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# Convex hull of two quadratic or a conic quadratic and a quadratic inequality

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**Abstract** In this paper we consider an aggregation technique introduced by Yıldıran [45] to study the convex hull of regions defined by two quadratic inequalities or by a conic quadratic and a quadratic inequality. Yıldıran [45] shows how to characterize the convex hull of open sets defined by two strict quadratic inequalities using Linear Matrix Inequalities (LMI). We show how this aggregation technique can be easily extended to yield valid conic quadratic inequalities for the convex hull of open sets defined by two strict quadratic inequalities or by a strict conic quadratic and a strict quadratic inequality. We also show that for sets defined by a strict conic quadratic and a strict quadratic inequality, under one additional containment assumption, these valid inequalities characterize the convex hull exactly. We also show that under certain topological assumptions, the results from the open setting can be extended to characterize the closed convex hull of sets defined with non-strict conic and quadratic inequalities.

**Keywords** Quadratic inequality, Conic quadratic inequality, Linear Matrix Inequality

## 1 Introduction

Development of strong valid inequalities or cutting planes such as Split cuts [19], Gomory Mixed Integer (GMI) cuts [27, 28], and Mixed Integer Rounding (MIR) cuts [34, 41, 42, 44] is one of the most important breakthroughs in the area of Mixed Integer Linear Programming (MILP) [17, 18, 20, 24]. Development of such strong valid inequalities has resulted in highly effective branch-and-cut algorithms [1, 12, 11, 29, 33].

There has recently been significant interest in extending the associated theoretical and computational results to the realm of Mixed Integer Conic Quadratic Programming (MICQP) [3, 13, 16, 21, 23, 25, 30, 35, 39, 40, 43]. Dadush et al. [22] study the split closure of a strictly convex body and characterize split cuts for ellipsoids. Atamtürk and Narayanan [4] study the extension of MIR cuts to sets defined by a single conic quadratic inequality and introduce conic MIR cuts which are linear inequalities derived from an extended formulation. Modaresi et al. [37] then characterize nonlinear split cuts for similar conic quadratic sets and also establish the relation between the split cuts and conic MIR cuts from [4]. Andersen and Jensen [2] also study similar conic quadratic sets as in [4] and derive nonlinear split cuts using the intersection points of the disjunctions and the conic set. Belotti et al. [6] study the families of quadratic surfaces

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having fixed intersections with two hyperplanes. Following the results in [6], Belotti et al. [5, 7] characterize disjunctive cuts for conic quadratic sets when the sets defined by the disjunctions are bounded and disjoint, or when the disjunctions are parallel. Modaresi et al. [36] characterize intersections cuts for several classes of nonlinear sets with specific structures, including conic quadratic sets. Bienstock and Michalka [9, 10] derive linear inequalities to characterize the convex hull of convex quadratic functions on the complement of a convex quadratic or polyhedral set and they also study the associated separation problem. Morán et al. [40] consider subadditive inequalities for general Mixed Integer Conic Programming and Kılınç-Karzan [31] studies minimal valid linear inequalities to characterize the convex hull of general conic sets with a disjunctive structure. Following the results in [31], Kılınç-Karzan and Yıldız [32] study the structure of the convex hull of a two-term disjunction applied to the second-order cone. Yıldız and Cornuéjols [46] study disjunctive cuts on cross sections of the second-order cone. Finally, Burer and Kılınç-Karzan [15] characterize the closed convex hull of sets defined as the intersection of a conic quadratic and a quadratic inequality that satisfy certain technical assumptions.

In this paper we study the convex hull of regions defined by two quadratic or by a conic quadratic and a quadratic inequality. The technique we use to characterize the convex hulls is an aggregation technique introduced by Yıldıran [45]. In particular, Yıldıran characterizes the convex hull of sets defined by two strict quadratic inequalities (i.e., intersection of two open quadratic sets) and obtains a Semidefinite Programming (SDP) representation of the convex hull using Linear Matrix Inequalities (LMI). Yıldıran also proposes a method to calculate the convex hull of two quadratics. In this paper we show that the SDP representation of the convex hull of two strict quadratics presented in [45] can be described by two strict conic quadratic inequalities. We also show that the aggregation technique in [45] can be easily extended to derive valid conic quadratic inequalities for the convex hull of sets defined by a strict conic quadratic and a strict quadratic inequality. We also show that under an additional containment assumption, the derived strict inequalities are sufficient to characterize the convex hull.

In addition to open sets defined with strict conic and quadratic inequalities, we also consider conic and quadratic sets defined with non-strict inequalities (i.e., sets defined as the intersection of two closed quadratic sets or a closed conic quadratic and a closed quadratic set). We note that the transition from open to closed setting is not trivial; however, we show that under certain topological assumptions, the strict inequality results directly imply their non-strict analogs. Note that a *lattice-free* set is defined as a set that does not contain any integer point in its interior. Therefore, the aggregation technique proposed in [45] provides a unified framework for generating *lattice-free* cuts for quadratic and conic quadratic sets. Moreover, as long as the lattice-free set can be described by a single quadratic inequality, such a framework is independent of the geometry of the lattice-free set.

The rest of this paper is organized as follows. In Section 2 we introduce some notation and provide the existing convex hull results from [45]. In Sections 3 and 4 we introduce the conic quadratic characterization of the convex hull of quadratic and conic quadratic sets and compare the results in this paper and those in [15]. We note that [15] contains similar results to those presented here and our main results have been developed independently. In Section 4.6 we compare and discuss these various results.

## 2 Notation, preliminaries, and existing convex hull results

We use the following notation. We let $e^i \in \mathbb{R}^n$ denote the $i$-th unit vector, $0_n$ be the zero vector, $[n] := \{1, \ldots, n\}$, and $\mathbb{S}^n$ denote symmetric matrices with $n$ rows and columns. For a matrix $\mathcal{P}$, we let $\pi^-(\mathcal{P})$ denote the number of negative eigenvalues of $\mathcal{P}$, $\pi^+(\mathcal{P})$ denote the number of positive eigenvalues of $\mathcal{P}$, and $\text{null}(\mathcal{P})$ denote its null space. We also let $\|x\|_2 := \sqrt{\sum_{i=1}^n x_i^2}$ denote the Euclidean norm of a given vector $x \in \mathbb{R}^n$. For a set $S \subseteq \mathbb{R}^n$, we let $\text{int}(S)$ be its interior, $\overline{S}$ be its closure, $\text{conv}(S)$ be its convex hull, $\overline{\text{conv}}(S)$ be the closure of its convex hull, and $S_\infty$ be its recession cone. Finally, for a set $\boldsymbol{\mathcal{S}} \in \mathbb{R}^{n+1}$, $\text{Proj}_x(\boldsymbol{\mathcal{S}})$ is the orthogonal projection of the set to the first $n$ variables.

In Sections 2 and 3 we follow the convention in [45] and define all sets using strict inequalities. However, in Section 4 all sets are defined by non-strict inequalities. This also allows us to compare our results with those in [15]. To simplify the exposition, we use the same notation for sets described by strict and non-strict inequalities; however, if we need to refer to sets defined by strict inequalities in Section 4, we use the interior to avoid any ambiguity.

### 2.1 Preliminaries

In this section we first define the quadratic sets that we study. We then provide some useful definitions and results from [45] that are relevant to our analysis. To save space, we do not provide the proofs of such results and we refer the reader to [45].

Our analysis is based on the work in [45] which studies the convex hull of open sets defined by two strict non-homogeneous quadratic inequalities. In particular, let

$$S := \left\{x \in \mathbb{R}^n \,:\, q_i < 0, \quad i = 0, 1\right\}, \tag{1}$$

where $q_i,\ i = 0, 1$ are quadratic polynomials of the form

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \mathcal{P} \begin{bmatrix} x \\ 1 \end{bmatrix} = x^T Q x + 2b^T x + \gamma, \tag{2}$$

where $\mathcal{P} = \begin{bmatrix} Q & b \\ b^T & \gamma \end{bmatrix} \in \mathbb{S}^{n+1}$, $Q \in \mathbb{S}^n, b \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$.

Note that [45] does not require the quadratic functions to satisfy any specific property. In particular, there is no requirement on the convexity or concavity of the quadratic functions defined in (2).

To characterize the convex hull of $S$, [45] considers the aggregated inequalities derived from the convex combinations of the two quadratics. More specifically, denote the pencil of quadratics induced by the convex combination of the two quadratic inequalities as

$$q_\lambda := (1 - \lambda) q_0 + \lambda q_1,$$

where $\lambda \in [0, 1]$. Similarly, define the associated symmetric matrix pencil

$$\mathcal{P}_\lambda := (1 - \lambda)\mathcal{P}_0 + \lambda \mathcal{P}_1,$$

and

$$Q_\lambda := (1 - \lambda) Q_0 + \lambda Q_1.$$

For a given quadratic pencil $q_\lambda$, define

$$S_\lambda := \left\{x \in \mathbb{R}^n \,:\, q_\lambda < 0\right\}.$$

The aggregation technique in [45] chooses $\lambda \in [0, 1]$ such that the aggregated inequalities give $\operatorname{conv}(S)$. The characterization of the sets $D$ and $E$, which are defined below, are crucial to the aggregation technique. Define

$$D := \{\lambda \in [0, 1] \,:\, (1 - \lambda) Q_0 + \lambda Q_1 \succeq 0\}$$

and

$$E := \left\{\lambda \in [0, 1] \,:\, \pi^- (\mathcal{P}_\lambda) = 1\right\}.$$

Note that $D$ is the collection of all $\lambda \in [0, 1]$ such that the associated quadratic set $S_\lambda$ is convex. On the other hand, $E$ is the collection of all $\lambda \in [0, 1]$ for which $\mathcal{P}_\lambda$ has exactly one negative eigenvalue. Therefore, $S_\lambda$ may be non-convex for some $\lambda \in E$. However, as shown in Theorem 2, two specific aggregated inequalities associated with $E$ admit a convex representation and these are enough to characterize $\operatorname{conv}(S)$. Throughout the paper, we use Lemma 2 in [45] which characterizes the structure of the set $E$ as follows.

**Lemma 1** *If $E \neq \emptyset$, then $E$ is the union of at most two disjoint connected intervals of the form*

$$E = [\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4],$$

*where $\lambda_i, \lambda_{i+1} \in [0,1]$ for $i \in \{1,3\}$ are generalized eigenvalues of the pencil $\mathcal{P}_\lambda$.*

If $E$ is a single connected interval, we denote $E = [\lambda_1, \lambda_2]$, for $\lambda_1, \lambda_2 \in [0,1]$. Also note that it is possible that the connected intervals of E are only single points. In such a case, we have $\lambda_i = \lambda_{i+1}$ for $i \in \{1,3\}$. Proposition 1 in [45] characterizes the relation between $D$ and $E$ as follows.

**Proposition 1** *If $S \neq \emptyset$, then $D$ is a closed interval contained in $E$.*

Therefore, if $E$ is composed of two disjoint connected intervals, Lemma 1 implies that $D \subseteq [\lambda_i, \lambda_{i+1}]$ for exactly one $i \in \{1,3\}$.

In what follows, we provide the convex hull results from [45]. In Section 2.2 we present the convex hull characterization of the homogeneous version of the quadratic set $S$ defined in (1). Section 2.3 then presents the convex hull characterization of $S$.

## 2.2 Homogeneous quadratic sets

Consider the homogeneous version of the quadratic function $q$ defined in (2) as

$$\boldsymbol{q} = y^T \mathcal{P} y, \tag{3}$$

where $y = \begin{bmatrix} x \\ x_0 \end{bmatrix} \in \mathbb{R}^{n+1}$. Also consider the homogeneous version of the quadratic set $S$ defined in (1) as

$$\boldsymbol{\mathcal{S}} := \left\{ y \in \mathbb{R}^{n+1} \ : \ \boldsymbol{q_i} < 0, \quad i = 0,1 \right\}. \tag{4}$$

Analogously, define the associated quadratic pencil $\boldsymbol{q_\lambda}$ as

$$\boldsymbol{q_\lambda} := (1-\lambda)\boldsymbol{q_0} + \lambda \boldsymbol{q_1},$$

where $\lambda \in [0,1]$. Also denote the homogeneous version of the set $S_\lambda$ as

$$\boldsymbol{\mathcal{S}_\lambda} := \left\{ y \in \mathbb{R}^{n+1} \ : \ \boldsymbol{q_\lambda} < 0 \right\}.$$

Throughout the paper, we use the following definitions.

**Definition 1** $\boldsymbol{\mathcal{C}} \subseteq \mathbb{R}^{n+1}$ is an open cone if for any $y \in \boldsymbol{\mathcal{C}}$ and $\alpha > 0$, we have $\alpha y \in \boldsymbol{\mathcal{C}}$.

We note that the above definition of a cone $\boldsymbol{\mathcal{C}}$ does not require $0 \in \boldsymbol{\mathcal{C}}$, and it also allows a non-convex set to be a cone.

**Definition 2** The symmetric reflection of $\boldsymbol{\mathcal{C}} \subseteq \mathbb{R}^{n+1}$ with respect to the origin is defined as $-\boldsymbol{\mathcal{C}} := \{-y \in \mathbb{R}^{n+1} \ : \ y \in \boldsymbol{\mathcal{C}}\}$.

**Definition 3** $\boldsymbol{\mathcal{C}} \subseteq \mathbb{R}^{n+1}$ is symmetric if $-\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}$.

Also define a linear hyperplane $\boldsymbol{\mathcal{H}} \subseteq \mathbb{R}^{n+1}$ with the associated normal vector $h \in \mathbb{R}^{n+1} \setminus \{0_{n+1}\}$ as

$$\boldsymbol{\mathcal{H}} := \left\{ y \in \mathbb{R}^{n+1} \ : \ h^T y = 0 \right\}.$$

One can see that $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{S}_\lambda}$ for $\lambda \in [0,1]$ are open symmetric cones. An important notion that we frequently use throughout the paper is the separation of an open symmetric cone which is given in the following definition.

**Definition 4** Consider an open symmetric non-empty cone $\mathcal{C} \subseteq \mathbb{R}^{n+1}$. If there exists a linear hyperplane $\mathcal{H} \subseteq \mathbb{R}^{n+1}$ such that $\mathcal{H} \cap \mathcal{C} = \emptyset$, we say $\mathcal{C}$ admits a separation (i.e., $\mathcal{H}$ is a separator of $\mathcal{C}$ or separates $\mathcal{C}$).

Denote the two halfspaces induced by the hyperplane $\mathcal{H}$ as

$$\mathcal{H}^+ := \left\{ y \in \mathbb{R}^{n+1} \,:\, h^T y > 0 \right\},$$

and

$$\mathcal{H}^- := \left\{ y \in \mathbb{R}^{n+1} \,:\, h^T y < 0 \right\}.$$

Therefore, a separator $\mathcal{H}$ induces two disjoint slices of the set $\mathcal{S}$ denoted by

$$\mathcal{S}^+ := \mathcal{H}^+ \cap \mathcal{S} \quad \text{and} \quad \mathcal{S}^- := \mathcal{H}^- \cap \mathcal{S}.$$

One can see that the resulting slices of $\mathcal{S}$ satisfy the following properties: (i) $\mathcal{S}^+ = -\mathcal{S}^-$, (ii) $\mathcal{S}^+ \cap \mathcal{S}^- = \emptyset$, and (iii) $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-$.

Another important definition that we need is the definition of a semi-convex cone.

**Definition 5** A semi-convex cone (SCC) is the disjoint union of two convex and open cones which are symmetric reflections of each other with respect to the origin.

An SCC is symmetric by definition. Moreover, an SCC always admits a unique separation. In other words, regardless of the separator we use to separate an SCC with, the associated disjoint slices will always be the same (i.e., after using any one of the valid hyperplanes for separation, the two pieces of the SCC are uniquely defined). This fact is formalized in Propositions 2 and 3 in [45] as follows.

**Proposition 2** *Let $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ be an open SCC. Assume that there exists a hyperplane $\mathcal{H}$ which separates $\mathcal{C}$. Then, $\mathcal{C}$ admits a unique separation, the slices of which are the convex connected components of $\mathcal{C}$.*

We also use the following useful proposition from [45].

**Proposition 3** *Consider an open symmetric non-empty cone given by*

$$\mathcal{C} := \left\{ y \in \mathbb{R}^{n+1} \,:\, y^T \mathcal{P} y < 0 \right\}.$$

*Then the following statements are equivalent:*

*(i) There exists a linear hyperplane which separates $\mathcal{C}$,*
*(ii) $\pi^-(\mathcal{P}) = 1$, and*
*(iii) $\mathcal{C}$ is an SCC.*

*Remark 1* Note that when $\pi^-(\mathcal{P}) = 1$, one can do the spectral decomposition of $\mathcal{P}$ as

$$\mathcal{P} = VV^T - uu^T,$$

for $u \in \mathbb{R}^{n+1}$ and $V \in \mathbb{R}^{(n+1)\times\pi^+(\mathcal{P})}$, where $\pi^+(\mathcal{P})$ represents the number of positive eigenvalues of $\mathcal{P}$. One can check that

$$\mathcal{H}_u := \left\{ y \in \mathbb{R}^{n+1} \,:\, u^T y = 0 \right\}$$

separates $\mathcal{C}$ and we call $\mathcal{H}_u$ a *natural* separator of $\mathcal{C}$.

Lemmas 4-7 in [45] imply the following theorem which characterizes the convex hull of any set of the form $\mathcal{S}$ defined by two homogeneous quadratic inequalities.

**Theorem 1** *Consider the non-empty open set* $\boldsymbol{\mathcal{S}}$ *defined in* (4) *and let* $\boldsymbol{\mathcal{H}}$ *be a separator of* $\boldsymbol{\mathcal{S}}$. *Then* $E \neq \emptyset$ *and exactly one of the connected components* $[\lambda_i, \lambda_{i+1}]$ *of* $E$ *is such that*

$$\boldsymbol{\mathcal{H}} \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda_i}} \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda_{i+1}}} = \emptyset.$$

*For such* $\lambda_i$ *and* $\lambda_{i+1}$ *we have that* $\boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda_i}} \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda_{i+1}}}$ *is an SCC,*

$$\operatorname{conv}\left(\boldsymbol{\mathcal{H}}^+ \cap \boldsymbol{\mathcal{S}}\right) = \boldsymbol{\mathcal{H}}^+ \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda_i}} \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda_{i+1}}}$$

*and there exists* $\boldsymbol{\mathcal{H}_s}$ *which separates both* $\boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda_i}}$ *and* $\boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda_{i+1}}}$ *such that*

$$\operatorname{conv}\left(\boldsymbol{\mathcal{H}}^+ \cap \boldsymbol{\mathcal{S}}\right) = \left(\boldsymbol{\mathcal{H}}_{\boldsymbol{s}}^+ \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda_i}}\right) \cap \left(\boldsymbol{\mathcal{H}}_{\boldsymbol{s}}^+ \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda_{i+1}}}\right).$$

### 2.3 Quadratic sets

Using the results from Theorem 1, the following theorem (Theorem 1 in [45]) characterizes the convex hull of any set of the form $S$ defined by two strict quadratic inequalities.

**Theorem 2** *Consider the non-empty open set* $S$ *defined in* (1). *If* $D = \emptyset$, *then* $\operatorname{conv}(S) = \mathbb{R}^n$. *Otherwise, let* $i \in \{1, 3\}$ *be such that* $[\lambda_i, \lambda_{i+1}]$ *is the unique connected component of* $E$ *such that* $D \subseteq [\lambda_i, \lambda_{i+1}]$. *For such* $\lambda_i$ *and* $\lambda_{i+1}$ *we have*

$$\operatorname{conv}(S) = S_{\lambda_i} \cap S_{\lambda_{i+1}}.$$

## 3 Conic quadratic characterization of convex hulls

In this section we first show that the convex hull characterizations presented in Section 2 can be described by two strict conic quadratic inequalities. Using results from Theorem 1, we then derive strict conic quadratic inequalities which provide a relaxation for the convex hull of sets defined as the intersection of a strict conic quadratic and a strict quadratic inequality. We also show that such valid inequalities characterize the convex hull exactly under an additional containment assumption.

### 3.1 Conic quadratic representation of convex hulls

In what follows, we show that each side of $\boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda_i}}$ and $\boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda_{i+1}}}$ can be described by a single conic quadratic inequality, where $[\lambda_i, \lambda_{i+1}]$ for $i \in \{1, 3\}$ is one of the connected components of $E$.

**Proposition 4** *Let* $\lambda \in [0, 1]$ *be such that* $\pi^-(\mathcal{P}_\lambda) = 1$ *and let* $\boldsymbol{\mathcal{H}}$ *be a separator of* $\boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}}$. *Then* $\boldsymbol{\mathcal{H}}^+ \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}}$ *can be described by a single strict conic quadratic inequality.*

*Proof* We have

$$\boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}} = \left\{y \in \mathbb{R}^{n+1} : y^T \mathcal{P}_\lambda y < 0\right\}.$$

Since $\pi^-(\mathcal{P}_\lambda) = 1$, using Proposition 3, one can see that $\boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}}$ is an SCC. Thus, using Remark 1, one can decompose $\mathcal{P}_\lambda$ as $\mathcal{P}_\lambda = VV^T - uu^T$ for the appropriately chosen matrix and vector $V$ and $u$. Therefore, we have

$$\boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}} = \left\{y \in \mathbb{R}^{n+1} : \left\|V^T y\right\|_2^2 < \left(u^T y\right)^2\right\}. \tag{5}$$

Let $\boldsymbol{\mathcal{H}_u}$ be the natural separator of $\boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}}$. Using Proposition 2, we have that $\boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}}$ admits a unique separation, that is,

$$\boldsymbol{\mathcal{H}}^+ \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}} = \boldsymbol{\mathcal{H}}_{\boldsymbol{u}}^+ \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}} \quad \text{or} \quad \boldsymbol{\mathcal{H}}^- \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}} = \boldsymbol{\mathcal{H}}_{\boldsymbol{u}}^- \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}}. \tag{6}$$

Therefore, from (5) and (6) we get

$$\boldsymbol{\mathcal{H}}^+ \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}} = \left\{ y \in \mathbb{R}^{n+1} \,:\, \left\| V^T y \right\|_2 < s\left( u^T y \right) \right\},$$

for some $s \in \{-1, 1\}$. □

A similar argument to the proof of Proposition 4 can be used to show that conv $\left(\boldsymbol{\mathcal{H}}^+ \cap \boldsymbol{\mathcal{S}}\right)$ given in Theorem 1 can be written as

$$\text{conv}\left(\boldsymbol{\mathcal{H}}^+ \cap \boldsymbol{\mathcal{S}}\right) = \boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda}_i} \cap \boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda}_{i+1}},$$

where

$$\boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda}_i} = \boldsymbol{\mathcal{H}}_i^+ \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}_i} \quad \text{and} \quad \boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda}_{i+1}} = \boldsymbol{\mathcal{H}}_{i+1}^+ \cap \boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}_{i+1}}, \tag{7}$$

$\boldsymbol{\mathcal{H}}_i^+ \in \left\{\boldsymbol{\mathcal{H}}_{\boldsymbol{u}_i}^+, \boldsymbol{\mathcal{H}}_{\boldsymbol{u}_i}^-\right\}$ and $\boldsymbol{\mathcal{H}}_{i+1}^+ \in \left\{\boldsymbol{\mathcal{H}}_{\boldsymbol{u}_{i+1}}^+, \boldsymbol{\mathcal{H}}_{\boldsymbol{u}_{i+1}}^-\right\}$, and where $\boldsymbol{\mathcal{H}}_{\boldsymbol{u}_i}$ and $\boldsymbol{\mathcal{H}}_{\boldsymbol{u}_{i+1}}$ are natural separators of $\boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}_i}$ and $\boldsymbol{\mathcal{S}}_{\boldsymbol{\lambda}_{i+1}}$, respectively. In particular, each of the sets $\boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda}_i}$ and $\boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda}_{i+1}}$ is described by a single strict conic quadratic inequality.

Similarly, conv $(S)$ given in Theorem 2 can be expressed as

$$\text{conv}\,(S) = K_{\lambda_i} \cap K_{\lambda_{i+1}},$$

where

$$K_{\lambda_i} = \left\{ x \in \mathbb{R}^n \,:\, \begin{bmatrix} x \\ 1 \end{bmatrix} \in \boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda}_i} \right\} \quad \text{and} \quad K_{\lambda_{i+1}} = \left\{ x \in \mathbb{R}^n \,:\, \begin{bmatrix} x \\ 1 \end{bmatrix} \in \boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda}_{i+1}} \right\}, \tag{8}$$

for $\boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda}_i}$ and $\boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda}_{i+1}}$ defined in (7). In particular, $K_{\lambda_i}$ and $K_{\lambda_{i+1}}$ can be described by a single strict conic quadratic inequality. An alternate way of obtaining such conic quadratic inequalities is to apply Schur's Lemma to a homogeneous version of the SDP representation of $S_{\lambda_i}$ and $S_{\lambda_{i+1}}$ given in Proposition A1 in [45].

### 3.2 Conic quadratic sets

In this section we aim to characterize the convex hull of sets defined by a strict conic quadratic and a strict quadratic inequality.

Using Theorem 1, we first derive valid conic quadratic inequalities for the convex hull of any set defined by a strict conic quadratic and a strict quadratic inequality. We then show that such valid inequalities characterize the convex hull exactly under an additional containment assumption.

We study open sets of the form

$$C := \left\{ x \in \mathbb{R}^n \,:\, L_0 < 0, \quad q_1 < 0 \right\}, \tag{9}$$

where $L_0 < 0$ is a strict conic quadratic inequality of the form

$$\left\| A_0 x - d_0 \right\|_2 < {a_0}^T x - g_0,$$

where $A_0 \in \mathbb{R}^{n \times n}, d_0, a_0 \in \mathbb{R}^n$, $g_0 \in \mathbb{R}$, and $q_1 < 0$ is a strict quadratic inequality of the form

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \mathcal{P}_1 \begin{bmatrix} x \\ 1 \end{bmatrix} = x^T Q_1 x + 2 b_1^T x + \gamma_1 < 0,$$

where $\mathcal{P}_1 = \begin{bmatrix} Q_1 & b_1 \\ b_1^T & \gamma_1 \end{bmatrix} \in \mathbb{S}^{n+1}$, $Q_1 \in \mathbb{S}^n, b_1 \in \mathbb{R}^n$, and $\gamma_1 \in \mathbb{R}$.

Our goal is to derive strong valid inequalities for conv $(C)$ and characterize the convex hull exactly when possible. Since we will use results from Theorem 1, we also need to consider the homogeneous version of the set $C$. Therefore, we define

$$\boldsymbol{\mathcal{C}} := \left\{ y \in \mathbb{R}^{n+1} \,:\, \boldsymbol{\mathcal{L}_0} < 0, \quad \boldsymbol{q_1} < 0 \right\}, \tag{10}$$

where $\boldsymbol{\mathcal{L}_0} < 0$ is a strict homogeneous conic quadratic inequality of the form

$$\|A_0 x - d_0 x_0\|_2 < {a_0}^T x - g_0 x_0,$$

and $\boldsymbol{q_1}$ is a quadratic function as defined in (3). By squaring both sides of the strict conic quadratic inequality $\boldsymbol{\mathcal{L}_0} < 0$, we define

$$\boldsymbol{\mathcal{S}}\,(\boldsymbol{C}) := \left\{ y \in \mathbb{R}^{n+1} \,:\, \boldsymbol{q_0} < 0, \quad \boldsymbol{q_1} < 0 \right\}, \tag{11}$$

where $\boldsymbol{q_0} = y^T \mathcal{P}_0 y$ such that $Q_0 = A_0^T A_0 - a_0 {a_0}^T$, $b_0 = -A_0^T d_0 + g_0 a_0$, and $\gamma_0 = d_0^T d_0 - {g_0}^2$. We also define the hyperplane

$$\boldsymbol{\mathcal{H}_0} := \left\{ y \in \mathbb{R}^{n+1} \,:\, (a_0, -g_0)^T\, y = 0 \right\}. \tag{12}$$

One can see that $\boldsymbol{\mathcal{H}_0}$ is a separator for $\boldsymbol{\mathcal{S}}\,(\boldsymbol{C})$,

$$\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{H}_0^+} \cap \boldsymbol{\mathcal{S}}\,(\boldsymbol{C}), \tag{13}$$

and

$$C = \operatorname{Proj}_x \left( \boldsymbol{\mathcal{H}_0^+} \cap \boldsymbol{\mathcal{S}}\,(\boldsymbol{C}) \cap \boldsymbol{\mathcal{E}^1} \right), \tag{14}$$

where $\boldsymbol{\mathcal{E}^1} := \left\{ (x, x_0) \in \mathbb{R}^{n+1} \,:\, x_0 = 1 \right\}$. In Proposition 5, we use (13) and (14) together with Theorem 1 to characterize conv $(C)$. We note that the proof of Proposition 5 is a direct adaptation of the proof of Theorem 1 in [45].

**Proposition 5** *Consider the non-empty open set $C$ defined in (9). Then exactly one of the connected components $[\lambda_i, \lambda_{i+1}]$ of $E$ is such that*

$$\boldsymbol{\mathcal{H}_0} \cap \boldsymbol{\mathcal{S}_{\lambda_i}} \cap \boldsymbol{\mathcal{S}_{\lambda_{i+1}}} = \emptyset, \tag{15}$$

*where $\boldsymbol{\mathcal{H}_0}$ is defined in (12). For such $\lambda_i$ and $\lambda_{i+1}$ we have that*

$$\operatorname{conv}(C) \subseteq K_{\lambda_i} \cap K_{\lambda_{i+1}}, \tag{16}$$

*where $K_{\lambda_i}$ and $K_{\lambda_{i+1}}$ are defined in (8). Furthermore, if $\boldsymbol{\mathcal{C}} \subseteq \boldsymbol{\mathcal{E}^+}$ for $\boldsymbol{\mathcal{E}} := \left\{ (x, x_0) \in \mathbb{R}^{n+1} \,:\, x_0 = 0 \right\}$, then (16) holds as equality.*

*Proof* Consider $\boldsymbol{\mathcal{C}}$, $\boldsymbol{\mathcal{S}}\,(\boldsymbol{C})$, and $\boldsymbol{\mathcal{H}_0}$ as defined in (10), (11), and (12), respectively. One can see that (15) directly follows from Theorem 1. To prove the containment in (16), recall from (13) and (14) that

$$\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{H}_0^+} \cap \boldsymbol{\mathcal{S}}\,(\boldsymbol{C})$$

and

$$C = \operatorname{Proj}_x \left( \boldsymbol{\mathcal{H}_0^+} \cap \boldsymbol{\mathcal{S}}\,(\boldsymbol{C}) \cap \boldsymbol{\mathcal{E}^1} \right),$$

where $\boldsymbol{\mathcal{E}}^{\mathbf{1}} := \left\{(x, x_0) \in \mathbb{R}^{n+1} \,:\, x_0 = 1\right\}$. Therefore, conv $(C)$ can be expressed as

$$\begin{aligned}
\operatorname{conv}(C) &= \left\{ x \in \mathbb{R}^n \,:\, \begin{bmatrix} x \\ 1 \end{bmatrix} = \sum_{j=1}^{n+1} \theta_j \begin{bmatrix} z_j \\ 1 \end{bmatrix},\ \sum_{j=1}^{n+1} \theta_j = 1,\ \theta_j \geq 0,\ \begin{bmatrix} z_j \\ 1 \end{bmatrix} \in \boldsymbol{\mathcal{C}},\ j \in [n+1] \right\} \\
&\subseteq \left\{ x \in \mathbb{R}^n \,:\, \begin{bmatrix} x \\ 1 \end{bmatrix} = \sum_{j=1}^{n+1} \theta_j \boldsymbol{z}_j,\ \sum_{j=1}^{n+1} \theta_j = 1,\ \theta_j \geq 0,\ \boldsymbol{z}_j \in \boldsymbol{\mathcal{C}},\ j \in [n+1] \right\} \qquad (17) \\
&= \left\{ x \in \mathbb{R}^n \,:\, \begin{bmatrix} x \\ 1 \end{bmatrix} \in \operatorname{conv}(\boldsymbol{\mathcal{C}}) \right\}, \\
&= \left\{ x \in \mathbb{R}^n \,:\, \begin{bmatrix} x \\ 1 \end{bmatrix} \in \boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda}_i} \cap \boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda}_{i+1}} \right\} = K_{\lambda_i} \cap K_{\lambda_{i+1}}, \qquad (18)
\end{aligned}$$

where the first equality holds by Carathéodory's Theorem, the first equality in (18) follows from Theorem 1, and where $i \in \{1, 3\}$ is an appropriate index evident from Theorem 1. The reverse containment in (17) trivially holds when $\boldsymbol{\mathcal{C}} \subseteq \boldsymbol{\mathcal{E}}^{+}$. ◻

## 4 Conic quadratic characterization of closed convex hulls

In this section we study conic and quadratic sets defined by non-strict inequalities instead of strict inequalities. In particular, whenever we refer to a previously defined set, such as $\boldsymbol{\mathcal{E}}^{+}$, $\boldsymbol{\mathcal{H}}^{+}$, $\boldsymbol{\mathcal{S}}$, $\boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda}_i}$, we redefine such a set by replacing strict inequalities with non-strict inequalities. In other words, unless stated explicitly, all sets in this section are closed and defined by non-strict inequalities. Working with non-strict inequalities requires the study of closed convex hulls instead of convex hulls. However, under certain topological assumptions, the strict inequality results directly imply non-strict analogs. One such assumption is (19) in the following lemma.

**Lemma 2** *Let $A$ and $B$ be two non-empty closed sets such that*

$$A \subseteq \overline{\operatorname{int}(A)} \qquad (19)$$

*and $B$ is convex. If* $\operatorname{conv}(\operatorname{int}(A)) \subseteq \operatorname{int}(B)$*, then* $\overline{\operatorname{conv}}(A) \subseteq B$ *and if* $\operatorname{conv}(\operatorname{int}(A)) = \operatorname{int}(B)$*, then* $\overline{\operatorname{conv}}(A) = B$*.*

*Proof* First note that (19) implies $A = \overline{\operatorname{int}(A)}$ and hence

$$\overline{\operatorname{conv}}(A) = \overline{\operatorname{conv}}\left(\overline{\operatorname{int}(A)}\right) = \overline{\operatorname{conv}}\left(\operatorname{int}(A)\right) = \overline{\operatorname{conv}}\left(\operatorname{conv}\left(\operatorname{int}(A)\right)\right). \qquad (20)$$

Furthermore,

$$B = \overline{\operatorname{int}(B)} = \overline{\operatorname{conv}}\left(\operatorname{int}(B)\right) \qquad (21)$$

because $B$ is closed and convex and $\operatorname{int}(B) \neq \emptyset$ (because $\operatorname{int}(A) \subseteq \operatorname{int}(B)$ and because (19) and $A \neq \emptyset$ imply $\operatorname{int}(A) \neq \emptyset$). The result then follows from (20)–(21) by taking the closed convex hull on both sides of the corresponding containment or equality. ◻

Note that if the set $A$ is non-empty, then Assumption (19) implies the non-emptiness of $\operatorname{int}(A)$. In the following subsections we show how Lemma 2 can be used to adapt the convex hull results from Sections 2 and 3 to the non-strict setting. We then give several examples that illustrate Assumption (19) and some characteristics of the closed convex hull results. Finally, considering sets defined by non-strict inequalities allows us to compare our results with those in [15].

### 4.1 Homogeneous quadratic sets

We consider homogeneous quadratic sets of the form

$$\boldsymbol{\mathcal{S}} := \left\{ y \in \mathbb{R}^{n+1} \; : \; \boldsymbol{q_i} \leq 0, \quad i = 0, 1 \right\}.$$

In this section with a slight abuse of notation, we say that the hyperplane $\boldsymbol{\mathcal{H}} \subseteq \mathbb{R}^{n+1}$ separates $\boldsymbol{\mathcal{S}}$ when $\boldsymbol{\mathcal{H}}$ is in fact a separator of $\operatorname{int}(\boldsymbol{\mathcal{S}}) = \left\{ y \in \mathbb{R}^{n+1} \; : \; \boldsymbol{q_i} < 0, \quad i = 0, 1 \right\}$. The following corollary characterizes the non-strict inequality version of Theorem 1 and follows directly from that theorem and Lemma 2.

**Corollary 1** *Let $\boldsymbol{\mathcal{S}} := \left\{ y \in \mathbb{R}^{n+1} \; : \; \boldsymbol{q_i} \leq 0, \quad i = 0, 1 \right\}$ be non-empty, $\boldsymbol{\mathcal{H}}$ be a separator of $\boldsymbol{\mathcal{S}}$, and $i \in \{1, 3\}$ be such that $[\lambda_i, \lambda_{i+1}]$ is the unique connected component of $E$ such that*

$$\boldsymbol{\mathcal{H}} \cap \boldsymbol{\mathcal{S}_{\lambda_i}} \cap \boldsymbol{\mathcal{S}_{\lambda_{i+1}}} = \emptyset.$$

*If*

$$\boldsymbol{\mathcal{H}}^+ \cap \boldsymbol{\mathcal{S}} \subseteq \overline{\operatorname{int}\left(\boldsymbol{\mathcal{H}}^+ \cap \boldsymbol{\mathcal{S}}\right)}, \tag{22}$$

*then*

$$\overline{\operatorname{conv}}\left(\boldsymbol{\mathcal{H}}^+ \cap \boldsymbol{\mathcal{S}}\right) = \boldsymbol{\mathcal{K}_{\lambda_i}} \cap \boldsymbol{\mathcal{K}_{\lambda_{i+1}}},$$

*where $\boldsymbol{\mathcal{K}_{\lambda_i}}$ and $\boldsymbol{\mathcal{K}_{\lambda_{i+1}}}$ are as in (7) defined with non-strict inequalities.*

Similar as before, if the set $\boldsymbol{\mathcal{S}}$ is non-empty, then Assumption (22) implies the non-emptiness of $\operatorname{int}(\boldsymbol{\mathcal{S}})$.

### 4.2 Conic quadratic sets

The following corollary characterizes the non-strict inequality version of Proposition 5 and follows directly from that proposition and Lemma 2.

**Corollary 2** *Let $C := \{x \in \mathbb{R}^n \; : \; L_0 \leq 0, \quad q_1 \leq 0\}$ be non-empty and $i \in \{1, 3\}$ be such that $[\lambda_i, \lambda_{i+1}]$ is the unique connected component of $E$ such that*

$$\boldsymbol{\mathcal{H}_0} \cap \boldsymbol{\mathcal{S}_{\lambda_i}} \cap \boldsymbol{\mathcal{S}_{\lambda_{i+1}}} = \emptyset,$$

*where $\boldsymbol{\mathcal{H}_0}$ is defined in (12). If*

$$C \subseteq \overline{\operatorname{int}(C)}, \tag{23}$$

*then*

$$\overline{\operatorname{conv}}(C) \subseteq K_{\lambda_i} \cap K_{\lambda_{i+1}}, \tag{24}$$

*where $K_{\lambda_i}$ and $K_{\lambda_{i+1}}$ are as in (8) defined with non-strict inequalities. Furthermore, if*

$$\boldsymbol{\mathcal{C}} \subseteq \boldsymbol{\mathcal{E}}^+, \tag{25}$$

*for $\boldsymbol{\mathcal{E}} := \left\{ (x, x_0) \in \mathbb{R}^{n+1} \; : \; x_0 = 0 \right\}$, then (24) holds as equality.*

Note that $\boldsymbol{\mathcal{C}} \subseteq \boldsymbol{\mathcal{E}}^+$ provides a sufficient condition under which (24) trivially holds as equality; however, equality in (24) may still hold even if $\boldsymbol{\mathcal{C}} \subseteq \boldsymbol{\mathcal{E}}^+$ is violated.

## 4.3 Quadratic sets

The following corollary characterizes the non-strict inequality version of Theorem 2 and follows directly from that theorem and Lemma 2.

**Corollary 3** *Let $S := \{x \in \mathbb{R}^n : q_i \le 0, \quad i = 0, 1\}$ be non-empty. If $D = \emptyset$, then $\overline{\operatorname{conv}}(S) = \mathbb{R}^n$. Otherwise, let $i \in \{1, 3\}$ be such that $[\lambda_i, \lambda_{i+1}]$ is the unique connected component of $E$ such that $D \subseteq [\lambda_i, \lambda_{i+1}]$. If*

$$S \subseteq \overline{\operatorname{int}(S)}, \tag{26}$$

*then*

$$\overline{\operatorname{conv}}(S) = K_{\lambda_i} \cap K_{\lambda_{i+1}},$$

*where $K_{\lambda_i}$ and $K_{\lambda_{i+1}}$ are as in (8) defined with non-strict inequalities.*

Obtaining $\lambda_i$ and $\lambda_{i+1}$ and checking Assumption (25) is relatively easy. For instance, to obtain $\lambda_i$ and $\lambda_{i+1}$ we calculate all generalized eigenvalues $\{\lambda_i\}_{i=1}^r$ of the pencil $\mathcal{P}_\lambda$ and order them such that $\lambda_i < \lambda_{i+1}$ for all $i \in [r-1]$. We can then construct $E$ by evaluating the number of negative eigenvalues of $\mathcal{P}_\lambda$ for all $\lambda = (\lambda_i + \lambda_{i+1})/2$ for $i \in [r-1]$. In contrast, checking topological Assumption (19) (or its specializations (22), (23) and (26)) can be significantly harder. Fortunately, as we show in the following subsection, the topological assumption can be easily verified for some specific geometric structures.

## 4.4 Verifying the topological assumption

In this section we give two lemmas that are useful when checking the topological Assumption (19). The first lemma shows that the Assumption (19) is automatically satisfied for a wide range of sets and the second lemma gives a sufficient condition that can often be easier to check than the original Assumption (19).

**Lemma 3** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be continuous functions and $K$ be a closed convex set or the complement of an open convex set. Then (19) holds for $A = \{(x, x_0) : f(x) \le x_0, \quad g(x) \le x_0\}$ and $A = \{(x, x_0) : f(x) \le x_0, \quad x \in K\}$.*

*Proof* The first case follows by noting that for any $(x, x_0) \in A$ and for every $\varepsilon > 0$, we have that $(x, x_0 + \varepsilon) \in \operatorname{int}(A)$. For the second case, note that for any $\bar{x} \in \operatorname{bd}(K)$, there exist $d \in \mathbb{R}^n$ such that $\bar{x} + \varepsilon d \in \operatorname{int}(K)$ for all sufficiently small $\varepsilon > 0$. Furthermore, $(\bar{x}, f(\bar{x})) = \lim_{\varepsilon \to 0} (\bar{x} + \varepsilon d, f(\bar{x} + \varepsilon d))$. Hence, it suffices to show that $(\bar{x} + \varepsilon d, f(\bar{x} + \varepsilon d)) \in \overline{\operatorname{int}(A)}$ for all sufficiently small $\varepsilon > 0$. This follows from noting that $(\bar{x} + \varepsilon d, f(\bar{x} + \varepsilon d) + \delta) \in \operatorname{int}(A)$ for all sufficiently small $\varepsilon > 0$ and for any $\delta > 0$. ☐

Sets of the form considered by Lemma 3 include a wide range of quadratic sets such as the intersection of a paraboloid with a general quadratic inequality. It also includes trust region problems and hence, together with Corollary 3, this lemma can be used to show that such problems are equivalent to simple convex optimization problems (e.g. [8, Corollary 8] and [15, Section 7.2])

**Lemma 4** *If $A = \bigcup_{i=1}^{l} A_i$ and $A_i$ satisfies (19) for each $i \in [l]$, then $A$ satisfies (19). In particular, if $A_i$ is convex and $\operatorname{int}(A_i) \neq \emptyset$ for each $i \in [l]$, then $A$ satisfies (19).*

*Proof* The first part follows from $A = \bigcup_{i=1}^{l} A_i \subseteq \bigcup_{i=1}^{l} \overline{\operatorname{int}(A_i)} \subseteq \overline{\operatorname{int}(A)}$. The second follows from the fact that (19) is naturally satisfied by convex sets with non-empty interiors as formalized in Lemma 2.1.6 in [26]. ☐

Sets considered by Corollaries 1–3 that are unions of convex sets include those constructed from two-term disjunctions such as the ones considered in [15, Section 6] and [2, 3, 4, 5, 6, 7, 9, 10, 13, 16, 21, 22, 23, 25, 30, 32, 36, 37, 46]. Such sets are the unions of two convex sets defined by a single quadratic or conic quadratic inequality and two linear inequalities. In the next sub-section we show that checking that these two convex sets have non-empty interior is often easy and that when one of the sets has an empty interior, the topological Assumption (19) can be violated.

One special case of the sets considered by Lemma 4 are those constructed from split disjunctions. The only restriction of Lemma 4 compared to the general split disjunctions is that Lemma 4 requires both sides of the splits to have non-empty interior, while this assumption is not needed for general split disjunctions. In particular, the works in [3, 4, 16, 21, 22, 23] study cuts from one-sided split disjunctions, and such cuts cannot be derived using Lemma 4. On the other hand, the works in [5, 6, 7, 32, 46] assume the non-emptiness of the interiors of both sides of the disjunctions in the derivation of the cuts, and such cuts can in fact be derived using Lemma 4. Finally, it should be noted that the assumption from [15, Section 6] that seems of relevance to Lemma 4, i.e., [15, Assumption 2], only assumes that a single set $A_i$ needs to have nonempty interior.

## 4.5 Illustrative examples

We now illustrate the results in this section through several examples. In particular, we show how the two inequalities in the closed convex hull or relaxation characterization may include one of the original inequalities, one or two new inequalities, or even a redundant inequality.

We begin with three examples for which the description of the closed convex hull only requires one additional inequality (i.e. one of the inequalities associated to $\lambda_i,\ \lambda_{i+1}$ is one of the original inequalities). In the first two examples, Corollaries 2 and 3 are able to prove that adding this additional inequality yields the closed convex hull. However, in the third example, Corollary 2 cannot prove that adding the additional inequality yields the closed convex hull even though it actually does.

*Example 1* Here we consider Example 3 in [38], which is given by

$$S_1 := \left\{x \in \mathbb{R}^3 \,:\, x_1^2 + x_2^2 - x_3 - 4 \le 0, \quad x_1^2 + x_2^2 - x_3^2 + 1 \le 0\right\}.$$

To check Assumption (26) of Corollary 3, first note that $S_1 = S_1' \cup S_1''$ for convex sets

$$S_1' := \left\{x \in \mathbb{R}^3 \,:\, x_1^2 + x_2^2 - x_3 - 4 \le 0, \quad \sqrt{x_1^2 + x_2^2 + 1} \le x_3\right\}$$

and

$$S_1'' := \left\{x \in \mathbb{R}^3 \,:\, x_1^2 + x_2^2 - x_3 - 4 \le 0, \quad \sqrt{x_1^2 + x_2^2 + 1} \le -x_3\right\}.$$

Furthermore, both sets have non-empty interiors (e.g. $(0,0,2) \in \operatorname{int}\left(S_1'\right)$ and $(0,0,-2) \in \operatorname{int}\left(S_1'\right)$). Hence, by Lemma 4, Assumption (26) is satisfied. We can also check that

$$E = \left[0, \frac{1}{21}\left(9 - 2\sqrt{15}\right)\right] \cup \left[\frac{1}{21}\left(9 + 2\sqrt{15}\right), 1\right]$$

and $D = \{0\}$ is contained in the first interval. Then, $\lambda_i = 0$ and $\lambda_{i+1} = \frac{1}{21}\left(9 - 2\sqrt{15}\right)$ and Corollary 3 yields

$$\overline{\operatorname{conv}}\left(S_1\right) = \left\{x \in \mathbb{R}^2 \,:\, \begin{array}{l} x_1^2 + x_2^2 - x_3 - 4 \le 0, \\ \qquad \sqrt{x_1^2 + x_2^2} \le \frac{1}{21}\sqrt{9 + 2\sqrt{15}}\left(\left(9 - 2\sqrt{15}\right) x_3 + \sqrt{15} + 6\right) \end{array}\right\}.$$

Because $\lambda_i = 0$ and $\lambda_{i+1} \notin \{0,1\}$, the first inequality given by Corollary 3 is one of the original inequalities and the second one is a new inequality, which we can check is non-redundant for the description of $\overline{\operatorname{conv}}\left(S_1\right)$.

*Example 2* Here we consider an example proposed by Burer and Kılınç-Karzan [14], which is given by

$$C_2 := \left\{ x \in \mathbb{R}^3 \ : \ \sqrt{x_1^2 + x_2^2} \le x_3, \quad (x_1 + x_3 - 3)(x_3 - 2) \le 0 \right\}.$$

The homogeneous version of this set is given by

$$\boldsymbol{\mathcal{C}_2} := \boldsymbol{\mathcal{H}_2^+} \cap \boldsymbol{\mathcal{S}_2} = \left\{ (x, x_0) \in \mathbb{R}^4 \ : \ \sqrt{x_1^2 + x_2^2} \le x_3, \quad x_3^2 + x_1 x_3 - 2x_1 x_0 - 5x_3 x_0 + 6x_0^2 \le 0 \right\},$$

for $\boldsymbol{\mathcal{H}_2^+} := \left\{ (x, x_0) \in \mathbb{R}^4 \ : \ x_3 \ge 0 \right\}$ and

$$\boldsymbol{\mathcal{S}_2} := \left\{ (x, x_0) \in \mathbb{R}^4 \ : \ x_1^2 + x_2^2 - x_3^2 \le 0, \quad x_3^2 + x_1 x_3 - 2x_1 x_0 - 5x_3 x_0 + 6x_0^2 \le 0 \right\}.$$

To check Assumption (23) of Corollary 2, first note that $C_2 = C_2' \cup C_2''$ for convex sets

$$C_2' := \left\{ x \in \mathbb{R}^3 \ : \ \sqrt{x_1^2 + x_2^2} \le x_3, \quad (x_1 + x_3 - 3) \le 0, \quad (x_3 - 2) \ge 0 \right\},$$

and

$$C_2'' := \left\{ x \in \mathbb{R}^3 \ : \ \sqrt{x_1^2 + x_2^2} \le x_3, \quad (x_1 + x_3 - 3) \ge 0, \quad (x_3 - 2) \le 0 \right\}.$$

Furthermore, both $C_2'$ and $C_2''$ have non-empty interiors. Hence, by Lemma 4 Assumption (23) is satisfied. We can also check that $E = [0, 8/9] \cup [1, 1]$ and $\boldsymbol{\mathcal{H}_2}$ only separates the set associated to the first interval. Then $\lambda_i = 0$ and $\lambda_{i+1} = 8/9$ and

$$\overline{\operatorname{conv}}\,(C_2) \subseteq \left\{ x \in \mathbb{R}^3 \ : \ \sqrt{x_1^2 + x_2^2} \le x_3^2, \quad \sqrt{(a x_1 + b x_3 + c)^2 + \frac{x_2^2}{9}} \le d x_1 + e x_3 + f \right\}, \tag{27}$$

where

$$\begin{aligned}
a &= \frac{1}{132}\left(53 + 3\sqrt{97}\right)\sqrt{\frac{1}{582}\left(53\sqrt{97} - 291\right)},\\
b &= \frac{1}{33}\left(20 + 3\sqrt{97}\right)\sqrt{\frac{1}{582}\left(53\sqrt{97} - 291\right)},\\
c &= \frac{1}{132}\left(-248 - 24\sqrt{97}\right)\sqrt{\frac{1}{582}\left(53\sqrt{97} - 291\right)},\\
d &= \frac{1}{132}\sqrt{\frac{1}{2} + \frac{53}{6\sqrt{97}}}\left(3\sqrt{97} - 53\right),\\
e &= \frac{1}{33}\sqrt{\frac{1}{2} + \frac{53}{6\sqrt{97}}}\left(3\sqrt{97} - 20\right),
\end{aligned}$$

and

$$f = \frac{1}{132}\sqrt{\frac{1}{2} + \frac{53}{6\sqrt{97}}}\left(248 - 24\sqrt{97}\right).$$

Finally, to check Assumption (25), first note that $\boldsymbol{\mathcal{C}_2} = \boldsymbol{\mathcal{C}_2'} \cup \boldsymbol{\mathcal{C}_2''}$ for convex sets

$$\boldsymbol{\mathcal{C}_2'} := \left\{ (x, x_0) \in \mathbb{R}^4 \ : \ \sqrt{x_1^2 + x_2^2} \le x_3, \quad (x_1 + x_3 - 3x_0) \le 0, \quad (x_3 - 2x_0) \ge 0 \right\},$$

and

$$\boldsymbol{\mathcal{C}_2''} := \left\{(x, x_0) \in \mathbb{R}^4 \,:\, \sqrt{x_1^2 + x_2^2} \le x_3, \quad (x_1 + x_3 - 3x_0) \ge 0, \quad (x_3 - 2x_0) \le 0\right\}.$$

The conic inequality of $\boldsymbol{\mathcal{C}_2'}$ implies $-x_1 - x_3 \le 0$, which together with its first linear inequality implies $x_0 \ge 0$. Similarly, the conic inequality of $\boldsymbol{\mathcal{C}_2''}$ implies $-x_3 \le 0$, which together with its second linear inequality implies $x_0 \ge 0$. Hence, Assumption (25) holds and Corollary 2 implies that (27) holds as equality.

*Example 3* Here we consider an example similar to those of Section 6.2 in [15], which is given by

$$C_3 := \left\{x \in \mathbb{R}^2 \,:\, |x_1| \le x_2, \quad (2x_1 + x_2 - 1)(-2x_1 - x_2 - 1) \le 0\right\}.$$

The homogeneous version of this set is given by

$$\boldsymbol{\mathcal{C}_3} := \boldsymbol{\mathcal{H}_3^+} \cap \boldsymbol{\mathcal{S}_3} = \left\{(x, x_0) \in \mathbb{R}^3 \,:\, |x_1| \le x_2, \quad (2x_1 + x_2 - x_0)(-2x_1 - x_2 - x_0) \le 0\right\},$$

where $\boldsymbol{\mathcal{H}_3^+} := \left\{(x, x_0) \in \mathbb{R}^3 \,:\, x_2 \ge 0\right\}$ and

$$\boldsymbol{\mathcal{S}_3} := \left\{(x, x_0) \in \mathbb{R}^3 \,:\, x_1^2 \le x_2^2, \quad (2x_1 + x_2 - x_0)(-2x_1 - x_2 - x_0) \le 0\right\}.$$

Similar to Example 2 we can check Assumption (23) of Corollary 2 through Lemma 4 as $C_3$ is the union of two convex sets with non-empty interior. We can also check that $E = [0, 1/4] \cup [1, 1]$ and $\boldsymbol{\mathcal{H}_3}$ only separates the set associated to the first interval. Then $\lambda_i = 0$ and $\lambda_{i+1} = 1/4$ and

$$\overline{\text{conv}}\,(C_3) \subseteq \left\{x \in \mathbb{R}^2 \,:\, |x_1| \le x_2, \quad 1 - x_1 - 2x_2 \le 0\right\}. \tag{28}$$

Using Corollary 2, we obtain the convex relaxation given in (28). Note that this relaxation is stronger than the trivial convex relaxation $\left\{x \in \mathbb{R}^2 \,:\, |x_1| \le x_2\right\}$ obtained by removing the non-convex inequality and keeping the convex inequality defining $C_3$. Furthermore, we can check that equality holds in (28) although Assumption (25) of Corollary 2 does not hold. Therefore, Corollary 2 cannot prove that the convex relaxation in (28) is in fact giving the closed convex hull.

For next pair of examples, we have that neither of the inequalities needed to describe the closed convex hull is one of the original inequalities.

*Example 4* Here we consider the set given by

$$\boldsymbol{\mathcal{S}_4} := \left\{(x, x_0) \in \mathbb{R}^3 \,:\, 2x_1^2 - x_2^2 - x_0^2 \le 0, \quad -x_1^2 + x_2^2 - x_0^2 \le 0\right\}.$$

One can see that $\boldsymbol{\mathcal{E}} := \left\{(x, x_0) \in \mathbb{R}^3 \,:\, x_0 = 0\right\}$ separates $\boldsymbol{\mathcal{S}_4}$. Let $\boldsymbol{\mathcal{S}_4^+} := \boldsymbol{\mathcal{E}}^+ \cap \boldsymbol{\mathcal{S}_4}$ and let $\mathcal{P}_0$ and $\mathcal{P}_1$ be the matrices associated with the quadratic inequalities. Assumption (22) of Corollary 1 can easily be checked using Lemma 3 or by noting that for every $(x, x_0) \in \boldsymbol{\mathcal{S}_4^+}$ and $\varepsilon > 0$ we have that $(x, x_0 + \varepsilon) \in \text{int}\left(\boldsymbol{\mathcal{S}_4^+}\right)$. We can also check that $E = [1/2, 2/3]$ and that $\boldsymbol{\mathcal{E}}$ separates the set associated to this interval. Then, $\lambda_i = 1/2$ and $\lambda_{i+1} = 2/3$ and Corollary 1 yields

$$\overline{\text{conv}}\left(\boldsymbol{\mathcal{S}_4^+}\right) = \left\{(x, x_0) \in \mathbb{R}^3 \,:\, |x_1| \le \sqrt{2}x_0, \quad |x_2| \le \sqrt{3}x_0\right\}.$$

In contrast to Examples 1–3, because $\lambda_i,\ \lambda_{i+1} \notin \{0, 1\}$, neither of the inequalities given by Corollary 1 is one of the original inequalities. We can also check that the two new inequalities given by Corollary 1 are non-redundant for the description of $\overline{\text{conv}}\left(\boldsymbol{\mathcal{S}_4^+}\right)$.

*Example 5* Consider the Example 1 in [45] and Example 2 in [38], which is given by

$$S_5 := \left\{ x \in \mathbb{R}^2 \,:\, x_1^2 - x_2^2 + 2x_1 + 2 \le 0, \quad -x_1^2 + x_2^2 + 2x_1 - 2 \le 0 \right\}.$$

We can check Assumption (26) of Corollary 3 through Lemma 4 by noting that $S_5$ is the union of two (non-convex) sets that satisfy Assumption (19). Alternatively, we can first note that if $x \in S_5$ satisfies both inequalities of $S_5$ strictly, then $x \in \operatorname{int}(S_5)$ and the Assumption (26) is trivially satisfied. Furthermore, if $x \in S_5$ satisfies one of the inequalities strictly, we can trivially perturb $x$ so that it remains in $S_5$ and satisfies both inequalities strictly. Hence, the only nontrivial check of the Assumption (26) is for points $x \in S_5$ that satisfy both inequalities of $S_5$ at equality. We can easily check that only two such points exist and each of them satisfy $(x_1 - \varepsilon, x_2) \in \operatorname{int}(S_5)$ for all sufficiently small $\varepsilon > 0$. We can also check that $E = [0, 1/2 - 1/(2\sqrt{2})] \cup [1/2, 1/2 + 1/(2\sqrt{2})]$ and $D = \{1/2\} \subseteq [1/2, 1/2 + 1/(2\sqrt{2})]$. Then, $\lambda_i = 1/2$ and $\lambda_{i+1} = 1/2 + 1/(2\sqrt{2})$ and Corollary 3 yields

$$\overline{\operatorname{conv}}\,(S_5) = \left\{ x \in \mathbb{R}^2 \,:\, x_1 \le 0, \quad |y| \le \sqrt{2} - x \right\}.$$

Again, because $\lambda_i,\ \lambda_{i+1} \notin \{0, 1\}$, neither of the inequalities given by Corollary 3 is one of the original inequalities. We can also check that the two new inequalities given by Corollary 3 are non-redundant for the description of $\overline{\operatorname{conv}}\,(S_5)$.

For the following example, we have that $\lambda_i = \lambda_{i+1}$, so Corollaries 1 and 2 yield a single inequality. In both cases, this single inequality is the convex inequality defining the original set. Hence, while the corollaries are applicable to construct convex relaxations, they only yield the trivial convex relaxation obtained by removing the non-convex inequality and keeping the convex inequality defining the original set. In the homogeneous case, Corollary 1 will still be useful because it proves that this trivial relaxation characterizes the convex hull, but in the non-homogeneous case, Corollary 2 will not be useful as it cannot characterize the convex hull which is strictly contained in the trivial relaxation.

*Example 6* Here we consider the homogeneous version of the example from Section 4.4 in [15], which is given by

$$\boldsymbol{\mathcal{C}_6} := \boldsymbol{\mathcal{H}_6^+} \cap \boldsymbol{\mathcal{S}_6} = \left\{ (x, x_0) \in \mathbb{R}^3 \,:\, |x_1| \le x_2, \quad x_1(x_2 - x_0) \le 0 \right\},$$

where $\boldsymbol{\mathcal{H}_6^+} := \left\{ (x, x_0) \in \mathbb{R}^3 \,:\, x_2 \ge 0 \right\}$ and $\boldsymbol{\mathcal{S}_6} := \left\{ (x, x_0) \in \mathbb{R}^3 \,:\, x_1^2 \le x_2^2, \quad x_1 x_2 - x_1 x_0 \le 0 \right\}$. Assumption (26) of Corollary 1 can easily be checked through Lemma 4 by noting that $\boldsymbol{\mathcal{C}_6}$ is the union of two convex sets with non-empty interior. We may hence use Corollary 1 to construct $\overline{\operatorname{conv}}\,(\boldsymbol{\mathcal{C}_6})$. For that note that $E = [0, 0] \cup [1, 1]$, and that $\boldsymbol{\mathcal{H}_6}$ only separates the set associated to the first interval. Hence, $\lambda_i = \lambda_{i+1} = 0$ and

$$\overline{\operatorname{conv}}\,(\boldsymbol{\mathcal{C}_6}) = \boldsymbol{\mathcal{K}_0} = \left\{ (x, x_0) \in \mathbb{R}^3 \,:\, |x_1| \le x_2 \right\}.$$

Finally, note that we trivially have $\overline{\operatorname{conv}}\,(\boldsymbol{\mathcal{C}_6}) \subseteq \left\{ (x, x_0) \in \mathbb{R}^3 \,:\, |x_1| \le x_2 \right\}$. However, the equality in this containment proven by Corollary 1 is not trivial. The non-homogeneous version of this example is given by

$$C_6 := \left\{ x \in \mathbb{R}^2 \,:\, |x_1| \le x_2, \quad x_1 x_2 - x_1 \le 0 \right\}.$$

While Corollary 2 shows that $\overline{\operatorname{conv}}\,(C_6) \subsetneq \left\{ x \in \mathbb{R}^2 \,:\, |x_1| \le x_2 \right\}$, equality does not hold in this containment. This aligns with the fact that (25) is violated and hence Corollary 2 is not applicable.

We end this section by considering an example where topological Assumption (19) fails and discussing one possible way to adapt the results in this paper to such a setting. This example also illustrates how one of the inequalities in the closed convex hull characterization may be redundant.

*Example 7* For any $\varepsilon \geq 0$, consider the generalization of the example from Section 4.5 in [15], which is given by

$$\boldsymbol{\mathcal{C}_7}(\varepsilon) := \boldsymbol{\mathcal{H}_7^+} \cap \boldsymbol{\mathcal{S}_7}(\varepsilon) := \left\{(x_1, x_0) \in \mathbb{R}^2 \,:\, |x_1| \leq x_0, \quad 2x_1x_0 - (2+\varepsilon)x_1^2 \leq 0\right\},$$

where $\boldsymbol{\mathcal{H}_7^+} := \left\{(x_1, x_0) \in \mathbb{R}^2 \,:\, x_0 \geq 0\right\}$ and $\boldsymbol{\mathcal{S}_7}(\varepsilon) := \left\{(x_1, x_0) \in \mathbb{R}^2 \,:\, x_1^2 \leq x_0^2, \quad 2x_1x_0 - (2+\varepsilon)x_1^2 \leq 0\right\}$. If we let $\mathcal{P}_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\mathcal{P}_1(\varepsilon) = \begin{bmatrix} -(2+\varepsilon) & 1 \\ 1 & 0 \end{bmatrix}$ be the matrices associated to $\boldsymbol{\mathcal{S}_7}(\varepsilon)$, we have that

$$E = \left[0, \frac{1}{2} - f(\varepsilon)\right] \cup \left[\frac{1}{2} + f(\varepsilon), 1\right],$$

where $f(\varepsilon) := \frac{1}{2}\sqrt{\frac{\varepsilon}{4+\varepsilon}}$. If $\varepsilon > 0$, then $E$ is composed of two intervals and we can check that $\boldsymbol{\mathcal{H}_7}$ only separates the sets associated to the first interval. The inequality associated to $\lambda_i = 0$ is the conic constraint $|x_1| \leq x_0$ and the one associated to $\lambda_{i+1} = \frac{1}{2} - f(\varepsilon)$ is dominated by this conic constraint and is hence redundant. We can also check that Assumption (22) is satisfied and then by Corollary 1, we have

$$\overline{\operatorname{conv}}\left(\boldsymbol{\mathcal{C}_7}(\varepsilon)\right) = \left\{(x_1, x_0) \in \mathbb{R}^2 \,:\, |x_1| \leq x_0\right\}. \tag{29}$$

In contrast, if $\varepsilon = 0$, we have that $E$ becomes the complete interval $[0, 1]$ and we instead obtain $\lambda_{i+1} = 1$. We can check that in this case (29) still holds, but the inequality associated to $\lambda_{i+1} = 1$ implies $x_1 \leq 0$, which removes a portion of the closed convex hull and is hence invalid. This aligns with the fact that Assumption (22) is not satisfied for $\varepsilon = 0$ and hence Corollary 1 cannot characterize relaxations of $\overline{\operatorname{conv}}\left(\boldsymbol{\mathcal{C}_7}(\varepsilon)\right)$.

The construction of $E$ in Lemma 2 in [45] explicitly considers the possibility of $E = [\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4]$ with $\lambda_2 = \lambda_3$ and relates the $\lambda_i$'s to the rank (and in particular singularity) of the pencil $\mathcal{P}_\lambda = (1-\lambda)\mathcal{P}_0 + \lambda\mathcal{P}_1$. However, special treatment of degenerate cases such as $\varepsilon = 0$ in this example is not considered in [45], since it is not required for the case of strict inequalities (indeed for the strict inequality version for $\varepsilon = 0$, the choice $\lambda_{i+1} = 1$ is correct). Recognizing such degenerate cases may allow relaxing the Assumption (22) in Corollary 1. However, achieving this will likely require adapting the proofs of some of the technical results from [45] or combining them with additional results. For instance, in this example maintaining $\lambda_{i+1} = \frac{1}{2} - f(\varepsilon)$ even for $\varepsilon = 0$ yields a correct characterization of $\overline{\operatorname{conv}}\left(\boldsymbol{\mathcal{C}_7}(\varepsilon)\right)$, so perhaps some type of perturbation analysis could resolve the issues with the non-compliance with Assumption (22).

### 4.6 Comparison to the closed convex hull characterizations by Burer and Kılınç-Karzan

The work in [15] studies the closed convex hull characterization of sets defined as the intersection of a conic quadratic and a quadratic inequality similar to those defined in (9) and (10) given by non-strict inequalities. The work in [15] studies a similar aggregation technique and identifies a set of assumptions that need to be verified in order to get the closed convex hull. Theorem 1 in [15] states the main result of the paper. In this section we do a comparison between the results in [15] and our work and highlight the similarities and differences of the two approaches.

In the language of this paper the first assumption in [15] is:

$$\mathcal{P}_0 \text{ has exactly one negative eigenvalue and } \boldsymbol{\mathcal{H}} \text{ is a separator of } \left\{y \in \mathbb{R}^{n+1} \,:\, \boldsymbol{q_0} \leq 0\right\}. \tag{A1}$$

Assumption (A1) simply formalizes the fact that [15] studies the intersection of a conic quadratic and a general quadratic inequality and hence is not an actual restriction in the context of [15]. Under Assumption (A1), the second assumption of [15] simply requires $\operatorname{int}(\boldsymbol{\mathcal{S}}) \neq \emptyset$. This assumption is shared by this paper and we denote it (A2). Note that Assumption (A2) is similar to the topological Assumption (22) as it requires the non-emptiness of $\operatorname{int}(\boldsymbol{\mathcal{S}})$; however, the topological Assumption (22) is more restrictive than

Assumption (A2) as it requires more than just the non-emptiness of the interior. The third assumption in [15] is a minor technical assumption on the singularity of $\mathcal{P}_0$ and $\mathcal{P}_1$ as follows: either (i) $\mathcal{P}_0$ is nonsingular, (ii) $\mathcal{P}_0$ is singular and $\mathcal{P}_1$ is positive definite on $\operatorname{null}(\mathcal{P}_0)$, or (iii) $\mathcal{P}_0$ is singular and $\mathcal{P}_1$ is negative definite on $\operatorname{null}(\mathcal{P}_0)$. We denote this assumption (A3) and show that this assumption seems to be mildly restrictive. Using Assumption (A3), [15] defines an $s \in [0,1]$ that allows then to describe the closed convex hull using conic quadratic inequalities associated to the pencils $\mathcal{P}_\lambda := (1-\lambda)\mathcal{P}_0 + \lambda\mathcal{P}_1$ at $\lambda = 0$ and $\lambda = s$. In particular, this forces one of the inequalities to be the original conic quadratic inequality, which is a natural choice in the context of [15]. Depending on the details of Assumption (A3), the choice of $s$ is either 0 or the minimum $s \in (0,1]$ such that the pencil $\mathcal{P}_s$ is singular. The last two assumptions of [15] are geometric assumptions on the inequalities used to describe the closed convex hull. To state these assumptions, let $\boldsymbol{\mathcal{H}}_{\boldsymbol{n_s}}$ be the natural separator of $\boldsymbol{\mathcal{S}_s} := \left\{y \in \mathbb{R}^{n+1} : \boldsymbol{q_s} < 0\right\}$ and let $\boldsymbol{\mathcal{K}_s} := \boldsymbol{\mathcal{H}}_{\boldsymbol{s}}^{+} \cap \boldsymbol{\mathcal{S}_s}$ for $\boldsymbol{\mathcal{H}}_{\boldsymbol{s}}^{+} \in \left\{\boldsymbol{\mathcal{H}}_{\boldsymbol{n_s}}^{+}, \boldsymbol{\mathcal{H}}_{\boldsymbol{n_s}}^{-}\right\}$ be defined analogously to $\boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda_i}}$ and $\boldsymbol{\mathcal{K}}_{\boldsymbol{\lambda_{i+1}}}$ in (7). With this notation, the homogeneous version of the geometric assumptions is

$$s = 1 \quad \text{or} \quad \boldsymbol{\mathcal{K}_s} \cap \boldsymbol{\mathcal{H}}_{\boldsymbol{n_s}} \cap \left\{y \in \mathbb{R}^{n+1} : \boldsymbol{q_1} < 0\right\} \neq \emptyset, \tag{A4}$$

while the non-homogeneous version is

$$\begin{gathered} s = 1 \quad \text{or} \quad \left\{\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathbb{R}^{n+1} : \begin{bmatrix} x \\ 0 \end{bmatrix} \in \boldsymbol{\mathcal{K}_s} \cap \boldsymbol{\mathcal{H}}_{\boldsymbol{n_s}}\right\} \cap \left\{y \in \mathbb{R}^{n+1} : \boldsymbol{q_1} < 0\right\} \neq \emptyset \\ \text{or} \\ \left\{\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathbb{R}^{n+1} : \begin{bmatrix} x \\ 0 \end{bmatrix} \in \boldsymbol{\mathcal{K}_s}\right\} \cap \left\{y \in \mathbb{R}^{n+1} : \boldsymbol{q_0} \leq 0\right\} \cap \boldsymbol{\mathcal{H}}^{+} \subseteq \left\{y \in \mathbb{R}^{n+1} : \boldsymbol{q_1} \leq 0\right\}. \end{gathered} \tag{A5}$$

With this notation, Theorem 1 in [15] can be written as follows.

**Theorem 3** *Let* $\boldsymbol{\mathcal{S}} := \left\{y \in \mathbb{R}^{n+1} : \boldsymbol{q_i} \leq 0, \quad i = 0,1\right\}$. *If Assumptions* (A1)–(A3) *hold, then there exists* $s \in [0,1]$ *such that*

$$\overline{\operatorname{conv}}\left(\boldsymbol{\mathcal{H}}^{+} \cap \boldsymbol{\mathcal{S}}\right) \subseteq \left\{y \in \mathbb{R}^{n+1} : \boldsymbol{q_0} \leq 0\right\} \cap \boldsymbol{\mathcal{H}}^{+} \cap \boldsymbol{\mathcal{K}_s}, \tag{30}$$

*where* $\boldsymbol{\mathcal{H}}$ *is a separator of* $\boldsymbol{\mathcal{S}}' := \left\{y \in \mathbb{R}^{n+1} : \boldsymbol{q_0} \leq 0\right\}$. *In such a case, the right hand side of* (30) *can be described by two conic quadratic inequalities. If additionally Assumption* (A4) *is satisfied, then* (30) *holds at equality.*

*If Assumptions* (A1)–(A3) *hold, then there exists* $s \in [0,1]$ *such that*

$$\overline{\operatorname{conv}}(G) \subseteq \left\{x \in \mathbb{R}^n : \begin{bmatrix} x \\ 1 \end{bmatrix} \in \boldsymbol{\mathcal{H}}^{+} \cap \boldsymbol{\mathcal{S}}'\right\} \cap \left\{x \in \mathbb{R}^n : \begin{bmatrix} x \\ 1 \end{bmatrix} \in \boldsymbol{\mathcal{K}_s}\right\}, \tag{31}$$

*for* $G = \left\{x \in \mathbb{R}^n : \begin{bmatrix} x \\ 1 \end{bmatrix} \in \boldsymbol{\mathcal{H}}^{+} \cap \boldsymbol{\mathcal{S}}\right\}$. *In such a case, the right hand side of* (31) *can be described by two conic quadratic inequalities. If additionally Assumptions* (A4)–(A5) *are satisfied, then* (31) *holds at equality.*

To compare Theorem 3 and the results in this paper, we begin by summarizing the assumptions needed by each proposition to construct convex relaxations and to characterize the closed convex hull for each class of set. These assumptions are shown in Table 1. We only include assumptions that are not trivially satisfied by the corresponding class (e.g., non-emptiness), and for the closed convex hull characterization we only include the additional assumptions needed on top of those required to construct a convex relaxation. For instance, Assumption (A1) is automatically satisfied for sets described by one conic quadratic and one quadratic inequality, therefore we do not include it as a requirement for Theorem 3 to construct a convex relaxation of these sets. Similarly, for sets described by homogeneous quadratic inequalities that satisfy topological Assumption (22), the convex relaxation obtained from Corollary 1 automatically characterizes the closed convex hull, therefore no additional assumptions are required (i.e., Corollary 1 either characterizes the closed convex hull or cannot give a non-trivial convex relaxation). Finally, we note that the assumptions

| Class of Set and Proposition | Assumptions for Convex Relaxation | Assumptions for Convex Hull |
|---|---|---|
| Homogeneous quadratic set $\boldsymbol{\mathcal{S}}$ using Theorem 3 | Assumptions (A1)–(A3) | Assumption (A4) |
| Homogeneous quadratic set $\boldsymbol{\mathcal{S}}$ using Corollary 1 | Topological Assumption (22) | – |
| Conic plus quadratic set $C$ using Theorem 3 | Assumptions (A2)–(A3) | Assumptions (A4)–(A5) |
| Conic plus quadratic set $C$ using Corollary 2 | Topological Assumption (23) | Containment Assumption (25) |
| General quadratic set $S$ using Theorem 3 | Assumptions (A1)–(A3) | Assumptions (A4)–(A5) |
| General quadratic set $S$ using Corollary 3 | Topological Assumption (26) | – |

**Table 1** Assumptions to obtain convex relaxations and closed convex hull characterizations for homogeneous quadratic set $\boldsymbol{\mathcal{S}} := \left\{y \in \mathbb{R}^{n+1} : \boldsymbol{q_i} \leq 0, \quad i = 0,1\right\}$, non-homogeneous conic quadratic plus quadratic set $C := \{x \in \mathbb{R}^n : L_0 \leq 0, \quad q_1 \leq 0\}$ and general non-homogeneous quadratic set $S := \{x \in \mathbb{R}^n : q_i \leq 0, \quad i = 0,1\}$.

for the convex relaxations are for the applicability of the propositions and do not guarantee that they will yield a useful relaxation. In particular, a set may satisfy the convex relaxation assumptions for a given proposition, but the proposition may only yield a trivial relaxation that can also be obtained without the proposition (cf. non-homogeneous case of Example 6).

We now compare Theorem 3 and the results in this paper using examples from Section 4.5. We begin by showing examples where Assumptions (A1) and (A3) restrict the applicability of Theorem 3 as compared to the results in this paper. We then show how Assumption (22) restricts the applicability of Corollary 1 as compared with Theorem 3 and how Assumption (25) restricts the applicability of Corollary 2 as compared with Theorem 3. Finally, we comment on the results of Section 7 in [15].

Note that in all of the examples below, the *trivial* relaxation given by the convex inequality $L_0 \leq 0$ (for non-homogeneous sets living in $\mathbb{R}^n$) or $\boldsymbol{\mathcal{L}_0} \leq 0$ (for homogeneous sets living in $\mathbb{R}^{n+1}$) used in the definition of the non-convex set remains valid, and it is assumed that this convex relaxation is immediately available in both Theorem 3 and the results in this paper.

To show how Assumption (A1) can be a tangible restriction when compared with the results in this paper we can use Examples 4 and 5 from Section 4.5. For Example 4, we have that Assumption (A1) is violated because neither $\mathcal{P}_0$ nor $\mathcal{P}_1$ have exactly one negative eigenvalue. Hence, Theorem 3 cannot characterize a relaxation for $\overline{\text{conv}}\left(\boldsymbol{\mathcal{S}_4^+}\right)$. For Example 5, we have that Assumption (A1) is violated, since there is no separator $\boldsymbol{\mathcal{H}}$ of the first homogeneous quadratic inequality which can be used to write $S_5$ as

$$S_5 = \left\{(x, x_0) \in \mathbb{R}^3 : \begin{bmatrix} x \\ 1 \end{bmatrix} \in \boldsymbol{\mathcal{H}}^+ \cap \boldsymbol{\mathcal{S}_3}\right\},$$

where $\boldsymbol{\mathcal{S}_5}$ is the homogeneous version of $S_5$. Hence, Theorem 3 cannot characterize a relaxation for $\overline{\text{conv}}\,(S_5)$. We note that considering cases beyond Assumption (A1) was out of the intended scope of [15]. Indeed, one important difference between Theorem 3 and Corollaries 1–3 is that the former only adds one inequality and the latter can add two inequalities. Adding one inequality is sufficient for the intended scope of [15], but two inequalities may be necessary for other cases such as Examples 4 and 5.

To show how technical Assumption (A3) can be mildly restrictive when compared with the results in this paper we can use the homogeneous version of Example 6 from Section 4.5. Because Assumption (A3) is violated, Theorem 3 does not lead to any relaxation stronger than the trivial relaxation, and it cannot prove that the closed convex hull is given by the trivial relaxation. In contrast, Corollary 1 proves that the trivial relaxation yields the closed convex hull. Note that if we instead consider the non-homogeneous version of Example 6, Assumption (A3) is technically not restrictive. Indeed, in this case, neither Theorem 3 nor Corollary 2 lead to any relaxation stronger than the trivial relaxation.

To show how Assumption (25) of Corollary 2 is restrictive as compared with Assumption (A5) of Theorem 3 we can use Example 3 from Section 4.5. Corollary 2 can show

$$\overline{\text{conv}}\,(C_3) \subseteq \left\{x \in \mathbb{R}^2 : |x_1| \leq x_2 \quad 1 - x_1 - 2x_2 \leq 0\right\}, \tag{32}$$

but since Assumption (25) is violated, it cannot prove that equality holds in (32). In contrast, Theorem 3 can construct the relaxation and prove the equality in (32).

To show how Assumption (22) from Corollary 1 can be restrictive when compared with Theorem 3 we can use Example 7 from Section 4.5 with $\varepsilon = 0$. Since Assumption (22) does not hold, the only relaxation for $\overline{\text{conv}}\,(\boldsymbol{\mathcal{C}_7})$ that Corollary 1 can characterize is the trivial relaxation $\left\{(x_1, x_0) \in \mathbb{R}^2 : |x_1| \le x_0\right\}$. Theorem 3 can also characterize this relaxation, but in a more systematic way that could provide non-trivial relaxations for other sets for which Assumption (22) fails. Analyzing how Theorem 3 characterizes this relaxation provides a convenient way to compare the technical results related to the selection of $s$ in [15] and $\lambda_i$ and $\lambda_{i+1}$ in [45]. For this, let $\mathcal{P}_0$ and $\mathcal{P}_0$ be the matrices defined in Example 7. The value $s$ from Theorem 3 is the minimum $s \in (0, 1]$ such that the pencil $(1 - s)\mathcal{P}_0 + s\mathcal{P}_1$ is singular, which corresponds to $s = \frac{1}{2} - f(\varepsilon)$ for $f$ defined in Example 7. For $\varepsilon > 0$, this $s$ is identical to $\lambda_{i+1}$ obtained by Corollary 1 which yields the relaxation for Example 7. In contrast, for $\varepsilon = 0$, we have $s = 1/2$ and Theorem 3 yields an inequality that is valid for $\overline{\text{conv}}\,(\boldsymbol{\mathcal{C}_7})$, while $\lambda_{i+1} = 1$ and Corollary 1 yields an invalid inequality. Hence, Theorem 3 seems to be less sensitive to the degeneracy issues caused by the violation of Assumption (22) that we discussed at the end of Example 7. We end the discussion of Example 7 by noting that for all $\varepsilon \ge 0$, we have that $\boldsymbol{\mathcal{K}_s}$ is dominated by the original conic inequality $|x_1| \le x_0$. This shows that, similarly to the results in this paper, Theorem 3 can also yield a redundant inequality $\boldsymbol{\mathcal{K}_s}$.

We note that for Examples 1 and 2, Theorem 3 yields the same results as Corollaries 2 and 3.

Finally, we consider the sets studied in Section 7 of [15]. This section develops simplifications of Assumptions A1–A5 for intersections of a conic section and a general quadratic constraint. All resulting sets correspond to the intersection of a convex quadratic inequality with a general quadratic inequality. The convex hull of the strict inequality version of all these sets can be characterized without any assumptions by Theorem 2. Similarly, characterizing the closed convex hull of the non-strict inequality versions through Corollary 3 only requires the sets to be contained in the closure of their interiors. Because this last assumption is not too restrictive, we can find examples where Corollary 3 can construct the closed convex hull of the intersections of a conic section and a general quadratic constraint, while the simplified assumptions from Section 7 of [15] do not hold. For instance, Example 3 in [38] shows how Corollary 3 yields the closed convex hull of a paraboloid intersected with a non-convex quadratic constraint. This example does not satisfy the simplified assumptions in Section 7 of [15]; however, it satisfies the more general Assumptions A1–A5. Hence there does not seem to be a major difference on the applicability of the two techniques on this class of problems.

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## References


1. Achterberg, T.: SCIP: solving constraint integer programs. Mathematical Programming Computation **1**, 1–41 (2009)
2. Andersen, K., Jensen, A.: Intersection cuts for mixed integer conic quadratic sets. In: M. Goemans, J. Correa (eds.) IPCO Conference, *LNCS*, vol. 7801, 37–48. Springer (2013)
3. Atamtürk, A., Narayanan, V.: Cuts for conic mixed-integer programming. In: M. Fischetti, D.P. Williamson (eds.) IPCO, *LNCS*, vol. 4513, 16–29. Springer (2007)
4. Atamtürk, A., Narayanan, V.: Conic mixed-integer rounding cuts. Mathematical Programming **122**, 1–20 (2010)
5. Belotti, P., Góez, J.C., Pólik, I., Ralphs, T.K., Terlaky, T.: A conic representation of the convex hull of disjunctive sets and conic cuts for integer second order cone optimization. In: M. Al-Baali, L. Grandinetti, A. Purnama (eds.) Numerical Analysis and Optimization, *Springer Proceedings in Mathematics & Statistics*, vol. 134, 1–35 (2015)
6. Belotti, P., Góez, J.C., Pólik, I., Ralphs, T.K., Terlaky, T.: On families of quadratic surfaces having fixed intersections with two hyperplanes. Discrete Applied Mathematics **161**(16), 2778–2793 (2013)
7. Belotti, P., Góez, J.C., Pólik, I., Ralphs, T.K., Terlaky, T.: Disjunctive conic cuts for mixed integer second order cone optimization. `http://coral.ie.lehigh.edu/~ted/files/papers/ConicCuts14.pdf` (2014)
8. Ben-Tal, A., Teboulle, M.: Hidden convexity in some nonconvex quadratically constrained quadratic programming. Mathematical Programming **72**(1), 51–63 (1996)

9. Bienstock, D., Michalka, A.: Strong formulations for convex functions over nonconvex sets. Optimization Online (2011). http://www.optimization-online.org/DB_HTML/2011/12/3278.html
10. Bienstock, D., Michalka, A.: Cutting-planes for optimization of convex functions over nonconvex sets. SIAM Journal on Optimization **24**(2), 643–677 (2014)
11. Bixby, R., Fenelon, M., Gu, Z., Rothberg, E., Wunderling, R.: Mixed-integer programming: a progress report. In: The sharpest cut: the impact of Manfred Padberg and his work, chap. 18, pp. 309–326. SIAM, Philadelphia, PA (2004)
12. Bixby, R., Rothberg, E.: Progress in computational mixed integer programming - a look back from the other side of the tipping point. Annals of Operations Research **149**, 37–41 (2007)
13. Bonami, P.: Lift-and-project cuts for mixed integer convex programs. In: Integer Programming and Combinatorial Optimization, pp. 52–64 (2011)
14. Burer, S., Kılınç-Karzan, F.: personal communication (2014)
15. Burer, S., Kılınç-Karzan, F.: How to convexify the intersection of a second order cone and a nonconvex quadratic. arXiv preprint arXiv:1406.1031, http://arxiv.org/abs/1406.1031 (2014)
16. Çezik, M.T., Iyengar, G.: Cuts for mixed 0-1 conic programming. Mathematical Programming **104**, 179–202 (2005)
17. Conforti, M., Cornuéjols, G., Zambelli, G.: Polyhedral approaches to mixed integer linear programming. 50 Years of Integer Programming 1958-2008 pp. 343–385 (2010)
18. Conforti, M., Cornuéjols, G., Zambelli, G.: Corner polyhedron and intersection cuts. Surveys in Operations Research and Management Science **16**, 105–120 (2011)
19. Cook, W.J., Kannan, R., Schrijver, A.: Chvátal closures for mixed integer programming problems. Mathematical Programming **47**, 155–174 (1990)
20. Cornuéjols, G.: Valid inequalities for mixed integer linear programs. Mathematical Programming **112**, 3–44 (2008)
21. Dadush, D., Dey, S.S., Vielma, J.P.: The Chvátal-Gomory closure of a strictly convex body. Mathematics of Operations Research **36**, 227–239 (2011)
22. Dadush, D., Dey, S.S., Vielma, J.P.: The split closure of a strictly convex body. Operations Research Letters **39**, 121 –126 (2011)
23. Dadush, D., Dey, S.S., Vielma, J.P.: On the Chvátal-Gomory closure of a compact convex set. Mathematical Programming **145**, 327–348 (2014)
24. Del Pia, A., Weismantel, R.: Relaxations of mixed integer sets from lattice-free polyhedra. 4OR: A Quarterly Journal of Operations Research **10**, 1–24 (2012)
25. Drewes, S.: Mixed integer second order cone programming. Ph.D. thesis, Technische Universität Darmstadt (2009)
26. Hiriart-Urruty, J-B., Lemaréchal, C.: Fundamentals of convex analysis. Springer Science & Business Media (2012)
27. Gomory, R.E.: Some polyhedra related to combinatorial problems. Linear Algebra and its Applications **2**, 451 – 558 (1969)
28. Gomory, R.E., Johnson, E.L.: Some continuous functions related to corner polyhedra. Mathematical Programming **3**, 23–85 (1972)
29. Johnson, E.L., Nemhauser, G.L., Savelsbergh, M.W.P.: Progress in linear programming-based algorithms for integer programming: An exposition. INFORMS Journal on Computing **12**, 2–23 (2000)
30. Kılınç, M.R., Linderoth, J., Luedtke, J.: Effective separation of disjunctive cuts for convex mixed integer nonlinear programs. Optimization Online (2010). http://www.optimization-online.org/DB_HTML/2010/11/2808.html
31. Kılınç-Karzan, F.: On minimal valid inequalities for mixed integer conic programs. Mathematics of Operations Research 41(2), 477–510 (2015)
32. Kılınç-Karzan, F., Yıldız, S.: Two-term disjunctions on the second-order cone. Mathematical Programming **154**(1-2), 463–491 (2015)
33. Lodi, A.: Mixed integer programming computation. In: M. Jünger, T. Liebling, D. Naddef, G. Nemhauser, W. Pulleyblank, G. Reinelt, G. Rinaldi, L. Wolsey (eds.) 50 Years of Integer Programming 1958-2008: From the Early Years to the State-of-the-Art, chap. 16, pp. 619–645. Springer-Verlag, New York (2010)
34. Marchand, H., Wolsey, L.: Aggregation and Mixed Integer Rounding to solve MIPs. Operations Research **49**, 363–371 (2001)
35. Modaresi, S.: Valid Inequalities and Reformulation Techniques for Mixed Integer Nonlinear Programming. Ph.D. thesis, University of Pittsburgh (2015)
36. Modaresi, S., Kılınç, M.R., Vielma, J.P.: Intersection cuts for nonlinear integer programming: Convexification techniques for structured sets. Mathematical Programming **155**, 575–611 (2016)
37. Modaresi, S., Kılınç, M.R., Vielma, J.P.: Split cuts and extended formulations for mixed integer conic quadratic programming. Operations Research Letters **43**(1), 10–15 (2015)

38. Modaresi, S., Vielma, J.P.: The power of a negative eigenvalue: Aggregation cuts for nonlinear integer programming. 2014 Mixed Integer Programming Workshop, July 21–24, 2014, Columbus, OH, Poster (2014). https://mip2014.engineering.osu.edu/sites/mip2014.engineering.osu.edu/files/uploads/Sina-Modaresi.pdf
39. Modaresi, S., Kılınç, M.R., Vielma, J.P.: Split Cuts for Conic Programming. Poster presented at the MIP 2012 Workshop at UC Davis (2012).
40. Moran R., D.A., Dey, S.S., Vielma, J.P.: A strong dual for conic mixed-integer programs. SIAM Journal on Optimization **22**(3), 1136–1150 (2012)
41. Nemhauser, G.L., Wolsey, L.A.: Integer and combinatorial optimization. Wiley (1988)
42. Nemhauser, G.L., Wolsey, L.A.: A recursive procedure to generate all cuts for 0-1 mixed integer programs. Mathematical Programming **46**, 379–390 (1990)
43. Stubbs, R.A., Mehrotra, S.: A branch-and-cut method for 0-1 mixed convex programming. Mathematical Programming **86**, 515–532 (1999)
44. Wolsey, L.A.: Integer Programming. Wiley (1998)
45. Yıldıran, U.: Convex hull of two quadratic constraints is an LMI set. IMA Journal of Mathematical Control and Information **26**, 417–450 (2009)
46. Yıldız, S., Cornuéjols, G.: Disjunctive cuts for cross-sections of the second-order cone. Operations Research Letters, **43**(4), 432–437 (2015)