Evaluating Approaches for Solving the Area Restriction Model in Harvest Scheduling

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Abstract

We survey three integer-programming approaches for solving the area restriction model (ARM) for harvest scheduling. We describe and analyze each of these approaches in detail, comparing them both from a modeling and computational point of view. In our analysis of these formulations as modeling tools we show how each can be extended to incorporate additional harvest scheduling concerns. In our computational analysis we illustrate the strengths and weaknesses of each formulation as a practical optimization tool by studying harvest scheduling in four North American forests.

Key Words: adjacency, ARM, green-up, optimization, integer-programming.

1. Introduction

Forest firms face the challenge of adapting current harvest scheduling methodologies in order to meet growing environmental concerns, particularly those of protecting species and natural habitat, while minimally foregoing economic gains. A common approach for incorporating such wildlife concerns consists of restricting the local extent of harvest activity by imposing a limit on the maximum contiguous area of clear-cut regions (Thompson et al 1973).

In order to model these types of problems, forest planners rely on spatially explicit models in order to represent forests and harvest scheduling plans. More specifically, Geographic Information Systems (GIS) are used to partition forests into stands representing sub-regions with homogeneous characteristics, and decisions are made at a stand level. In this way, the design of a harvest schedule is abstracted as a combinatorial optimization problem, with decision variables corresponding to stands for potential harvest at a particular time, and where the objective may consist in maximizing profits or other objectives, subject to operational and environmental constraints. The constraint we are concerned with in this article consists of prohibiting contiguous groups of stands from exceeding the prescribed clear-cut area limit.

Approaches to solve this problem initially were relatively simple, and relied on an experienced planner to manually pre-define disjoint feasible "clusters" (contiguous groups of stands) each satisfying the maximum contiguous area restriction. This was done in such a way that if two clusters were adjacent, then their combined area would exceed the maximum opening size restriction. Thus, the harvest scheduling problem reduced to selecting, among these pre-defined clusters, a subset which would maximize revenue, subject to the constraint that no two adjacent clusters be simultaneously selected. Clearly this approach was highly dependent on how the clusters were pre-defined. Murray (1999) referred

to this approach as the unit restriction model (URM). A number of exact formulation approaches for this model have been pursued: Barahona et al (1992) and Weintraub et al (1994) proposed a column generation scheme; Murray and Church (1996, 1997) proposed using cliques to directly solve a set packing problem; and, Hoganson and Borges (1998) proposed a dynamic programming approach. Heuristic and meta-heuristic approaches have also been proposed, including Tabu search (Murray and Church 1995) and simulated annealing (O'Hare 1989, Nelson and Brodie 1990, Murray and Church 1995).

More recently, Hokans (1983) and Lookwood and Moore (1993) proposed incorporating the construction of clusters from stands in the decision model, as opposed to having these clusters be pre-defined. Murray (1999) referred to this approach as the area restriction model (ARM). Murray and Weintraub (2002) empirically showed that solution profit value can be improved through the addition of cluster-forming in the ARM. However, the ARM is a significantly more complex problem. Until recently, most solution approaches were heuristic or meta-heuristic in nature (see Hokans 1983, Lockwood and Moore 1993, Barrett et al 1998, Clark et al 2000, Richards and Gunn 2000, Boston and Bettinger 2002, Caro et al 2003). Lately, however, several exact integer programming formulations have been proposed for solving the problem. One formulation, which we henceforth call the Path Formulation, is based on enumerating all possible ways in which a harvesting cluster may be infeasible, and defining constraints which prohibit each of these infeasibilities (see McDill et al. 2002, Crowe et al. 2003, Gunn and Richards 2005). A second formulation, which we henceforth call the Cluster Packing Formulation, is based on defining variables for all possible feasible harvesting clusters and defining constraints which prevent any selected pair from being overlapping or adjacent (see McDill et al. 2002, Murray et. al 2004, Goycoolea et al. 2005, Martins et al. 2005, Vielma et al. 2005). A third formulation, which we call the Bucket Formulation, is based on defining buckets a-priori, and then assigning stands to each bucket so that each bucket represents a harvested cluster (see Constantino et al. 2006).

In this paper we review and compare these three mixed integer programming formulations for the ARM, and discuss some basic extensions. We consider both modeling and computational issues. In terms of modeling, we discuss different spatial considerations which can be incorporated into the formulations, such as green-up, minimum-size, maximum average-size, shape constraints and others. In terms of computational issues our focus is on performance.

It is well known that most integer programming problems can be formulated in different ways, and that the properties of these different formulations can significantly affect their performance. One such issue is the formulation size. Another, less trivial issue is tightness of the formulation. There are several ways of defining the notion tightness. Given that we consider formulations that are defined on different variables; we compare their LP relaxation values. That is, the optimal value obtained when solving the formulation without imposing the integrality constraints. This defines a natural bound on the optimal value, and indicates how good a proxy the relaxation value is to the actual optimal. This is useful both as a way of validating solutions found with heuristic methods, and is crucial for the effectiveness of Branch and Bound algorithms.

For more information, see Nemhauser and Wolsey (1988). A final measure that we consider is the amount of time it takes to find an optimal (or near-optimal solution). If this cannot be achieved in a reasonable amount of time, we consider instead what is the best solution found after an allotted time.

This paper is organized as follows. In Section 2 we introduce the ARM and the notation which will be employed. We then describe the three integer programming formulations of the ARM. We discuss implementation issues relating to these formulations, and compare the theoretical quality of the linear programming relaxation bounds they provide. In Section 3 we discuss extensions of the ARM and how these may be incorporated into the integer programming formulations. In Section 4 we present computational results obtained by testing these approaches using four forest planning applications. We focus on analyzing the size of the different formulations, the strength of the linear programming relaxation bounds they provide, the difficulty of solving each formulation to optimality, and the effect of incorporating model extensions. Finally, in Section 5 the relative advantages of each approach are discussed and possible future directions of research are noted.

2. The Area Restriction Model (ARM)

Throughout this article we convene on the following notation in order to formally describe the problem. Let S be the number of stands in the forest, and let T represent the number of time periods considered. Assume that each time period represents Y years. To each stand s we assign the following attributes:

- a_s : area of stand s,
- $p_{s,t}$: revenue obtained when harvesting stand s in time t,
- $\alpha_{s,t}$: volume of timber obtained when harvesting stand s in time t,
- g_s : initial age of stand s (i.e. the age of stand s in time t=1).

Let E represent all pairs of adjacent stands. That is, if r,s are adjacent stands, then $\{r,s\} \in E$. Let $N(s)$ represent the set of all stands which are adjacent to s. That is, stand r belongs to set $N(s)$ if and only if r is adjacent to s, or equivalently, if $\{r,s\} \in E$.

To illustrate these concepts consider Figure 1.

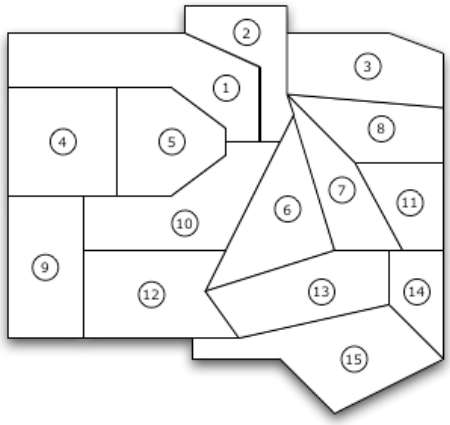


Figure 1

In this example, $S=15$. If we define two stands as being adjacent if they touch, then we have that the pair $\{1,4\}$ is in E but that the pair $\{1,9\}$ is not. Note also for this example that $N(4) = \{1,5,9,10\}$.

We say that a set C of contiguous stands defines a *cluster* of stands. For each cluster C we define its total area as $a_C = \sum_{s \in C} a_s$. Likewise, we define the revenue obtained from harvesting cluster C at time t as $p_{C,t} = \sum_{s \in C} p_{s,t}$. Finally, let $\alpha_{C,t} = \sum_{s \in C} \alpha_{s,t}$ equal the volume of timber obtained when harvesting cluster C at time t.

The Area Restriction Model, or ARM, consists of selecting, for each time period, a set of stands to be harvested so as to maximize revenue subject to the following constraints:

Volume yield constraints: A typical requirement in forestry operations is that a "forest produces a non-declining even flow of timber" (Buongiorno and Gilless, 2003, p. 70) or a "reasonable yield pattern" (Ware and Clutter, 1971, page 436). For each time period t, the amount of timber extracted should be no more than U_t times (and no less than L_t times) what was extracted in the previous time period. Variants of this constraint might consider imposing minimum and maximum volume requirements in different time periods.

Average ending age constraints: The average age of standing timber at the end of the planning horizon should be at least $\overline{G}$ years.

Harvest-once constraints: Each stand should be harvested at most once during the planning horizon. Here it is implicitly assumed that we are only concerned with first-harvest and that replanted stands will not reach a profitable age in the planning horizon.

Maximum clear-cut size constraints: No contiguous group of stands harvested in a same time period may exceed an area limit of A.

Henceforth we say that a cluster C is *feasible* if it satisfies $a_C \leq A$. Otherwise we say that C is *infeasible*. Using this definition, the maximum clear-cut constraint can be equivalently described as prohibiting infeasible clusters from being harvested in all of the time periods. Note that the definition of a feasible cluster coincides with the definition of g*eneralized management unit* as proposed by McDill et al. (2002).

The maximum clear-cut size constraint can be visualized using the hypothetical forest illustrated in Figure 1. For simplicity, assume that all stands have an area of 10 hectares, that a maximum opening size of $A = 30$ hectares is stipulated, and that we are considering a single time period. Given this, the cluster defined by stands 4, 5, and 9 is feasible. However, if any additional neighboring stand were added, the resulting cluster would be infeasible. For example, cluster {4,5,9,10} is infeasible, as it has a total area of 40 hectares. In this way, if we momentarily ignore volume-flow and average ending age constraints, it can be seen that set of shaded stands illustrated in Figure 2 define a feasible solution.

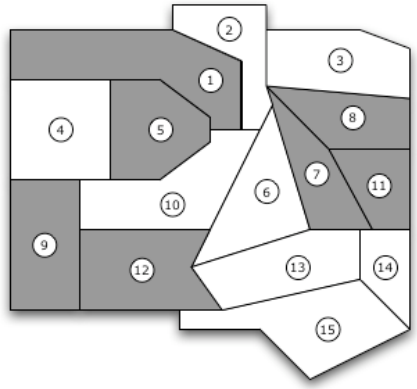


Figure 2

We now describe three different integer programming formulations which have been proposed for this problem.

The focus of this article is on these maximum clear-cut size constraints.

2.1 The Path Formulation

The Path Formulation, originally proposed in McDill et al. 2002, works by defining for each stand s and each time period t, a binary variable $y_{s,t}$ taking value 1 if stand s is to be harvested in period t, and value 0 if not. In order to see how the maximum clear-cut size condition can be imposed using these variables, let us consider again the hypothetical forest of the previous section, and a candidate solution in which the stands {1,2,3,4,5} are all harvested in time period $t=1$. Since the total area of this cluster is 50, and the maximum clear-cut size is $A=30$, this cluster would be infeasible. Thus, in order to ensure a feasible solution, we need to prohibit that all these stands be harvested simultaneously in $t=1$. For this, it suffices to add the constraint:

$$y_{1,1} + y_{2,1} + y_{3,1} + y_{4,1} + y_{5,1} \leq 4$$

This idea can be generalized in the following manner: Let Λ represent the set of all infeasible clusters. That is, $(C \in \Lambda)$, if and only if, C corresponds to a contiguous set of stands such that $(a_C > A)$. In order to prohibit all infeasible clusters from being selected add the family of constraints:

$$\sum_{s \in C} y_{s,t} \leq |C| - 1 \qquad \forall C \in \Lambda, \forall t = 1, \ldots, T$$

A concern regarding this formulation is that the set Λ of infeasible clusters may be exponentially large, thus rendering this formulation impractical. However, it is possible to greatly reduce the size of this formulation by utilizing only those constraints which are strictly necessary. Consider two infeasible clusters C and D such that $C \subseteq D$. It is easy to see that by prohibiting the harvest of cluster C, the harvest of cluster D is also prohibited. Going back to the previous example, observe that to prevent {1,2,3,4,5} from being harvested in t=1 it would suffice to add the constraint:

$$y_{1,1} + y_{2,1} + y_{3,1} + y_{4,1} \leq 3$$

Formally, we can generalize this concept as follows. Define a cluster $C \in \Lambda$ to be *minimally infeasible* if C is infeasible and if for every $s \in C$ either C-{s} is a feasible cluster or C- {s} is not contiguous. That is, if removing any stand from C we obtain another cluster which is either feasible or the resulting set of stands is disconnected. If C is minimally infeasible, then C cannot contain any other infeasible clusters. Thus, if we define $\Lambda^+ = \{ C \in \Lambda : C$ is minimally infeasible$\}$ we can instead use the so-called Path Inequalities:

$$\sum_{s \in C} y_{s,t} \leq |C| - 1 \qquad \forall C \in \Lambda^+, \forall t = 1, \ldots, T$$

This leads us to the Path Formulation of the ARM (McDill et al. 2002):

$$\max \sum_{t=1}^{T} \sum_{s=1}^{S} p_{s,t} y_{s,t} \qquad (1)$$

Subject to

$$\sum_{t=1}^{T} y_{s,t} \leq 1 \qquad \forall s = 1, \ldots, S \qquad (2)$$

$$\sum_{s \in C} y_{s,t} \leq |C| - 1 \qquad \forall C \in \Lambda^+, \forall t = 1, \ldots, T \qquad (3)$$

$$\sum_{s=1}^{S} \alpha_{s,t+1} y_{s,t+1} \leq U_t \sum_{s=1}^{S} \alpha_{s,t} y_{s,t} \qquad \forall t = 1, \ldots, T-1 \qquad (4)(a)$$

$$\sum_{s=1}^{S} \alpha_{s,t+1} y_{s,t+1} \geq L_t \sum_{s=1}^{S} \alpha_{s,t} y_{s,t} \qquad \forall t = 1, \ldots, T-1 \qquad (4)(b)$$

$$\sum_{s=1}^{S} a_s \left(g_s + T \cdot Y - \sum_{t=1}^{T} (t \cdot Y + g_s) y_{s,t} \right) \geq \overline{G} \sum_{s=1}^{S} a_s \qquad (5)$$

$$y_{s,t} \in \{0,1\} \qquad \forall s = 1, \ldots, S, \forall t = 1, \ldots, T \qquad (6)$$

The objective function, (1), consists in maximizing the total harvest revenue. Constraints (2) impose that each stand be harvested at most once during the planning horizon. Constraints (3) impose the maximum clear-cut size condition. Constraints (4) impose the volume flow constraints. Constraints (5) impose the average ending age condition. Constraints (6) impose integrality on decision variables.

We say that set K of stands defines a *clique* if every pair of stands $r,s \in K$ are adjacent to each other. A clique K is *maximal*, if no other clique contains it. Observe that the sets {1,4,5,10} and {6,7,13} in Figure 1 each define a maximal clique. Let Π denote the set of all maximal cliques. Crowe et al (2003) suggest adding the following family of constraints in order to strengthen the problem formulation:

$$\sum_{s \in K} a_s y_{s,t} \leq A \qquad \forall K \in \Pi, \forall t = 1,\ldots,T \quad (7)$$

Observe that constraints (7) need only be defined for those cliques whose total area exceeds the prescribed limit. Computational results presented in Crowe et al (2003) suggest that this strengthening approach is not very effective in practice. However, by building on this idea, Gunn and Richards (2005) propose another way of tightening the Path formulation by adding only one constraint per stand. The basic idea is based on the following observation: if a stand s is harvested in period t, then the total area adjacent to stand s which can be harvested is bounded by $(A - a_s)$. The constraints they propose are as follows:

$$\sum_{r \in N(s)} a_r y_{r,t} \leq M_s (1 - y_{s,t}) + (A - a_s) \quad \forall s = 1,\ldots,S, \forall t = 1,\ldots,T \quad (8)$$

As an example, consider again the forest depicted in Figure 1. For stand 4, time period t, this constraint would take form:

$$10y_{1,t} + 10y_{5,t} + 10y_{9,t} + 10y_{10,t} \leq M_4 (1 - y_{4,t}) + 20$$

Gunn and Richards (2005) describe different ways by which to compute suitable coefficients M_s, as well as a lifting algorithm by which to strengthen these constraints. Gunn and Richards (2005) observe that by replacing constraints (3) with constraints (8), they were able to generate optimal feasible solutions for several test cases. However, they point out that such an approach will not always yield feasible solutions. Consider the forest depicted in Figure 3, and assume that the area of stands 1, 2, 3 and 4 is 20 hectares each. Further, assume that $A = 60$.

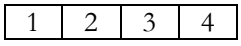


Figure 3

A cluster consisting of stands 1, 2, 3, and 4 exceeds the maximum opening size, yet no constraint of type (8) excludes it.
Regardless, constraints (8) can be effective for improving the performance of the Path Formulation when combined with constraints (3), or, constraints (8) can be used alone to

generate approximate solutions (when individual stand sizes are large relative to the maximum clear-cut size) as proposed by Gunn and Richards (2005).

An important issue pertaining to the use of the Path Formulation is the enumeration of the minimally infeasible clusters which are needed to define constraints (3). McDill et al. (2002) propose what they call the Path Algorithm for achieving this. In Appendix **I** we present an alternative algorithm which is easier, and which we have found to work better in practice. In the appendix we also discuss how this algorithm can be modified so as to generate the set of all maximal cliques and other sets of stands.

2.2 The Cluster Packing Formulation

An alternative way to formulate the ARM problem is based on focusing on feasibilities rather than infeasibilities. This can be achieved by defining a 0-1 decision variable for every feasible cluster (taking value 1 if we should select it and 0 otherwise), and defining constraints in such a way as to ensure that the selected clusters are disjoint and separated one from another.

More formally, for each feasible cluster C and every time period t, let binary variable $x_{C,t}$ indicate if cluster C is to be harvested in time period t or not. We say that two feasible clusters C and D are *incompatible* if they intersect or are adjacent. The Cluster Packing model imposes the maximum area constraint by forcing selected clusters into being compatible with one another.

A natural way to achieve this consists in explicitly prohibiting incompatibility by using pair-wise constraints (McDill et al 2002, Goycoolea et al 2005). That is, if two feasible clusters C and D are incompatible one could impose, for each time period t, constraints of the form:

$$x_{C,t} + x_{D,t} \le 1$$

.

Let Ω represent the set of all feasible clusters. For each set of stands U, let $\Omega(U)$ represent the set of all feasible clusters containing at least one stand in U and define $\Omega(s) = \Omega(\{s\})$ for each stand s. The Pairwise Cluster Packing formulation for the ARM is defined as follows:

$$\max \sum_{t=1}^{T} \sum_{C \in \Omega} p_{C,t} x_{C,t} \qquad (9)$$

Subject to

$$\sum_{t=1}^{T} \sum_{C \in \Omega(s)} x_{C,t} \le 1 \qquad \forall s = 1,\ldots,S \qquad (10)$$

$$x_{C,t} + x_{D,t} \le 1 \qquad \forall C, D \in \Omega \text{ with } C, D \text{ incompatible.}\ \forall t = 1,\ldots,T \qquad (11)$$

$$\sum_{C \in \Omega} \alpha_{C,t+1} x_{C,t+1} \le U_t \sum_{C \in \Omega} \alpha_{C,t} x_{C,t} \qquad \forall t = 1,...,T-1 \qquad (12)(a)$$

$$\sum_{C \in \Omega} \alpha_{C,t+1} x_{C,t+1} \ge L_t \sum_{C \in \Omega} \alpha_{C,t} x_{C,t} \qquad \forall t = 1,...,T-1 \qquad (12)(b)$$

$$\sum_{s=1}^{S} a_s \left(g_s + T \cdot Y - \sum_{t=1}^{T} \left(t \cdot Y + g_s \right) \sum_{C \in \Omega(s)} x_{C,t} \right) \ge \overline{G} \sum_{s=1}^{S} a_s \qquad (13)$$

$$x_{C,t} \in \{0,1\} \qquad \forall C \in \Omega, \forall t = 1,\ldots,T \qquad (14)$$

The objective function, (9), is to maximize total harvest revenue. Constraints (10) impose that each stand should be harvested at most once during the planning horizon. Constraints (11) impose that incompatible pairs of clusters should not be selected in a same time period, (12) impose the volume flow constraints. Constraints (13) impose that the average ending age condition. Constraints (14) impose integrality on decision variables. If we replace constraints (14) with constraint,

$$0 \leq x_{C,t} \leq 1 \qquad \forall C \in \Omega, \forall t = 1,\ldots,T \quad (14)(b)$$

we obtain what is called the LP relaxation of this formulation. In Appendix **I** we describe how to algorithmically generate the set Ω required for formulating the problem.

Observe that there may exist two feasible clusters whose union defines yet another feasible cluster. This can happen if the clusters are disjoint, adjacent to each other and if their combined area does not exceed the prescribed maximum. A common concern is that constraints (11) would prohibit these from being simultaneously harvested. It should be noted that this is not a problem because the cluster defined by taking the union of these smaller clusters also has a variable associated to it.

Though this formulation is valid, it is possible to obtain others which, building upon the same idea, provide much tighter linear programming bounds.

To see this, consider a pair of adjacent stands *r,s*. That is, a pair $\{r,s\}$ in the set E. These pairs will henceforth be referred to as *edges*. In a feasible ARM solution there can be at most one selected cluster intersecting the edge $\{r,s\}$. In fact, if there are two clusters intersecting the set $\{r,s\}$ then either they both contain a same stand, or otherwise, they are disjoint but adjacent to each other. In both cases, they are incompatible. From the preceding analysis, the following family of constraints is valid for the ARM (see Martins et al, 2005):

$$\sum_{C \in \Omega(\{r,s\})} x_{C,t} \leq 1 \qquad \forall \{r,s\} \in E, \forall t = 1,\ldots,T \quad (11)(b)$$

If we replace constraints (11) with constraints (11)(b) we obtain what we call the Edge Cluster Packing formulation, which is tighter than the Pairwise Cluster Packing formulation.

An even tighter formulation was developed by Martins et al (2005) and Goycoolea et al (2005). As before, let Π denote the set of all maximal cliques in the forest. Constraints (11)(b) can be substituted with constraints:

$$\sum_{C \in \Omega(K)} x_{C,t} \leq 1 \qquad \forall K \in \Pi, \forall t = 1,\ldots,T \quad (11)(c)$$

The reason for this is that two clusters intersect a common edge if and only if they intersect a common clique. When this substitution is used we obtain the Clique Cluster Packing Formulation. Going back to the example depicted in Figure 1, {6,7,13} defines a clique. This means that among the feasible clusters {6,7,13},

{2,6,10}, {3,8,7}, {7,13,14}, {6,12}, {13,14} and all others that intersect {6,7,13}, only one can be chosen. Recall that an algorithm by which to enumerate all maximal cliques is described in Appendix **I**.

The Clique Cluster Packing formulation is not only tighter than the Edge Cluster Packing formulation. Results detailed in Goycoolea et al (2005) show that the Clique Cluster Packing formulation is substantially faster to solve. Goycoolea et al (2005) also introduced a methodology called "constraint projection" for deriving additional families of valid inequalities for this problem (see Nemhauser and Wolsey 1988). Finally, Goycoolea et al (2005) describe a methodology for strengthening constraints (11)(c) by a clique-lifting procedure. Computational tests, however, indicate that these constraints are very effective, and do not need such strengthening.

In the computational results section we will see that in the linear programming relaxation bounds afforded by the Cluster Packing formulation are strictly better than those afforded by the Path formulation in all but one of the instances tested (For example, in Table 5, see El Dorado with $T=1$).

A natural question is whether or not the Cluster Packing formulation can have a worse linear programming relaxation bound than the Path formulation. The answer, remarkably, is no. That the LP relaxation value of the Path formulation will always be larger or equal to that of the Cluster Packing formulation is proved as a mathematical theorem in Appendix II. That in practice it is very often strictly larger can be seen in the computational results.

In important implication of this result is that the Clique-Cluster Packing formulation is better suited than the Path formulation as a tool for validating heuristics and estimating quality of solution values.

2.3 The Bucket Formulation

Consider an instance of ARM and observe that in any given time period, one can never simultaneously harvest more clusters than there are stands in the forest. This naturally follows from the fact that harvested clusters must be disjoint and non-adjacent to each other. This suggests yet a third way in which the ARM can be formulated.

For each stand i define a "bucket" B_i. The idea is that stands should be assigned to these buckets in such a way that (a) each stand is assigned to at most one bucket, (b) the total area of stands assigned to a same bucket does not exceed the prescribed maximum, and (c) buckets should be non-adjacent to each other. Consider such an assignment of stands to buckets and consider any given bucket. If the bucket is non-empty, and the stands in the bucket are all connected to each other, then the bucket defines a feasible cluster. Otherwise, break-up the stands of the bucket into connected components. Given condition (b), each such component defines a feasible cluster. Finally, due to condition (c) all of the clusters obtained from the stand-to-bucket assignment will be compatible with each other, and so, one obtains feasible ARM solutions through these assignments. Moreover, it is easy to see that given any feasible ARM solution it is possible to construct such a stand-to-bucket assignment by simply letting each feasible cluster define a bucket.

Let us now formalize this idea for multiple-time periods by means of an integer programming model. For each stand s, period t and bucket B_i define a binary decision variable y_s^{it} indicating whether or not stand s will be assigned to bucket B_i in time period t. For each $e \in E$, $t = 1,\ldots, T$, and $i=1,\ldots,S$ define a binary decision variable w_e^{it} indicating if at least one of the end-points of edge e has been assigned to bucket B_i in period t.

Using these variables it is possible to use the following integer programming formulation for the ARM problem (see Constantino et al. 2006):

$$\max \sum_{t=1}^{T}\sum_{s=1}^{S}\sum_{i=1}^{S} p_{s,t} y_s^{it} \qquad (15)$$

Subject to

$$\sum_{t=1}^{T}\sum_{i=1}^{S} y_s^{it} \le 1 \qquad \forall s = 1,\ldots,S \qquad (16)$$

$$\sum_{i=1}^{S} w_e^{it} \le 1 \qquad \forall e \in E, \forall t = 1,\ldots,T \qquad (17)$$

$$y_s^{it} - w_e^{it} \le 0 \qquad \forall e \in E, \forall s \in e, \forall i = 1,\ldots,S, \forall t = 1,\ldots,T \qquad (18)$$

$$\sum_{s=1}^{S} a_s y_s^{it} \le A \qquad \forall i = 1,\ldots,S \qquad (19)$$

$$\sum_{i=1}^{S}\sum_{s=1}^{S} \alpha_{s,t+1} y_s^{i,t+1} \le U_t \sum_{i=1}^{S}\sum_{s=1}^{S} \alpha_{s,t} y_s^{i,t} \qquad \forall t = 1,\ldots,T-1 \qquad (20)(a)$$

$$\sum_{i=1}^{S}\sum_{s=1}^{S} \alpha_{s,t+1} y_s^{i,t+1} \ge L_t \sum_{i=1}^{S}\sum_{s=1}^{S} \alpha_{s,t} y_s^{i,t} \qquad \forall t = 1,\ldots,T-1 \qquad (20)(b)$$

$$\sum_{s=1}^{S} a_s \left(g_s + T \cdot Y - \sum_{t=1}^{T} (t \cdot Y + g_s) \sum_{i=1}^{S} y_s^{it} \right) \ge \overline{G} \sum_{s=1}^{S} a_s y_s^{it} \qquad (21)$$

$$y_s^{it} \in \{0,1\} \qquad \forall i = 1,\ldots,S, \forall s = 1,\ldots,S, \forall t = 1,\ldots,T \qquad (22)$$

$$w_e^{it} \ge 0 \qquad \forall e \in E, \forall i = 1,\ldots,S, \forall t = 1,\ldots,S \qquad (23)$$

The objective function, (15), is to maximize the total harvest revenue. Constraints (16) impose that each stand can be assigned at most to a single bucket throughout the planning horizon. Constraints (17) impose that clusters are pair wise non-adjacent. This is achieved by imposing that at most one bucket intersects each edge in the forest. Constraints (18) impose the relationship between the y and w variables. That is, each time a stand is assigned to a bucket, the variable corresponding to the edges which it touches must be triggered. Constraints (19) impose that the total area of stands in a bucket cannot exceed A. Constraints (20) impose the volume flow condition and constraints (21) impose the average ending age condition. Constraints (22) impose integrality of the y variables. Constraints (23) impose non-negativity of the w variables. Note that it is not necessary to impose integrality for the w variables, as a feasible schedule will be fully determined by the y variables.

As with the Cluster Packing Formulation, it is observed in Constantino et al. (2006) that a tighter formulation can be obtained replacing edges by cliques. For this, define a binary decision variables W_K^{it} for each $i=1,\ldots,S$, each $t = 1,\ldots,T$, and each $K \in \Pi$, indicating if at least one of the stands in clique K has been assigned to bucket B_i in period t. Then, substitute constraints (17), (18), and (23) by:

$$\sum_{i=1}^{S} W_K^{it} \leq 1 \qquad \forall K \in \Pi, \forall t = 1,\ldots,T \qquad (17)(b)$$

$$y_s^{it} - W_K^{it} \leq 0 \qquad \forall K \in \Pi, \forall s \in K, \forall i = 1,\ldots,S, \forall t = 1,\ldots,T \qquad (18)(b)$$

$$W_K^{it} \geq 0 \qquad \forall K \in \Pi, \forall i = 1,\ldots,S, \forall t = 1,\ldots,T \qquad (23)(b)$$

A very important feature of this formulation is that the number of rows and columns is independent of the maximum clear-cut size A. As the value of A increases all that needs change in this formulation is the right-hand side of constraints (19). In the Path formulation, the number of constraints increases exponentially with A, and in the Cluster Packing formulation, the number of columns increases exponentially with A. This makes the Bucket formulation especially significant for problems in which the average stand size relative to the maximum clear-cut size is small.

Despite this fact, however, the formulation can still be very large in practice. In order to make up for this, Constantino et al. (2006) propose a series of pre-processing ideas to make the formulation practical to real applications. Without going into all the details, we briefly summarize the two key ideas for effective pre-processing:

- Assume that the clusters defined from bucket B_i will only be made up of stands s_j with $j = i,\ldots,S$. In this way, each cluster will be uniquely represented by the bucket corresponding to its lowest stand index.
- Consider a bucket L_i and time period t. Only define y_s^{it} variables for stands s such that there exists a feasible cluster C containing both s and i. In this way, stands that are too far away from s will not be included in bucket B_i.

3 Incorporating additional spatial constraints into the ARM

There are many possible extensions to the basic ARM. Such extensions are often fundamentally important to different planning applications. In this section we discuss several such extensions and ways of incorporating them into the three different formulations.

3.1 Harvest costs

While income from timber sales can be approximated as a summation of the volumes from all harvested stands, costs are not necessarily linear. There are usually fixed costs involved in harvesting an area, such as establishing a work force, setting up operations and moving equipment (Weintraub et al. 1999). Other costs include fencing, which depends on the perimeter length of the harvested area. McDill et al (2002) point out that the cost savings achieved by joint management of adjacent stands can be significant.

In the Cluster Packing approach it is straightforward to consider fixed costs or non- linearities because costs for each cluster are derived a priori. It is not clear how this issue can be imposed in the other formulations in a simple way.

3.2 Average clear-cut size constraints.

In certain cases, imposing average clear-cut size constraints can provide much flexibility. For example, instead of imposing clusters *never* exceed a size of 40 hectares, it could be imposed that in *average* they never do so. Recent voluntary initiatives, such as that of the American Forest and Paper Association (2001), promote the use of such average constraints, as opposed to strict area limitations.

Let $\overline{A_t}$ equal the maximum allowed average clear-cut size for time period t. A bound on the average clear-cut size can be imposed using the following constraint in the Cluster Packing Formulation (see Murray et al. 2004):

$$\sum_{C \in \Omega} (a_{C,t} - \overline{A_t}) x_{C,t} \leq 0 \qquad \forall t = 1,\ldots,T \quad (24)$$

Observe that imposing (24) in the cluster packing formulation still requires that a maximum area be imposed when defining the feasible cluster variables.

This constraint can also be imposed in the bucket formulation by use of the following constraints:

$$\begin{aligned} \sum_{s=1}^{S} \sum_{i=1}^{S} a_s y_s^{it} &\leq \overline{A_t} \sum_{i=1}^{S} y_i^{it} && \forall t = 1,\ldots,T \\ y_s^{it} - y_i^{it} &\leq 0 && \forall i = 1,\ldots,S, \forall t = 1,\ldots,T \end{aligned} \quad (25)$$

The validity of this constraint follows immediately from the observation that $\sum_{i=1}^{S} y_i^{it}$ defines a lower bound on the total number of clusters selected for harvesting. Observe that an advantage of the bucket formulation over the cluster packing formulation in terms of this constraint is that a strict maximum area requirement need not be imposed in the bucket formulation.

It is not clear how this constraint can be imposed in the path formulation.

3.4 Green-Up constraints

Green-Up constraints extend maximum clear-cut area constraints to multiple-time periods. That is, they impose that once an area is harvested, adjacency conditions should still be enforced for that area during a certain amount of time which is called a green-up period. The length of this period depends on the species that are replanted, and the length of each period. For example, the American Forest and Paper Association (2001) promotes adopting "green-up requirements, under which past clear-cut harvest areas must have trees at least 3 years old or 5 feet high … before adjacent areas may be clear-cut."

We distinguish between two types of green-up constraints, namely dynamic and static green-up constraints. The main difference between the two is that dynamic green-up constraints enforce the adjacency condition at the stand level, whereas static green-up constraints enforce the condition at the cluster level. It is important to note that this is a fundamental difference that has not been made explicit in other articles. For example, the notion of green-up employed by Goycoolea et al. 2005 and Constantino et al. 2005 is that

of static green-up. However, the notion of green-up used by Gunn and Richards 2005b, Barrett and Giles 2000 and Davis et. al. 2001 is that of dynamic green-up.

Consider a green-up period equal to Δ. That is, if a stand is harvested in period t, it will be considered in clear-cut state for all periods in $\{t, \ldots, t + \Delta - 1\}$.

Dynamic green-up constraints limit the combined area of contiguous stands in clear-cut state independent of the moment in which the stands were harvested. Dynamic green-up constraints can be easily implemented by adding prescription variables (see Gunn and Richards 2005b, Barrett and Giles 2000 and Davis et. al. 2001). Under our assumptions of only one possible treatment, the formulation obtained has two sets of variables. One indicates when a stand is harvested and the other when a stand is in clear-cut state. The details for implementing this formulation in the ARM model are as follows. We first introduce additional binary variables $z_{s,t}$ for each stand s and each period t and then modify some of the constraints. The idea of these variables is that they should indicate if stand s is harvested in time t, whereas the variables $y_{s,t}$ and $x_{C,t}$ should indicate if a stand or cluster is in clear-cut state during period t.

For the Path Formulation constraints (2) need to be replaced by,

$$y_{s,t} = \sum_{q=t-\Delta+1}^{t} z_{s,q} \qquad \forall t = 1,\ldots,T, \forall s = 1,\ldots,S \quad (26)$$

For the Cluster Packing formulations constraints (10) need to be replaced by,

$$\sum_{C \in \Omega(s)} x_{C,t} = \sum_{q=t-\Delta+1}^{t} z_{s,q} \qquad \forall t = 1,\ldots,T, \forall s = 1,\ldots,S \quad (27)$$

For both formulations it is necessary to add constraints,

$$\sum_{t=1}^{T} z_{s,t} \le 1 \qquad \forall s = 1,\ldots,S \quad (28)$$

In addition, changes need to be made to both models, since the objective function, volume flows, and average ending age constraints should be stated in terms of the z variables as opposed to the y and x variables. For example, objective functions (1) and (9) should be replaced by,

$$\max \sum_{t=1}^{T} \sum_{s=1}^{S} p_{s,t} z_{s,t} \qquad (29)$$

Modifying the Bucket Formulation is analogous to modifying the Path Formulation.

Static green-up constraints (Goycoolea et al. 2005, Constantino et. al. 2005) are different in the sense that they mimic the effect of green-up constraints over the URM model. The main difference with the dynamic green-up approach is that in static green-up forces all contiguous stands in a clear-cut state are to be harvested in the same time period. A methodological advantage of this is that green-up constraints can be easily imposed at the cluster level without the introduction of any additional variables.
Static green-up constraints can be implemented in the Clique Cluster Packing formulation by extending the clique inequalities to:

$$\sum_{C \in \Omega(K)} \sum_{q=t}^{t+\Delta-1} x_{C,q} \le 1 \qquad \forall K \in \Pi, \forall t = 1 \ldots T-1 \quad (30)$$

in the Path Formulation by adding,

$$\sum_{u \in N(s)} \sum_{q=t-\Delta+1, q \neq t}^{t+\Delta-1} y_{u,q} \leq M(1 - y_{s,t}) \qquad \forall s = 1,\ldots,S, \forall t = 1,\ldots,T \quad (31)$$

(where M is some suitable large number), and in the Bucket Formulation by extending (17)(b) to:

$$\sum_{i=1}^{S} \sum_{q=t}^{t+\Delta-1} W_K^{it} \leq 1 \qquad \forall K \in \Pi, \forall t = 1,\ldots,T$$

We now illustrate the difference between the dynamic and static green-up constraints using the following example. Again consider Figure 3 that consists of a narrow forest made up of 4 stands. Assume that in this example the green up requirement is of two time periods, that all stands span ten hectares, and that the maximum clear-cut size is twenty hectares. Thus, if stand 1 is harvested in period 1, stand 2 in period 2, stand 3 in period 3, and stand 4 in period 4, we observe that for *t=1* there will be a single clear-cut cluster {1}, for *t=2* there will be a single clear-cut cluster {1,2}, for t=3 a clear-cut cluster {2,3}, for *t=4* a clear-cut cluster {3,4}, and for *t=5* a clear-cut cluster {4}. Given that in all time periods the clusters satisfy the maximum clear-cut size condition, the solution is feasible for the dynamic green-up constraints. However, this same solution would not be valid for the static green-up constraints.

It is interesting to note that the introduction of such green-up constraints and the choice between the dynamic or static versions can have important consequences in terms of spatial analysis. In fact, many of the specialized Cluster Packing constraints have to be carefully considered. This is because the shapes of clear-cut areas will be constantly changing in time when using dynamic green-up constraints. For example, consider the average clear-cut size constraints described in Section 3.2. Should the average size be considered in terms of the clear-cut clusters as they evolve in time? Or should the average size only be considered in terms of when clusters are actually harvested? How should fixed costs be defined when using dynamic green-up constraints?

3.3 Restricting the set of feasible areas

In certain applications it may be of importance to restrict the actual shapes of harvested clusters. For example long, elongated clusters may be undesirable from an operational point of few, and clusters with holes in the middle may be undesirable in terms of wildlife protection. In some cases it may even be desirable to limit the *minimum* size of a clear-cut area (Andalaft et al. 2003). This condition is applied when determining fixed costs is difficult. In the Cluster Packing formulation all such constraints are easily imposed by discarding clusters having non-desirable shapes or characteristics through a preprocessing phase. It is not clear how this issue can be imposed in the other formulations in a simple way.

3.4 Using alternative definitions of adjacency

It has long been recognized that adjacency conditions can be interpreted in a variety of ways. The classic approach is merely to infer adjacency when two stands share a common border or edge (point-touch adjacency). However, adjacency could be defined based upon many conceived notions of proximity, both spatial and aspatial (for example, see Walters 1996). There are several reasons why it might be desirable to consider different definitions of adjacency. As it has been pointed out before, when using point-touch adjacency definitions, maximum clear-cut size constraints tend to leave managed forests very

fragmented, reducing it to small, disconnected patches, each having a reduced interior area (Gustafson and Crow 1998, Borges and Hoganson 2000, Rebain and McDill 2003). While some species might prefer early successional stage habitats (which often occur in forest edges), it is well known that this can be very detrimental to many others.

Part of the problem is due to the fact that a single definition of adjacency is used in current models to simultaneously model two different aspects of harvest scheduling. On the one hand it is used to define when a group of stands make up a cluster, and in the other, it is used to define how far apart different clusters should be from each other. Point-touch adjacency metrics seem a reasonable measure with which to define clusters. This is because (for operational reasons) one would like the stands which make up a common cluster to be tightly packed together. However, it seems a poor choice for defining cluster incompatibility as one would like clusters well separated one from another, possible by a wide buffer.

An alternative to point-touch definitions of adjacency for use in defining cluster incompatibility would consist in defining clusters to be adjacent when they are within a certain distance of each other; say, M meters (distance adjacency). In this way, if we consider a single time period model it is easy to see that harvested clusters will be separated one from another by a wide buffer, and that the diameter of the standing forest will be at least M meters wide at all points; except, possibly, near the edges. If in addition it is imposed that clusters should be M meters away from the edge of the forest, then the resulting harvest plans will be such that the standing forest is entirely connected and that the diameter condition will hold throughout. In this way we would obtain a forest where the un-harvested region is connected by corridors of width M meters throughout. By properly combining distance adjacency metrics with green-up constraints it is possible to ensure that these connectivity conditions extend in time.

Goycoolea et al. (2005) define this variation of the ARM, which considers two simultaneous definitions of adjacency, as the Extended Area Restriction Model (EARM). As they point out, formulating this model using the cluster packing formulation is trivial. In fact, it is simply a matter of defining the clusters with one definition, and using the latter to define the cliques which define the compatibility constraints. It is not clear how this issue can be imposed in the path and bucket formulations.

4 Computational Tests

4.1 Introduction

In order to compare the performance of the different integer programming approaches, tests were conducted on four forest regions. The goal of the experiments was to assess:

(a) The size of the different formulations. We focus on the number of columns, rows, and non-zeroes resulting in the formulation of real forest instances. Special attention is paid to how the size changes relative to the maximum clear-cut size condition.
(b) The strength of the linear programming relaxation bounds. The proximity of the linear programming relaxation bound to the value of the best known feasible solution is compared. Special attention is paid to how this proximity changes with the number of time periods considered, and with the use of volume flow, average ending age constraints and green-up constraints.

(c) The amount of time required to solve problems with each formulation. The time required to solve each problem to optimality is compared, as well as the time required to find feasible solutions having a reasonable gap (one percent).

The first data set, El Dorado, corresponds to a United States national forest region in northern California. Sets Shullkel and Lemon Creek, correspond to Canadian national forest regions in Nova Scotia and British Columbia. The last set, NBCL5 corresponds to a publicly owned tract of industrial forest region in New Brunswick. Each data set contains information describing the age, timber volume, and area for each stand. All of these data sets, with the exception of Lemon Creek[1], are available at the FMOS (2006) web. Furthermore note that a choice of maximum clearcut size of 16.19 hectares was chosen instead of 48.56 hectares for Shullkel and Lemon Creek. Similarly a choice of 32.37 hectares was chosen for NBCL5. This is because these instances have many stands which are very small. Thus, limiting the maximum clearcut size assures that the problems could be formulated in at most 2 GB of RAM. In Table 1 some basic information concerning these instances is summarized.

Instance	Stands	Total Area [hectares]	Min Stand Area [hectares]	Avg. Stand Area [hectares]	Max Stand Area [hectares]	Maximum Clearcut Size [hectares]
El Dorado	1,363	21,147.03	4.05	15.52	47.09	48.56
Shullkel	1,039	4,498.75	0.13	4.33	112.36	16.19
Lemon Creek	6,675	40,814.17	2.84	6.11	97.74	16.19
NBCL5	5,881	60,393.1	0.99	10.27	99.95	32.37

Table 1. Problem instance information.

These forests were selected because their characteristics make them induce relatively hard to solve ARM models. For example, El Dorado has a fairly homogeneous age distribution which usually results in harder ARM models (McDill and J. Braze. 2000, McDill et. al 2002). Computational results should then be considered illustrative of types of forests.

In order to carry out the tests, two stands were defined as adjacent if they touched in a line. The revenue was assumed proportional to the timber volume for each stand. For multiple time-period runs a discount rate of 3% was applied to each period. When using volume flow constraints, we defined $L_t = 1 - 15/100$, and $U_t = 1 + 15/100$, for all $t = 1,\ldots,T$. Unless specifically stated otherwise, no green-up constraints were imposed. When using average ending age constraints, the average ending age was required to be of 40 years. Stands having an area larger than the maximum imposed limit were ignored. Note that the Path formulation strengthening of Crowe et al (2003) and Gunn and Richards (2005) as described in Section 2.1 were not used.

All runs were made on a Pentium IV (Xeon) running at 2.40 ghz and with 2 GB of RAM running Linux. All programs were written in the C++ programming language, and ILOG CPLEX 9.0 was utilized for all linear and integer programs solves. Summaries of the runs can be found in Tables 3-9 and in

[1] The original Lemon Creek data contained so many stands that none of the one period formulations could be generated even when using a computer with 12 GB of RAM. For this reason the data set was modified by aggregating stands together until no stand had an area below 2.83 hectares. This modified instance is what will be denoted as Lemon Creek from here on.

Graph 1. Unless otherwise noted, all runs were made using the default CPLEX settings and a maximum time limit of 10,000 seconds (≈2.8 hours) was imposed. Finally, note that the times reported do not take into account the amount of time taken to build the problem, e.g. the time required to enumerate all clusters and cliques and set up the formulation. However, the enumeration algorithms described in Section 2 ran very fast. In fact, completely formulating (e.g. enumerating clusters, cliques, and defining the constraints) each of the problems took less than one minute for most combinations of instances and approaches. More specifically, the average formulation time for the Path, Clique-Cluster Packing and Bucket formulations was 28.97, 23.39, and 73.08 seconds respectively.

4.2 Size of the different formulations.

Our first experiment consisted in formulating our four test cases using the different formulations, and comparing the resulting sizes. As a preliminary step it was necessary to enumerate all of the feasible and minimally infeasible clusters, as well as all of the maximal cliques for each forest. In Table 2 we list the number of each of the enumerated objects when using the maximum clear-cut size areas indicated in Table 1.

Instance	Stands	F. Clusters	M. I. Clusters	M. Cliques
El Dorado	1,363	21,411	20,629	2,033
Shullkel	1,039	29,630	12,889	1,093
Lemon Creek	6,675	87,336	175,743	10,766
NBCL5	5,881	94,912	52,136	5,967

Table 2. Number of clusters and cliques.

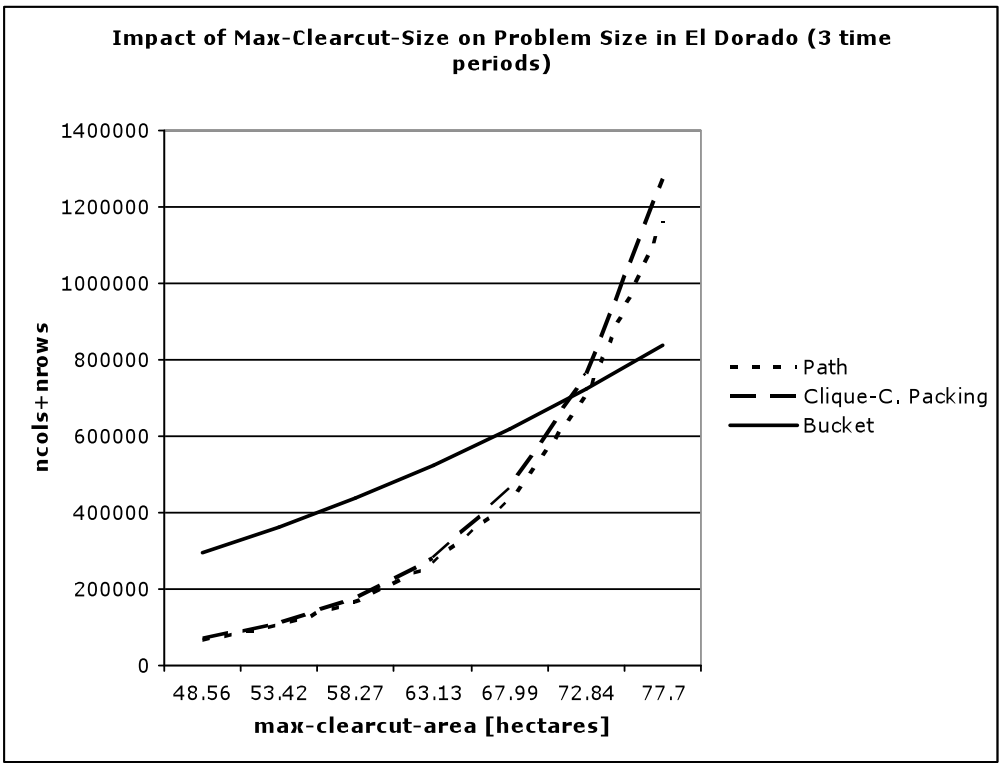
As can be seen, the number of clusters (both feasible and minimally infeasible) per stand is roughly 10-30, and the number of maximal cliques is 1-2 per stand. Further, it can be observed that the number of feasible clusters is similar to the number of minimally infeasible clusters in each forest. Thus, it would be expected that the total number of rows and columns should be very similar for the Path and Clique-Cluster Packing formulations. This is confirmed in Table 3, where we describe the number of rows, columns and non-zero coefficients for each of the three formulations when considering the specific case in which there are three time periods, and both volume and ending age constraints.

Instance	PATH			CLIQUE-C. PACKING			BUCKET		
	Cols	Rows	NZ	Cols	Rows	NZ	Cols	Rows	NZ
El Dorado	4089	63255	297952	64233	7467	1280747	114309	264987	760437
Shullkel	3117	39711	233113	88890	4323	1822338	60264	135111	418464
L.Creek	20025	533909	2359277	262008	38978	5234423	684171	1579148	4435775
NBCL5	17643	162294	915292	284736	23787	5352639	294981	633042	1942489

Table 3. Size for 3 periods.

For example, consider instance El Dorado. It can be seen that the total number of rows and colums for the cluster formulation is 71,700, and for the Path formulation 67,344. On the other hand, it can be seen that the Bucket Formulation results in much larger problems for these tests. In El Dorado the number of rows and columns adds up to 379,296. It can also be seen that the cluster formulation is by far the densest formulation (in terms of non-zeroes).

Our next experiment consisted in increasing the maximum clear-cut size to see how this affects problem size. As should be expected, problem size increased exponentially with both the Path and Clique-Cluster Packing formulations. Increasing the size of the maximum clear-cut condition by 60% resulted in an increased problem size of roughly 500%. Not surprisingly, problem size increased linearly with the Bucket Formulation. That the Bucket Formulation grows at all is due to the fact that the pre-processing becomes less effective as the maximum clear-cut size value is increased. This can be observed in the case of El Dorado (3 time periods, with volume flow and average ending age constraints) in Graph 1. By observing the graph we can see that if the maximum clear-cut size increases to 72.84 hectares or more, then the Bucket formulation becomes the most compact (in terms of total number of columns and rows) of the three. It is interesting to observe how the average cluster size grows with the maximum clear-cut size condition. Table 4 shows that in the specific case of El Dorado, the size of feasible clusters nearly doubles when increasing the average clear-cut size condition by 60%.



Graph 1. Impact of maximum clear-cut size parameter.

Maximum clearcut size (hectares)	Average cluster size	Maximum cluster size (number of stands)
48.56	3.57926	7
53.42	4.01706	8
58.27	4.46242	8
63.13	4.91272	9
67.99	5.36555	10
72.84	5.81671	10

77.7	6.26803	11

Table 4. Impact of maximum clear-cut size parameter (El Dorado).

4.3 Strength of the Linear Programming Relaxation.

Our second experiment consisted in solving the linear programming relaxation of each test case with 1, 3, and 5 time periods. We also solved For El Dorado with 12 time periods. For each problem we considered up to four variants; one without using volume flow, ending age or any green-up constraints (labeled as "*T*", indicating the number of time periods considered), one using both volume flow and average ending age constraints ("*T*, volume+age"), and two using volume flow, average ending age, and dynamic green up constraints ("*T*, volume+age, green-up", where the last parameter indicates the length of the green-up period). For each run we recorded the time it took to solve the problem (root lp time) and in addition, the value of the best solution obtained. Then, for each instance we kept the best solution found among all the different runs. For each run we compared the value of the lp relaxation to the value of the best known feasible solution for the corresponding instance (root lp gap). If *x* is the value of the best known solution for a given instance, and *r* is the value of the lp relaxation obtained using a given formulation, then the corresponding root lp gap would have value *((r/x)-1)*100*. For example, the best known feasible solution for El Dorado (*T=1*) has value *x=2154060* and the lp relaxation for the Bucket formulation has value *r = 2249236.2*. Thus, the value of root lp gap is (2249236.2/2154060-1)*100 = 4.4184. The results are illustrated in Table 5.

		Path		Clique-C. Packing		Bucket	
	Number of time periods, and extensions used in the formulation.	Root lp gap (%)	Root lp time (s)	Root lp gap (%)	Root lp time (s)	Root lp gap (%)	Root lp time (s)
El Dorado	T=1	5.81%	0.84	0.16%	0.57	4.42 %	802.71
	T=3	5.67%	17.9	0.57%	22.86	3.51 %	10000+
	T=3, volume+age	0.74%	12.11	0.08%	8.35	0.47%	444.21
	T=3, volume+age+greenup=2	7.16%	12.22	0.38%	6.97	5.11%	10000+
	T=5	3.21%	92.91	0.44%	31.77	2.01%	8916.97
	T=5, volume+age	0.25%	13.05	0.05%	13.19	0.13%	394.81
	T=5, volume+age+greenup=2	3.37%	247.7	0.96%	215.79	2.63%	10000+
	T=5, volume+age+greenup=3	8.31%	538.1	0.52%	500.92	5.16%	10000+
	T=12	0.79%	41.38	0.08%	33.68	0.53%	1541.38
	T=12, volume+age	0.47%	16.12	0.46%	31.61	0.48%	706
	T=12, volume+age+greenup=2	5.89%	621.43	5.74%	120.85	5.86%	10000+
	T=12, volume+age+greenup=3	12.44%	1125.19	10.83%	2259.42	11.98%	10000+
Shulkell	T=1	2.16%	0.29	0.02%	0.37	4.00%	33.5
	T=3	1.06%	3.54	0.06%	2.97	0.77%	107.55
	T=3, volume+age	0.38%	7.06	0.03%	9.85	0.55%	148.7
	T=3, volume+age+greenup=2	7.75%	22.18	0.13%	27.72	7.14%	5644.02
	T=5	0.34%	14.71	0.02%	4.49	0.25%	82.47
	T=5, volume+age	0.03%	12.67	0.01%	14.08	0.08%	89.8
	T=5, volume+age+greenup=2	1.16%	45.41	0.37%	109.2	1.29%	4138.67
	T=5, volume+age+greenup=3	7.45%	155.86	0.15%	103.1	6.78%	10000+
Lemon	T=1	14.16%	24.26	0.17%	5.32	5.71%	10000+
	T=3	7.24%	52.24	1.23%	29.46	3.72%	10000+
	T=3, volume+age	1.65%	381.12	0.68%	2303.17	1.18%	10000+
	T=5	3.75%	11012.59	0.61%	1759.53	2.51%	10000+
	T=5, volume+age	0.25%	2021.5	0.08%	1555.07	0.19%	10000+
NBC	T=1	0.78%	1.58	0.01%	1.5	2.49%	286.34
	T=3	0.21%	13.71	0.01%	10.21	0.42%	64.6
	T=3, volume+age	0.04%	120.4	0.01%	31	0.08%	267.5
	T=3, volume+age+greenup=2	0.34%	63.05	0.02%	54.07	0.75%	1285.87

	T=5	0.07%	48.29	0.00%	14.43	0.18%	176.59
	T=5, volume+age	0.01%	267.42	0.01%	66.82	0.02%	504.05
	T=5, volume+age+greenup=2	0.09%	1122.17	0.01%	32.73	0.12%	496.51
	T=5, volume+age+greenup=3	0.24%	1936.1	0.03%	108.8	0.43%	1368.18

Table 5. Strength of linear programming relaxations and time required to solve.

Several observations arise:

It can be seen that the cluster formulation yields, by far, the tightest root lp gaps. For example, consider the Lemon Creek instance (*T=3*,volume+age). The root lp gap for the Path formulation is 1.65% and for the Bucket formulation this value is 1.18%. However, the value for the Clique-Cluster Packing formulation is 0.68%. In general, however, all three formulations perform rather well with the averages gaps (over all test cases) for the Path, Clique-Cluster Packing and Bucket formulation being 3.13%, 0.73% and 2.45% respectively.

In Theorem 1 it was proved that the lp relaxation gap of the Clique-Cluster Packing formulation could never be less tight than that of the Path formulation. However, no such relationship was established between the Bucket formulation and any other. From the table it can be seen that neither the Path nor Bucket formulation is stronger than the other. In fact, for El Dorado (*T=5*) it can be seen that the Bucket formulation is tighter than the Path formulation. However, for Shulkell (*T=1*) it can be seen that the converse holds. On the other hand, the table suggests that it may be true that the Clique-Cluster Packing formulation is always tighter than the Bucket formulation (this is observed in every instance considered).

The time required to solve the LP relaxation of the Bucket formulation is considerably longer than that required to solve the Path and Clique-Cluster Packing formulations. We attempted to solve the same instances with the barrier method (as opposed to the default cplex settings), and though several problems took considerably less time to solve, others took much longer – thus making it unclear which method was the better. It would seem this difficulty should be further studied.

As the number of time periods increases, the root lp gaps tend to decrease in the Path and Bucket formulations. A clear example is the El Dorado problem with the Path formulation, where the root lp gap values for *T=1,T=3,T=5* and *T=12* are 5.81%, 5.67%, 3.21% and 0.79%. The Clique-Cluster Packing formulation, on the other hand, seems to peak when *T=3*. For example, consider the Lemon Creek instance, in which the lp root gaps for *T=1,T=3*, and *T=5* are 0.17%, 1.23%, and 0.61% respectively.

Adding the volume and ending age constraints decreases the lp root gap values in all formulations. For example, in the Path formulation, observe that El Dorado (*T=5*) has root lp gap 3.21%, and that El Dorado (T=5, volume+age) has root lp gap 0.25%. However, adding the extended green-up constraints (greenup=2, greenup=3) increases the root lp gaps in all formulations. In El Dorado (*T=3*, volume+age) it can be seen that the Path formulation has root lp gap 0.74%. However, when green up constraints are introduced, this value goes up to 7.16%.

The impact of imposing green-up constraints seems to be much less severe in the Clique-Cluster Packing formulation than in the Path and Bucket

formulations. For example, consider the Shulkell (*T=5*, volume+age) instance. The root lp gaps for the Path, Clique-Cluster Packing, and Bucket formulations are 0.34%, 0.02% and 0.25% respectively. However, when introducing greenup=3 constraints, these values increase to 7.45%, 0.15% and 6.78% respectively.

4.4 Solving the integer programming problem.

Our third experiment consisted in solving the integer programming problem (with a time limit of 10,000 seconds) for each of the problem variants considered in Section 4.3. The main tool used for measuring performance was the gap. Given an incomplete run of a branch and bound algorithm, let *zu* be the value of the current upper bound and *zl* be the value of the best known feasible solution. We compute gap as *(zu/zl-1)*100*. For each optimization run we recorded the time at which the first feasible solution within a provable one percent of optimality was obtained ("First 1% solution", in seconds) and the best provable gap obtained after the 10,000 second time limit expired ("Final gap / Opt time"). In those cases in which a problem was solved to optimality within the time limit, the number of seconds required to complete the solve is indicated in parenthesis. If no feasible solution was found, or no feasible 1% solution was found, we indicate this with a dash in the corresponding column. The results obtained are summarized in Table 6.

		Path		Clique-C. Packing		Bucket	
		First 1% solution	Final gap (Opt time)	First 1% solution	Final gap (Opt time)	First 1% solution	Final gap (Opt time)
El Dorado	T=1	-	2.07%	2.93	(4.3 s)	-	2.58%
	T=3	-	3.62%	171.94	(5417.4s)	-	15.26%
	T=3, volume+age	27.19	0.13%	73.42	0.01%	1433.18	0.44%
	T=3, volume+age+greenup=2	-	7.52%	173.48	(1732.6)	-	-
	T=5	-	1.71%	269.99	0.09%	-	11.38%
	T=5, volume+age	36.02	0.03%	101.72	0.07%	841.14	0.23%
	T=5, volume+age+greenup=2	-	2.77%	2379.93	0.80%	-	-
	T=5, volume+age+greenup=3	-	8.68%	1817.17	0.19%	-	-
	T=12	73.61	0.28%	109.48	(192.31 s)	3533.57	0.79%
	T=12, volume+age	1684.58	0.45%	-	1.98%	-	2.69%
	T=12,volume+age+greenup=2	-	5.79%	-	5.72%	-	-
	T=12, volume+age+greenup=3	-	-	-	10.69%	-	-
Shulkell	T=1	19.21	(6315.2 s)	0.9	(1.0 s)	-	1.80%
	T=3	37.04	0.36%	12.82	(22.8 s)	1376.34	0.50%
	T=3, volume+age	120.03	0.11%	74.45	(497.3 s)	3960.02	0.47%
	T=3,volume+age+greenup=2	-	4.92%	38.44	(1190.9 s)	-	-
	T=5	16.26	0.06%	17.53	(28.2 s)	148.93	0.12%
	T=5, volume+age	17.12	(786.6 s)	31	(2878.1 s)	223.39	0.19%
	T=5,volume+age+greenup=2	-	1.01%	655.44	0.27%	-	-
	T=5, volume+age+greenup=3	-	6.04%	340.99	0.07%	-	-
Lemon Creek	T=1	-	13.50%	17.75	(35.4 s)	-	-
	T=3	-	7.35%	9850	0.91%	-	-
	T=3, volume+age	-	3.06%	7143.51	0.62%	-	-
	T=5	-	5.19%	8544.3	0.41%	-	-
	T=5, volume+age	8831.06	0.52%	1716.68	0.06%	-	-
NBCL	T=1	11.9	0.22%	4.15	(4.4 s)	-	1.14%
	T=3	20.41	0.04%	30.56	(65.9 s)	286.89	0.18%
	T=3, volume+age	132.78	(235.7 s)	55.21	(250.3 s)	1334.25	0.05%
	T=3,volume+age+greenup=2	260.66	0.14%	208.19	(474 s)	-	-
	T=5	53.6	0.02%	43.25	(71.8 s)	216.08	0.07%

	T=5, volume+age	340.68	(3948.79 s)	104.22	(1103.7 s)	999.91	(3302.3 s)
	T=5,volume+age+greenup=2	1341.34	0.06%	293.78	0.01%	2067.39	0.48%
	T=5, volume+age+greenup=3	2086.22	0.16%	703.52	0.02%	-	-

Table 6. Solving the integer programming problem.

Several observations:

The Clique-Cluster Packing formulation significantly outperformed the Path and Bucket formulations in problems without volume, ending age, and green-up constraints, having managed to solve most of these to optimality. The other formulations had a very difficult time even finding solutions within one percent on these problems. Example: For El Dorado (*T=1*), the Clique-Cluster Packing formulation found a 1% solution in 2.93 seconds and solved the problem to optimality in 4.3 seconds. However, after 10,000 seconds the Path formulation had only managed to establish a solution within 2.07% of optimality, and the Bucket formulation a solution within 2.58% of optimality.

The performance of the Clique-Cluster Packing formulation relative to that of the Path formulation is most significant when the difference between the LP relaxation bounds is large (see El Dorado, *T=1*). On the other hand, when the difference between the LP relaxation bounds is small, the performance of both methods is more similar, with the Path formulation sometimes performing better (see Shulkell, *T=5*, volume+age).

Problems with green-up constraints proved considerably harder to solve for all of the formulations. For the Lemon Creek problem we were simply unable to obtain feasible solutions with any of the formulations.

In problems with no green up constraints, but with volume and age constraints, the relative performance of the Path and Clique-Cluster Packing formulations seems to depend on the number of time periods. In fact, for *T=1* instances the Clique-Cluster Packing formulation seems to perform better, yet for *T=5* instances (and the *T=12* instance) the Path formulation seems to be the top performer.

In most of the runs, the Bucket formulation did not perform as well as the other two. In fact, in more than half of the runs it was unable to even find a feasible solution. This was very surprising; especially when comparing these results to those reported in Constantino et al. (2006), where very different results are presented for the specific case of El Dorado. This discrepancy led us to contact the authors of Constantino et al. (2006). We found that our implementations differed in many aspects. First, we had used different parameters for the discount rates, volume flows and average ending age constraints. More importantly, however, we found that the LP formulations we were ultimately solving were very different in terms of constraints and number of variables. The reason for this turned out to be that the pre-processing routine was dependent on the way stands in the forest was numbered. This suggests that permuting the ordering of the stands could lead to alternative formulation pre-processing schemes.

5. Final Remarks

In this paper we have reviewed the three major integer programming formulations which have been proposed for the Area Restriction Model (ARM) and have discussed how different spatial concerns can be addressed using each approach. In addition, through computational tests, we have shown how these

methodologies perform on a set of real problems of small to medium size. It is important to keep in mind that due to the limited sample size of our experiments, the results should be regarded as being illustrative rather than as statistically significant in their conclusions.

All of these formulations, which have been developed during the last decade, constitute an important first step in being able to solve the ARM to optimality. However, it seems clear from our analysis that several important issues need to be resolved before these methodologies can be extended to larger problems. Some suggested research directions follow.

In Table 5 it can be seen that all three formulations provide very tight linear programming bounds. However, solving these formulations to optimality via branch and bound is painfully slow (see Table 6). Though much research has been conducted in proposing customized heuristics for these problems, no efforts have been pursued which combine such heuristics (lower bounds) and linear programming to validate their quality (upper bounds). The tight bounds obtained in Section 4.3 suggest that this may be a fruitful procedure in practice. After all, solving the LP relaxation of these problems is very fast, and it is natural to assume that the LP relaxation of much larger problems can be solved. If the bound is as tight on these problems, and good customized heuristics are developed, there may be no need to do branch-and-bound to obtain provably good solutions.

As shown in Section 4.2, all three integer programming formulations result in extremely large problems. Larger integer programming problems are hard to solve, primarily because they require enormous amounts of memory to store. With current methodologies, problems with over 30,000 stands, or problems in which feasible clusters average more than ten stands each, seem largely intractable. It would seem that in order to solve such problems decomposition-based integer programming methodologies, such as cutting plane or column generation algorithms, are necessary.

As shown in Section 4.3 and Section 4.4 it would seem that further research in computationally effective ways of implementing the Bucket Formulation are necessary. This approach is very promising as an alternative way of dealing with problems where stands are very small relative to the maximum allowed clear-cut area. So far, our computational experiments are largely disappointing. It has been noted that one way of dealing with this issue is through more advanced preprocessing schemes. This follows from the observation that the CPLEX preprocessor eliminates a large amount of variables and constraints even after the pre-processing steps described in Section 2.3. Some work in this direction is currently being pursued by Mills and McDill 2006.

As observed in Sections 4.3 and 4.4, incorporating green-up constraints has a very adverse effect on the linear programming relaxation gaps. This in turn, makes the integer programming problems much harder to solve. Deriving new classes of strong valid inequalities might be an important direction in being able to better solve this variant of the ARM.

Incorporating old-growth stands into optimization-based forest harvest planning is a key issue which to date has not been successfully tackled. The Cluster based approach of Martins et al. (2005) and the Path based approach of Rebain and McDill (2003) are both still very exploratory with only small problem instances being solvable to date. Both studies raise very relevant

questions, provide insightful analysis of obtained solutions and suggest a need for further studies. Interesting methodological approaches to these problems are also that of Ohman (2000) and Wei and Hoganson (2006) It is not clear how these methodologies can be made to work better, or if instead, there is another way of adapting current ARM formulations to consider this type of constraint.

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Appendix I. Enumerating sets in a forest graph.

In order to successfully implement any of the integer programming formulations described in this article, it is critical that an efficient algorithm for enumerating sets in a graph be used. In the case of the Path Formulation, for example, it is necessary to enumerate all minimally infeasible clusters in order to define constraints (3). In the case of the Clique Cluster Packing formulation it is necessary to enumerate all feasible clusters in order to define the variables, and all maximal cliques in order to define constraints (32)(c). Finally, in the case of the strengthened Bucket Formulation it is necessary to enumerate all maximal cliques in order to define the clique-variables W and constraints (17)(b), (18)(b), and (23)(b).

In Algorithm 1 we describe an algorithm for enumerating all minimally infeasible clusters which can be defined from a forest. This algorithm works recursively, starting from clusters comprised of a single stand, and growing them one stand at a time until the maximum area is met. Note that this algorithm is, in its original form, very similar to the one used by Rebain and McDill 2003.

More specifically, the algorithm starts by defining all feasible clusters composed of a single stand and putting them in a set called S[1] (see lines 02-04). Then, a recursion is initiated (line 05). At the beginning of the k-th iteration, the set S[k-1] will contain all feasible clusters composed of exactly (k-1) stands. From these, the set S[k] is built by adding, to each cluster in S[k-1] all stands which are contiguous to it and which do not make it grow beyond the required limit (see lines 09-22). When adding a stand results in an infeasible cluster we test if this cluster is minimally infeasible, and if so, store it (lines 15-17). The recursion ends when we can no longer grow any clusters and remain feasible.

Algorithm 1.

In order for this algorithm to run fast, special attention must be paid in steps 12 and 15 in

```
01. begin
02.   for v∈V do
03.     S[1]:=S[1] ∪ {v};
04.   end for.
05.   for k=1 to ∞ do
06.     if S[k] is empty then
07.       terminate;
08.     end if.
09.     for C∈S[k] do
10.       for v∈N(C) do
11.         D:=C∪{v};
12.         if D is feasible and D ∉ S[k+1] then
13.           S[k+1]:=S[k+1] ∪{D};
14.         end if.
15.         else D is minimally infeasible and D∉Λ+ then
16.           Λ+:=Λ+U{D};
17.         end else.
18.       end for.
19.     end for.
20.   end for.
21. end.
```

order to check if $D \notin S[k+1]$ or if $D \notin \Lambda^+$. If simple enumeration and comparison is utilized, then the algorithm may run prohibitively slow. One way of speeding this up could be by use of hash tables or binary trees (see Cormen 1990) to check for repetitions. Most modern programming languages (such as C++ and Python) include special data structures (such as "sets" in C++) which automatically check for repetitions. In our computational tests we found that using the "set" class to define the sets S[k], rather than brute-force comparison, for handling repetitions allowed us to reduce the running time of this algorithm from many hours to a few seconds in most instances.

Note that it is easy to modify Algorithm 1 so as to generate all possible cliques. In fact, it is simply a matter of deleting lines 12-17, and replacing these with

the following instructions:

After completing a run, each of the sets S[k] will contain cliques of size k. Note that

```
12b. if D is a clique then
13b.   Mark C as "non-maximal";
14b.   if D∉S[k+1] then
15b.     S[k+1]:=S[k+1]U{D};
16b.   end if.
17b. end if.
```

not all of these will be maximal. However, if a clique is not maximal it will have been marked in step 13b, thus they can easily be identified.

Appendix II. Comparing strength of the Path and Cluster Packing formulations.

In this section, we prove the following Theorem and Corollary.

Theorem 1. Let x be a non-negative vector in $R^{|\Omega|xT}$. Define $y_{s,t} = \sum_{C \in \Omega(s)} x_{C,t}$ for each stand s and each t=1,…,T. If x satisfies constraints (10), (11)(b), (12), (13) and (14)(b) (that is, if x is in the LP relaxation of the Edge Cluster Packing formulation) then y satisfies constraints (2), (3), (4), (5) and (6)(b) (that is, y is in the LP relaxation of the Path Formulation). Further, we have that,

$$\sum_{t=1}^{T} \sum_{C \in \Omega} p_{C,t} x_{C,t} = \sum_{t=1}^{T} \sum_{s=1}^{S} p_{s,t} y_{s,t}$$

Corollary. The Cluster Packing formulation is tighter than the Path formulation. That is, the value obtained by solving the LP-relaxation of the Edge Cluster Packing formulation can never be strictly greater than that obtained from the Path Formulation.

Proof of the Corollary: Given any solution x valid for the linear programming relaxation of the Edge Cluster Packing formulation, by defining $y_{s,t} = \sum_{C \in \Omega(s)} x_{C,t}$, we obtain a solution y valid for the LP relaxation of the Path formulation which has the same objective function value. Thus, the optimal value of the linear programming relaxation to the Edge Cluster Packing formulation must have a smaller or equal value than that of its Path counterpart, as it is optimizing over a subset of possible objective values.

Proof of the Theorem: To provide a compact proof we need to use the language of graph theory (Diestel, R. 2006). We will work in the graph of stands with vertex set V=1,…,S and edge set E given by all pairs of adjacent stands.

For every set of stands U, let E(U) denote the set of all edges in E having both end-points in U. Similarly, for every set of edges F ⊆ E, let V(F) denote the set of all nodes in V which are an end-point to some edge in F.

Let L ⊆ E define a *tree* in G, if (a) L is connected in G, and (b) L contains no cycles. Note that a tree L always satisfies |L| = |V(L)| - 1. A well known result is as follows: If a set U ⊆ V is connected in G, then, there exists a tree L ⊆ E(U), such that V(L) = U.

Recall that Ω represents set of all feasible clusters, and Λ the set of all infeasible clusters. Further, recall that for a set U ⊆ V we have that Ω(U) represents the set of all feasible clusters intersecting U. That is, $\Omega(U) = \{ C \in \Omega : C \cap U \neq \emptyset \}$.

Let y be a non-negative vector in $R^{|V|}$ and let x be a non-negative vector in $R^{|\Omega|}$.

Lemma 1. Assume that for all v∈V, the vectors x and y satisfy the following identity: $y_v = \sum_{C \in \Omega(v)} x_C$. For each v∈V consider a scalar α_v and for each C ∈ Ω, assume that $\alpha_C = \sum_{v \in C} \alpha_v$. Consider a set U ⊆ V. Then,

$$\sum_{C\in\Omega(U)} \alpha_C x_C = \sum_{v\in U} \alpha_v y_v \,.$$

Proof of Lemma 1.

$$\sum_{v\in U} \alpha_v y_v = \sum_{v\in U}\left(\alpha_v\left(\sum_{C\in\Omega(v)} x_C\right)\right) = \sum_{C\in\Omega(U)}\left(x_C\left(\sum_{v\in C}\alpha_v\right)\right) = \sum_{C\in\Omega(U)} \alpha_C x_C$$

□

Lemma 2. Assume that for all v∈V, the vectors x and y satisfy the following identity: $y_v = \sum_{C\in\Omega(v)} x_C$.

$$\text{If } \sum_{C\in\Omega(\{uv\})} x_C \le 1 \quad \forall\{u,v\}\in E \qquad \text{then } \sum_{v\in U} y_v \le |U|-1 \quad \forall U\in\Lambda\,.$$

Proof of Lemma 2.
Consider an infeasible cluster U∈Λ.

$$\sum_{v\in U} y_v = \sum_{v\in U}\left(\sum_{\{C:v\in V\}} x_C\right) = \sum_{C\in\Omega} |C\cap U|\, x_C$$

On the other hand, given that U is connected, there exists a tree L⊆ E(U), such that V(L) = U. Adding constraints (11)(b) over the edges in L, one obtains,

$$\sum_{\{uv\}\in L}\left(\sum_{C\in\Omega(\{uv\})} x_C\right) \le |L|$$

which is equivalent to,

$$\sum_{C\in\Omega} x_C\,|\{u,v\}\in L : u\in C \text{ or } v\in C\}| \le |L| = |U|-1$$

Thus, it suffices to prove that for all feasible clusters C∈Ω we have:

$$|\{u,v\}\in L : u\in C \text{ or } v\in C\}| \;\ge\; |C\cap U|$$

However, observe that $\{\{u,v\}\in L : u\in C \text{ or } v\in C\}\}$ can be partitioned into a disjoint family of trees L_1 ... L_k having the following properties:

- There is no edge in L joining L_i and L_j for all i ≠ j.
- Defining $C_i = V(L_i) \cap C$, then, $C \cap U$ can be partitioned into the sets C_i.
- For each of the trees L_i , there exists at least one edge with an end-point in C_i and the other end-point in $U - C_i$ (This is because U is connected, and because $U - C_i$ is non-empty). This implies that, $|V(L_i)| \ge |C_i| + 1$.

Putting everything together, it is easy to see that,

$$|\{u,v\}\in L : u\in C \text{ or } v\in C\}| = \sum_{i=1}^{k}|L_i| = \sum_{i=1}^{k}\left(|V(L_i)|-1\right) \ge \sum_{i=1}^{k}|C_i| = |C\cap U|$$

□

Proof. From Lemma 1 it is easy to see that since (10), (12), and (13) hold, then (2), (4), and (5) hold. From Lemma 2 it is easy to see that since (11)(b), then (3) holds.