Split Cuts and Extended Formulations for Mixed Integer Conic Quadratic Programming

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Abstract

We study split cuts and extended formulations for Mixed Integer Conic Quadratic Programming (MICQP) and their relation to Conic Mixed Integer Rounding (CMIR) cuts. We show that CMIR is a linear split cut for the polyhedral portion of an extended formulation of a quadratic set and it can be weaker than the nonlinear split cut of the same quadratic set. However, we also show that families of CMIRs can be significantly stronger than the associated family of nonlinear split cuts.

Keywords: Split Cut, Conic MIR, Mixed Integer Conic Quadratic Programming

1. Introduction

Split cuts [12], Gomory Mixed Integer (GMI) cuts [17], and Mixed Integer Rounding (MIR) cuts [23, 24] are some of the most effective valid inequalities for Mixed Integer Linear Programming (MILP) [8]. While they are known to be equivalent [15, 24], each of them provide different advantages and insights. In particular, the split cuts construction shows that they are a particular case of disjunctive cuts [3] and hence have a straightforward extension to Mixed Integer Nonlinear Programming (MINLP). The study of split cuts for MINLP is still much more limited than for MILP; however, there has been significant work on the computational use of split cuts in MINLP [9, 11, 16, 18, 25] and a recent surge of theoretical developments [1, 2, 4, 5, 14, 19, 21, 22]. In particular, several formulas for split cuts for Mixed Integer Conic Quadratic Programming (MICQP) have

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been recently developed [1, 4, 5, 14, 21]. While the resulting cuts are strong nonlinear inequalities, adding these cuts to the continuous relaxation of the MICQP can significantly increase its solution time, which could negate the effectiveness of the cuts. One potential solution is to use linearizations of these cuts [9, 18], but in this case there is a strong trade-off between their strength and the computational burden of generating them. An alternative approach was introduced by Atamtürk and Narayanan [2] who use the polyhedral portion of a nonlinear *extended formulation* (i.e., a formulation with auxiliary variables) to construct an inexpensive, but potentially strong, linear cut they denote the Conic MIR (CMIR). In this paper we attempt to broaden our understanding of split cuts for MINLP by providing a precise link between the CMIRs and split cuts for quadratic sets. In particular, this link provides a possible solution to the trade-off between the strength and computational burden resulting from adding the cuts to the relaxation.

Our first contribution is to show that the CMIR is a linear split cut for the polyhedral portion of the nonlinear extended formulation from [2]. Through this equivalence, we can extend the most general version of the CMIRs to the case of variables with unrestricted signs which was not previously possible. Our second contribution is to give a precise relation between the CMIR and nonlinear split cuts for quadratic sets. In particular, we show that, since the CMIR construction does not consider any quadratic information, a single CMIR can be weaker than a single nonlinear split cut. However, we also show that when families of split cuts and CMIRs are considered, CMIRs can provide a significant advantage over nonlinear split cuts by exploiting their common extended formulation. To the best of our knowledge, this is the first illustration of how the power of an extended formulation can improve the strength of a cutting plane procedure in MINLP.

The rest of the paper is structured as follows. In Section 2 we introduce some notation and describe previous results on CMIRs and split cuts for MINLP. In Section 3 we establish the equivalency between CMIRs and linear split cuts for an extended formulation. Finally, in Section 4 we compare the strength of nonlinear split cuts and CMIRs.

2. Notation and Previous Work

We let $e^i \in \mathbb{R}^n$ and $I \in \mathbb{R}^{n\times n}$ denote the i-th unit vector and the identity matrix where we omit dimension n if evident from the context. We also let $\|x\|_2 := \sqrt{\sum_{i=1}^n x_i^2}$ denote the Euclidean norm of $x \in \mathbb{R}^n$ and $|x| \in \mathbb{R}^n$ be the vector whose components are the absolute value of the components of $x \in \mathbb{R}^n$. In addition, for $a \in \mathbb{R}$ we let $(a)^+ := \max\{0, a\}$ and $\lfloor a \rfloor := \max\{k \in \mathbb{Z} : k \leq a\}$, and we let $[n] := \{1, \ldots, n\}$. Finally, for notational convenience, we define split cuts while identifying a single set of integer variables $x \in \mathbb{Z}^n$ and three sets of continuous variables $y \in \mathbb{R}^p$, $t \in \mathbb{R}^m$, and $t_0 \in \mathbb{R}$.

Definition 1. *Let $K \subseteq \mathbb{R}^{n+p+m+1}$ be a closed convex set and $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. A* split cut *for K is any valid inequality for*

$$K^{\pi,\pi_0} := \operatorname{conv}\left(\{(x, y, t, t_0) \in K : \pi^T x \leq \pi_0\} \cup \{(x, y, t, t_0) \in K : \pi^T x \geq \pi_0 + 1\}\right)$$

for some $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. If $\pi = e^i$ for some $i \in [n]$, we refer to (π, π_0) as an elementary disjunction *and to the obtained cuts as* elementary split cuts.

Because $K^{\pi,\pi_0} \supseteq \operatorname{conv}(K \cap (\mathbb{Z}^n \times \mathbb{R}^{p+m+1}))$, split cuts are valid inequalities for $K \cap (\mathbb{Z}^n \times \mathbb{R}^{p+m+1})$. For MILP, where K is a rational polyhedron, K^{π,π_0} is also a polyhedron and we only need linear split cuts. In contrast, if K is a general closed convex set, K^{π,π_0} is only closed and convex [14]. However, for special classes of K, we can characterize the nonlinear split cuts that need to be added to K to obtain K^{π,π_0} [1, 4, 5, 14, 19, 21]. For instance, the following proposition from [21] characterizes split cuts for conic quadratic sets of the form

$$C := \left\{(x, t_0) \in \mathbb{R}^{n+1} : \|B(x - c)\|_2 \leq t_0\right\}, \tag{1}$$

where C is in fact an affine transformation of the Quadratic cone $\{(x, t_0) \in \mathbb{R}^{n+1} : \|x\|_2 \leq t_0\}$.

Proposition 1. *Let $B \in \mathbb{R}^{n \times n}$ be an invertible matrix, $c \in \mathbb{R}^n$, $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, and C be as defined in (1). If $\pi^T c \notin (\pi_0, \pi_0 + 1)$, then $C^{\pi,\pi_0} = C$. Otherwise, there exist $\bar{B} \in \mathbb{R}^{n \times n}$ and $\bar{c} \in \mathbb{R}^n$ such that*

$$C^{\pi,\pi_0} = \left\{(x, t_0) \in C : \left\|\bar{B}(x - c) + \bar{c}\right\|_2 \leq t_0\right\}.$$

Proposition 1 shows that the single split cut for C is $\left\|\bar{B}(x - c) + \bar{c}\right\|_2 \leq t_0$ which is of the same class as the inequality describing C. However, this inequality can be too expensive computationally and it can be preferable to add linear cuts instead. One way to achieve this is to add a finite number of linearizations of the nonlinear cuts. Such linearizations can be algorithmically obtained even in the absence of nonlinear cut formulas. Two examples of this are the algorithms introduced in [9, 18] to generate disjunctive inequalities for convex MINLPs.

A completely different linearization scheme was introduced by Atamtürk and Narayanan [2] for the general conic quadratic set given by

$$M_+ := \left\{(x, y, t_0) \in \mathbb{R}^{n+p+1} \;:\; \|Ax + Gy - b\|_2 \le t_0, \quad x \ge 0, \quad y \ge 0\right\},$$

for rational matrices and vectors $A \in \mathbb{Q}^{m\times n}$, $G \in \mathbb{Q}^{m\times p}$, and $b \in \mathbb{Q}^m$. Instead of considering valid inequalities for $\operatorname{conv}(M_+ \cap (\mathbb{Z}^n \times \mathbb{R}^{p+1}))$ directly, using auxiliary variables $t \in \mathbb{R}^m$, they first introduce the nonlinear extended formulation of M_+ given by

$$|Ax + Gy - b| \le t, \quad x \ge 0, \quad y \ge 0, \quad \|t\|_2 \le t_0, \tag{2}$$

so that, if $P_+ := \{(x, y, t) \in \mathbb{R}^{n+p+m} \;:\; |Ax + Gy - b| \le t, \quad x \ge 0, \quad y \ge 0\}$ and $\operatorname{Proj}_{(x,y,t_0)}$ is the projection onto the (x, y, t_0) space, then

$$M_+ = \operatorname{Proj}_{(x,y,t_0)}\left(\left\{(x, y, t, t_0) \in \mathbb{R}^{n+p+m+1} \;:\; \|t\|_2 \le t_0, \quad (x, y, t) \in P_+\right\}\right).$$

They then exploit the fact that P_+ is a polyhedron to generate a class of valid inequalities they denote the *Conic MIR* (CMIR). The first version of the CMIR is a simple but strong cut for a four variable and one constraint version of P_+.

Proposition 2 (Simple CMIR)**.** *Let $b_0 \in \mathbb{R}$, $f = b_0 - \lfloor b_0 \rfloor$,*

$$S_0 := \left\{(x, y, t_0) \in \mathbb{R}^4 \;:\; |x + y_1 - y_2 - b_0| \le t_0, \quad y_1, y_2 \ge 0\right\},$$

and let the simple CMIR be the inequality given by

$$(1 - 2f)(x - \lfloor b_0 \rfloor) + f \le t_0 + y_1 + y_2. \tag{3}$$

The simple CMIR is valid for $\operatorname{conv}(S_0 \cap (\mathbb{Z} \times \mathbb{R}^2_+ \times \mathbb{R}_+))$ *and furthermore*

$$\operatorname{conv}(S_0 \cap (\mathbb{Z} \times \mathbb{R}^2_+ \times \mathbb{R}_+)) = \{(x, y, t_0) \in S_0 \;:\; (3)\}.$$

The simple CMIR is a linear inequality, but Atamtürk and Narayanan show that it can induce nonlinear inequalities in the (x, t_0) space through (2).

Lemma 1 (Nonlinear CMIR)**.** *Let $T_0 := \{(x, y, t_0) \in \mathbb{R}^3 \;:\; \sqrt{(x - b_1)^2 + y^2} \le t_0\}$, $P_0 := \left\{(x, y, t) \in \mathbb{R}^4 : |x - b_1| \le t_1,\ |y| \le t_2\right\}$, $b_1 \in \mathbb{R}$, and $f = b_1 - \lfloor b_1 \rfloor$. Then the simple CMIR for $|x - b_1| \le t_1$ is given by*

$$(1 - 2f)(x - \lfloor b_1 \rfloor) + f \le t_1, \tag{4}$$

$\operatorname{conv}(T_0 \cap (\mathbb{Z} \times \mathbb{R}^2)) = T_0^{e^1,\lfloor b_1 \rfloor}$*, and*

$$\begin{aligned} T_0^{e^1,\lfloor b_1 \rfloor} &= \left\{(x, y, t_0) \in T_0 \;:\; \sqrt{((1 - 2f)(x - \lfloor b_1 \rfloor) + f)^2 + y^2} \le t_0\right\} \\ &= \operatorname{Proj}_{(x,y,t_0)}\left(\left\{(x, y, t, t_0) \in \mathbb{R}^5 \;:\; (x, y, t) \in P_0, \quad \|t\|_2 \le t_0, \quad (4)\right\}\right). \end{aligned}$$

Atamtürk and Narayanan follows the traditional linear MIR procedure [23, 24] to get CMIRs for M_+ and develop a super-additive version of the CMIR. Their most general version results in the following family of cuts.

Theorem 1 (Super-additive CMIR)**.** *Let* $a, v \in \mathbb{R}^n, g, w \in \mathbb{R}^p, h, u \in \mathbb{R}^m$, $S_+ := \{(x, y, t) \in \mathbb{R}^{n+p+m} : \left|a^T x + g^T y + h^T t - b_0\right| \leq u^T t + v^T x + w^T y, \quad x, y, t \geq 0\}$ *be a relaxation of* P_+ *and let* $\varphi_f(a) = -a + 2(1 - f)\left(\lfloor a \rfloor + \frac{(a - \lfloor a \rfloor - f)^+}{1-f}\right)$*. Then for any* $\alpha \neq 0$ *and* $f_\alpha = b_0/\alpha - \lfloor b_0/\alpha \rfloor$*, a valid cut for* S_+ *and* P_+ *is*

$$\sum_{j=1}^{n} \varphi_{f_\alpha}(a_j/\alpha)x_j - \varphi_{f_\alpha}(b_0/\alpha) \leq \left((u + |h|)^T t + (w + |g|)^T y + v^T x\right)/|\alpha|. \quad (5)$$

We let a super-additive CMIR be any cut of this form obtained for some relaxation S_+, which can be constructed through various aggregation procedures. Finally, with regards to its relation to the traditional linear MIR, Atamtürk and Narayanan use the aggregation to show that every MIR is a CMIR. In Section 3 we show that these two cuts are in fact equivalent.

3. Conic MIR and Linear Split Cuts

We now show that CMIRs are equivalent to linear split cuts for P_+, which are in turn equivalent to traditional linear MIRs for P_+. Through this equivalence, we extend all CMIRs to the case of variables with unrestricted signs and show that such extension follows naturally from the simple CMIR. To show the equivalence between linear split cuts and super-additive conic MIRs, we need the following well-known characterization of split cuts for a polyhedron T (e.g. [26]).

Proposition 3. *Let* $T := \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p : Cx + Dy \leq d\}$ *for* $C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times p}, d \in \mathbb{R}^m$*, and let* $\mu \in \mathbb{R}^m$ *be such that* $C^T\mu = \pi \in \mathbb{Z}^n$ *and* $D^T\mu = 0 \in \mathbb{R}^p$*. Also let* $f = \mu^T d - \lfloor \mu^T d \rfloor$*. Then every split cut for* T *is of the form*

$$|\mu|^T (Cx + Dy - d) + (1 - 2f)\left(\pi^T x - \left\lfloor \mu^T d \right\rfloor\right) + f \leq 0.$$

Using this proposition, we show that every linear split cut for P_+ can be obtained from the simple CMIR and that every CMIR is a split cut.

Theorem 2. *Every non-dominated split cut for* P_+ *is of the form*

$$(1 - 2f)\left(\pi^T x - \lfloor \mu^T b \rfloor\right) + f \leq |\mu|^T t + |\lambda|^T x + |\gamma|^T y, \quad (6)$$

for some $\mu \in \mathbb{R}^m, \lambda \in \mathbb{R}^n$, $\gamma \in \mathbb{R}^p$*, and* $\pi \in \mathbb{Z}^n$ *such that* $A^T\mu - \lambda = \pi$, $G^T\mu - \gamma = 0$*, and* $f = \mu^T b - \lfloor \mu^T b \rfloor$*. Furthermore, every super-additive CMIR for* P_+ *is equivalent or dominated by a split cut of this form.*

Proof. We first prove formula (6) using Proposition 3. We have $P_+ = \{(x, y, t) \in \mathbb{R}^{n+p+m} : \hat{C}x + \hat{D}\begin{bmatrix} y \\ t \end{bmatrix} \le \hat{d}\}$, where

$$\hat{C} = \begin{bmatrix} A \\ -A \\ -I \\ 0 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} G & -I \\ -G & -I \\ 0 & 0 \\ -I & 0 \end{bmatrix}, \text{ and } \hat{d} = \begin{bmatrix} b \\ -b \\ 0 \\ 0 \end{bmatrix}.$$

Let $\hat{\mu} = (\mu_1^T, \mu_2^T, \lambda^T, \gamma^T)^T$, where $\mu_1, \mu_2 \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}^p$. $\hat{D}^T\mu = 0$ implies $G^T(\mu_1 - \mu_2) - \gamma = 0$ and $\mu_1 = -\mu_2$. Furthermore, $\hat{C}^T\mu = \pi$ implies $A^T(\mu_1 - \mu_2) - \lambda = \pi$. Let $\mu = \mu_1 - \mu_2$ and the result then follows from Proposition 3.

Let

$$C = \begin{bmatrix} \frac{a^T}{2\alpha} - \frac{v^T}{2|\alpha|} \\ -\frac{a^T}{2\alpha} - \frac{v^T}{2|\alpha|} \\ -I \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} \frac{g^T}{2\alpha} - \frac{w^T}{2|\alpha|} & \frac{h^T}{2\alpha} - \frac{u^T}{2|\alpha|} \\ -\frac{g^T}{2\alpha} - \frac{w^T}{2|\alpha|} & -\frac{h^T}{2\alpha} - \frac{u^T}{2|\alpha|} \\ 0 & 0 \\ -I & 0 \\ 0 & -I \end{bmatrix} \text{ and } d = \begin{bmatrix} \frac{b_0}{2\alpha} \\ -\frac{b_0}{2\alpha} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

so that $S_+ = \left\{(x, y, t) \in \mathbb{R}^{n+p+m} : Cx + D\begin{bmatrix} y \\ t \end{bmatrix} \le d\right\}$ is a relaxation of P_+. Now let $f_\alpha = b_0/\alpha - \lfloor b_0/\alpha \rfloor$, $\mu = (1, -1, \lambda^T, g^T/\alpha, h^T/\alpha)^T$ where $\lambda_j = a_j/\alpha - \lfloor a_j/\alpha \rfloor$ if $a_j/\alpha - \lfloor a_j/\alpha \rfloor < f_\alpha$, and $\lambda_j = -(1 - a_j/\alpha + \lfloor a_j/\alpha \rfloor)$ if $a_j/\alpha - \lfloor a_j/\alpha \rfloor \ge f_\alpha$. Then, by Proposition 3, we obtain the split cut for S_+ given by

$$\begin{aligned} &\sum_{j \in [n]: a_j/\alpha - \lfloor a_j/\alpha \rfloor < f_\alpha} \left(-\frac{a_j}{\alpha} + 2(1 - f_\alpha)\lfloor a_j/\alpha \rfloor\right) x_j \\ &\quad + \sum_{j \in [n]: a_j/\alpha - \lfloor a_j/\alpha \rfloor \ge f_\alpha} \left(-\frac{a_j}{\alpha} + 2(1 - f_\alpha)\lfloor a_j/\alpha \rfloor + 2\left(a_j/\alpha - \lfloor a_j/\alpha \rfloor - f_\alpha\right)\right) x_j \\ &\quad - \sum_{j=1}^{n} \frac{v_j}{|\alpha|} x_j - \sum_{j=1}^{p} \frac{w_j + |g_j|}{|\alpha|} y_j - \sum_{j=1}^{p} \frac{u_j + |h_j|}{|\alpha|} t_j \le 2(1 - f_\alpha)\lfloor b_0/\alpha \rfloor - b_0/\alpha. \end{aligned}$$

The cut above is precisely super-additive CMIR (5). The result follows by noting that since $P_+ \subseteq S_+$, then any split cut for S_+ is also a split cut for P_+. □

From Theorem 2, we have that a natural extension of the super-additive CMIR to the case of variables with unrestricted signs is to consider split cuts. While we can also consider cases with partial non-negativity requirements, because of space limitations, we here focus on the set with no non-negativity constraints given by $M := \left\{(x, y, t_0) \in \mathbb{R}^{n+p+1} \;:\; \|Ax + Gy - b\|_2 \le t_0\right\}$. As before, we let the polyhedral portion of the extended formulation of M be

$$P := \{(x, y, t) \in \mathbb{R}^{n+p+m} \;:\; |Ax + Gy - b| \le t\}.$$

We can extend the CMIR to this setting through the following theorem.

Theorem 3. *Every non-dominated split cut for P is of the form*

$$(1 - 2f)\left(\pi^T x - \lfloor \mu^T b \rfloor\right) + f \le |\mu|^T t, \tag{7}$$

for some $\mu \in \mathbb{R}^m$ such that $A^T \mu = \pi \in \mathbb{Z}^n$, $G^T \mu = 0$, and $f = \mu^T b - \lfloor \mu^T b \rfloor$.

Proof. Follows from Proposition 3. □

From (6) and (7), we can see that all split cuts for P_+ and P can be obtained from the simple CMIR (3) and some simple aggregation procedures.

4. Comparison between Cuts

Through Lemma 1, Atamtürk and Narayanan show that using an extended formulation analog to (2), the effect of the simple CMIR on the (x, y, t_0) space is equivalent to that of a conic split cut from Proposition 1. We now study to what extent this holds for more general settings. We first study containment relations between the sets obtained by adding nonlinear split cuts and CMIRs to some specific regions bounded by a single conic quadratic inequality. To consider more general sets, we then compare the strength of the bounds generated by the two classes of cuts on some quadratic integer programming problems. In both cases, it will be convenient to use the following direct corollary that specializes Theorem 3 to the polyhedral portion of the analog of extended formulation (2) for $C := \left\{(x, t_0) \in \mathbb{R}^{n+1} : \|B(x - c)\|_2 \le t_0\right\}$, which is of the form

$$L := \left\{(x, t) \in \mathbb{R}^{2n} : |B(x - c)| \le t\right\}. \tag{8}$$

Corollary 1. *Let $B \in \mathbb{R}^{n\times n}$ be an invertible matrix, $c \in \mathbb{R}^n$, $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, and L be as defined in (8). If $\pi^T c \notin (\pi_0, \pi_0 + 1)$, then $L^{\pi,\pi_0} = L$. Otherwise*

$$L^{\pi,\pi_0} = \left\{(x, t) \in L : (1 - 2f)\left(\pi^T x - \lfloor \pi^T c \rfloor\right) + f \le |\mu|^T t\right\},$$

where $\mu \in \mathbb{R}^n$ is the unique solution to $B^T \mu = \pi \in \mathbb{Z}^n$ and $f = \pi^T c - \lfloor \pi^T c \rfloor$.

4.1. Containment Relations

Because $C = \mathrm{Proj}_{(x,t_0)}\left(\left\{(x,t,t_0) \in \mathbb{R}^{2n+1} \;:\; (x,t) \in L,\ \|t\|_2 \le t_0\right\}\right)$, it is natural to compare the strength of the CMIRs (i.e., linear split cuts) for L and the nonlinear split cuts for C from Proposition 1. As discussed, Lemma 1 shows that these cuts can sometimes be equivalent. However, the following proposition shows that this is true only for very specific structures and that a single nonlinear split cut for C is at least as strong as (and many times stronger than) the CMIR associated to the same disjunction.

Proposition 4. *Let $B \in \mathbb{R}^{n\times n}$ be invertible, $c \in \mathbb{R}^n$, $(\pi,\pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, and*

$$MIR^{\pi,\pi_0} := \left\{(x,t,t_0) \in \mathbb{R}^{2n+1} \;:\; (x,t) \in L^{\pi,\pi_0}, \quad \|t\|_2 \le t_0\right\}.$$

Then $C^{\pi,\pi_0} \subsetneq \mathrm{Proj}_{(x,t_0)}(MIR^{\pi,\pi_0})$. The containment holds as equality if $B = I$ and $\pi = e^i$ for some $i \in [n]$, but can otherwise be strict even for $n = 2$.

Proof. We begin with proving the containment. Let $(\bar{x},\bar{t}_0) \in C^{\pi,\pi_0}$. There exist $\alpha \in [0,1]$, $(x^0,t_0^0) \in C$, and $(x^1,t_0^1) \in C$ such that $(\bar{x},\bar{t}_0) = \alpha(x^0,t_0^0) + (1-\alpha)(x^1,t_0^1)$, $\pi^T x^0 \le \pi_0$, and $\pi^T x^1 \ge \pi_0 + 1$. Let $t^0 := \left|B(x^0 - c)\right|$, $t^1 := \left|B(x^1 - c)\right|$, and $\bar{t} := \alpha t^0 + (1-\alpha)t^1$. Then $(\bar{x},\bar{t}_0) = \mathrm{Proj}_{(x,t_0)}((\bar{x},\bar{t},\bar{t}_0))$ and $(\bar{x},\bar{t}) \in L^{\pi,\pi_0}$. It then only remains to show that $\|\bar{t}\|_2 \le \bar{t}_0$, which follows from

$$\begin{aligned}\|\bar{t}\|_2 &= \left\|\alpha t^0 + (1-\alpha)t^1\right\|_2 \le \alpha\left\|t^0\right\|_2 + (1-\alpha)\left\|t^1\right\|_2 \\ &= \alpha\left\|B(x^0 - c)\right\|_2 + (1-\alpha)\left\|B(x^1 - c)\right\|_2 \le \alpha t_0^0 + (1-\alpha)t_0^1 = \bar{t}_0. \quad (9)\end{aligned}$$

Now we show that the containment holds as equality for $B = I$ and $\pi = e^i$ for some $i \in [n]$. Using Corollary 1, we have

$$MIR^{\pi,\pi_0} = \left\{(x,t,t_0) \in \mathbb{R}^{2n+1} \;:\; |x - c| \le t,\ \|t\|_2 \le t_0,\ (1-2f_i)(x_i - \lfloor c_i \rfloor) + f_i \le t_i\right\},$$

where $f_i = c_i - \lfloor c_i \rfloor$. Furthermore, one can check that MIR^{π,π_0} does not change by replacing $(1-2f_i)(x_i - \lfloor c_i \rfloor) + f_i \le t_i$ with $|(1-2f_i)(x_i - \lfloor c_i \rfloor) + f_i| \le t_i$. Thus, $\mathrm{Proj}_{(x,t_0)}(MIR^{\pi,\pi_0})$ is defined by the original constraint $\|x - c\|_2 \le t_0$ and

$$\sqrt{\sum_{j\in[n]\,:\,j\neq i}\left(x_j - c_j\right)^2 + ((1-2f_i)(x_i - \lfloor c_i \rfloor) + f_i)^2} \le t_0. \quad (10)$$

Also using Corollary 5 in [21], the split cut associated to C^{π,π_0} is

$$\sqrt{\sum_{j\in[n]\,:\,j\neq i}\left(x_j - c_j\right)^2 + (a(x_i - c_i) + b)^2} \le t_0, \quad (11)$$

where $a = \lfloor c_i \rfloor + \lfloor c_i \rfloor + 1 - 2c_i = 1 - 2f_i$ and $b = -2(\lfloor c_i \rfloor - c_i)(\lfloor c_i \rfloor + 1 - c_i) = 2f_i(1 - f_i)$. The result then follows by noting that (10) and (11) are equivalent.

Finally, we show that the containment is strict for $n = 2$, $B = I$, $c = (1/4, 0)^T$, $\pi = (1, 1)^T$, and $\pi_0 = 0$. Again using Corollary 5 in [21], one can check that after a few simplifications, the corresponding split cut is

$$\sqrt{(3x_1 - x_2)^2 + (3x_2 - x_1 + 1)^2} \leq 4t_0, \tag{12}$$

and using Corollary 1, the corresponding CMIR cut is $x_1 + x_2 + 1/2 \leq 2(t_1 + t_2)$. Let $(\bar{x}, \bar{t}, \bar{t}_0) = (-0.082, 0.922, 0.337, 0.928, 1)$. We have that $(\bar{x}, \bar{t}, \bar{t}_0) \in MIR^{\pi,\pi_0}$, but $(\bar{x}, \bar{t}_0)$ violates the split cut (12). □

While a single CMIR can be weaker than the corresponding nonlinear split cut, a family of CMIRs sharing the same extended formulation can be significantly stronger than the associated family of nonlinear split cuts. This can be illustrated by considering split cuts for C (see Proposition 7 for a result along this line). However, the behavior is more dramatic for an ellipsoid given by

$$E := \{x \in \mathbb{R}^n \ : \ \|B(x - c)\|_2 \leq r\},$$

where $B \in \mathbb{R}^{n \times n}$ is an invertible matrix, $c \in \mathbb{R}^n$, and $r \in \mathbb{R}_+$. As formalized in the following straightforward lemma, an ellipsoid can be described as projections of linear sections of either the cone C defined in (1), a paraboloid Q of the form

$$Q := \left\{(x, s_0) \in \mathbb{R}^{n+1} : \|B(x - c)\|_2^2 \leq s_0\right\},$$

and the extended formulation associated to the CMIR, which provides a way of comparing the strength of several cuts.

Lemma 2. *Let $B \in \mathbb{R}^{n \times n}$ be an invertible matrix, $c \in \mathbb{R}^n$, and $r \in \mathbb{R}_+$. Then*

$$\begin{aligned} E &= \operatorname{Proj}_x(\{(x, t_0) \in C \ : \ t_0 = r\}) = \operatorname{Proj}_x(\{(x, s_0) \in Q \ : \ s_0 = r^2\}) \\ &= \operatorname{Proj}_x\left\{(x, t, t_0) \in \mathbb{R}^{2n+1} \ : \ (x, t) \in L, \quad \|t\|_2 \leq t_0, \quad t_0 = r\right\}. \end{aligned}$$

The CMIR and nonlinear split cuts for C and Q (characterized in [21]) can be used to induce valid inequalities for $E \cap \mathbb{Z}^n$ through the same linear section of Lemma 2. However, the construction of these cuts does not exploit the structure induced by the section and they hence cannot be expected to always achieve the full strength of the nonlinear split cuts for E studied in [4, 14, 21]. The following proposition shows that this is indeed the case and that the cut with the weakest effect on E is the CMIR.

Proposition 5. *Let $B \in \mathbb{R}^{n\times n}$ be an invertible matrix, $c \in \mathbb{R}^n$, $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, and $r \in \mathbb{R}_+$. Then*

$$\begin{aligned} E^{\pi,\pi_0} &\subseteq \mathrm{Proj}_x\left(\left\{(x, s_0) \in Q^{\pi,\pi_0} \ : \ s_0 = r^2\right\}\right) \\ &\subseteq \mathrm{Proj}_x(\{(x, t_0) \in C^{\pi,\pi_0} \ : \ t_0 = r\}) \\ &\subseteq \mathrm{Proj}_x\left(\{(x, t, t_0) \in MIR^{\pi,\pi_0} \ : \ t_0 = r\}\right). \end{aligned} \tag{13}$$

All containments can be simultaneously strict even for $n = 2$.

Proof. The last containment follows from Proposition 4 by noting that $C^{\pi,\pi_0} \subseteq \mathrm{Proj}_{(x,t_0)}(MIR^{\pi,\pi_0})$.

We now prove the first containment. If $\bar{x} \in E^{\pi,\pi_0}$, then there exist x^1, x^2 such that $\bar{x} = \alpha x^1 + (1-\alpha)x^2$ for some $\alpha \in [0, 1]$ and

$$\left\|B(x^1 - c)\right\|_2 \le r,\ \pi x^1 \le \pi_0 \quad \text{and} \quad \left\|B(x^2 - c)\right\|_2 \le r,\ \pi x^2 \ge \pi_0 + 1,$$

which implies that (x^1, s^*) and (x^2, s^*) - where $s^* = r^2$ - satisfy, respectively

$$\left\|B(x^1 - c)\right\|_2^2 \le s^*,\ \pi x^1 \le \pi_0 \quad \text{and} \quad \left\|B(x^2 - c)\right\|_2^2 \le s^*,\ \pi x^2 \ge \pi_0 + 1.$$

Therefore, $\alpha(x^1, s^*) + (1-\alpha)(x^2, s^*) = (\bar{x}, s^*) = (\bar{x}, r^2)$ belongs to $\{(x, s_0) \in Q^{\pi,\pi_0} \ : \ s_0 = r^2\}$, and thus $\bar{x}$ belongs to the projection of this set on the x-space.

The fact that the second set is contained in the third set can be proved as follows. If $\bar{x}$ belongs to the second set, then $(\bar{x}, r^2) \in \{(x, s_0) \in Q^{\pi,\pi_0} \ : \ s_0 = r^2\}$ which implies that there exist $(x', s'), (x'', s'')$ such that $(\bar{x}, r^2) = \alpha(x', s') + (1 - \alpha)(x'', s'')$ for some $\alpha \in [0, 1]$ and

$$\left\|B\,(x' - c)\right\|_2^2 \le s',\ \pi x' \le \pi_0 \quad \text{and} \quad \left\|B\,(x'' - c)\right\|_2^2 \le s'',\ \pi x'' \ge \pi_0 + 1.$$

We can therefore conclude that $(x', r' = \sqrt{s'})$ and $(x'', r'' = \sqrt{s''})$ satisfy

$$\left\|B\,(x' - c)\right\|_2 \le r',\ \pi x' \le \pi_0 \quad \text{and} \quad \left\|B\,(x'' - c)\right\|_2 \le r'',\ \pi x'' \ge \pi_0 + 1.$$

As the function $f(x) = \sqrt{x}$ is a concave function for $x \ge 0$, we have

$$r = f(r^2) = f(\alpha s' + (1-\alpha)s'') \ge \alpha f(s') + (1-\alpha)f(s'') = \alpha r' + (1-\alpha)r''.$$

Now, replacing r' by a larger number r'_+, we still have $\|B\,(x' - c)\|_2 \le r'_+$; we can choose r'_+ such that $r = \alpha r'_+ + (1-\alpha)r''$, so that $(\bar{x}, r) \in \{(x, t_0) \in C^{\pi,\pi_0} \ : \ t_0 = r\}$.

Finally, we show that all three containments are strict for $n = 2$, $B = I$, $c = (1/4, 0)^T$, $\pi = (1, 1)^T$, $\pi_0 = 0$, and $r = 1$. The last strict containment follows by considering the example previously provided in the proof of Proposition 4. Using Corollaries 4 and 6 in [21], one can check that after a few simplifications, the corresponding split cuts associated to E^{π,π_0} and Q^{π,π_0} are given by

$$|x_2 - x_1 + 1/4| \leq \left(\left(\sqrt{23} - \sqrt{31}\right)(x_1 + x_2) + \sqrt{31}\right)/4 \tag{14}$$

and

$$(x_2 - x_1 + 1/4)^2 + (x_1 + x_2)/2 + 1/16 \leq 2s_0, \tag{15}$$

respectively. The first two strict containments then follow from noting that $(-0.082, 0.903)$ belongs to $\text{Proj}_x(\{(x, s_0) \in Q^{\pi,\pi_0} : s_0 = 1\})$ but violates the split cut (14), and $(-0.082, 0.911)$ belongs to $\text{Proj}_x(\{(x, t_0) \in C^{\pi,\pi_0} : t_0 = 1\})$ but violates the split cut (15) for $s_0 = 1$. □

For the effect of a single cut on E, the CMIR is the weakest in Proposition 5. However, several CMIRs combined through a common extended formulation (i.e., with a single set of auxiliary variables $t \in \mathbb{R}^n$) can be significantly stronger than even the associated family of split cuts for E. This effectively sidesteps the three potentially strict containments in (13). For instance, the following proposition shows that elementary CMIRs are enough to show emptiness of the convex hull of integer points of an ellipsoid with no lattice points, while this cannot be done even with all the nonlinear split cuts (elementary and non-elementary) of E.

Proposition 6. *Let $n \geq 2$, $r = 1/2$, $B = I$, and $c_i = 1/2$ for all $i \in [n]$ so that $E \cap \mathbb{Z}^n = \emptyset$. Then*

$$\emptyset = \text{Proj}_x\left(\left\{(x, t, t_0) \in \bigcap\nolimits_{i=1}^{n} MIR^{e^i, \lfloor c_i \rfloor} : t_0 = r\right\}\right) \subsetneq \bigcap\nolimits_{(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Z}} E^{\pi,\pi_0} = \{c\}.$$

Proof. For the first equality, note that

$$\bigcap\nolimits_{i=1}^{n} MIR^{e^i, \lfloor c_i \rfloor} = \{(x, t, t_0) : \|t\|_2 \leq t_0, |x_i - 1/2| \leq t_i, 1/2 \leq t_i \ \forall\, i \in [n]\},$$

and every point in this set has $t_0 \geq \sqrt{n}/2 > r = 1/2$ for the assumed $n \geq 2$.

For the last equality, we first prove left to the right containment. This follows by noting that the set obtained by adding all the elementary split cuts is equal to $\{c\}$. In particular, the intersection of E with $x_i = 1$ (or $x_i = 0$) is exactly $(1, \ldots, 1)/2 \pm e_i/2$ and $E^{e_i,0}$ is exactly the convex hull of these two points.

The reverse containment directly follows from Lemma 1 in [13], where they show that as long as a convex set contains points where one component is 0/1, and all other components are 1/2, then $\{c\}$ is contained in the split closure of this set (while Lemma 1 in [13] is stated for polyhedra, the extension to general convex sets is straightforward). The strict containment above then follows automatically. □

4.2. Bound Strength

The quadratic integer programming problem that we consider is the Closest Vector Problem (CVP) [10, 20] which aims to find the element in an integer lattice that is closest (with respect to the Euclidean distance) to a given target vector not in the lattice. CVP can be equivalently formulated as

$$\min_x \{\|B(x-c)\|_2 \;:\; x \in \mathbb{Z}^n\} \tag{16}$$

or

$$\min_x \left\{\|B(x-c)\|_2^2 \;:\; x \in \mathbb{Z}^n\right\}, \tag{17}$$

where $B \in \mathbb{R}^{n\times n}$ is an invertible matrix whose columns compose the basis of the lattice and $c \in \mathbb{R}^n$ (the target vector is Bc in this case). As noted in [6, 7, 21], because $\operatorname{conv}(\mathbb{Z}^n) = \mathbb{R}^n$, to effectively use cuts in CVP we need the equivalent reformulations of (16) and (17) given by

$$\min_{x,t_0} \{t_0 \;:\; (x,t_0) \in C, \quad x \in \mathbb{Z}^n\} \tag{18}$$

for $C := \{(x,t_0) \in \mathbb{R}^{n+1} : \|B\,(x-c)\|_2 \le t_0\}$, and

$$\min_{x,s_0} \{s_0 \;:\; (x,s_0) \in Q, \quad x \in \mathbb{Z}^n\} \tag{19}$$

for $Q := \{(x,s_0) \in \mathbb{R}^{n+1} : \|B\,(x-c)\|_2^2 \le s_0\}$. We can then strengthen these formulations by adding split cuts for C and Q. However, using techniques similar to the proof of Proposition 5, we can show that adding split cuts for Q to (19) is always equal or better than adding split cuts for C to (18). For this reason, we only compare the strength of nonlinear split cuts for Q to the strength of the CMIRs. In this context, we consider the extended formulation given by

$$\min_{x,t,t_0} \left\{t_0^2 \;:\; |B(x-c)| \le t, \quad \|t\|_2 \le t_0, \quad t \in \mathbb{R}^n_+,\ t_0 \in \mathbb{R}_+,\ x \in \mathbb{Z}^n\right\}, \tag{20}$$

which can be strengthened by adding CMIR cuts. Similarly to Proposition 4, we can show that a single split cut for Q added to (19) is at least as strong as the corresponding CMIR added to (20). However, as formalized in the following proposition, there are examples where just elementary CMIR cuts can provide a bound that is arbitrarily better than that obtained by all split cuts for Q.

Proposition 7. *Let $B = I$ and $c_i = 1/2$ for all $i \in [n]$. Then*

$$n/4 \quad = \min_x \left\{\|x - c\|_2^2 \ : \ x \in \mathbb{Z}^n\right\} = \min_{x,t,t_0} \left\{t_0^2 \ : \ (x, t, t_0) \in \bigcap_{i=1}^n MIR^{e^i,\lfloor c_i \rfloor}\right\},$$

while $1/4 \geq \min_{x,s_0} \left\{s_0 \ : \ (x, s_0) \in \bigcap_{(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Z}} Q^{\pi,\pi_0}\right\}$.

Proof. The first equality is straightforward. From the proof of Proposition 6, we have that $t_0 \geq \sqrt{n}/2$ for any $(x, t, t_0) \in \bigcap_{i=1}^n MIR^{e^i,\lfloor c_i \rfloor}$, which proves the second equality. To prove the inequality above, we show that $(\bar{x}, \bar{s}_0)$ given by $\bar{s}_0 = 1/4$ and $\bar{x}_i = 1/2$ for all $i \in [n]$ satisfies all the quadratic split cuts. For this, note that using Corollary 4 in [21], the quadratic split cut with x replaced by $\bar{x}$ is given by

$$-\Big(\pi_0 + 1 - (1/2) \sum_{i=1}^n \pi_i\Big)\Big(\pi_0 - (1/2) \sum_{i=1}^n \pi_i\Big)/\|\pi\|_2^2 \leq s_0.$$

Then the only interesting cases are those with $\pi_0 < (1/2) \sum_{i=1}^n \pi_i < \pi_0 + 1$, for which the cut reduces to $(1/4 \, \|\pi\|_2^2) \leq s_0$. The strongest of these cuts is $1/4 \leq s_0$ which is satisfied by $(\bar{x}, \bar{s}_0)$. □

Note that the example in Proposition 7 is very specific. In fact, our preliminary computational experiments show that for randomly generated CVP instances, the integrality gaps obtained by adding quadratic split cuts and CMIRs are roughly the same. It seems that using an extended formulation is enough to compensate for the lack of non-polyhedral information in the generation of CMIR cuts; however, it does not provide an advantage in general.

Finally, while CVP provides a simple and clean setting to compare the strength of cuts, no class of cuts seems to provide a computational advantage for solving these problems. We are currently exploring the effectiveness of these cuts on more practical MICQPs.

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References

[1] K. Andersen and A. Jensen. Intersection cuts for mixed integer conic quadratic sets. In M. Goemans and J. Correa, editors, *IPCO*, volume 7801 of *LNCS*, pages 37–48, 2013.

[2] A. Atamtürk and V. Narayanan. Conic mixed-integer rounding cuts. *Math. Program.*, 122:1–20, 2010.

[3] E. Balas. Disjunctive programming: Properties of the convex hull of feasible points. *Discrete Appl. Math.*, 89:3–44, 1998.

[4] P. Belotti, J. C. Góez, I. Pólik, T. K. Ralphs, and T. Terlaky. A conic representation of the convex hull of disjunctive sets and conic cuts for integer second order cone optimization, 2012. Optimization Online, http://www.optimization-online.org/DB_HTML/2012/06/3494.html.

[5] P. Belotti, J. C. Góez, I. Pólik, T. K. Ralphs, and T. Terlaky. On families of quadratic surfaces having fixed intersections with two hyperplanes. *Discrete Appl. Math.*, 161:2778 – 2793, 2013.

[6] D. Bienstock and A. Michalka. Strong formulations for convex functions over nonconvex sets. *Optimization Online*, 2011. Optimization Online, http://www.optimization-online.org/DB_HTML/2011/12/3278.html.

[7] Daniel Bienstock and Alexander Michalka. Cutting-planes for optimization of convex functions over nonconvex sets. *SIAM Journal on Optimization*, 24 (2):643–677, 2014.

[8] R.E. Bixby, M. Fenelon, Z. Gu, E. Rothberg, and R. Wunderling. Mixed-integer programming: a progress report. In *The sharpest cut: the impact of Manfred Padberg and his work*, chapter 18, pages 309–326. SIAM, Philadelphia, PA, 2004.

[9] P. Bonami. Lift-and-project cuts for mixed integer convex programs. In O. Günlük and G. J. Woeginger, editors, *IPCO*, volume 6655 of *LNCS*, pages 52–64, 2011.

[10] C. Buchheim, A. Caprara, and A. Lodi. An effective branch-and-bound algorithm for convex quadratic integer programming. *Math. Program.*, 135: 369–395, 2012.

[11] M. T. Çezik and G. Iyengar. Cuts for mixed 0-1 conic programming. *Math. Program.*, 104:179–202, 2005.

[12] W. J. Cook, R. Kannan, and A. Schrijver. Chvátal closures for mixed integer programming problems. *Math. Program.*, 47:155–174, 1990.

[13] Gérard Cornuéjols and Yanjun Li. On the rank of mixed 0, 1 polyhedra. *Mathematical Programming*, 91(2):391–397, 2002.

[14] D. Dadush, S. S. Dey, and J. P. Vielma. The split closure of a strictly convex body. *Oper. Res. Lett.*, 39:121 –126, 2011.

[15] S. Dash, O. Günlük, and A. Lodi. MIR closures of polyhedral sets. *Math. Program.*, 121:33–60, 2010.

[16] S. Drewes. *Mixed Integer Second Order Cone Programming*. PhD thesis, Technische Universität Darmstadt, 2009.

[17] R. E. Gomory. Some polyhedra related to combinatorial problems. *Linear Algebra and its Applications*, 2:451 – 558, 1969.

[18] M. R. Kılınç, J. Linderoth, and J. Luedtke. Effective separation of disjunctive cuts for convex mixed integer nonlinear programs, 2010. Optimization Online, http://www.optimization-online.org/DB_HTML/2010/11/2808.html.

[19] F. Kılınç-Karzan. On minimal valid inequalities for mixed integer conic programs, 2013. Optimization Online, http://www.optimization-online.org/DB_HTML/2013/06/3936.html.

[20] Daniele Micciancio and Shafi Goldwasser. *Complexity of lattice problems: a cryptographic perspective*, volume 671. Springer, 2002.

[21] Sina Modaresi, Mustafa R. Kılınç, and Juan Pablo Vielma. Intersection cuts for nonlinear integer programming: Convexification techniques for structured sets, 2013. arXiv preprint arXiv:1302.2556, http://arxiv.org/abs/1302.2556.

[22] Diego A Moran R, Santanu S Dey, and Juan Pablo Vielma. A strong dual for conic mixed-integer programs. *SIAM Journal on Optimization*, 22(3): 1136–1150, 2012.

[23] G. L. Nemhauser and L. A. Wolsey. *Integer and combinatorial optimization*. Wiley, 1988.

[24] G. L. Nemhauser and L. A. Wolsey. A recursive procedure to generate all cuts for 0-1 mixed integer programs. *Math. Program.*, 46:379–390, 1990.

[25] R. A. Stubbs and S. Mehrotra. A branch-and-cut method for 0-1 mixed convex programming. *Math. Program.*, 86:515–532, 1999.

[26] J. P. Vielma. A constructive characterization of the split closure of a mixed integer linear program. *Oper. Res. Lett.*, 35:29–35, 2007.