly convex body $C \subseteq \mathbb{R}^n$

Step 1 Construct a finite set $S \subset \mathbb{Z}^n$ such that:

      (C1) $\mathrm{CGC}_S(C) \subseteq C$.

      (C2) $\mathrm{CGC}_S(C) \cap \mathrm{bd}(C) \subseteq \mathbb{Z}^n$.

Step 2 Update S with any vector $a \in \mathbb{Z}^n$ such that the CG cut $\langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor$ separates a point of $\mathrm{CGC}_S(C) \setminus \mathrm{CGC}(C)$ and repeat until no such a exists.

To show that Step 1 can be accomplished, we use the separation result to cover the boundary $\mathrm{bd}(C)$ of C with a possibly infinite number of open sets that are associated with the CG cuts. Then, if there are no integral points on the boundary of C, we use compactness of the boundary of C to obtain a finite sub-cover that yields a finite number of CG cuts that separate every point on the boundary of C. If there are integer points on the boundary, then for every $z \in \mathrm{bd}(C) \cap \mathbb{Z}^n$ we use CG cuts to build a polyhedral cone with vertex z which cuts off all the boundary points in some neighborhood around z. In this way, we are able to find a finite set of CG cuts to separate all the non-integral points on the boundary. We do this formally in Proposition 4.1.

To show that Step 2 terminates finitely, we simply show that the set of CG cuts that separate at least one point in $\mathrm{CGC}_S(C) \setminus \mathrm{CGC}(C)$ is finite. We do this formally in Proposition 4.2.

We note that the separation of non-integral points using CG cuts on the boundary of C, required in Step 1 of Figure 1, is not straightforward. A natural first approach to separate a non-integral point u on the boundary of C is to use $a \in (N_C(u) \cap \mathbb{Z}^n) \setminus \{\mathbf{0}\}$. Then $\langle a, u \rangle = \sigma_C(a)$ and if $\sigma_C(a) \notin \mathbb{Z}$ then CG cut $\langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor$ separates u. This approach can fail either because $N_C(u) \subset (\mathbb{R}^n \setminus \mathbb{Z}^n) \cup \{\mathbf{0}\}$ or because $\sigma_C(a) \in \mathbb{Z}$ for every $a \in N_C(u) \cap \mathbb{Z}^n$. This is illustrated by the following two examples.

EXAMPLE 2.1 *Let $C := \left\{ x \in \mathbb{R}^2 : \|x\| \leq 1 \right\}$ and $u = (1/2, \sqrt{3}/2)^T \in \mathrm{bd}(C)$. Then $N_C(u) = \mathrm{cone}(\{u\}) \subset (\mathbb{R}^n \setminus \mathbb{Z}^n) \cup \{\mathbf{0}\}$.*

EXAMPLE 2.2 *Let* $C := \{x \in \mathbb{R}^2 : \|x\| \le 5\}$ *and* $u = (25/13, 60/13)^T \in \mathrm{bd}(C)$. *Then* $N_C(u) = \mathrm{cone}(\{u\}) = \mathrm{cone}\left(\{(5, 12)^T\}\right)$ *and* $\sigma_C(a) \in \mathbb{Z}$ *for every* $a \in N_C(u) \cap \mathbb{Z}^n$.

Fortunately, for both examples we can select alternative left hand sides a' for which the associated CG cut will separate u. For instance, for Example 2.1 we can use $a' = (1, 1)$ for which $\sigma_C(a') = \sqrt{2}$. In Section 3 we will show there exists a systematic method to obtain this alternative left hand side.

2.2 Proof Outline of Theorem 2.3   To prove Theorem 2.3 we again use the procedure in Figure 1 with C replaced by $C \cap P$. However, this time we cannot achieve $\mathrm{CGC}_S(C \cap P) \cap \mathrm{bd}(C \cap P) \subseteq \mathbb{Z}^n$ in Step 1 because not all non-integral points on the boundary of a rational polyhedron can be separated by a CG cut. For instance, fractional points in the relative interior of a facet with integral extreme points cannot be separated. For this reason we replace condition (C2) by $\mathrm{CGC}_S(C \cap P) \cap \mathrm{bd}(C \cap P) \subseteq \mathrm{CGC}(C \cap P)$. To achieve this new condition we show that every point in $\mathrm{bd}(C \cap P) \setminus \mathrm{CGC}(C \cap P)$ can be separated by a finite number of CG cuts. For this, we divide $\mathrm{bd}(C \cap P)$ into points in $\mathrm{bd}(C) \cap P$ and points in $C \cap F$ where F is a facet of P. The separation argument for the first case is the same as that for Theorem 2.2 and for the second case we apply induction on the dimension of $C \cap P$ by noting that $C \cap F$ is also the intersection of a strictly convex body and a rational polyhedron. The arguments for Step 2 are identical to those of Theorem 2.2.

3. Separation   As mentioned in Section 2.1, a key step in the proof of Theorem 2.2 is to show that if $C \subseteq \mathbb{R}^n$ is a strictly convex body then every $u \in \mathrm{bd}(C) \setminus \mathbb{Z}^n$ can be separated by a CG cut. An initial strategy to achieve this is to take $s \in (N_C(u) \cap \mathbb{Z}^n) \setminus \{\mathbf{0}\}$ such that $\sigma_C(s) \notin \mathbb{Z}$ to obtain the CG cut $\langle s, x \rangle \le \lfloor \sigma_C(s) \rfloor$ which separates u. However, as illustrated in Examples 2.1 and 2.2 this can fail either because

(i) $N_C(u) \subset (\mathbb{R}^n \setminus \mathbb{Z}^n) \cup \{\mathbf{0}\}$,

(ii) or $\sigma_C(s) \in \mathbb{Z}$ for every $s \in N_C(u) \cap \mathbb{Z}^n$.

A natural solution for case (i) is to approximate some $s \in N_C(u)$ by a sequence $\{s^i\}_{i \in \mathbb{N}} \subseteq \mathbb{Z}^n$ such that $\bar{s}^i \xrightarrow{i \to \infty} \bar{s}$, where $\bar{a} = a/\|a\|$, and hope that $\langle s^i, u \rangle > \lfloor \sigma_C(s^i) \rfloor$ for some i. This solution will in fact work for both cases but we will need the sequence $\{s^i\}_{i \in \mathbb{N}}$ to additionally comply with the following two properties

(P1) $\lim_{i \to +\infty} \langle s^i, u \rangle - \sigma_C(s^i) = 0$

(P2) $\lim_{i \to +\infty} F(\sigma_C(s^i)) = \delta > 0$. (A weaker condition like $\limsup_{i \to +\infty} F(\sigma_C(s^i)) > 0$ is sufficient, but we will verify the stronger condition),

where $F(r) = r - \lfloor r \rfloor$.

A sequence with these properties will directly yield a separating CG cut because $\langle s^i, u \rangle - \lfloor \sigma_C(s^i) \rfloor = \langle s^i, u \rangle - \sigma_C(s^i) + F(\sigma_C(s^i))$. However, the existence of such a sequence requires a proof, as the conditions do not hold for every sequence such that $\bar{s}^i \xrightarrow{i \to \infty} \bar{s}$. For instance, let $D(H_{s^i, \sigma_C(s^i)}, u) := \frac{|\langle s^i, u \rangle - \sigma(s^i)|}{\|s^i\|}$ be the distance between hyperplane $H_{s^i, \sigma_C(s^i)} := \{x \in \mathbb{R}^n : \langle s^i, x \rangle = \sigma(s^i)\}$ and u. Then $\bar{s}^i \xrightarrow{i \to \infty} \bar{s}$ implies $\lim_{i \to +\infty} D(H_{s^i, \sigma_C(s^i)}, u) = 0$. However, this last condition is weaker than (P1) when $\|s^i\| \to +\infty$, which is necessary for complying with $s^i \in \mathbb{Z}^n$ when $\lambda s \notin \mathbb{Z}^n$ for every $\lambda > 0$. In fact $\|s^i\| \to +\infty$ will be useful condition to have even when $s \in \mathbb{Z}^n$.

The next example illustrates the need for conditions (P1)–(P2) and the fact that they are not automatically satisfied by every sequence such that $\bar{s}^i \xrightarrow{i \to \infty} \bar{s}$.

EXAMPLE 3.1 (CONTINUATION OF EXAMPLE 2.2) *Let* $C := \{x \in \mathbb{R}^2 : \|x\| \le 5\}$ *and* $u = (25/13, 60/13)^T \in \mathrm{bd}(C)$. *Then* $N_C(u) = \mathrm{cone}(\{u\}) = \mathrm{cone}\left(\{(5, 12)^T\}\right)$ *and* $\sigma_C(a) \in \mathbb{Z}$ *for every* $a \in N_C(u) \cap \mathbb{Z}^n$.

We can select $s = (5, 12)^T$ *and approximate* s *with sequence* $\{s^i\}_{i \in \mathbb{N}}$ *given by* $s^i = (65i^2, 26i + 156i^2)^T$[1]. *This sequence complies with* $\bar{s}^i \to \bar{s}$ *and* $D(H_{s^i, \sigma_C(s^i)}, u) \to 0$. *However,* $\langle s^i, u \rangle - \sigma_C(s^i) < -1$ *for all* i

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[1]Note that $s^i = (13 \times 5i^2, 26i + 13 \times 15i^2)^T$. We could have alternatively used $s^i = (i^2, 26i + 15i^2)^T$, but the first option will yield something closer to the construction in our general propositions.

so $\langle s^i, x \rangle \leq \lfloor \sigma(s^i) \rfloor$ *will never separate* u. *This illustrates the need for condition (P1) and the fact that (P1) is not necessarily satisfied by any sequences such that* $\bar{s}^i \xrightarrow{i \to \infty} \bar{s}$.

A sufficient condition for (P1) for Euclidean balls, such as C *in this example, is* $\left\| \bar{s}^i - \bar{s} \right\|^2 \in o(1/\|s^i\|)$. *We hence need a sequence for which* $\bar{s}^i$ *converges to* $\bar{s}$ *faster than the growth of* $\|s^i\|$. *One such sequence is given by* $s^i = (65i, 26 + 156i)^T$. *Unfortunately, although it complies with (P1), we have that for this new sequence* $\langle s^i, u \rangle - \sigma_C(s^i) \leq -F(\sigma(s^i))$ *and hence* $\langle s^i, u \rangle \leq \lfloor \sigma(s^i) \rfloor$ *for all* i. *However, we can comply with condition (P2) without losing (P1) by slightly perturbing the last sequence to obtain* $s^i = (65i, 25 + 156i)^T$. *For this last sequence we finally have* $\langle s^3, u \rangle > \lfloor \sigma(s^3) \rfloor$.

A general way to obtain a sequence $\{s^i\}_{i \in \mathbb{N}}$ complying with (P1)–(P2) is to pick any $s \in N_C(u)$ and construct a simple perturbation of the modified simultaneous Diophantine approximation of s given by the following theorem.

THEOREM 3.1 (DIRICHLET) *Let* $s \in \mathbb{R}^n$. *There exists* $\left\{p^i, q_i\right\}_{i \in \mathbb{N}} \subseteq \mathbb{Z}^n \times \mathbb{Z}$ *such that* $1 \leq q_i \leq i^n$ *for all* $i \in \mathbb{N}$, $\max_{1 \leq j \leq n} \left| p_j^i - q_i s_j \right| \leq \frac{1}{i}$ *and* $\lim_{i \to \infty} q_i = +\infty$.

Note that Theorem 3.1 is usually written without condition $\lim_{i \to \infty} q_i = +\infty$. However, this additional condition is always satisfied for $s \in \mathbb{R}^n \setminus \mathbb{Q}^n$ and can be easily enforced for $s \in \mathbb{Q}^n$ (If $s = \frac{1}{q} p$ where $(p, q) \in \mathbb{Z}^n \times \mathbb{Z}$, then for sufficiently large i set $p^i = ip$ and $q_i = iq$). To show compliance with (P1)–(P2) we will additionally need the following known property of strictly convex bodies, which we prove in the appendix for completeness.

LEMMA 3.1 *Let* $C \subseteq \mathbb{R}^n$ *be a strictly convex body. Let* $m_C : S^{n-1} \to \mathrm{bd}(C)$ *be such that*

$$m_C(v) = \arg\max_{x \in C} \langle v, x \rangle$$

Then m_C *is a well-defined and continuous function on* S^{n-1}.

LEMMA 3.2 *Let* $C \subseteq \mathbb{R}^n$ *be a strictly convex body,* $u \in \mathrm{bd}(C)$ *and* $s \in N_C(u) \setminus \{0\}$. *Let* $s^i = p^i + w$, *where* $\{p^i, q_i\}_{i \in \mathbb{N}}$ *is the simultaneous Diophantine approximation of* s *given by Theorem 3.1 and* $w \in \mathbb{R}^n \setminus \{0\}$. *Then for any* $\varepsilon > 0$, *there exists* $N_\varepsilon \geq 0$ *such that for all* $i \geq N_\varepsilon$ *we have*

  (i) $\left\| p^i - q_i s \right\| \leq \varepsilon$.

  (ii) $\left\| \bar{s}^i - \bar{s} \right\| \leq (1 + \varepsilon) \frac{\|w\|}{\|s^i\|}$.

  (iii) $\sigma_C(s^i) \geq \langle s^i, u \rangle \geq \sigma_C(s^i) - \varepsilon$.

  (iv) $\left| \sigma_C(s^i) - \langle q_i s + w, u \rangle \right| \leq \varepsilon$.   *Additionally if* $\langle s, u \rangle \in \mathbb{Z}$ *and* $\langle w, u \rangle \notin \mathbb{Z}$, *then* $\left| F\left(\sigma_C(s^i) \right) - F\left(\langle q_i s + w, u \rangle \right) \right| \leq \varepsilon$.

PROOF.

  (i) Follows directly from $\left\| p^i - q_i s \right\| \leq \sqrt{n} \max_{1 \leq j \leq n} \left| p_j^i - q_i s_j \right|$ and Theorem 3.1.

  (ii) To prove this point we will use the fact that for $a, b \in \mathbb{R}^n \setminus \{0\}$ the geometric-arithmetic mean inequality is equivalent to $\left\| \frac{a}{\|a\|} - \frac{b}{\|b\|} \right\| \leq \frac{\|a - b\|}{\sqrt{\|a\| \|b\|}}$. In effect we have that

$$\|a\| \|b\| \leq \frac{1}{2} \langle a, a \rangle + \frac{1}{2} \langle b, b \rangle \Leftrightarrow \|a\| \|b\| - \langle a, b \rangle \leq \frac{1}{2} (\langle a, a \rangle - 2\langle a, b \rangle + \langle b, b \rangle)$$

$$\Leftrightarrow \|a\| \|b\| - \langle a, b \rangle \leq \frac{1}{2} \|a - b\|^2$$

$$\Leftrightarrow 2 \left(1 - \frac{\langle a, b \rangle}{\|a\| \|b\|} \right) \leq \frac{\|a - b\|^2}{\|a\| \|b\|}$$

$$\Leftrightarrow \left\| \frac{a}{\|a\|} - \frac{b}{\|b\|} \right\| \leq \frac{\|a - b\|}{\sqrt{\|a\| \|b\|}}. \tag{3}$$

Now, since $\|p^i\| \geq \|q_i s\| - \|p^i - q_i s\|$, $\|s\| > 0$ and $\|p^i - q_i s\| \xrightarrow{i \to \infty} 0$ we get that $\|p^i\| \xrightarrow{i \to \infty} \infty$. Similarly, since $\|s^i\| = \|p^i + w\| \geq \|p^i\| - \|w\|$ we get that $\|s^i\| \xrightarrow{i \to \infty} \infty$. Then for any $\delta > 0$ there exists N such that for all $i \geq N$, $\delta \|s^i\| \geq \|w\| + \|p^i - q_i s\|$ and $\delta \|w\| \geq \|p^i - q_i s\|$. Hence for $i \geq N$, we deduce that

$$\|q_i s\| \geq \|s^i\| - \|s^i - q_i s\| \geq \|s^i\| - \|p^i - q_i s\| - \|w\| \geq (1 - \delta) \|s^i\|.$$

By additionally using (3) with $a = s^i$ and $b = q_i s$ we finally obtain that for $i \geq N$, $\delta < 1$ we have

$$\|\bar{s}^i - \bar{s}\| = \left\| \frac{s^i}{\|s^i\|} - \frac{q_i s}{\|q_i s\|} \right\| \leq \frac{\|s^i - q_i s\|}{\sqrt{\|s^i\| \|q_i s\|}} \leq \frac{\|w\| + \|p^i - q_i s\|}{\sqrt{(1-\delta)} \|s^i\|} \leq \left(\frac{1+\delta}{1-\delta} \right) \frac{\|w\|}{\|s^i\|},$$

and the result follows by taking $\delta = \frac{\varepsilon}{2+\varepsilon}$.

(iii) By part (ii) and continuity of m_C we have that for any $\varepsilon > 0$ there exist N such that for all $i \geq N$ we have $\|\bar{s}^i - \bar{s}\| \leq 2\frac{\|w\|}{\|s^i\|}$ and $\|m_C(\bar{s}^i) - m_C(\bar{s})\| \leq \frac{\varepsilon}{2\|w\|}$. Also, by the definition of m_C, we have that $\sigma_C(s^i) = \langle s^i, m_C(\bar{s}^i) \rangle \geq \langle s^i, m_C(\bar{s}) \rangle = \langle s^i, u \rangle$ and $\sigma_C(\bar{s}) = \langle \bar{s}, m_C(\bar{s}) \rangle \geq \langle \bar{s}, m_C(\bar{s}^i) \rangle$. Now we see that

$$\sigma_C(s^i) - \langle s^i, u \rangle = \langle s^i, m_C(\bar{s}^i) \rangle - \langle s^i, m_C(\bar{s}) \rangle$$
$$= \|s^i\| \left(\langle \bar{s}^i, m_C(\bar{s}^i) \rangle - \langle \bar{s}^i, m_C(\bar{s}) \rangle \right)$$
$$\leq \|s^i\| \left(\langle \bar{s}^i, m_C(\bar{s}^i) \rangle - \langle \bar{s}^i, m_C(\bar{s}) \rangle + \langle \bar{s}, m_C(\bar{s}) \rangle - \langle \bar{s}, m_C(\bar{s}^i) \rangle \right)$$
$$= \|s^i\| \langle \bar{s}^i - \bar{s}, m_C(\bar{s}^i) - m_C(\bar{s}) \rangle$$
$$\leq \|s^i\| \|\bar{s}^i - \bar{s}\| \|m_C(\bar{s}^i) - m_C(\bar{s})\|$$
$$\leq \|s^i\| \left(2\frac{\|w\|}{\|s^i\|} \right) \left(\frac{\varepsilon}{2\|w\|} \right) = \varepsilon$$

as needed.

(iv) For the first part we simply note that by parts (i) and (iii) we have that for every $\varepsilon > 0$ there exists N such that for all $i \geq N$ we have

$$\sigma_C(s^i) \geq \langle s^i, u \rangle = \langle p^i, u \rangle + \langle w, u \rangle = \langle p^i - q_i s, u \rangle + \langle q_i s, u \rangle + \langle w, u \rangle$$
$$\geq -\|p^i - q_i s\| \|u\| + \langle q_i s, u \rangle + \langle w, u \rangle \geq \langle q_i s + w, u \rangle - \varepsilon,$$

and

$$\sigma_C(s^i) \leq \langle s^i, u \rangle + \varepsilon = \langle p^i, u \rangle + \langle w, u \rangle + \varepsilon = \langle p^i - q_i s, u \rangle + \langle q_i s, u \rangle + \langle w, u \rangle + \varepsilon$$
$$\leq \|p^i - q_i s\| \|u\| + \langle q_i s, u \rangle + \langle w, u \rangle \leq \langle q_i s + w, u \rangle + 2\varepsilon.$$

For the second part we have that, under the additional assumptions, $F(\langle q_i s + w, u \rangle) = F(\langle w, u \rangle) \in (0, 1)$ is independent of i. Then, there exists sufficiently small ε_0 such that for every $\varepsilon \leq \varepsilon_0$

$$\lfloor \langle q_i s + w, u \rangle \rfloor < \langle q_i s + w, u \rangle - \varepsilon < \langle q_i s + w, u \rangle = \lfloor \langle q_i s + w, u \rangle \rfloor + F(\langle w, u \rangle)$$
$$< \langle q_i s + w, u \rangle + \varepsilon \leq \lfloor \langle q_i s + w, u \rangle \rfloor + 1.$$

Together with the first part we have that for every $\varepsilon \leq \varepsilon_0$ there exists N such that for all $i \geq N$

$$\lfloor \langle q_i s + w, u \rangle \rfloor < \langle q_i s + w, u \rangle - \varepsilon < \sigma_C(s^i) < \langle q_i s + w, u \rangle + \varepsilon \leq \lfloor \langle q_i s + w, u \rangle \rfloor + 1.$$

Because $F(\cdot)$ is affine in $(\lfloor \langle q_i s + w, u \rangle \rfloor, \lfloor \langle q_i s + w, u \rangle \rfloor + 1)$ we finally obtain that

$$F(\langle q_i s + w, u \rangle) - \varepsilon < F(\sigma_C(s^i)) < F(\langle q_i s + w, u \rangle) + \varepsilon.$$

$\square$

By suitably scaling s, the sequence $\{s^i\}_{i \in \mathbb{N}}$ described in Lemma 3.2 will satisfy (P1) and (P2). In particular part (iii) of Lemma 3.2 implies that $\{s^i\}_{i \in \mathbb{N}}$ satisfies (P1). Part(iv) of Lemma 3.2 will be used to verify (P2). Using this construction we can prove the desired separation results for non-integral points in the boundary of C and a slightly stronger result for points that are additionally close to integral points.

PROPOSITION 3.1 *Let $C \subseteq \mathbb{R}^n$ be a strictly convex body. Take $u \in \mathrm{bd}(C)$.*

    S1 If $u \notin \mathbb{Z}^n$, then there exists $a \in \mathbb{Z}^n$ such that $\langle a, u \rangle > \lfloor \sigma_C(a) \rfloor$.

    S2 If $u \in \mathbb{Z}^n$, then for every $v \in \mathbb{R}^n \setminus \mathrm{int}(T_C(u))$, $v \neq 0$, there exists $a \in \mathbb{Z}^n$ such that

$$\langle a, u \rangle = \lfloor \sigma_C(a) \rfloor \quad and \quad \langle a, v \rangle > 0.$$

PROOF.

    (i) Let $s \in N_C(u) \setminus \{0\}$. By possibly scaling s by a positive scalar, we may assume that $\langle s, u \rangle \in \mathbb{Z}$. Since $u \notin \mathbb{Z}^n$, there exists l, $1 \leq l \leq n$, such that $u_l \notin \mathbb{Z}$. Let $s^i = p^i + e^l$, for $i \geq 0$, where $\{p^i, q_i\}_{i \in \mathbb{N}}$ is the simultaneous Diophantine approximation of s given by Theorem and e^l is the lth unit vector. Because $q_i \in \mathbb{Z}$, $\langle s, u \rangle \in \mathbb{Z}$ and $\langle e^l, u \rangle = u_l \notin \mathbb{Z}$ we have that $\delta := F\left(\langle q_i s + e^l, u \rangle\right) = F(u_l) > 0$ is independent of i. Then using Lemma 3.2 for $w = e^l$ and $\varepsilon < \delta/3$ we have that there exists i such that $\langle s^i, u \rangle - \sigma_C(s^i) > -\delta/3$ and $F\left(\sigma_C(s^i)\right) > (2/3)\delta$. The result follows by setting $a = s^i$ and noting that $\langle s^i, u \rangle - \lfloor \sigma_C(s^i) \rfloor = \langle s^i, u \rangle - \sigma_C(s^i) + F\left(\sigma_C(s^i)\right) > (1/3)\delta > 0$.

    (ii) Since $u \in \mathrm{bd}(C)$ and $v \in \mathbb{R}^n \setminus \mathrm{int}(T_C(u))$, $v \neq 0$, there exists $s \in N_C(u) \setminus \{0\}$ such that $\langle s, v \rangle \geq 0$. Again by possibly scaling s, we may assume that $\langle s, u \rangle \in \mathbb{Z}$. Let $s^i = p^i + w$, for $i \geq 0$, where $\{p^i, q_i\}_{i \in \mathbb{N}}$ is the simultaneous Diophantine approximation of s given by Theorem and $w \in \mathbb{Z}^n$ is any integer vector such that $\langle w, v \rangle \geq \frac{2}{3}$. Then using Lemma 3.2 for $\varepsilon < \min\left\{\frac{1}{3\|v\|}, \frac{1}{2}\right\}$ we have that there exists i such that

$$\sigma_C(s^i) \geq \langle s^i, u \rangle \geq \sigma_C(s^i) - \frac{1}{2} \tag{4}$$

and

$$\left\|p^i - q_i s\right\| \leq \frac{1}{3\|v\|}. \tag{5}$$

Because $\langle s^i, u \rangle \in \mathbb{Z}$ we have that (4) implies $\lfloor \sigma_C(s^i) \rfloor = \langle s^i, u \rangle$. Furthermore, together with $\langle s^i, v \rangle = \langle p^i - q_i s, v \rangle + q_i \langle s, v \rangle + \langle w, v \rangle$, we have that (5), $\langle s, v \rangle \geq 0$ and $\langle w, v \rangle \geq \frac{2}{3}$ imply that $\langle s^i, v \rangle \geq -\frac{1}{3} + 0 + \frac{2}{3} > 0$. The result then follows by setting $a = s^i$.

    $\square$

REMARK 3.1 *In the proof of part (iii) of Lemma 3.2 we obtained as a partial result that*

$$\sigma_C\left(s^i\right) - \langle s^i, u \rangle \leq \left\|s^i\right\| \left\|\bar{s}^i - \bar{s}\right\| \left\|m_C(\bar{s}^i) - m_C(\bar{s})\right\|$$

If we could prove that $\left\|s^i\right\| \left\|\bar{s}^i - \bar{s}\right\| \xrightarrow{i \to \infty} 0$ instead of the weaker result in part (ii) of Lemma 3.2, we would obtain condition S1 for any convex body. Of course this cannot hold as it would imply that CG cuts can separate fractional points in the relative interior of the facets of an integral polytope, which clearly cannot happen. However, we should note that this impossibility is only due to the use of nonzero perturbation w as we do have that $\left\|p^i\right\| \left\|\bar{p}^i - \bar{s}\right\| \xrightarrow{i \to \infty} 0$ and hence Diophantine approximation does give us condition (P1) with $s^i = p^i$ for any convex set. In contrast, to obtain condition (P1) and condition (P2) we need continuity of m_C.

We end this section by proving the following corollary of Proposition 3.1 that we will need for the proof of Proposition 4.1.

COROLLARY 3.1 *Let $C \subseteq \mathbb{R}^n$ be a convex body for which condition S2 of Proposition 3.1 holds for every $u \in \mathrm{bd}(C) \cap \mathbb{Z}^n$. Then, for each $u \in \mathrm{bd}(C) \cap \mathbb{Z}^n$ there exists a finite set $S_u \subseteq \mathbb{Z}^n$ such that*

    (i) $T'(u) := \{x \in \mathbb{R}^n : \langle c, x \rangle \leq 0, c \in S_u\}$ is a polyhedral cone,

$$c \in \mathbb{Z}^n \quad and \quad \lfloor \sigma_C(c) \rfloor = \langle c, u \rangle \quad \forall\, c \in S_u \tag{6}$$

    and

$$T'(u) \subseteq \mathrm{int}(T_C(u)) \cup \{0\}. \tag{7}$$

(*ii*) *There exists an open neighborhood $\mathcal{N}$ of u such that*

$$\mathcal{N} \cap \mathrm{bd}(C) \cap (u + T'(u)) = \{u\}$$

(*iii*) $\mathrm{CGC}_{S_u}(C) = u + T'(u)$.

PROOF.

(i) Examine $K = S^{n-1} \setminus \mathrm{int}(T_C(u))$. Since $\mathrm{int}(T_C(u))$ is an open subset of $\mathbb{R}^n$ (as $\mathrm{int}(C) \neq \emptyset$) we have that K is a closed subset of the sphere, and hence K is compact. For each $v \in K$, we note that $v \in \mathbb{R}^n \setminus \mathrm{int}(T_C(u))$, $v \neq 0$, therefore by condition S2 of Proposition 3.1, there exists $c^v \in \mathbb{Z}^n$ such that

$$\lfloor \sigma(c^v) \rfloor = \langle c^v, u \rangle \quad \text{and} \quad \langle c^v, v \rangle > 0$$

Let $U_v = \{w \in K : \langle c^v, w \rangle > 0\}$. Clearly $v \in U_v$ and U_v is an open subset of K (relative to the subspace topology). Therefore the collection $\{U_v\}_{v \in K}$ is an open cover of K. Hence by compactness of K there exists a finite subcover $\{U_{v_i}\}_{i=1}^k \subseteq \{U_v\}_{v \in K}$ such that $K \subseteq \cup_{i=1}^k U_{v_i}$. Let $S_u = \{c^{v_i} : 1 \leq i \leq k\}$, where $c^{v_i} \in \mathbb{Z}^n$ is defined as above. By construction $T'(u)$ is a polyhedral cone and S_u satisfies (6), hence we need only verify that $T'(u)$ satisfies (7), for which we prove its contrapositive. Assume that $v \in \mathbb{R}^n \setminus \mathrm{int}(T_C(u))$, $v \neq 0$. Then note that $\bar{v} \in K$, and hence there exists i, $1 \leq i \leq k$, such that $\bar{v} \in U_{v_i}$. Therefore

$$\langle c^{v_i}, \bar{v} \rangle > 0 \Rightarrow \langle c^{v_i}, v \rangle > 0 \Rightarrow v \notin T'(u)$$

since $c^{v_i} \in S_u$ as needed.

(ii) By translating C to $C - u$, we may assume that $u = 0$, and so $T'(u) = T'(0)$, $T_C(u) = T_C(0)$ and $N_C(u) = N_C(0)$. For notational convenience, we now denote $T'(0)$ as T', $T_C(0)$ as T_C and $N_C(0)$ as N_C. Hence we need to show that there exists an open neighborhood $\mathcal{N}$ of 0 such that

$$\mathcal{N} \cap \mathrm{bd}(C) \cap T' = \{0\}.$$

Because T' is a polyhedral cone there exists vectors $\{v^i\}_{i=1}^r \subseteq T' \setminus \{0\}$ such that

$$T' = \left\{ x \in \mathbb{R}^n : \exists \mu \in \mathbb{R}_+^r \text{ s.t. } x = \sum_{i=1}^r \mu_i v^i \right\}.$$

By (7) we also have that $T' \setminus \{0\} \subseteq \mathrm{int}(T_C) = \bigcup_{\lambda > 0} \lambda \, \mathrm{int}(C)$ (For the last equality see for example section A.5.3 of [20]). Then, by scaling them appropriately, we may assume that $\{v^i\}_{i=1}^r \subseteq \mathrm{int}(C)$ to obtain

$$\mathrm{conv}\left(\{0, v^1, \ldots, v^r\} \right) \cap \mathrm{bd}(C) = \{0\} \tag{8}$$

by using the fact that $\mathrm{int}(C) \cup \{0\}$ is a convex set. Now, let $c \in N_C$, $c_0 = \max_{i=1}^r \langle c, v^i \rangle$, $\mathcal{N} = \{x \in \mathbb{R}^n : \langle c, x \rangle > c_0\}$ and for $x \in \mathcal{N} \cap T'$ let $\mu \in \mathbb{R}_+^r$ be such that $x = \sum_{i=1}^r \mu_i v^i$. Then $c_0 < \langle c, x \rangle = \sum_{i=1}^r \mu_i \langle c, v^i \rangle \leq c_0 \sum_{i=1}^r \mu_i$. Because $v^i \in \mathrm{int}(T_C)$ and N_C is the polar of T_C we have that $c_0 < 0$ and hence $\sum_{i=1}^r \mu_i \leq 1$. Then $x \in \mathrm{conv}\left(\{0, v^1, \ldots, v^r\} \right)$ and hence $\mathcal{N} \cap T' \subseteq \mathrm{conv}\left(\{0, v^1, \ldots, v^r\} \right)$. Together with (8) this gives the desired result.

(iii) This part holds by construction.

$\square$

4. CG Closure of a Strictly Convex Body To prove Theorem 2.2 we first show that Step 1 of Figure 1 can be achieved. We assume that $\mathrm{int}(C) \cap \mathrm{CGC}(C) \neq \emptyset$ as the alternative case is trivial for strictly convex bodies. However instead of requiring strict convexity of convex body C we simply require boundary separation conditions (S1) and (S2). These conditions are satisfied by every strictly convex body by Proposition 3.1, but they may also be satisfied by some convex bodies that are not strictly convex.

PROPOSITION 4.1 *Let C be a convex body such that $\mathrm{int}(C) \cap \mathrm{CGC}(C) \neq \emptyset$. If conditions (S1) and (S2) of Proposition 3.1 hold for C, then there exists a finite set $S \subseteq \mathbb{Z}^n$ such that conditions (C1) and (C2) in Step 1 of Figure 1 hold.*

PROOF. Since C is a bounded set, let $\mathcal{I} := \mathrm{bd}(C) \cap \mathbb{Z}^n$ be the finite (and possibly empty) set of integer points on the boundary of C. For each $u \in \mathcal{I}$, let $\mathcal{N}_u$ be the neighborhood of u, S_u be the finite set and $T'(u)$ be the polyhedral cone from Corollary 3.1. Let $\mathcal{D} := \mathrm{bd}(C) \setminus \bigcup_{u \in \mathcal{I}} \mathcal{N}_u$. Observe that $\mathcal{D} \cap \mathbb{Z}^n = \emptyset$ by construction and that $\mathcal{D}$ is compact since it is obtained from a compact set $\mathrm{bd}(C)$ by removing a finite number of open sets. Now, for any $a \in \mathbb{Z}^n$ let $O(a) := \{x \in \mathcal{D} : \langle a, x \rangle > \lfloor \sigma(a) \rfloor\}$ be the set of points of $\mathcal{D}$ that are separated by the CG cut $\langle a, x \rangle \leq \lfloor \sigma(a) \rfloor$. This set is open with respect to $\mathcal{D}$. Furthermore, by Conditions S1 and the construction of $\mathcal{D}$, we have that $\mathcal{D} \subseteq \bigcup_{a \in \mathcal{A}} O(a)$ for a possibly infinite set $\mathcal{A} \subseteq \mathbb{Z}^n$. However, since $\mathcal{D}$ is a compact set we have that there exists a finite subset $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $\mathcal{D} \subseteq \bigcup_{a \in \mathcal{A}_0} O(a)$. Let $S := \mathcal{A}_0 \cup \bigcup_{u \in \mathcal{I}} S_u$, then, by construction of S and Corollary 3.1, $\mathrm{CGC}_S(C) \cap \mathrm{bd}(C) \subseteq \mathbb{Z}^n$ and hence (C2) holds. To verify (C1) we show that $w \notin \mathrm{CGC}_S(C)$ for any $w \in \mathbb{R}^n \setminus C$. Let $v \in \mathrm{int}(C) \cap \mathrm{CGC}_S(C)$, then there exists $\lambda \in (0, 1)$, such that $u = \lambda w + (1 - \lambda) v \in \mathrm{bd}(C)$. If $u \notin \mathbb{Z}^n$, then there exists a CG cut corresponding to an integer vector in $\mathcal{A}_0$ that separates u. However, since this CG cut does not separate v and u is a convex combination of w and v, it must separate w. Thus $w \notin \mathrm{CGC}_S(C)$. If $u \in \mathbb{Z}^n$, then by Corollary 3.1 (iii), there exists a CG cut corresponding to an integer vector in S_u that separates w and therefore $w \notin \mathrm{CGC}_S(C)$. $\qquad\square$

Conditions (C1) and (C2) are sufficient for achieving Step 2 of Figure 1. However, we show that Step 2 of Figure 1 can be achieved if slightly weaker conditions are satisfied as well. Furthermore, we also show that these weaker conditions are in fact sufficient for the polyhedrality of the CG closure of a convex body even if strict convexity is not satisfied.

PROPOSITION 4.2 *Let C be a convex body. If there exists a finite set $S \subseteq \mathbb{Z}^n$ such that*

    (C1') $\mathrm{CGC}_S(C) \subseteq C$.

    (C2') $\mathrm{CGC}_S(C) \cap \mathrm{bd}(C) \subseteq \mathrm{CGC}(C)$.

then $\mathrm{CGC}(C)$ is a rational polytope.

PROOF. Let $\mathrm{ext}(\mathrm{CGC}_S(C))$ be the set of vertices of the polytope $\mathrm{CGC}_S(C)$. Because of (C2') we have that any CG cut that separates a point $u \in \mathrm{CGC}_S(C) \setminus \mathrm{CGC}(C)$ must also separate a point in $\mathrm{ext}(\mathrm{CGC}_S(C)) \setminus \mathrm{bd}(C)$. It is then sufficient to show that the set of CG cuts that separates some point in $\mathrm{ext}(\mathrm{CGC}_S(C)) \setminus \mathrm{bd}(C)$ is finite.

Because $\mathrm{ext}(\mathrm{CGC}_S(C)) \setminus \mathrm{bd}(C) \subseteq C \setminus \mathrm{bd}(C) = \mathrm{int}(C)$ and $|\mathrm{ext}(\mathrm{CGC}_S(C))| < \infty$ we have that there exists $\varepsilon > 0$ such that

$$\varepsilon B^n + v \subseteq C \quad \forall v \in \mathrm{ext}(\mathrm{CGC}_S(C)) \setminus \mathrm{bd}(C) \tag{9}$$

where $B^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Now take $v \in \mathrm{ext}(\mathrm{CGC}_S(C))$ and take $a \in \mathbb{Z}^n$ such that $\|a\| \geq \frac{1}{\varepsilon}$. Now note that

$$\lfloor \sigma_C(a) \rfloor \geq \sigma_C(a) - 1 \geq \sigma_{v + \varepsilon B^n}(a) - 1 = \langle v, a \rangle + \varepsilon \|a\| - 1 \geq \langle v, a \rangle$$

Hence the CG cut associated with a does not cut off v. Then $\mathrm{CGC}(C) = \mathrm{CGC}_{S \cup S'}(C)$ for $S' := \mathbb{Z}^n \cap \frac{1}{\varepsilon} B^n$. Since $|S'| < \infty$, the claim follows. $\qquad\square$

Note that condition (C1') is identical to (C1) and (C2') is equivalent to (C2) for strictly convex sets. The extra generality of (C2') will be useful when dealing with the CG closure of the intersection of a strictly convex body and a rational polyhedron.

With these two propositions the proof of Theorem 2.2 is as follows.

PROOF OF THEOREM 2.2. We divide the proof into the following cases

    (i) $\mathrm{CGC}(C) = \emptyset$.

    (ii) $\mathrm{CGC}(C) \neq \emptyset$ and $\mathrm{CGC}(C) \cap \mathrm{int}(C) = \emptyset$.

    (iii) $\mathrm{CGC}(C) \cap \mathrm{int}(C) \neq \emptyset$.

For the first case, the result follows directly. For the second case, by Proposition 3.1 and the strict convexity of C, we have that $|\mathrm{bd}(C) \cap \mathbb{Z}^n| = 1$ and $\mathrm{CGC}(C) = \mathrm{bd}(C) \cap \mathbb{Z}^n$ so the result again follows directly. For the third case the result follows from Propositions 3.1, 4.1 and 4.2. $\qquad\square$

5. CG Closure of the Intersection of a Strictly Convex Set and a Rational Polyhedron
Before considering intersections with general rational polyhedra we first concentrate on intersections with rational affine subspaces. To achieve this we will need the following well known theorem (e.g. see page 46 of [30]).

THEOREM 5.1 (INTEGER FARKAS'S LEMMA) *Let A be a rational matrix and b be a rational vector. Then the system $Ax = b$ has integral solutions if and only if $\langle y, b \rangle$ is integer whenever y is a rational vector and $A^T y$ is an integer vector.*

Using this result we can characterize the CG closure of convex sets that are not full dimensional.

PROPOSITION 5.1 *Let $C \subseteq \mathbb{R}^n$ be a closed convex set such that $\mathrm{aff}(C) = W + w$, where $w \in \mathbb{Q}^n$ and W is a rational subspace with $\dim(W) = k$. Then one of the following conditions is satisfied:*

(*i*) $\mathrm{aff}(C) \cap \mathbb{Z}^n = \emptyset$ *and* $\mathrm{CGC}(\mathrm{aff}(C)) = \emptyset$.

(*ii*) *There exist an affine map $L : \mathbb{R}^n \to \mathbb{R}^k$ such that*

$$\mathrm{CGC}(C) = L^{-1}(\mathrm{CGC}(L(C))) \cap \mathrm{aff}(C),$$

where $L^{-1}(A) := \{x \in \mathbb{R}^n \, : \, L(x) \in A\}$.

PROOF. Suppose $\mathrm{aff}(C) \cap \mathbb{Z}^n = \emptyset$. By Theorem 5.1 we have that there exists $a \in \mathbb{Z}^n$ and $a_0 \in \mathbb{Q} \setminus \mathbb{Z}$ such that $\mathrm{aff}(C) \subset \{x \in \mathbb{R}^n \, : \, \langle a, x \rangle = a_0\}$. Then, $\langle a, x \rangle \leq \lfloor a_0 \rfloor$ and $\langle a, x \rangle \geq \lceil a_0 \rceil$ are valid CG cuts for C and we obtain $\mathrm{CGC}(C) = \emptyset$. Now, suppose $\mathrm{aff}(C) \cap \mathbb{Z}^n \neq \emptyset$. We then may assume that $w \in \mathbb{Z}^n$ and $W = \{Ax + w \, : \, x \in \mathbb{R}^k\}$, where $A \in \mathbb{Z}^{n \times k}$ is a rank k matrix. Then there exists an unimodular matrix $U \in \mathbb{Z}^{n \times n}$ and a non-singular matrix $B \in \mathbb{Z}^{k \times k}$ such that $\begin{bmatrix} B \\ \mathbf{0}_{n-k \times k} \end{bmatrix} = UA$, where $\mathbf{0}_{n-k \times k} \in \mathbb{R}^{n-k \times k}$ is the all zeros matrix (See for example Corollary 4.3b in page 49 of [30]). We can then take $L(x) = P_k(U(x - w))$ where P_k is the projection onto the first k variables. This transformation gives the desired result because of the following useful properties of the CG closure.

(i) If $C \subseteq \mathbb{R}^k$ and $\mathbf{0}_{n-k}$ is the all zeros vector in $\mathbb{R}^{n-k}$ then $\mathrm{CGC}(C \times \mathbf{0}_{n-k}) = \mathrm{CGC}(C) \times \mathbf{0}_{n-k}$.

(ii) If $C \subseteq \mathbb{R}^n$ is a closed convex set and $w \in \mathbb{Z}^n$ then $\mathrm{CGC}(C - w) = \mathrm{CGC}(C) - w$.

(iii) If $C \subseteq \mathbb{R}^n$ is a closed convex set and $U \in \mathbb{Z}^{n \times n}$ is an unimodular matrix then $U \, \mathrm{CGC}(C) = \mathrm{CGC}(UC)$.

The first two properties are direct. For the third one we can use the fact that $U^{-T}\mathbb{Z}^n = \mathbb{Z}^n$ to see that

$$\begin{aligned}
U \, \mathrm{CGC}(C) &= \bigcap_{a \in \mathbb{Z}^n} \left\{ x \in \mathbb{R}^m \, : \, \langle a, U^{-1}x \rangle \leq \lfloor \sigma_C(a) \rfloor \right\} \\
&= \bigcap_{a \in \mathbb{Z}^n} \left\{ x \in \mathbb{R}^m \, : \, \langle U^{-T}a, x \rangle \leq \lfloor \sigma_C(a) \rfloor \right\} \\
&= \bigcap_{a \in \mathbb{Z}^n} \left\{ x \in \mathbb{R}^m \, : \, \langle a, x \rangle \leq \lfloor \sigma_C (U^T a) \rfloor = \lfloor \sigma_{UC}(a) \rfloor \right\} = \mathrm{CGC}(UC).
\end{aligned}$$

$\square$

Using Proposition 5.1 the polyhedrality of the CG closure of a non full dimensional convex set C is equivalent to the polyhedrality of full dimensional convex set $L(C)$. For instance, if $L(C)$ is a strictly convex set we can use Theorem 2.2 to deduce the polyhedrality of $L(C)$ and C. In particular we obtain the following corollary.

COROLLARY 5.1 *Let C be a strictly convex body and let V be an affine rational subspace. Then the CG closure of $V \cap C$ is a rational polytope.*

For the intersection with a general rational polyhedron P we need to understand the role of each face of P in the construction of the CG closure of $P \cap C$. Specifically, we would like to be able to replace P by $P \cap C$ in the following lemma which is proven in page 340 of [30].

LEMMA 5.1 *If F is a face of rational polyhedron P, then $\mathrm{CGC}(F) = \mathrm{CGC}(P) \cap F$.*

To generalize Lemma 5.1 we will need to understand the support function of $C \cap P$ in relation to the support functions of C and P. In other words, we need a Farka's Lemma type result for constraints $x \in C$ and $x \in P$, which requires some type of constraint qualification. If P and C satisfy constraint qualification $P \cap \mathrm{int}(C) \neq \emptyset$, then it allows us to use the following known result. For completeness, we present a short proof of this result in the appendix.

PROPOSITION 5.2 *Let C be a closed convex set and P be a polyhedron such that $P \cap \mathrm{int}(C) \neq \emptyset$. Then*

$$\sigma_{P \cap C}(a) \leq \sigma_P(a_P) + \sigma_C(a_C) \quad . \tag{10}$$

for all $a_P, a_C \in \mathbb{R}^n$ such that $a = a_P + a_C$. Furthermore, for every $a \in \mathbb{R}^n$ there exists $a_P, a_C \in \mathbb{R}^n$ such that (10) *holds at equality.*

LEMMA 5.2 *Let $C \subseteq \mathbb{R}^n$ be a strictly convex body and let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. Then if $\dim(C \cap P) \geq 1$ (i.e. $C \cap P$ is not empty or a single point), we have that $P \cap \mathrm{int}(C) \neq \emptyset$.*

     PROOF.   By assumption $\dim(P \cap C) \geq 1$ we have that there exists $x, y \in P \cap C$ such that $x \neq y$ and $z := \frac{1}{2}(x + y) \in P$. Since C is a strictly convex body we have that $z \in \mathrm{int}(C)$.   □

     With these results we obtain the following generalization of Lemma 5.1 and a direct corollary that describes the intersection of the CG closure of $C \cap P$ with the boundary of $C \cap P$.

PROPOSITION 5.3 *Let C be a strictly convex body and P be a rational polyhedron. If F is any nonempty face of P, then for any $S \subseteq \mathbb{Z}^n$ there exists $S' \subseteq \mathbb{Z}^n$ such that $\mathrm{CGC}_{S'}(P \cap C) \cap F \subseteq \mathrm{CGC}_S(F \cap C)$ and hence $\mathrm{CGC}(P \cap C) \cap F = \mathrm{CGC}(F \cap C)$. If we additionally have $\mathrm{CGC}(F \cap C) = \mathrm{CGC}_S(F \cap C)$ for some $S \subseteq \mathbb{Z}^n$ with $|S| < \infty$, then we also have that there exists $S' \subseteq \mathbb{Z}^n$ with $|S'| < \infty$ such that $\mathrm{CGC}_{S'}(P \cap C) \cap F = \mathrm{CGC}(F \cap C)$.*

     PROOF.

     To prove that for any $S \subseteq \mathbb{Z}^n$ there exists $S' \subseteq \mathbb{Z}^n$ such that $\mathrm{CGC}_{S'}(P \cap C) \cap F \subseteq \mathrm{CGC}_S(F \cap C)$ we show that for any CG cut $\langle a, x \rangle \leq \lfloor \sigma_{F \cap C}(a) \rfloor$ for $F \cap C$, there exists a CG cut $\langle \hat{a}, x \rangle \leq \lfloor \sigma_{P \cap C}(\hat{a}) \rfloor$ for $P \cap C$ such that if $u \in F \cap C$ and $\langle a, u \rangle > \lfloor \sigma_{F \cap C}(a) \rfloor$ then $\langle \hat{a}, u \rangle > \lfloor \sigma_{P \cap C}(\hat{a}) \rfloor$. The proof is divided into two cases.

     First consider the case where $\dim(F \cap C) = 0$ or $F \cap C = \emptyset$. If $F \cap C = \emptyset$, then the result is evident. Otherwise $F \cap C = \{v\}$ fore some v such that either $v \in \mathrm{bd}(C)$ or $F = \{v\}$. In either case, if $v \notin \mathbb{Z}^n$ there exists a CG cut for $C \cap P$ that separates v. In the first case (i.e. $v \in \mathrm{bd}(C)$), by Proposition 3.1, there is a CG cut for C and in the second case (i.e. $F = \{v\}$) there is a CG cut for P by Lemma 5.1 and the fact that if $v \notin \mathbb{Z}^n$ then $\mathrm{CGC}(\{v\}) = \emptyset$. Thus $\mathrm{CGC}(F \cap C) = \mathrm{CGC}(P \cap C) \cap F = \emptyset$. On the other hand if $v \in \mathbb{Z}^n$, then $\mathrm{CGC}(F \cap C) = F \cap C = \mathrm{CGC}(P \cap C) \cap F$.

     Now assume that $\dim(F \cap C) \geq 1$. By Lemma 5.2, we have $F \cap \mathrm{int}(C) \neq \emptyset$. Therefore, by Proposition 5.2 there exists a_F and a_C such that $\sigma_{F \cap C}(a) = \sigma_F(a_F) + \sigma_C(a_C)$ and $a = a_F + a_C$. Since P is a rational polyhedron, let $P = \{x \in \mathbb{R}^n : A^{\leq} x \leq b^{\leq}, A^{=} x \leq b^{=}\}$ and $F = \{x \in \mathbb{R}^n : A^{\leq} x \leq b^{\leq}, A^{=} x = b^{=}\}$ where $A^{\leq}, A^{=}, b^{\leq}, b^{=}$ are integral. By the nonemptyness of F there exists $y^{\leq}, y^{=}$ such that

$$(y^{\leq})^T A^{\leq} + (y^{=})^T A^{=} = (a_F)^T \tag{11}$$

$$\langle y^{\leq}, b^{\leq} \rangle + \langle y^{=}, b^{=} \rangle = \sigma_F(a_F) \tag{12}$$

$$y^{\leq} \geq 0. \tag{13}$$

Consider $\hat{a}_F$ and $\hat{r}_F$ defined as $\hat{a}_F := (y^{\leq})^T A^{\leq} + (y^{=} - \lfloor y^{=} \rfloor)^T A^{=}$ and $\hat{r}_F := \langle y^{\leq}, b^{\leq} \rangle + \langle y^{=} - \lfloor y^{=} \rfloor, b^{=} \rangle$, where $\lfloor \cdot \rfloor$ is taken componentwise. Then observe that, because $y^{\leq} \geq 0$ and $(y^{=} - \lfloor y^{=} \rfloor) \geq 0$, $\langle \hat{a}_F, x \rangle \leq \hat{r}_F$ is non-negative linear combination of the inequalities defining P and hence is a valid inequality for P.

     Let $\hat{a} = a - (\lfloor y^{=} \rfloor)^T A^{=}$ and $\hat{r} = \sigma_{F \cap C}(a) - \langle \lfloor y^{=} \rfloor, b^{=} \rangle$. Then observe that $\hat{a} \in \mathbb{Z}^n$ and $\hat{a} = \hat{a}_F + a_C$ and $\hat{r} = \hat{r}_F + \sigma_C(a_C)$. Therefore, by Proposition 5.2, $\sigma_{C \cap P}(\hat{a}) \leq \hat{r}$ and thus $\langle \hat{a}, x \rangle \leq \hat{r}$ is a valid inequality

for $C \cap P$. Finally, observe that if $u \in F \cap C$, then

$$
\begin{aligned}
\langle a, u \rangle - \lfloor \sigma_{F \cap C}(a) \rfloor &= \langle \hat{a}, u \rangle + \langle \lfloor y^= \rfloor, A^= u \rangle - \lfloor \hat{r} + \langle \lfloor y^= \rfloor, b^= \rangle \rfloor \\
&= \langle \hat{a}, u \rangle - \lfloor \hat{r} \rfloor + \langle \lfloor y^= \rfloor, A^= u \rangle - \langle \lfloor y^= \rfloor, b^= \rangle \\
&= \langle \hat{a}, u \rangle - \lfloor \hat{r} \rfloor \le \langle \hat{a}, u \rangle - \lfloor \sigma_{C \cap P}(\hat{a}) \rfloor.
\end{aligned}
$$

The rest of the assertions follows from the fact that $\mathrm{CGC}(F \cap C) \subseteq F$ and that for any $S' \subset \mathbb{Z}^n$ we have $\mathrm{CGC}(F \cap C) \subseteq \mathrm{CGC}_{S'}(P \cap C) \cap F$ because every CG cut for $P \cap C$ is also a CG cut for $F \cap C$. □

COROLLARY 5.2 *Let C be a strictly convex set and P a rational polyhedron such that $C \cap P$ is full dimensional. Let $\{F_i\}_{i=1}^m$, denote the facets of P, then*

$$
\mathrm{CGC}(C \cap P) \cap \mathrm{bd}(C \cap P) = (\mathbb{Z}^n \cap \mathrm{bd}(C) \cap P) \cup \cup_{i=1}^m \mathrm{CGC}(F_i \cap C)
$$

PROOF. We first note that if $C \cap P$ is full dimensional then $\mathrm{bd}(C \cap P) = (\mathrm{bd}(C) \cap P) \cup (\mathrm{bd}(P) \cap C)$. By Proposition 3.1 we have that $\mathrm{CGC}(C) \cap \mathrm{bd}(C) = \mathbb{Z}^n \cap \mathrm{bd}(C)$ and hence $\mathrm{CGC}(C \cap P) \cap \mathrm{bd}(C) \cap P = \mathbb{Z}^n \cap \mathrm{bd}(C) \cap P$. Now, because P is full dimensional we have that $\mathrm{bd}(P) = \bigcup_{i=1}^m F_i$ and hence $\mathrm{CGC}(C \cap P) \cap \mathrm{bd}(P) = \bigcup_{i=1}^m F_i \cap \mathrm{CGC}(C \cap P) = \bigcup_{i=1}^m \mathrm{CGC}(C \cap F_i)$, where the last equality is obtained using Proposition 5.3. The result then follows from $\mathrm{CGC}(C \cap F_i) \cap C = \mathrm{CGC}(C \cap F_i)$ for all i. □

Using these results the proof of Theorem 2.3 is as follows.

PROOF OF THEOREM 2.3. The proof is by induction on the dimension of $C \cap P$. The base case is when the dimension is 0 and in this case $\mathrm{CGC}(C \cap P)$ is trivially a polyhedron ($\mathrm{CGC}(C \cap P) = C \cap P$ if $C \cap P \in \mathbb{Z}^n$ and $\mathrm{CGC}(C \cap P) = \emptyset$ otherwise).

Let $W = \mathrm{aff}\, C \cap P$. Without loss of generality, we may redefine $C := C \cap W$ and $P := P \cap W$ (Therefore $\dim(P) = \dim(C) = \dim(P \cap C)$). For the induction argument we have that, by Proposition 5.1, we may assume that $C \cap P$ is full-dimensional. By Theorem 2.2 , $\mathrm{CGC}(C)$ is a polyhedron. Let $S_0 \subseteq \mathbb{Z}^n$ be a finite set such that $\mathrm{CGC}_{S_0}(C) = \mathrm{CGC}(C)$. Now, let $\{F_i\}_{i=1}^m$, denote the facets of P. Then by the induction hypothesis, we have that $\mathrm{CGC}(C \cap F_i)$ is a polyhedron for each i. For each $S_i \subseteq \mathbb{Z}^n$ such that $\mathrm{CGC}_{S_i}(C \cap F_i) = \mathrm{CGC}(C \cap F_i)$, let $S_i' \subseteq \mathbb{Z}^n$ be the set given by Proposition 5.3 such that $\mathrm{CGC}_{S_i'}(C \cap P) \cap F_i = \mathrm{CGC}_{S_i}(C \cap F_i) = \mathrm{CGC}(C \cap F_i)$. Finally, let $S = S_0 \cup S_P \cup \bigcup_{i=1}^m S_i'$ where S_P is the finite set such that $P = \{x \in \mathbb{R}^n : \langle a, x \rangle \le \sigma_P(a) \,\forall a \in S_P\}$ (Observe that S_P exists because P is a rational polyhedron).

Noting than $\mathrm{CGC}_{S_0}(C) \subseteq C$ we obtain that $\mathrm{CGC}_S(C \cap P) \subseteq C \cap P$. Moreover, by Proposition 3.1 we have that $\mathrm{CGC}_{S_0}(C) \cap \mathrm{bd}(C) \subseteq \mathbb{Z}^n$ so

$$
\mathrm{CGC}_S(C \cap P) \cap \mathrm{bd}(C \cap P) = (\mathrm{CGC}_S(C \cap P) \cap \mathrm{bd}(C) \cap P) \cup (\mathrm{CGC}_S(C \cap P) \cap C \cap \mathrm{bd}(P)) \tag{14}
$$

$$
= (\mathrm{CGC}_S(C \cap P) \cap \mathrm{bd}(C) \cap P) \cup \bigcup_{i=1}^m (\mathrm{CGC}_S(C \cap P) \cap C \cap F_i) \tag{15}
$$

$$
= (\mathbb{Z}^n \cap \mathrm{bd}(C) \cap P) \cup \bigcup_{i=1}^m \mathrm{CGC}(C \cap F_i) \tag{16}
$$

$$
= \mathrm{CGC}(C \cap P) \cap \mathrm{bd}(C \cap P) \tag{17}
$$

where the last two containments follow from the definition of S, Proposition 5.3 and Corollary 5.2. Then S complies with the hypothesis of Proposition 4.2 and hence $\mathrm{CGC}(C \cap P)$ is a polyhedron. □

Appendix A. Omitted Proofs PROOF OF PROPOSITION 2.1. By Corollary C.3.1.2 in [20] we have that

$$
C = \bigcap_{a \in \mathbb{R}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \le \sigma(a)\}.
$$

It then suffices to show that

$$
\bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \le \sigma(a)\} \subseteq \bigcap_{a \in \mathbb{Q}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \le \sigma(a)\} \subseteq \bigcap_{a \in \mathbb{R}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \le \sigma(a)\}.
$$

The first containment follows from positive homogeneity of $\sigma_C(\cdot)$ and the second follows from density of $\mathbb{Q}^n$ in $\mathbb{R}^n$ and continuity of $\sigma_C(\cdot)$ and $\langle \cdot, x \rangle$. □

PROOF OF LEMMA 3.1.   Take $v \in S^{n-1}$. Since C is compact, the linear form $\langle v, . \rangle$ achieves its maximum over C. By linearity of $\langle v, . \rangle$ and since $v \neq 0$, any maximizer must be contained in $\mathrm{bd}(C)$. To show that m_C is well-defined, we need only guarantee that this maximum is unique. Assume then that $x, y \in \mathrm{bd}(C)$, $x \neq y$, such that

$$\langle v, x \rangle = \langle v, y \rangle = \sigma_C(v)$$

Then note that the line $[x, y] \subseteq \mathrm{bd}(C)$, since C is convex and $\langle v, z \rangle \leq \sigma_C(v)$ for all $z \in C$ implies $\langle v, z \rangle = \sigma_C(v)$ for all $z \in [x, y]$. But by the strict convexity of C, we have that $(x, y) \subseteq \mathrm{int}(C)$, a clear contradiction. Hence the form $\langle v, . \rangle$ has a unique maximizer as needed.

Now let $\{v^i\}_{i \in \mathbb{N}}$ be a sequence of vectors in S^{n-1} such that $v^i \xrightarrow{i \to \infty} v$. Let $x = m_C(v)$. To show that m_C is continuous it suffices to show that $m_C(v^i) \xrightarrow{i \to \infty} m_C(v) = x$. Assume not, then for some open neighborhood $\mathcal{N}$ of x, there exists a subsequence $\{v^{\alpha_i}\}_{i \in \mathbb{N}}$ such that $m_C(v^{\alpha_i}) \notin \mathcal{N}$ for all $i \in \mathbb{N}$. Note that the sequence $\{m_C(v^{\alpha_i})\}_{i \in \mathbb{N}}$ is an infinite sequence on a compact set $\mathrm{bd}(C)$. Hence there exists a convergent subsequence $\{m_C(v^{\beta_i})\}_{i \in \mathbb{N}}$ with limit $y \in \mathrm{bd}(C)$. Since by construction, $m_C(v^{\beta_i}) \notin \mathcal{N}$, we have that $\lim_{i \to \infty} m_C(v^{\beta_i}) = y \notin \mathcal{N}$. Since $x \in \mathcal{N}$, we have that $x \neq y$. Now we see that

$$\langle v, y \rangle = \left\langle \lim_{i \to \infty} v^{\beta_i}, \lim_{i \to \infty} m_C(v^{\beta_i}) \right\rangle = \lim_{i \to \infty} \left\langle v^{\beta_i}, m_C(v^{\beta_i}) \right\rangle$$
$$\geq \lim_{i \to \infty} \left\langle v^{\beta_i}, x \right\rangle = \langle v, x \rangle \geq \langle v, y \rangle$$

Now since $x = m_C(v)$, we have that $y \in \mathrm{bd}(C)$ is a maximizer of the form $\langle v, . \rangle$. But by strict convexity this maximizer is unique, and since $x \neq y$, we get a contradiction. Hence m_C is a continuous function as claimed.   □

PROOF OF PROPOSITION 5.2.   For a convex set C let

$$\mathrm{i}_C(x) := \begin{cases} 0 & x \in C \\ +\infty & o.w. \end{cases}$$

be its indicator and for convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ let $f^*(a) := \sup\{\langle a, x \rangle - f(x) : x \in \mathrm{dom}(f)\}$ be its conjugate or Legendre-Fenchel transform. Now, let $g_1 = \mathrm{i}_C$ and $g_2 = \mathrm{i}_P$ so that $(g_1)^* = \sigma_C$, $(g_2)^* = \sigma_P$ and $\mathrm{i}_{C \cap P} = g_1 + g_2$. Because $P \cap \mathrm{int}(C) \neq \emptyset$ we have that $\mathrm{int}(\mathrm{dom}(g_1)) \cap \mathrm{dom}(g_2) \neq \emptyset$ and we hence have qualification assumption (2.3.Q.jj') in page 228 of [20]. We can then use Theorem E.2.3.2 in [20] to obtain that $\sigma_{P \cap C}(a) = (\mathrm{i}_{C \cap P})^*(a) = \inf\{\sigma_P(a_P) + \sigma_C(a_C) : a = a_P + a_C\}$ and that for every $a \in \mathbb{R}^n$ there exists $a_P, a_C \in \mathbb{R}^n$ such that $\sigma_{P \cap C}(a) = \sigma_P(a_P) + \sigma_C(a_C)$.   □

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