A STRONG DUAL FOR CONIC MIXED-INTEGER PROGRAMS*

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Abstract. Mixed-integer conic programming is a generalization of mixed-integer linear programming. In this paper, we present an extension of the duality theory for mixed-integer linear programming (see [4], [11]) to the case of mixed-integer conic programming. In particular, we construct a subadditive dual for mixed-integer conic programming problems. Under a simple condition on the primal problem, we show that strong duality holds.

Key words. Mixed-integer nonlinear programming, Conic programming, Duality, Cutting planes

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1. Introduction. One of the fundamental goals of optimization theory is the study of structured techniques to obtain bounds on the optimal objective function value for a given class of optimization problems. For a minimization (resp. maximization) problem, upper (resp. lower) bounds on the optimal objective function value are provided by points belonging to the feasible region. Dual bounds, that is, lower (resp. upper) bounds on the optimal objective function value for a minimization (resp. maximization) problem are typically obtained by constructing various types of dual optimizations problems whose feasible solutions provide these bounds. We will say that a minimization (resp. maximization) problem is finite if its feasible region is nonempty and the objective function is bounded from below (resp. above). A strong dual is typically characterized by two properties:

1. The primal program is finite if and only if the dual program is finite.
2. If the primal and the dual are finite, then the optimal objective function values of the primal and dual are equal.

In the case of linear programming problems and more generally for conic (convex) optimization problems the dual optimization problem is well understood and plays a key role in various algorithmic devices [2]. The subadditive dual for mixed-integer linear programs is also well understood [5, 6, 7, 9, 13]. In this paper, we evaluate the possibility of extending the subadditive dual to the case of mixed-integer conic programs.

The rest of the paper is organized as follows. In Section 2, we present the necessary notation, definitions and the statement of our main result. In Section 3, we verify the basic weak duality result, that is, the fact that the dual feasible solutions produce valid bounds. Apart from weak duality, the proof of strong duality relies on the following additional three results: (i) The finiteness of the primal being equivalent to the finiteness of its continuous relaxation. (ii) Strong duality for conic programs. (iii) The possibility of constructing a subadditive function defined over $\mathbb{R}^m$ such that it is dual feasible and matches the value function of the primal on a relevant subset of $\mathbb{R}^m$. In Section 4, we develop and present (in the case of conic duality) these preliminary results. In Section 5, we present the proof of the strong duality result. In particular,

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in Section 5.1, we present a sufficient condition for the finiteness of the primal program being equivalent to the finiteness of the dual program. In Section 5.2, we prove that under this sufficient condition, if the primal and dual are finite, then their optimal values must be equal. In Section 6, we discuss valid subadditive inequalities for conic mixed-integer programs. Finally, in Section 7 we present the dual for an alternative form of the primal conic mixed-integer program.

2. Notation, definitions and main result. Let $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Let $K \subseteq \mathbb{R}^m$ be a full-dimensional, closed and pointed cone. A conic vector inequality is defined as follows ([2]):

Definition 2.1 (Conic vector inequality). *For $a, b \in \mathbb{R}^m$, $a \succeq_K b$ if and only if $a - b \in K$. In addition, we write $a \succ_K b$ whenever $a - b \in \text{int}(K)$.*

A mixed-integer conic programming problem (the primal optimization problem) is an optimization problem of the following form:

$$(\mathcal{P}) \begin{cases} z^* = \inf \quad c^t x \\ \qquad s.t. \quad Ax \succeq_K b \\ \qquad\qquad x_i \in \mathbb{Z},\ \forall i \in \mathcal{I}, \end{cases}$$

where $\mathcal{I} = \{1, \dots, n_1\} \subseteq \{1, \dots, n\}$ is the set of indices of integer variables.

Notice that problems of the form of $(\mathcal{P})$ are a generalization of mixed-integer linear programming problems, which are recovered by setting $K = \mathbb{R}^m_+$. Hence, a natural way of defining a dual optimization problem for mixed-integer conic programming is to generalize the well-known subadditive dual for mixed-integer linear programming (see, for example, [4] and [11]). Consequently, to define the dual of $(\mathcal{P})$, we first present some notation and definitions that are slight variations of those necessary to define the subadditive dual for mixed-integer linear programming problems.

Definition 2.2 (Subadditive). *Let $\mathcal{S} \subseteq \mathbb{R}^m$. A function $g : \mathcal{S} \mapsto \mathbb{R} \cup \{-\infty\}$ is said to be subadditive if for all $u, v \in \mathcal{S}$ such that $u + v \in \mathcal{S}$, the inequality $g(u + v) \leq g(u) + g(v)$ holds.*

Definition 2.3 (Nondecreasing w.r.t. K). *Let $\mathcal{S} \subseteq \mathbb{R}^m$. A function $g : \mathcal{S} \mapsto \mathbb{R} \cup \{-\infty\}$ is said to be nondecreasing w.r.t. K if for all $u, v \in \mathcal{S}$ such that $u \succeq_K v$, the inequality $g(u) \geq g(v)$ holds.*

We define the subadditive dual problem for $(\mathcal{P})$ as follows:

$$(\mathcal{D}) \begin{cases} \rho^* = \sup \quad g(b) \\ \qquad s.t. \quad g\left(A^i\right) = -g\left(-A^i\right) = c_i, \quad \forall i \in \mathcal{I} \\ \qquad\qquad \bar{g}\left(A^i\right) = -\bar{g}\left(-A^i\right) = c_i, \quad \forall i \in \mathcal{C} \\ \qquad\qquad g(0) = 0 \\ \qquad\qquad g \in \mathcal{F}, \end{cases}$$

where $\mathcal{C} = \{n_1 + 1, \dots, n\}$ is the set of indices of continuous variables, A^i denotes the ith column of A, for a function $g : \mathbb{R}^m \mapsto \mathbb{R}$ we write $\overline{g}(d) = \limsup_{\delta\to 0^+} \frac{g(\delta d)}{\delta}$ and $\mathcal{F} = \{g : \mathbb{R}^n \mapsto \mathbb{R} : g \text{ is subadditive and nondecreasing w.r.t. } K\}$.

Notice that when $K = \mathbb{R}^m_+$, we retrieve the subadditive dual for a mixed-integer linear programming problem. In this paper, we generalize the subadditive dual for mixed-integer linear programming as presented in Section II.3.3 of [11]. In [4] a different form of primal is used, and as a consequence, the form of the dual presented in that paper is slightly different from the dual shown in [11]. In the mixed-integer linear case both approaches are equivalent.

In the subadditive duality theory for mixed-integer linear programming ($K = \mathbb{R}^m_+$), a sufficient condition to have strong duality is the rationality of the data defining the problem, that is, $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$. The main result of this paper is to show that strong duality for mixed-integer conic programming holds under the following technical condition

$$\text{there exists } \hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \text{ such that } A\hat{x} \succ_K b. \quad (*),$$

where $n_2 = n - n_1$. We state the main result formally next.

THEOREM 2.4 (Strong duality). *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$, then*

1. *$(\mathcal{P})$ is finite if and only if $(\mathcal{D})$ is finite.*
2. *If $(\mathcal{P})$ is finite, then there exists a function g^* feasible for $(\mathcal{D})$ such that $g^*(b) = z^*$ and consequently $z^* = \rho^*$.*

Condition $(*)$ in the case of mixed-integer conic programs plays the same role as the assumption of rational data in the case of mixed-integer programs in the proof of the strong duality result. In particular, we will see in Section 4.1 that both are sufficient conditions for the finiteness of the corresponding convex mixed-integer problem being equivalent to the finiteness of its continuous relaxation.

3. Weak duality. As in the case of mixed-integer linear programming, weak duality is a straightforward consequence of the definition of the subadditive dual. We first require a well-known result relating g and $\bar{g}$ when g is a subadditive function.

THEOREM 3.1 ([5], [8], and [11]). *If $g : \mathbb{R}^m \mapsto \mathbb{R}$ is a subadditive function such that $g(0) = 0$, then $\forall\, d \in \mathbb{R}^m$ with $\overline{g}(d) < \infty$ and $\forall\, \lambda \geq 0$ we have that $g(\lambda d) \leq \lambda\overline{g}(d)$.*

PROPOSITION 3.2 (Weak duality). *For all $x \in \mathbb{R}^n$ feasible for $(\mathcal{P})$ and for all $g : \mathbb{R}^m \mapsto \mathbb{R}$ feasible for $(\mathcal{D})$, we have that $g(b) \leq c^t x$.*

Proof. Let $u, v \geq 0$ such that $x = u - v$. We have

$$\begin{aligned}
g(b) &\leq g(Ax)\\
&= g(Au - Av)\\
&= g\left(\sum_{i=1}^{n} A^i u_i + \sum_{i=1}^{n} (-A^i) v_i\right)\\
&= g\left(\sum_{i\in\mathcal{I}} A^i u_i + \sum_{i\in\mathcal{I}} (-A^i) v_i + \sum_{i\in\mathcal{C}} A^i u_i + \sum_{i\in\mathcal{C}} (-A^i) v_i\right)\\
&\leq \sum_{i\in\mathcal{I}} g(A^i u_i) + \sum_{i\in\mathcal{I}} g(-A^i v_i) + \sum_{i\in\mathcal{C}} g(A^i u_i) + \sum_{i\in\mathcal{C}} g(-A^i v_i)\\
&\leq \sum_{i\in\mathcal{I}} g(A^i) u_i + \sum_{i\in\mathcal{I}} g(-A^i) v_i + \sum_{i\in\mathcal{C}} \overline{g}(A^i) u_i + \sum_{i\in\mathcal{C}} \overline{g}(-A^i) v_i\\
&= \sum_{i=1}^{n} c_i u_i + \sum_{i=1}^{n} (-c_i) v_i\\
&= c^t x.
\end{aligned}$$

The first inequality relies on the fact that x satisfies $Ax \succeq_K b$ and g is nondecreasing w.r.t. K and the second inequality relies on the fact that g is subadditive. The third inequality is based on the subadditivity of g, the fact that $g(0) = 0$ and Theorem 3.1. ∎

We obtain the following corollary of Proposition 3.2.

COROLLARY 3.3.

1. *If the primal problem* $(\mathcal{P})$ *is unbounded, then the dual problem* $(\mathcal{D})$ *is infeasible.*
2. *If the dual problem* $(\mathcal{D})$ *is unbounded, then the primal problem* $(\mathcal{P})$ *is infeasible.*

4. Preliminary results for proving strong duality.

4.1. Finiteness of a convex mixed-integer problem being equivalent to the finiteness of its continuous relaxation. In this section, we study a sufficient condition for the finiteness of the primal $(\mathcal{P})$ being equivalent to the finiteness of its continuous relaxation. This condition is required to show that the primal program is finite if and only if the dual program is finite.

The main result of this section is a sufficient condition for property

$$\inf\{c^t x \,:\, x \in B\} = -\infty \Leftrightarrow \inf\{c^t x \,:\, x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\} = -\infty \tag{4.1}$$

to hold in the context of general convex mixed-integer optimization, that is, when the feasible region of the primal is of the form $B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$, where $B \subseteq \mathbb{R}^n$ is a closed convex set and $n = n_1 + n_2$.

The next example shows that property (4.1) is not always satisfied, not even when the feasible set is a polyhedron.

EXAMPLE 4.1. *Consider the polyhedral set* $B_1 = \{x \in \mathbb{R}^2 \,:\, x_2 - \sqrt{2}x_1 = 0\}$ *and let the objective function be given by* $c = (1, \sqrt{2})$*. In this case* $B_1 \cap \mathbb{Z}^2 = \{0, 0\}$*, so* $\inf\{c^t x \,:\, x \in B_1 \cap \mathbb{Z}^n\} = 0$*. On the other hand,* $\inf\{c^t x \,:\, x \in B_1\} = -\infty$*. Therefore, the integer programming problem has finite optimal objective function value, but its relaxation has unbounded objective function value.*

When the feasible set is a polyhedron, a well-known sufficient condition for property (4.1) to be true is that the polyhedron is defined by rational data. However, the following example shows that (4.1) is not necessarily true when the convex set B is full-dimensional, $\operatorname{conv}(B \cap \mathbb{Z}^n)$ is a polyhedron and B is conic quadratic representable using rational data.

EXAMPLE 4.2. *Consider the set*

$$\begin{aligned} B_2 = \operatorname{conv}(&\{x \in \mathbb{R}^3 \,:\, x_3 = 0, x_1 = 0, x_2 \geq 0\} \\ &\cup \{x \in \mathbb{R}^3 \,:\, x_3 = 0.5, x_2 \geq x_1^2\} \\ &\cup \{x \in \mathbb{R}^3 \,:\, x_3 = 1, x_1 = 0, x_2 \geq 0\}). \end{aligned}$$

Notice B_2 *is full-dimensional, closed (the sets defining* B_2 *have the same recession cone) and is defined by rational data. Observe that* $\operatorname{conv}(B_2 \cap \mathbb{Z}^3) = \{x \in \mathbb{R}^3 \,:\, x_1 = 0, x_2 \geq 0, 0 \leq x_3 \leq 1\}$ *is a polyhedron. Since we have* $\inf\{x_1 \,:\, x \in B_2\} = -\infty$ *and* $\inf\{x_1 \,:\, x \in B_2 \cap \mathbb{Z}^3\} = 0 > -\infty$*, the set* B_2 *does not satisfy property (4.1).*

Finally, it can be shown that the set B_2 *is conic quadratic representable using rational data, that is, there exists a rational matrix* A *and a rational vector* b *such that*

$$B_2 = \left\{ x \in \mathbb{R}^3 \,:\, \exists u \; A \begin{pmatrix} x \\ u \end{pmatrix} \succeq_K b \right\},$$

where K *is a direct product of Lorentz cones (see [2]).*

Before we state the sufficient condition for property (4.1) to hold, we give some definitions and preliminary results that will be needed to prove the validity of this condition.

A linear subspace $L \subseteq \mathbb{R}^n$ is said to be a rational linear subspace if there exists a basis of L formed by rational vectors. A convex set $B \subseteq \mathbb{R}^n$ is called lattice-free, if $\text{int}(B) \cap \mathbb{Z}^n = \emptyset$. A lattice-free convex set $B \subseteq \mathbb{R}^n$ is called maximal lattice-free convex set if does not exist a lattice-free convex set $B' \subseteq \mathbb{R}^n$ satisfying $B \subsetneq B'$.

The following result is from [10]. See also [1] for a related result.

THEOREM 4.3 ([10]). *Every lattice-free convex set is contained in some maximal lattice-free convex set. A full-dimensional lattice-free convex set $B \subseteq \mathbb{R}^n$ is a maximal lattice-free convex set if and only if B is a polyhedron of the form $B = P + L$, where P is a polytope, L is a rational linear subspace and every facet of B contains a point of $\mathbb{Z}^n$ in its relative interior.*

We require a Corollary of Theorem 4.3.

COROLLARY 4.4. *Let $B \subseteq \mathbb{R}^n$ be a full-dimensional convex set. Let $n_1 + n_2 = n$. If $\text{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$, then there exists a polytope $P \subseteq \mathbb{R}^n$ and a rational subspace $L \subseteq \mathbb{R}^n$ such that $Q = P + L$ satisfies $\text{int}(Q) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$ and $B \subseteq Q$.*

Proof. Let $p : \mathbb{R}^n \mapsto \mathbb{R}^{n_1}$ denote the projection on to the first n_1 components and $\text{int}_{\mathbb{R}^{n_1}}(p(B))$ denote the interior of $p(B)$ with respect to $\mathbb{R}^{n_1}$. By Theorem 6.6 of [12] and since $\text{rel.int}(B) = \text{int}(B)$, we obtain that $\text{rel.int}(p(B)) = p(\text{rel.int}(B)) = p(\text{int}(B))$. Thus, $\dim(p(B)) = n_1$. Therefore, $\text{int}_{\mathbb{R}^{n_1}}(p(B)) = \text{rel.int}(p(B)) = p(\text{int}(B))$.

We show next that $\text{int}_{\mathbb{R}^{n_1}}(p(B)) \cap \mathbb{Z}^{n_1} = \emptyset$. Since $\text{int}_{\mathbb{R}^{n_1}}(p(B)) = p(\text{int}(B))$, if $x \in \text{int}_{\mathbb{R}^{n_1}}(p(B))$, then there exists $y \in \mathbb{R}^{n_2}$ such that $(x, y) \in \text{int}(B)$. Hence, since $\text{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$, we obtain that $x \notin \mathbb{Z}^{n_1}$. Thus, $p(B)$ is a full-dimensional lattice-free convex set of $\mathbb{R}^{n_1}$. Therefore, by Theorem 4.3, there exists a polytope $P_1 \subseteq \mathbb{R}^{n_1}$ and a rational subspace $L_1 \subseteq \mathbb{R}^{n_1}$ such that $Q_1 = P_1 + L_1$ satisfies $\text{int}_{\mathbb{R}^{n_1}}(Q_1) \cap \mathbb{Z}^{n_1} = \emptyset$ and $p(B) \subseteq Q_1$.

Now,

$$\begin{aligned} B &\subseteq p(B) \times \mathbb{R}^{n_2} \\ &\subseteq Q_1 \times \mathbb{R}^{n_2} \\ &= (P_1 + L_1) \times \mathbb{R}^{n_2} \\ &= [P_1 \times \{0\}] + [L_1 \times \mathbb{R}^{n_2}]. \end{aligned}$$

So, by considering, $P = P_1 \times \{0\}$, $L = L_1 \times \mathbb{R}^{n_2}$ and $Q = P + L$, and observing that $\text{int}(Q) = \text{int}_{\mathbb{R}^{n_1}}(Q_1) \times \mathbb{R}^{n_2}$, we arrive at the desired conclusion. □

The sufficient condition for (4.1) to hold is stated in the following result. The proof of this result is modified from a result in [3].

PROPOSITION 4.5. *Let $n_1 + n_2 = n$ and let $B \subseteq \mathbb{R}^n$ be a convex set such that $\text{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$. Then*

$$\inf\{c^t x : x \in B\} = -\infty \Leftrightarrow \inf\{c^t x : x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\} = -\infty.$$

Proof.

($\Leftarrow$) Clearly, if $\inf\{c^t x : x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\} = -\infty$, then we must have that $\inf\{c^t x : x \in B\} = -\infty$.

($\Rightarrow$) Suppose $\inf\{c^t x : x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})\} = d > -\infty$. We will show that $\inf\{c^t x : x \in B\} > -\infty$. Assume for a contradiction that $\inf\{c^t x : x \in B\} = -\infty$. Consider the set $B^{\leq} = B \cap \{x \in \mathbb{R}^n : c^t x \leq d\}$. Notice that since $\text{int}(B) \cap (\mathbb{Z}^{n_1} \times$

$\mathbb{R}^{n_2}) \neq \emptyset$, we obtain that B is a full-dimensional set. Also, by assumption, we have that $B \nsubseteq \{x \in \mathbb{R}^n : c^t x \geq d\}$. Therefore, $\mathrm{int}(B) \cap \{x \in \mathbb{R}^n : c^t x < d\} \neq \emptyset$. This implies that $\mathrm{int}(B^{\leq}) = \mathrm{int}(B) \cap \{x \in \mathbb{R}^n : c^t x < d\} \neq \emptyset$ and thus $B^{\leq}$ is a full-dimensional set.

Moreover, we have that $\mathrm{int}(B^{\leq}) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$, since $\mathrm{int}(B^{\leq}) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = (\mathrm{int}(B^{\leq}) \cap B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \subseteq \mathrm{int}(B^{\leq}) \cap (B \cap \{x \in \mathbb{R}^n : c^t x \geq d\}) \subseteq \{x \in \mathbb{R}^n : c^t x < d\} \cap \{x \in \mathbb{R}^n : c^t x \geq d\} = \emptyset$. Hence, by Corollary 4.4 there exists a full-dimensional polyhedron $Q = \{x \in \mathbb{R}^n : a_k^t x \leq b_k,\ k \in \{1, \ldots, q\}\}$ such that $Q = P + L$, where P is a polytope and L a rational linear subspace, $\mathrm{int}(Q) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$ and $B^{\leq} \subseteq Q$.

Since $\mathrm{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$, we obtain that $B \nsubseteq Q$. Therefore, there exists $x_0 \in B \setminus Q$, that is, $x_0 \in B$ and $a_j^t x_0 > b_j$, for some $j \in \{1, \ldots, q\}$. Notice that, since $B^{\leq} \subseteq Q$, we have that $x_0 \notin B^{\leq}$. Thus, we obtain that $c^t x_0 > d$.

On the other hand, since $Q \subseteq \{x \in \mathbb{R}^n : a_j^t x \leq b_j\}$, we have that $\sup\{a_j^t x : x \in Q\} < \infty$. Therefore, since $\text{rec.cone}(Q) = L$, we must have $a_j^t r = 0$, for all $r \in \text{rec.cone}(Q)$. Hence, $\inf\{a_j^t x : x \in Q\} > -\infty$, implying that there exists $M > 0$ such that $Q \subseteq \{x \in \mathbb{R}^n : a_j^t x \geq b_j - M\}$.

Let $\{x_n\}_{n=1}^{\infty} \subseteq B^{\leq}$ such that $\lim_{n\to\infty} c^t x_n = -\infty$ and let $\lambda_n \in (0, 1]$ such that $y_n = (1 - \lambda_n)x_0 + \lambda_n x_n$ satisfies $c^t y_n = d$.

Notice that

$$\begin{aligned} a_j^t y_n - b_j &= (1 - \lambda_n) a_j^t x_0 + \lambda_n a_j^t x_n - b_j \\ &\geq (1 - \lambda_n)(a_j^t x_0 - b_j) - \lambda_n M \\ &= (a_j^t x_0 - b_j) - \lambda_n [(a_j^t x_0 - b_j) + M], \end{aligned} \tag{4.2}$$

where the inequality follows from the fact that $\{x_n\}_{n=1}^{\infty} \subseteq B^{\leq} \subseteq Q \subseteq \{x \in \mathbb{R}^n : a_j^t x \geq b_j - M\}$.

On the other hand, by definition of λ_n, we have that $\lambda_n = \frac{d - c^t x_0}{c^t x_n - c^t x_0}$ and thus $\lim_{n\to\infty} \lambda_n = 0$. Hence, by (4.2), for sufficiently large N, we obtain that $a_j^t y_N > b_j$. Also, since B is a convex set and y_N is a convex combination of $x_0, x_N \in B$ we obtain that $y_N \in B$. Thus, $y_N \in B^{\leq} \subseteq Q$, a contradiction with $a_j^t y_N > b_j$. Therefore, we must have $\inf\{c^t x : x \in B\} > -\infty$. ∎

The condition that there exists a mixed-integer feasible solution in the interior of the continuous relaxation is crucial for Proposition 4.5. This is illustrated in Example 4.1 and Example 4.2, where B_1 and B_2, respectively, do not satisfy property (4.1). Finally, observe that the converse of Proposition 4.5 is not true; consider any lattice-free rational unbounded polyhedron.

4.2. Strong duality for conic programming. In mixed-integer linear programming, the proof of strong duality for the corresponding subadditive dual relies on the existence of a strong duality result for linear programming. Unfortunately, unlike the case of linear programming, strong duality for conic programming requires some additional assumptions. Therefore, it is not surprising that we require the extra condition $(*)$ to prove strong duality for mixed-integer conic programming. We recall the duality theorem for conic programming ([2]).

THEOREM 4.6 (Duality for conic programming). *Let $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Let $K \subseteq \mathbb{R}^m$ be a full-dimensional, closed and pointed cone. Denote $K_* = \{y \in \mathbb{R}^m : y^t x \geq 0,\ \forall x \in K\}$, the dual cone of K. Then:*

1. *(*Weak duality*) For all $x \in \{x \in \mathbb{R}^n : Ax \succeq_K b\}$ and $y \in \{y \in \mathbb{R}^m : A^t y = c,\ y \succeq_{K_*} 0\}$ we have that $b^t y \leq c^t x$.*

2. *(*Strong duality*) If there exists $\hat{x} \in \mathbb{R}^n$ such that $A\hat{x} \succ_K b$ and $\inf\{c^t x : Ax \succeq_K b\} > -\infty$, then*

$$\inf\{c^t x : Ax \succeq_K b\} = \max\{b^t y : A^t y = c,\ y \succeq_{K_*} 0\}.$$

4.3. Value function of $(\mathcal{P})$**.** In this section we study some properties of the value function of $(\mathcal{P})$. The motivation is the following: we will verify that the value function satisfies all the properties of the feasible functions of the dual $(\mathcal{D})$, except that it might not be defined over all of $\mathbb{R}^m$.

We begin with some notation. For $u \in \mathbb{R}^m$, let $S(u) = \{x \in \mathbb{R}^n : Ax \succeq_K u,\ x_i \in \mathbb{Z},\ \forall\, i \in \mathcal{I}\}$ be the feasible region with right-hand-side u and let $P(u) = \{x \in \mathbb{R}^n : Ax \succeq_K u\}$ be its continuous relaxation. Let $\Omega = \{u \in \mathbb{R}^m : S(u) \neq \emptyset\}$. Notice that since $0 \in S(0)$, we have that $\Omega \neq \emptyset$.

Definition 4.7 (Value function of $(\mathcal{P})$). *The value function of a mixed-integer conic program is the function $f : \Omega \mapsto \mathbb{R} \cup \{-\infty\}$, defined as*

$$f(u) = \inf\{c^t x \,:\, x \in S(u)\}.$$

We show next some basic properties of the value function.

Proposition 4.8. *Let $f : \Omega \mapsto \mathbb{R} \cup \{-\infty\}$ be the value function of $(\mathcal{P})$, then*

1. *f is subadditive on Ω.*
2. *f is nondecreasing w.r.t. K on Ω.*
3. *If $f(0) = 0$, then $\underline{f}(A^i) = -\underline{f}(-A^i) = c_i$, $\forall\, i \in \mathcal{I}$.*
4. *If $f(0) = 0$, then $\overline{f}(A^i) = -\overline{f}(-A^i) = c_i$, $\forall\, i \in \mathcal{C}$.*
5. *Let $u \in \Omega$. If $f(u) > -\infty$, then $f(0) = 0$.*

Proof.

1. Let $u_1, u_2 \in \Omega$, $x_1 \in S(u_1)$ and $x_2 \in S(u_2)$. By additivity of $\succeq_K$, we have that $x_1 + x_2 \in S(u_1 + u_2)$. This implies that $c^t x_1 + c^t x_2 \geq f(u_1 + u_2)$. By taking the infimum over $x_i \in S(u_i)$, $i = 1, 2$ we conclude that $f(u_1) + f(u_2) \geq f(u_1 + u_2)$.
2. Let $u_1, u_2 \in \Omega$, $u_1 \succeq_K u_2$. Let $x \in S(u_1)$. By transitivity of $\succeq_K$, we have that $x \in S(u_2)$. Therefore, $S(u_1) \subseteq S(u_2)$, and thus we obtain that $f(u_1) \geq f(u_2)$.
3. For $i \in \mathcal{I}$ and $\alpha \in \{-1, 1\}$, since the vector $x_i = \alpha,\ x_j = 0, \forall j \neq i$ is feasible for $(\mathcal{P})$ with right-hand-side $b = \alpha A^i$, we have that $f(\alpha A^i) \leq \alpha c_i$. Since f is subadditive, we obtain $0 = f(0) \leq f(\alpha A^i) + f(-\alpha A^i)$. Therefore, $\alpha c_i \leq -f(-\alpha A^i) \leq f(\alpha A^i) \leq \alpha c_i$, and thus we obtain that $f(\alpha A^i) = \alpha c_i$. Equivalently, for $i \in \mathcal{I}$, we have that $f(A^i) = c_i$ and $f(-A^i) = -c_i$.
4. For $i \in \mathcal{C}$ and $\delta \in \mathbb{R}$, since the vector $x_i = \delta,\ x_j = 0, \forall j \neq i$ is feasible for $(\mathcal{P})$ with right-hand-side $b = \delta A^i$, we have that $f(\delta A^i) \leq \delta c_i$. Since f is subadditive, we obtain $0 = f(0) \leq f(\delta A^i) + f(-\delta A^i)$. Therefore, $\delta c_i \leq -f(-\delta A^i) \leq f(\delta A^i) \leq \delta c_i$, so, $f(\delta A^i) = \delta c_i$. This implies that $\overline{f}(A^i) = \limsup_{\delta \to 0^+} \frac{f(\delta A^i)}{\delta} = c_i$ and $\overline{f}(-A^i) = \limsup_{\delta \to 0^+} \frac{f(\delta(-A^i))}{\delta} = -c_i$. Therefore, for $i \in \mathcal{C}$, $\overline{f}(A^i) = c_i$ and $\overline{f}(-A^i) = -c_i$.
5. We verify the contrapositive of this statement. Assume $f(0) < 0$. Then there exists $\bar{x} \in S(0)$ such that $c^t \bar{x} < 0$. For all $\lambda \in \mathbb{Z}_+$, we have that $\lambda \bar{x} \in S(0)$ and $c^t(\lambda \bar{x}) = \lambda c^t \bar{x} < 0$. Let $x \in S(u)$. By additivity of $\succeq_K$, we have that $x + \lambda \bar{x} \in S(u)$ for all $\lambda \in \mathbb{Z}_+$. Since $c^t(x + \lambda \bar{x}) = c^t x + \lambda c^t \bar{x}$, we obtain that $f(u) = -\infty$.

□

Since the value function f might not be defined over $\mathbb{R}^m$, it is not necessarily a feasible solution to the dual. In the next section, we will construct a new function that is equal to f on b, is finite-valued over $\mathbb{R}^m$, and continues to satisfy all the conditions of the dual $(\mathcal{D})$. The following corollary of Proposition 4.8 and the subsequent two propositions are crucial in this construction.

Corollary 4.9. *Let $K_1 \subseteq \mathbb{R}^p$, $K_2 \subseteq \mathbb{R}^q$ be full-dimensional closed and pointed convex cones. Let $S(u,v) = \{(x,y) \in \mathbb{R}^{(p+q)} : Ax + Cy \succeq_{K_1} u,\ Fy \succeq_{K_2} v,\ x_i \in \mathbb{Z}, \forall\ i \in \mathcal{I}_p,\ y_i \in \mathbb{Z}, \forall\ i \in \mathcal{I}_q\}$, where $\mathcal{I}_p \subseteq \{1,\dots,p\}$, $\mathcal{I}_q \subseteq \{1,\dots,q\}$, and let $\Omega_p = \{u \in \mathbb{R}^p : S(u,0) \neq \emptyset\}$. Let*

$$G(u,v) = \inf\{c^t x + d^t y : (x,y) \in S(u,v)\}.$$

Consider $g : \Omega_p \mapsto \mathbb{R}$ defined as $g(u) = G(u,0)$. Then

1. *g is subadditive on Ω_p.*
2. *g is nondecreasing w.r.t. K_1 on Ω_p.*
3. *If $G(0,0) = 0$, then $g(A^i) = -g(-A^i) = c_i$, $\forall\ i \in \mathcal{I}_p$ and $\overline{g}(A^i) = -\overline{g}(-A^i) = c_i$, $\forall\ i \in \{1,\dots,p\} \setminus \mathcal{I}_p$.*

Proof.

1. Observe that, by (1.) of Proposition 4.8, G is subadditive on its domain. Let $u_1, u_2 \in \Omega_p$. Then

$$g(u_1 + u_2) = G[(u_1,0) + (u_2,0)] \leq G(u_1,0) + G(u_2,0) = g(u_1) + g(u_2),$$

where the inequality is a consequence of G being subadditive.

2. Observe that, by (2.) of Proposition 4.8, G is nondecreasing w.r.t. $K_1 \times K_2$ on its domain. Also, $u_1 \succeq_{K_1} u_2$ if and only if $(u_1,0) \succeq_{K_1 \times K_2} (u_2,0)$. Therefore, if $u_1 \succeq_{K_1} u_2$, then $g(u_1) = G(u_1,0) \geq G(u_2,0) = g(u_2)$, as desired.
3. If $G(0,0) = 0$, then, by (3.) and (4.) of Proposition 4.8, for $\alpha \in \{-1,1\}$ we obtain that $g(\alpha A^i) = G(\alpha A^i, 0) = \alpha c_i$, $\forall\ i \in \mathcal{I}_p$ and $\overline{g}(\alpha A^i) = \overline{G}(\alpha A^i, 0) = \alpha c_i$, $\forall\ i \in \{1,\dots,p\} \setminus \mathcal{I}_p$.

□

The next proposition states a sufficient condition for $\Omega = \mathbb{R}^m$ to hold, that is, $S(b) \neq \emptyset$ for all $b \in \mathbb{R}^m$.

Proposition 4.10. *If there exists $\hat{x} \in \mathbb{R}^n$ such that $A\hat{x} \succ_K 0$, then $\forall\ b \in \mathbb{R}^m$, there exists $x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $Ax \succ_K b$.*

Proof. Let $b \in \mathbb{R}^m$. It is sufficient to prove that there exists $x \in \mathbb{Z}^n$ such that $Ax \succ_K b$. We will show this next. Since $A\hat{x} \succ_K 0$, there exists $\varepsilon > 0$ such that $B(A\hat{x}, \varepsilon) \subseteq K$, where $B(A\hat{x}, \varepsilon)$ is the open ball of radius ε around $A\hat{x}$. Therefore, by continuity of Ax and density of $\mathbb{Q}^n$ in $\mathbb{R}^n$, there exists $q \in \mathbb{Q}^n$ such that $Aq \in B(A\hat{x}, \varepsilon)$. This implies, by a suitable positive scaling of q, that there exists $z \in \mathbb{Z}^n$ such that $Az \in \text{int}(K)$. Hence, there exists $\delta > 0$ such that $B(Az, \delta) \subseteq K$. For $M \in \mathbb{N}$ sufficiently large, we obtain that $Az - \frac{b}{M} \in B(Az, \delta) \subseteq K$. Thus, scaling by $M > 0$, we obtain that $A(Mz) - b \in \text{int}(K)$, that is, $A(Mz) \succ_K b$, as desired. □

The following result gives a condition such that if the primal is finite for some right-hand-side b, then it is also is finite for every right-hand-side $u \in \Omega$.

Proposition 4.11. *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$ and $f(b) > -\infty$, then $\forall\ u \in \Omega$ we have that $\inf\{c^t x : x \in P(u)\} > -\infty$. In particular, $\forall\ u \in \Omega$, we have that $f(u) > -\infty$.*

Proof. Since $\{x \in \mathbb{R}^n : Ax \succ_K b\} = \text{int}(P(b))$, we have that $\text{int}(P(b)) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$. Therefore, since $f(b) > -\infty$, by Proposition 4.5, we obtain that $\inf\{c^t x : x \in P(b)\} > -\infty$. This implies, by (2.) of Theorem 4.6, that the set $\{y : A^t y = c,\ y \succeq_{K_*} 0\}$ is nonempty.

Let $u \in \Omega$ and let $\bar{y} \in \{y : A^t y = c,\ y \succeq_{K_*} 0\}$. By (1.) of Theorem 4.6 we obtain that $\inf\{c^t x : x \in P(u)\} \geq u^t \bar{y}$, as required. ☐

5. Strong duality. For ease of exposition, we recall next the main result of this paper.

THEOREM 2.4 (Strong duality). *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$, then*

1. *$(\mathcal{P})$ is finite if and only if $(\mathcal{D})$ is finite.*
2. *If $(\mathcal{P})$ is finite, then there exists a function g^* feasible for $(\mathcal{D})$ such that $g^*(b) = z^*$ and, consequently, $z^* = \rho^*$.*

We will prove Theorem 2.4 in the following two subsections.

5.1. Finiteness of the primal being equivalent to the finiteness of the dual. In this section we present a sufficient condition for the following to hold: $(\mathcal{P})$ is finite if and only if $(\mathcal{D})$ is finite. Observe that essentially we need conditions under which the converse of Corollary 3.3 holds.

We begin by showing that 'a part' of the converse of Corollary 3.3 holds generally. The proof is modified from a result in [4].

PROPOSITION 5.1. *If the primal problem is infeasible, then the dual is unbounded or infeasible.*

Proof. If the dual problem is feasible, then we need to verify that it is unbounded. Define $G : \mathbb{R}^m \mapsto \mathbb{R}$ as $G(d) = \min\{x_0 : Ax + x_0 d \succeq_K d,\ x_i \in \mathbb{Z},\ i \in \mathcal{I},\ x_0 \in \mathbb{Z}_+\}$. Notice $G(d) \in \{0, 1\}$ for all $d \in \mathbb{R}^m$, because $x = 0$ and $x_0 = 1$ is always a feasible solution. We have that $G(d) = 0$ if and only if $\{x : Ax \succeq_K d,\ x_i \in \mathbb{Z},\ i \in \mathcal{I}\} \neq \emptyset$. Hence, for $d_1, d_2 \in \mathbb{R}^m$, $G(d_1) = G(d_2) = 0$ implies $G(d_1 + d_2) = 0$. Therefore, we obtain that G is subadditive. Also, for $d_1, d_2 \in \mathbb{R}^m$ such that $d_1 \succeq_K d_2$, we have that $G(d_1) = 0$ implies $G(d_2) = 0$. Hence, we obtain that G is nondecreasing w.r.t. K. For $i \in \mathcal{I}$, $\alpha \in \{-1, 1\}$, we obtain that $G(\alpha A^i) = 0$, because $x_i = \alpha,\ x_j = 0, \forall j \neq i$ is a feasible solution when $d = \alpha A^i$. Similarly, for $i \in \mathcal{C}$, we have that $\overline{G}(A^i) = \overline{G}(-A^i) = 0$. Moreover, $G(0) = 0$ and since the primal is infeasible, we obtain that $G(b) = 1$.

Let g be a feasible solution for the dual. Then $g + \lambda G$ is also a feasible solution for the dual for all $\lambda \geq 0$. Since $[g + \lambda G](b) = g(b) + \lambda$ for all $\lambda \geq 0$, we conclude that the dual is unbounded. ☐

PROPOSITION 5.2. *If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$, then $(\mathcal{P})$ is finite if and only if $(\mathcal{D})$ is finite.*

Proof.

($\Leftarrow$) Assume that the dual is finite. Then, by Proposition 5.1, we obtain that the primal is feasible. Thus, Corollary 3.3 implies that the primal is finite.

($\Rightarrow$) Assume that the primal is finite. Corollary 3.3 implies that if the dual is feasible then the dual cannot be unbounded. We next verify that dual is indeed feasible.

First observe that since $\mathrm{int}(P(b)) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$ and $f(b) > -\infty$ hold, by the application of Proposition 4.5, we obtain that $\inf\{c^t x : Ax \succeq_K b\} > -\infty$.

Since $\inf\{c^t x : Ax \succeq_k b\}$ is finite and there exists $\hat{x}$ such that $A\hat{x} \succ_K b$, by the application of (2.) of Theorem 4.6, we obtain that the set $\{y \in \mathbb{R}^m : A^t y = c,\ y \succeq_{K_*} 0\}$ is nonempty. Let $\hat{y} \in \{y \in \mathbb{R}^m : A^t y = c,\ y \succeq_{K_*} 0\}$. Then the function $g(u) = \hat{y}^t u$ is a feasible solution of $(\mathcal{D})$, so the dual problem is feasible. ☐

Notice that Proposition 5.2 gives a proof for (1.) of Theorem 2.4. In the next section, we will refine the second half of the proof of Proposition 5.2, to show that when there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$ and the primal is finite, not only is the dual finite, but also its optimal objective function value is equal to that of the primal.

5.2. Feasible optimal solution for $(\mathcal{D})$**.** In this section, we construct a feasible optimal solution for the dual problem $(\mathcal{D})$. The next proposition shows how this can be done.

PROPOSITION 5.3. *If there exists* $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ *such that* $A\hat{x} \succ_K b$ *and* $(\mathcal{P})$ *is finite, then there exists a function* $g^* : \mathbb{R}^m \mapsto \mathbb{R}$*, feasible for* $(\mathcal{D})$ *such that* $g^*(b) = z^*$ *and consequently* $z^* = \rho^*$*.*

Proof. Let f be the value function of $(\mathcal{P})$. By Proposition 4.11, we obtain that $f(u) > -\infty$ for all $u \in \Omega$. Therefore, $f : \Omega \mapsto \mathbb{R}$ is a well defined function.

If $\Omega = \mathbb{R}^n$, then by Proposition 4.8, $f : \mathbb{R}^n \mapsto \mathbb{R}$ is feasible for the dual $(\mathcal{D})$. Moreover, by definition of f, $f(b) = z^*$. Thus, by considering $g^* = f$, we obtain that $g^*(b) = z^*$, as desired.

If $\Omega \subsetneq \mathbb{R}^n$, then we will use f to construct $g^* : \mathbb{R}^m \mapsto \mathbb{R}$ feasible for the dual $(\mathcal{D})$ such that $g^*(b) = f(b)$.

For $\lambda \geq 0$ denote $f_R(\lambda b) = \inf\{c^t x \,:\, x \in P(\lambda b)\}$. By Proposition 4.5, we have that $f_R(b) > -\infty$. Therefore, since $f_R(\lambda b) = \lambda f_R(b)$ we obtain that $f_R(\lambda b) = \lambda f_R(b) > -\infty$, for all $\lambda \geq 0$. Notice also that for all $y \in \mathbb{Z}$ we have that $yb \in \Omega$, implying that $f(yb) > -\infty$ for all $y \in \mathbb{Z}$.

Denote $X(u) = \{(x, y) \in \mathbb{R}^{n+1} \,:\, Ax - yb \succeq_K u, y \geq 0,\ x_i \in \mathbb{Z}, \forall\ i \in \mathcal{I},\ y \in \mathbb{Z}\}$. Now we will show how to construct g^*. We have to consider three cases. For each of these three cases we give a different construction of g^* and show that $g^*(0) = 0$ and $g^*(b) = f(b)$.

Case 1: $f(b) \geq 0$ and $f_R(b) \geq 0$. Define

$$g^*(u) = \inf\{c^t x + [f(b) - f_R(b)]y \,:\, (x, y) \in X(u)\}.$$

First we prove that $g^*(0) = 0$. Let $(x, y) \in X(0)$. Then we have

$$\begin{aligned} c^t x + [f(b) - f_R(b)]y &\geq f(yb) + f(b)y - f_R(b)y \\ &= [f(yb) - f_R(yb)] + f(b)y \\ &\geq 0. \end{aligned}$$

By considering the feasible solution $(0, 0)$, with objective value 0, we conclude that $g^*(0) = 0$.

Now we prove that $g^*(b) = f(b)$. Let $(x, y) \in X(b)$ with $y \geq 1$. Then we have

$$\begin{aligned} c^t x + [f(b) - f_R(b)]y &\geq f((y+1)b) + f(b)y - f_R(b)y \\ &= [f((y+1)b) - f_R((y+1)b)] + f_R(b) + f(b)y \\ &\geq f(b)y \\ &\geq f(b). \end{aligned}$$

On the other hand, notice that $(x, 0) \in X(b)$ if and only if $x \in S(b)$. For $(x, 0) \in X(b)$ we have $c^t x + [f(b) - f_R(b)]0 = c^t x$. Therefore, by taking the infimum over $(x, 0) \in X(b)$, we conclude that $g^*(b) = f(b)$.

Case 2: $f(b) \leq 0$ and $f_R(b) \leq 0$. In this case, define

$$g^*(u) = \inf\{c^t x - 2f_R(b)y \,:\, (x, y) \in X(u)\}.$$

First we prove that $g^*(0) = 0$. Let $(x, y) \in X(0)$. Then we have

$$\begin{aligned} c^t x - 2 f_R(b) y &\geq f(yb) - f_R(b) y - f_R(b) y \\ &= [f(yb) - f_R(yb)] - f_R(b) y \\ &\geq 0. \end{aligned}$$

By considering the feasible solution $(0, 0)$, with objective value 0, we conclude that $g^*(0) = 0$.

Now we prove that $g^*(b) = f(b)$. Let $(x, y) \in X(b)$ with $y \geq 1$. Then we have

$$\begin{aligned} c^t x - 2 f_R(b) y &\geq f((y+1)b) - f_R(b) y - f_R(b) y \\ &= [f((y+1)b) - f_R((y+1)b)] - f_R(b)(y-1) \\ &\geq 0 \\ &\geq f(b). \end{aligned}$$

For $(x, 0) \in X(b)$ we have $c^t x - 2 f_R(b) 0 = c^t x$. Therefore, by taking the infimum over $(x, 0) \in X(b)$, we conclude that $g^*(b) = f(b)$.

Case 3: $f(b) \geq 0$ and $f_R(b) \leq 0$. In this case, define

$$g^*(u) = \inf\{c^t x + [f(b) - 2 f_R(b)] y \,:\, (x, y) \in X(u)\}.$$

First we prove that $g^*(0) = 0$. Let $(x, y) \in X(0)$. Then we have

$$\begin{aligned} c^t x + [f(b) - 2 f_R(b)] y &\geq f(yb) - f_R(b) y + [f(b) - f_R(b)] y \\ &= [f(yb) - f_R(yb)] + [f(b) - f_R(b)] y \\ &\geq 0. \end{aligned}$$

By considering the feasible solution $(0, 0)$, with objective value 0, we conclude that $g^*(0) = 0$.

Now we prove that $g^*(b) = f(b)$. Let $(x, y) \in X(b)$ with $y \geq 1$. Then we have

$$\begin{aligned} c^t x + [f(b) - 2 f_R(b)] y &\geq f((y+1)b) - f_R(b) y + [f(b) - f_R(b)] y \\ &= [f((y+1)b) - f_R((y+1)b)] + f_R(b) + [f(b) - f_R(b)] y \\ &\geq f_R(b)(1 - y) + y f(b) \\ &\geq y f(b) \\ &\geq f(b). \end{aligned}$$

For $(x, 0) \in X(b)$ we have $c^t x + [f(b) - 2 f_R(b)] 0 = c^t x$. Therefore, by taking the infimum over $(x, 0) \in X(b)$, we conclude that $g^*(b) = f(b)$.

We show next that in all the three cases described above, we have that g^* is feasible for the dual $(\mathcal{D})$. Observe that since $A\hat{x} - b \succ_K 0;\ 1 > 0$, by the application of Proposition 4.10, we obtain that $X(u) \neq \emptyset$ for all $u \in \mathbb{R}^m$. Moreover, since $g^*(0) = 0$, we have that $g^*(u) > -\infty$ for all $u \in \mathbb{R}^m$ (Proposition 4.11). Thus, we have defined a function $g^* : \mathbb{R}^m \mapsto \mathbb{R}$. Finally, by Corollary 4.9, observe that g^* satisfies all the constraints of the dual $(\mathcal{D})$. In conclusion, g^* is feasible for the dual $(\mathcal{D})$ and $g^*(b) = f(b) = z^*$, thus completing the proof. $\square$

Notice that Proposition 5.3 gives a proof for (2.) of Theorem 2.4.

6. Valid inequalities. A valid inequality for the feasible region of the primal $(\mathcal{P})$ (that is, $S(b)$) is a linear inequality $\pi^t x \geq \pi_0$ such that for all $x \in S(b)$, we have $\pi^t x \geq \pi_0$. In the case of mixed-integer linear programs defined by rational data ($K = \mathbb{R}^m_+$, $A \in \mathbb{Q}^{m\times n}$, and $b \in \mathbb{Q}^m$) it can be shown that all interesting valid inequalities $\pi^t x \geq \pi_0$ for $S(b)$ are of the form

$$\sum_{i\in\mathcal{I}} g(A^i)x_i + \sum_{i\in\mathcal{C}} \overline{g}(A^i)x_i \geq g(b),$$

where g is feasible for the dual $(\mathcal{D})$ with $c = \pi$ and $g(b) \geq \pi_0$ (see [4], [5] and [7]).

Similarly, in the case of mixed-integer conic programming we can use subadditive functions that are nondecreasing with respect to K to generate valid inequalities. In particular, Proposition 3.2 (Weak duality) and Theorem 2.4 (Strong duality) yield the following corollary.

Corollary 6.1.

1. *Assume that the problems $(\mathcal{P})$ and $(\mathcal{D})$ are both feasible. If g is feasible for the dual $(\mathcal{D})$, then the inequality $\sum_{i\in\mathcal{I}} g(A^i)x_i + \sum_{i\in\mathcal{C}} \overline{g}(A^i)x_i \geq g(b)$ is a valid inequality for $S(b)$.*
2. *Assume there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ such that $A\hat{x} \succ_K b$. Given a valid inequality $\pi^t x \geq \pi_0$ for $S(b)$, then there exists $g \in \mathcal{F}$ satisfying $g(0) = 0$, $g(A^i) = -g(-A^i) = \pi_i$ $\forall i \in \mathcal{I}$, $\overline{g}(A^i) = -\overline{g}(-A^i) = \pi_i$ $\forall i \in \mathcal{C}$, and $g(b) \geq \pi_0$. Then, $\sum_{i\in\mathcal{I}} g(A^i)x_i + \sum_{i\in\mathcal{C}} \overline{g}(A^i)x_i \geq g(b)$ is a valid inequality for $S(b)$ that is equivalent to or dominates $\pi^t x \geq \pi_0$.*

In [13], it is shown that in the case of pure integer linear programs, given a rational left-hand-side matrix A, there exists a finite set of subadditive functions that yields the subadditive dual for any choice of the right-hand-side b. Such a result is unlikely in the integer conic setting since, in general, the convex hull of feasible points is not necessarily a polyhedron. The following example illustrates this behavior.

Example 6.2. *Let $S \subseteq \mathbb{R}^2$ be the epigraph of the parabola $x_2 = x_1^2$, that is, $S = \{x \in \mathbb{R}^2 \,:\, x_1^2 \leq x_2\}$. It is well-known that S is a conic quadratic representable set ([2]). Indeed, we have*

$$S = \left\{ x \in \mathbb{R}^2 \,:\, \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \succeq_{L^3} \begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right\},$$

where $L^3 = \{x \in \mathbb{R}^3 \,|\, \sqrt{x_1^2 + x_2^2} \leq x_3\}$ is the Lorentz cone in $\mathbb{R}^3$.

On the other hand, we have that $\operatorname{conv}(S \cap \mathbb{Z}^2)$ is a nonpolyhedral closed convex set; also see [3]. In fact, we have

$$\operatorname{conv}(S \cap \mathbb{Z}^2) = \bigcap_{n\in\mathbb{Z}} \{x \in \mathbb{R}^2 \,:\, x_2 - (2n+1)x_1 \geq -(n^2+n)\},$$

where all these inequalities define facets of $\operatorname{conv}(S \cap \mathbb{Z}^2)$.

By (2.) of Corollary 6.1, we have that for all $n \in \mathbb{Z}$, there exists a subadditive function $g_n : \mathbb{R}^3 \mapsto \mathbb{R}$, such that g_n is nondecreasing w.r.t. L^3, $g_n(0) = 0$, for $\alpha \in \{-1, 1\}$

$$g_n\left[\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right] = -\alpha(2n+1), \quad g_n\left[\alpha \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right] = \alpha, \text{ and } g_n\left[\begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}\right] \geq -(n^2+n).$$

Moreover, for this particular case, we explicitly present these functions. For all $n \in \mathbb{Z}$*, the function* g_n *can be defined as* $g_n(y) = \lceil \mu_n^t y \rceil$*, where* $\mu_n \in L^3$ *is given by* $\mu_n = (-(2n+1), 1-(n^2+n), 1+(n^2+n))^t$.

Therefore, we conclude that we can write the convex hull of the integer points in S *in terms of an infinite number of linear inequalities given by subadditive functions that are nondecreasing w.r.t.* L^3*, that is*

$$\operatorname{conv}(S \cap \mathbb{Z}^2) = \bigcap_{n \in \mathbb{Z}} \left\{ x \in \mathbb{R}^2 \,:\, g_n \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] x_1 + g_n \left[\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right] x_2 \geq g_n \left[\begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right] \right\}.$$

Finally, notice that since $\operatorname{conv}(S \cap \mathbb{Z}^2)$ *is a nonpolyhedral set, it cannot be described in terms of a finite number of linear inequalities.*

7. Primal problems with particular structure. It is sometimes convenient to write the dual for specially structured problems like the ones with some nonnegative variables, with some equality constraints, etc. Finding the appropiate version of the dual and showing that it satisfies strong duality, using the results of this paper, is a relatively simple exercise. We illustrate this for problems with some nonnegative variables. This problem is given by

$$(\mathcal{P}') \begin{cases} z' = \inf & c^t x \\ \quad s.t. & Ax \succeq_K b \\ & x_i \in \mathbb{R}_+, \forall i \in \mathcal{J} \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I}. \end{cases}$$

A subadditive dual for $(\mathcal{P}')$ is given by

$$(\mathcal{D}') \begin{cases} \rho' = \sup & g(b) \\ \quad s.t. & g\left(A^i\right) \leq c_i, & \forall i \in \mathcal{I} \cap \mathcal{J} \\ & \bar{g}\left(A^i\right) \leq c_i, & \forall i \in \mathcal{C} \cap \mathcal{J} \\ & g\left(A^i\right) = -g\left(-A^i\right) = c_i, & \forall i \in \mathcal{I} \setminus \mathcal{J} \\ & \bar{g}\left(A^i\right) = -\bar{g}\left(-A^i\right) = c_i, & \forall i \in \mathcal{C} \setminus \mathcal{J} \\ & g(0) = 0 & \\ & g \in \mathcal{F}. & \end{cases}$$

We formally state this result as a corollary of Theorem 2.4.

COROLLARY 7.1.

1. *(Weak duality) For all* $x \in \mathbb{R}^n$ *feasible for* $(\mathcal{P}')$ *and for all* $g : \mathbb{R}^m \mapsto \mathbb{R}$ *feasible for* $(\mathcal{D}')$*, we have* $g(b) \leq c^t x$.
2. *(Strong duality) If there exists* $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ *such that* $A\hat{x} \succ_K b$ *and* $\hat{x}_i > 0$ *for all* $i \in \mathcal{J}$*, then*
   (a) $(\mathcal{P}')$ *is finite if and only if* $(\mathcal{D}')$ *is finite.*
   (b) *If* $(\mathcal{P}')$ *is finite, then there exists a function* g^* *feasible for* $(\mathcal{D}')$ *such that* $g^*(b) = z'$ *and consequently* $z' = \rho'$.

Proof. The proof of weak duality is just a slight modification of the proof of weak duality for $(\mathcal{P})$ and $(\mathcal{D})$ (Proposition 3.2). We now verify the strong duality result.

We want to show that a strong dual of $(\mathcal{P}')$ is given by $(\mathcal{D}')$. Let $q = |\mathcal{J}|$ and let $l : \mathcal{J} \mapsto \{1, \ldots, q\}$ be a bijection. Notice that we can write $(\mathcal{P}')$ in the form of $(\mathcal{P})$ as follows

$$(\mathcal{P}'')\begin{cases} z^* = \inf & c^t x \\ \quad s.t. & \begin{bmatrix} A \\ E \end{bmatrix} x \succeq_{K\times\mathbb{R}^q_+} \begin{pmatrix} b \\ 0 \end{pmatrix} \\ & x_i \in \mathbb{Z},\ \forall i \in \mathcal{I}, \end{cases}$$

where $E \in \mathbb{R}^{q\times n}$ is the matrix whose columns are defined next: for $i \notin \mathcal{J}$ the column is the 0 vector and for $i \in \mathcal{J}$, the column is $e_{l(i)}$ (the $l(i)$th vector of the canonical basis of $\mathbb{R}^q$).

By Theorem 2.4, the problem

$$(\mathcal{D}'')\begin{cases} \rho^* = \sup & G(b,0) \\ \quad s.t. & G\left(A^i, e_{l(i)}\right) = -G\left(-A^i, -e_{l(i)}\right) = c_i, \quad \forall i \in \mathcal{I}\cap\mathcal{J} \\ & \bar{G}\left(A^i, e_{l(i)}\right) = -\bar{G}\left(-A^i, -e_{l(i)}\right) = c_i, \quad \forall i \in \mathcal{C}\cap\mathcal{J} \\ & G\left(A^i, 0\right) = -G\left(-A^i, 0\right) = c_i, \quad \forall i \in \mathcal{I}\setminus\mathcal{J} \\ & \bar{G}\left(A^i, 0\right) = -\bar{G}\left(-A^i, 0\right) = c_i, \quad \forall i \in \mathcal{C}\setminus\mathcal{J} \\ & G(0) = 0 \\ & G : \mathbb{R}^{m+q} \mapsto \mathbb{R}\ s.t.\ G \text{ is subadditive} \\ & \text{and nondecreasing w.r.t } K\times\mathbb{R}^q_+, \end{cases}$$

is a strong dual for $(\mathcal{P}'')$.

Let $G^* : \mathbb{R}^{m+q} \mapsto \mathbb{R}$ be the function given by Theorem 2.4, that is, G^* is feasible for $(\mathcal{D}'')$ and $G^*(b,0) = z^*$. For $x \in \mathbb{R}^m$, define $g^*(x) = G^*(x,0)$. Notice that $g^*(0) = G^*(0,0) = 0$. Also, since G^* is subadditive, we obtain that g^* is subadditive. Since $x \succeq_K y$ implies $(x,0) \succeq_{K\times\mathbb{R}^q_+} (y,0)$, we have that g is nondecreasing w.r.t K. For $i \in \mathcal{I}\cap\mathcal{J}$, since G^* is nondecreasing w.r.t $K\times\mathbb{R}^q_+$, we have that $g^*(A^i) = G^*(A^i,0) \le G^*(A^i, e_{l(i)}) = c_i$. Similarly, for $i \in \mathcal{C}\cap\mathcal{J}$ and $\delta \ge 0$, we have that $g^*(\delta A^i) \le G^*(\delta A^i, \delta e_{l(i)})$. Hence, by definition of $\bar{g^*}$ and $\bar{G^*}$, we obtain that $\bar{g^*}(A^i) \le \bar{G^*}(A^i, e_{l(i)}) = c_i$. Therefore, we have that g^* is feasible for $(\mathcal{D}')$.

Finally, since $g^*(b) = G^*(b,0) = z^*$, we conclude that $(\mathcal{D}')$ is a strong dual of $(\mathcal{P}')$, as desired. ∎

We note here that Corollary 7.1 allows us to consider a somewhat simpler form of dual for the problem $(\mathcal{P}')$ than the one given directly by Theorem 2.4. In particular, the feasible functions of $(\mathcal{D}')$ have a domain of smaller dimension than the feasible functions of $(\mathcal{D})$ and some of the constraints in $(\mathcal{D}')$ are less restrictive than the corresponding constraints in $(\mathcal{D})$.

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