A Constructive Characterization of the Split Closure of a Mixed Integer Linear Program

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December 23, 2005

Abstract. Two independent proofs of the polyhedrality of the split closure of Mixed Integer Linear Program have been previously presented. Unfortunately neither of these proofs is constructive. In this paper, we present a constructive version of this proof. We also show that split cuts dominate a family of inequalities introduced by Köppe and Weismantel.

Keywords: Mixed-integer linear programming; Lattice bases; Cutting planes; Split Cuts; Split Closure

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1 Introduction

In 1990 Cook, Kannan and Schrijver [8] introduced a family of cuts for a Mixed Integer Linear Program (MILP) which they called split cuts. These cuts are a special case of Balas' disjunctive cuts [4] which arise from a particular two term disjunction. Split cuts are also related to intersection cuts introduced by Balas in 1971 [3]. A precise correspondence between split cuts and intersection cuts has been established for $0-1$ MILPs by Balas and Perregaard [5] and for general MILPs by Andersen, Cornuejols and Li [1, 2].

The split closure of a MILP is the convex set defined by the intersection of all of its split cuts. Cook, Kannan and Schrijver [8] proved that the split closure of a MILP is a polyhedron. Andersen, Cornuejols and Li [1, 2] have given an alternate proof of this fact. Unfortunately, neither of these proofs is constructive in the sense that they do not provide a method for constructing the split closure for a given MILP.

Another family of cutting planes related to split cuts is the one introduced by Köppe and Weismantel in 2004 [9]. This family of cuts is based on a mixed integer version of the Farkas' Lemma and they were related to split cuts by Bertsimas and Weismantel in 2005 [6].

By using an algebraic characterization of split cuts introduced by Caprara and Letchford [7] we are able to show that every cut from [9] is dominated by the split cut to which it is related. Furthermore, by using this relationship and a result from [1, 2] we are able to construct a finite set of split cuts that define the split closure, hence providing a constructive proof of its polyhedrality. The key step of this proof is using the characterization from [7] to note that every non-dominated split cut for a particular relaxation of a MILP can be associated to an element in a lattice introduced in [9].

The rest of the paper is organized as follows. In section 2 we introduce some notation, the algebraic characterization of split cuts from [7] and some results from [1, 2] we will use later. Then, in section 3 we present a simplified characterization of split cuts for a particular relaxation of the MILP. Finally, in section 4 we use this simplified characterization show that the cutting planes introduced in [9] are dominated by split cuts and develop the constructive proof of the polyhedrality of the split closure.

2 Split Cuts

We study the feasible region of a Mixed Integer Linear Programming (MILP) problem given by

$$P_I := \{x \in P \subseteq \mathbb{R}^n : x_j \in \mathbb{Z} \quad \forall\, j \in N_I\}$$

where $N = \{1, \ldots, n\}$, $N_I \subseteq N$ and P is a rational polyhedron given by

$$P := \{x \in \mathbb{R}^n : Ax \leq b\}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $M = \{1, \ldots, m\}$, $r = \operatorname{rank}(A)$ and $a_{i\cdot}$ corresponds to row i of A. We will assume that $P \neq \emptyset$, but we will not assume $r = n$ allowing for P to contain a line. We also allow for P to be not full dimensional.

Now, let

$$\mathcal{B}_r^* := \{B \subseteq M \ : \ |B| = r \text{ and } \{a_{i\cdot}\}_{i\in B} \text{ are linearly independent}\}.$$

Then, for every $B \in \mathcal{B}_r^*$ we define the following relaxation of P

$$P(B) := \{x \in \mathbb{R}^n \ : \ a_{i\cdot}^T x \leq b_i \quad \forall\, i \in B\}.$$

Note that $\mathcal{B}_n^*$ corresponds to the bases of P, so for simplicity we will refer to $B \in \mathcal{B}_r^*$ as a basis even when $r < n$, noting that in this later case, feasible bases will not define extreme points of P. In any case we will define $\bar{x}(B)$ to be a particular, but arbitrarily selected, solution to $a_{i\cdot}^T x = b_i$, $\forall\, i \in B$.

We will study *split disjunctions* $D(\pi, \pi_0)$ of the form $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$ where $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$. We denote the set of points satisfying split disjunction $D(\pi, \pi_0)$ as

$$F_{D(\pi,\pi_0)} := \{x \in \mathbb{R}^n \ : \ \pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1\}$$

and $\text{conv}(P \cap F_{D(\pi,\pi_0)})$ as the *disjunctive hull* defined by P and $D(\pi, \pi_0)$. Similarly for $B \in \mathcal{B}_r^*$ we define the *basic disjunctive hull* defined by B and $D(\pi, \pi_0)$ as $\text{conv}(P(B) \cap F_{D(\pi,\pi_0)})$.

We say that a disjunction $D(\pi, \pi_0)$ is valid for P_I if $P_I \subseteq F_{D(\pi,\pi_0)} \subsetneq \mathbb{R}^n$. We are interested in the following set of disjunctions, which are always valid for P_I.

$$\Pi_0^n(N_I) := \{(\pi, \pi_0) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} \,:\, \pi_j = 0,\, j \notin N_I\}$$

We also define the projection of $\Pi_0^n(N_I)$ into the π variables as

$$\Pi^n(N_I) := \{\pi \in \mathbb{Z}^n \setminus \{0\} \,:\, \pi_j = 0,\, j \notin N_I\}.$$

With this, the *split closure* of P_I is defined as

$$SC := \bigcap_{(\pi,\pi_0)\in\Pi_0^n(N_I)} \text{conv}(P \cap F_{D(\pi,\pi_0)}).$$

Similarly, for $B \in \mathcal{B}_k^*$ we define the *basic split closure* as

$$SC(B) := \bigcap_{(\pi,\pi_0)\in\Pi_0^n(N_I)} \text{conv}(P(B) \cap F_{D(\pi,\pi_0)}).$$

A *split cut* is an inequality valid for SC and hence valid for P_I. Similarly a *basic split cut* is an inequality valid for $SC(B)$ for some $B \in \mathcal{B}_r^*$. It is known that basic split cuts are exactly the same as intersection cuts (see, for example, [1, 2]).

If $\delta^T x \leq \delta_0$ and $\gamma^T x \leq \gamma_0$ are two inequalities valid for SC, we will say that $\delta^T x \leq \delta_0$ is *dominated* by $\gamma^T x \leq \gamma_0$ if and only if

$$\{x \in P \,:\, \gamma^T x \leq \gamma_0 \quad\} \subseteq \{x \in P \,:\, \delta^T x \leq \delta_0\}.$$

Similarly, if the inequalities are valid for $SC(B)$ for some $B \in \mathcal{B}_r^*$, we will say that $\delta^T x \leq \delta_0$ is *dominated* by $\gamma^T x \leq \gamma_0$ if and only if

$$\{x \in P(B) \,:\, \gamma^T x \leq \gamma_0 \quad\} \subseteq \{x \in P(B) \,:\, \delta^T x \leq \delta_0\}.$$

In particular, we will say that a split cut or basic split cut $\delta^T x \leq \delta_0$ is *non-trivial* if and only if it is not dominated by the trivial inequality $\mathbf{0}^T x \leq 1$. In other words, non-trivial split cuts and non-trivial basic split cuts are cuts that are not already valid for P and $P(B)$ respectively.

Any split cut $\delta^T x \leq \delta_0$ is a valid inequality for $\text{conv}(P \cap F_{D(\pi,\pi_0)})$ for some $(\pi, \pi_0) \in \Pi_0^n(N_I)$, so we will concentrate on characterizing split cuts for a fixed, but arbitrary $(\pi, \pi_0) \in \Pi_0^n(N_I)$. We will refer to these split cuts for a fixed $(\pi, \pi_0) \in \Pi_0^n(N_I)$ as a split cut for $D(\pi, \pi_0)$. Similarly we will also talk about basic split cuts for $D(\pi, \pi_0)$ and B for some basis $B \in \mathcal{B}_r^*$.

For any $(\pi, \pi_0) \in \Pi_0^n(N_I)$ we have that $P \cap F_{D(\pi,\pi_0)}$ is the disjoint union of sets $P_1(\pi, \pi_0)$ and $P_2(\pi, \pi_0)$ given by

$$\begin{aligned} P_1(\pi, \pi_0) &:= \{x \in \mathbb{R}^n : Ax \leq b, \quad \pi^T x \leq \pi_0\} \\ P_2(\pi, \pi_0) &:= \{x \in \mathbb{R}^n : Ax \leq b, \quad -\pi^T x \leq -\pi_0 - 1\} \end{aligned}$$

either of which could be empty. Note that $P_1(\pi, \pi_0)$ and $P_2(\pi, \pi_0)$ respectively correspond to the *Left* and *Right* polyhedra of [7].

The following characterization is constructed by using Farkas' Lemma and the fact that an inequality is valid for $\text{conv}(P \cap F_{D(\pi,\pi_0)})$ if and only if it is valid for $P_1(\pi, \pi_0)$ and $P_2(\pi, \pi_0)$. This approach was first proposed by Cook, Kannan and Schrijver [8], but was not used to give a characterization of split cuts. This characterization can also be seen as a slight simplification of the reverse polar characterization of Balas [4] for the particular case of split disjunctions. It has been implicitly used for $0-1$ MILPs by Balas and Perregaard [5] and for general MILPs by Andersen, Cornuejols and Li [1, 2], but to the best of our knowledge it was explicitly introduced for the first time by Caprara and Letchford [7].

Proposition 1. *Let $(\pi, \pi_0) \in \Pi_0^n(N_I)$. Then, for any $\mu_0^1, \mu_0^2 \in \mathbb{R}_+$ and $\mu^1, \mu^2 \in \mathbb{R}_+^m$ which are solutions to the system*

$$\sum_{i \in M} \mu_i^2 a_{i\cdot} - \sum_{i \in M} \mu_i^1 a_{i\cdot} = \pi \tag{1}$$

$$\sum_{i \in M} \mu_i^2 b_i - \sum_{i \in M} \mu_i^1 b_i - \mu_0^2 = \pi_0 \tag{2}$$

$$\mu_0^1 + \mu_0^2 = 1 \tag{3}$$

we have that $\delta(\mu, \pi, \pi_0)^T x \leq \delta_0(\mu, \pi, \pi_0)$ given by

$$\delta(\mu, \pi, \pi_0) := \mu_0^1 \pi + \sum_{i \in M} \mu_i^1 a_{i\cdot} = -\mu_0^2 \pi + \sum_{i \in M} \mu_i^2 a_{i\cdot}$$

$$\delta_0(\mu, \pi, \pi_0) := \mu_0^1 \pi_0 + \sum_{i \in M} \mu_i^1 b_i = -\mu_0^2 (\pi_0 + 1) + \sum_{i \in M} \mu_i^2 b_i$$

is a split cut for $D(\pi, \pi_0)$. Conversely, any non-trivial split cut for $D(\pi, \pi_0)$ is dominated by $\delta(\mu, \pi, \pi_0)^T x \leq \delta_0(\mu, \pi, \pi_0)$ for some $\mu_0^1, \mu_0^2 \in \mathbb{R}_+$ and $\mu^1, \mu^2 \in \mathbb{R}_+^m$ which are solutions to (1)–(3) also satisfying

$$\mu_0^2 \in (0, 1) \tag{4}$$

$$\mu_i^1 \cdot \mu_i^2 = 0 \qquad \forall i \in M. \tag{5}$$

The validity of this characterization can be established from [4], in which a higher dimensional characterization of $\mathrm{conv}(P \cap F_{D(\pi,\pi_0)})$ is introduced. For proofs without using this higher dimensional characterization see [7] and [10].

We will also use the following results from [1, 2].

Proposition 2. *Let $(\pi, \pi_0) \in \Pi_0^n(N_I)$. Then*

$$\mathrm{conv}(P \cap F_{D(\pi,\pi_0)}) = \bigcap_{B \in \mathcal{B}_r^*} \mathrm{conv}(P(B) \cap F_{D(\pi,\pi_0)}).$$

Corollary 1. $SC = \bigcap_{B \in \mathcal{B}_r^*} SC(B)$.

Given that $\mathcal{B}_r^*$ is finite, from Corollary 1 we have that proving that SC is a polyhedron reduces to proving that $SC(B)$ is a polyhedron for all $B \in \mathcal{B}_r^*$. This was exactly the approach used in [1, 2] to prove the polyhedrality of SC. In this same spirit we now study the characterization of basic split cuts to give a constructive proof of the polyhedrality of $SC(B)$.

3 Basic Split Cuts

In this section we study basic split cuts for a particular basis $B \in \mathcal{B}_r^*$. We will use Proposition 1 to give a family of inequalities that contains all non-dominated basic split cuts for B and does not explicitly depend on (π, π_0).

For a particular basis $B \in \mathcal{B}_r^*$ we denote by $\bar{B}$ the $r \times n$ submatrix of A defined by this basis and by $\bar{b}$ the corresponding r-dimensional righthand side.

By defining $y^- = \max\{-y, 0\}$, $y^+ = \max\{y, 0\}$ and $f(y) = y - \lfloor y \rfloor$ for any $y \in \mathbb{R}$ and assuming that these operations together with $|y|$ are applied component wise for $y \in \mathbb{R}^r$ we have that

Proposition 3. *For $B \in \mathcal{B}_r^*$ and $\bar{\mu} \in \mathbb{R}^r$ define the following inequalities:*

1. $\delta^1(\bar{\mu}, B)^T x \le \delta_0^1(\bar{\mu}, B)$ *given by*
$$(\bar{\mu}^-)^T(\bar{B}x - \bar{b}) + (1 - f(\bar{\mu}^T \bar{b}))(\bar{\mu}^T \bar{B}x - \lfloor \bar{\mu}^T \bar{b} \rfloor) \le 0 \tag{6}$$

2. $\delta^2(\bar{\mu}, B)^T x \le \delta_0^2(\bar{\mu}, B)$ *given by*
$$(\bar{\mu}^+)^T(\bar{B}x - \bar{b}) - f(\bar{\mu}^T \bar{b})(\bar{\mu}^T \bar{B}x - \lfloor \bar{\mu}^T \bar{b} \rfloor) + f(\bar{\mu}^T \bar{b}) \le 0 \tag{7}$$

3. $\delta^3(\bar{\mu}, B)^T x \le \delta_0^3(\bar{\mu}, B)$ *given by*
$$\frac{1}{2}\Big(|\bar{\mu}|^T(\bar{B}x - \bar{b}) + (1 - 2f(\bar{\mu}^T \bar{b}))(\bar{\mu}^T \bar{B}x - \lfloor \bar{\mu}^T \bar{b} \rfloor) + f(\bar{\mu}^T \bar{b})\Big) \le 0 \tag{8}$$

Also, for $\pi \in \Pi^n(N_I)$ define the following linear system over $\bar{\mu} \in \mathbb{R}^r$
$$\bar{B}^T \bar{\mu} = \pi. \tag{9}$$

Then

1. *If $\bar{\mu}$ is a solution to* (9) *then, for all $k \in \{1, 2, 3\}$ we have that $\delta^k(\bar{\mu}, B)^T x \le \delta_0^k(\bar{\mu}, B)$ is valid for* $\operatorname{conv}(P(B) \cap F_{D(\pi, \lfloor \bar{\mu}^T \bar{b} \rfloor)})$. *Furthermore, $\delta^1(\bar{\mu}, B) = \delta^2(\bar{\mu}, B) = \delta^3(\bar{\mu}, B)$ and $\delta_0^1(\bar{\mu}, B) = \delta_0^2(\bar{\mu}, B) = \delta_0^3(\bar{\mu}, B)$.*

2. *If $\bar{\mu}$ is the unique solution to* (9) *and $\bar{\mu}^T \bar{b} \notin \mathbb{Z}$ then for any $k \in \{1, 2, 3\}$*
$$\begin{aligned} \operatorname{conv}(P(B) \cap F_{D(\pi, \lfloor \bar{\mu}^T \bar{b} \rfloor)}) &= \{x \in P(B) \,:\, \delta^k(\bar{\mu}, B)x \le \delta_0^k(\bar{\mu}, B)\} \\ &\subsetneq P(B). \end{aligned} \tag{10}$$

3. *If* (9) *is infeasible or the unique solution $\bar{\mu}$ to* (9) *is such that $\bar{\mu}^T \bar{b} \in \mathbb{Z}$ then*
$$\operatorname{conv}(P(B) \cap F_{D(\pi, \lfloor \bar{\mu}^T \bar{b} \rfloor)}) = P(B).$$

4. $\operatorname{conv}(P(B) \cap F_{D(\pi, \pi_0)}) = P(B)$ *for all $\pi_0 \ne \lfloor \bar{\mu}^T \bar{b} \rfloor$.*

Proof. To prove this proposition we will use Proposition 1 for the special case $P = P(B)$ for the given $B \in \mathcal{B}_r^*$. For this case, condition (5) of Proposition 1 for (π, π_0) allow us to combine μ^1 and μ^2 into $\bar{\mu} \in \mathbb{R}^r$. We can then write conditions (1)–(3),(4) of Proposition 1 in variables $\bar{\mu}$ and the original μ_0^1, μ_0^2 as:

$$\bar{B}^T \bar{\mu} = \sum_{i \in B} \bar{\mu}_i a_{i\cdot} = \pi \tag{11}$$
$$\bar{\mu}^T b = \sum_{i \in B} \bar{\mu}_i b_i = \pi_0 + \mu_0^2 \tag{12}$$
$$\begin{aligned} \mu_0^1 + \mu_0^2 &= 1 \\ \mu_0^2 &\in (0, 1) \end{aligned} \tag{13}$$

Then, from (11),(12),(13) and Proposition 1 we have that for $P = P(B)$ there can be a non-trivial split cut for $D(\pi, \pi_0)$ only if (9) is feasible and the unique solution $\bar{\mu}$ to (9) is such that $\bar{\mu}^T \bar{b} \notin \mathbb{Z}$ and $\pi_0 = \lfloor \bar{\mu}^T \bar{b} \rfloor$. This proves parts 3 and 4 by noting that, besides the original constraints of $P(B)$, the only necessary inequalities for the description of $\operatorname{conv}(P(B) \cap F_{D(\pi, \pi_0)})$ are non-trivial split cuts.

Now, if $\bar{\mu}$ is a solution to (9), by the identities $\mu_i^1 = \bar{\mu}_i^-$, $\mu_i^2 = \bar{\mu}_i^+$, $\mu_0^2 = f(\bar{\mu}^T \bar{b})$, $\mu_0^1 = 1 - f(\bar{\mu}^T \bar{b})$ and $\pi_0 = \lfloor \bar{\mu}^T \bar{b} \rfloor$ we have that $\delta^k(\bar{\mu}, B) = \delta(\mu, \pi, \pi_0)$ and $\delta_0^k(\bar{\mu}, B) = \delta_0(\mu, \pi, \pi_0)$ for $k \in \{1, 2\}$. Hence for

$k \in \{1,2\}$ we have that $\delta^k(\bar{\mu},B)^T x \le \delta_0^k(\bar{\mu},B)$ is valid for $\operatorname{conv}(P(B) \cap F_{D(\pi,\pi_0)})$. By noting that (8) is the average of (6) and (7) we have part 1.

Finally, we already know that under the conditions of part 2 the first equality of (10) holds. To prove that the strict containment holds we will show that for any $k \in \{1,2,3\}$, $\delta^k(\bar{\mu})\bar{x}(B) > \delta_0^k(\bar{\mu})$. First, by multiplying $\bar{B}\bar{x}(B) = \bar{b}$ by $\bar{\mu}^T$ and using $\bar{\mu}^T\bar{b} \notin \mathbb{Z}$ we have that $\bar{\mu}^T\bar{B}\bar{x}(B) - \lfloor \bar{\mu}^T\bar{b} \rfloor > 0$. By multiplying this last strict inequality by $(1 - f(\mu^T\bar{b})) > 0$ and $\bar{B}\bar{x}(B) = \bar{b}$ by $\mu^- \ge 0$ and adding them together we have that $\delta^1(\bar{\mu})\bar{x}(B) > \delta_0^1(\bar{\mu})$. The result follows from the equivalence of $\delta^k(\bar{\mu},B)^T x \le \delta_0^k(\bar{\mu},B)$ for $k \in \{1,2,3\}$.

□

We note that parts 2–4 of Proposition 3 can be seen to be equivalent to known properties of intersection cuts which are usually stated and proved with respect to (π,π_0) instead of $\bar{\mu}$. These details are explored in [10].

Using Proposition 3 for $B \in \mathcal{B}_r^*$ and $\bar{\mu} \in \mathbb{R}^r$ the unique solution to (9) for some $\pi \in \Pi^n(N_I)$ we define $\delta(\bar{\mu},B)^T x \le \delta_0(\bar{\mu},B)$ to be the inequality defined by $\delta^k(\bar{\mu},B)^T x \le \delta_0^k(\bar{\mu},B)$ for any $k \in \{1,2,3\}$. We note that $\delta(\bar{\mu},B)^T x \le \delta_0(\bar{\mu},B)$ does not explicitly depend on (π,π_0), but it does implicitly depend on π.

4 Mixed Integer Lattices and Polyhedrality of the Split Closure

In this section we show that every non-dominated basic split cut for a particular basis $B \in \mathcal{B}_r^*$ can be associated to an element of the lattice used by [9]. Then we construct a finite set of inequalities defining $SC(B)$.

We start by summarizing the results from [9] in Proposition 4. For this we let $\bar{B}_I \in \mathbb{R}^{r \times |N_I|}$ and $\bar{B}_C \in \mathbb{R}^{r \times (n - |N_I|)}$ be the submatrices of $\bar{B}$ corresponding to the integer and the continuous variables of P_I respectively and for $\bar{\mu} \in \mathbb{R}^r$ we let $\lceil \bar{\mu}^- \rceil \in \mathbb{R}^r$ be the vector with components $\lceil \bar{\mu}_i^- \rceil$. We also use the following definition of a lattice.

Definition 1. *Let $\{v^i\}_{i \in \mathcal{V}} \subseteq \mathbb{R}^r$ be a finite set of linear independent vectors. The* lattice *generated by $\{v^i\}_{i \in \mathcal{V}}$ is $\mathcal{L} := \{\mu \in \mathbb{R}^r : \mu = \sum_{i \in \mathcal{V}} k_i v^i \quad k_i \in \mathbb{Z}\}$. The set $\{v^i\}_{i \in \mathcal{V}}$ is called a* basis *of $\mathcal{L}$.*

Proposition 4. *For every $B \in \mathcal{B}_r^*$*

1. *$\mathcal{L}(B) := \{\bar{\mu} \in \mathbb{R}^r : {\bar{B}_I}^T \bar{\mu} \in \mathbb{Z}^{|N_I|}, \quad {\bar{B}_C}^T \bar{\mu} = 0\}$ is a lattice.*
2. *If $\bar{\mu} \in \mathcal{L}(B)$ is such that $\bar{\mu}^T b \notin \mathbb{Z}$ then the inequality defined by*

$$\lceil \bar{\mu}^- \rceil^T (\bar{B}x - \bar{b}) + (1 - f(\bar{\mu}^T\bar{b}))(\bar{\mu}^T\bar{B}x - \lfloor \bar{\mu}^T\bar{b} \rfloor) \le 0 \tag{14}$$

is valid for $\{x \in P(B) : x_j \in \mathbb{Z} \, \forall \, j \in N_I\}$. Furthermore this inequality is not satisfied by $\bar{x}(B)$.

Proof. See Proposition 2 and Theorem 3 of [9] for the case $r = n$. The case $r < n$ is analogous. □

Bertsimas and Weismantel [6] related split cuts to inequality (14) by showing that every $\bar{\mu} \in \mathcal{L}(B)$ such that $\bar{\mu}^T b \notin \mathbb{Z}$ induces a valid split disjunction for P_I. We will now see that in fact, the only split disjunctions necessary for the description of $SC(B)$ are the ones induced by elements of $\mathcal{L}(B)$.

We have that $\mathcal{L}(B)$ precisely corresponds to all the $\bar{\mu} \in \mathbb{R}^r$ such that $\bar{\mu}$ is the unique solution to (9) for B and some $\pi \in \Pi^n(N_I)$. Hence, every non-dominated basic split cuts for B is associated to an element in $\mathcal{L}(B)$. Furthermore, from Proposition 3 we get the following characterization of $SC(B)$.

Proposition 5. *For every $B \in \mathcal{B}_r^*$ and $\bar{\mu} \in \mathcal{L}(B)$ we have that $\delta(\bar{\mu},B)^T x \le \delta_0(\bar{\mu},B)$ is a valid inequality for $SC(B)$. Furthermore we have that*

$$SC(B) = \bigcap_{\substack{\bar{\mu} \in \mathcal{L}(B) \\ \bar{\mu}^T\bar{b} \notin \mathbb{Z}}} \{x \in P(B) : \delta(\bar{\mu},B)^T x \le \delta_0(\bar{\mu},B)\}.$$

We then have that every element of $\mathcal{L}(B)$ that is associated to a cut given by (14) is also associated to a non-dominated basic split cut for B. Furthermore we have that

Proposition 6. *Let $\bar{\mu} \in \mathcal{L}(B)$ be such that $\bar{\mu}^T b \notin \mathbb{Z}$ then cut (14) for $\bar{\mu}$ is dominated by the basic split cut $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$.*

Proof. The result follows directly from representation (6) of $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ and the facts that $\bar{B}x - \bar{b} \leq 0$ for all $x \in P(B)$ and that $\lceil \bar{\mu}^- \rceil \geq \bar{\mu}^-$. □

We will now construct a finite subset of $\mathcal{L}(B)$ that suffices to characterize $SC(B)$. To do this we will need to study the intersection of $\mathcal{L}(B)$ with each orthant separately. For any $\sigma \in \{0,1\}^r$ let

$$\mathcal{L}(B, \sigma) := \{\mu \in \mathcal{L}(B) \,:\, (-1)^{\sigma_i} \mu_i \geq 0, \quad \forall\, i \in \{1, \ldots, r\}\}$$

be the intersection of $\mathcal{L}(B)$ with the orthant defined by σ. We then have that

$$\mathcal{L}(B) = \bigcup_{\sigma \in \{0,1\}^r} \mathcal{L}(B, \sigma)$$

Now, for each $\sigma \in \{0,1\}^r$, we construct a finite subset of $\mathcal{L}(B, \sigma)$, such that the finite union of these sets suffices to characterize $SC(B)$. To do this we will need the following lemma

Lemma 1. *Let $\sigma \in \{0,1\}^r$ and let $\bar{\mu} \in \mathcal{L}(B, \sigma)$ with $\bar{\mu} = \alpha + \beta$ for $\alpha, \beta \in \mathcal{L}(B, \sigma)$ such that $\beta^T b \in \mathbb{Z}$. Then $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ is dominated by $\delta(\alpha, B)^T x \leq \delta_0(\alpha, B)$.*

Proof. We will use representation (8) of $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ and $\delta(\alpha, B)^T x \leq \delta_0(\alpha, B)$. First note that

$$\lfloor \bar{\mu}^T b \rfloor = \lfloor \alpha^T b \rfloor + \beta^T b \tag{15}$$
$$f(\bar{\mu}^T b) = f(\alpha^T b). \tag{16}$$

Then

$$\begin{aligned}
2(\delta(\bar{\mu}, B)^T x - \delta_0(\bar{\mu}, B)) &= |\bar{\mu}|^T (\bar{B}x - \bar{b}) + (1 - 2f(\bar{\mu}^T \bar{b}))(\bar{\mu}^T \bar{B}x - \lfloor \bar{\mu}^T \bar{b} \rfloor) \\
&\quad + f(\bar{\mu}^T \bar{b}) \\
&= |\alpha + \beta|^T (\bar{B}x - \bar{b}) \\
&\quad + (1 - 2f(\alpha^T \bar{b}))(\alpha^T \bar{B}x - \lfloor \alpha^T \bar{b} \rfloor + \beta^T \bar{B}x - \beta^T \bar{b}) \\
&\quad + f(\alpha^T \bar{b}) \\
&= |\alpha|^T (\bar{B}x - \bar{b}) + (1 - 2f(\alpha^T \bar{b}))(\alpha^T \bar{B}x - \lfloor \alpha^T \bar{b} \rfloor) \\
&\quad + f(\alpha^T \bar{b}) + |\beta|^T (\bar{B}x - \bar{b}) \\
&\quad + (1 - 2f(\alpha^T \bar{b}))(\beta^T \bar{B}x - \beta^T \bar{b})
\end{aligned}$$

where the first equality follows from using representation (8) of $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ scaled by 2, the second one follows from $\bar{\mu} = \alpha + \beta$ and (15)–(16) and the last equality follows from the fact when α and β are on the same orthant $|\alpha + \beta| = |\alpha| + |\beta|$.

Then, by using representation (8) of $\delta(\alpha, B)^T x \leq \delta_0(\alpha, B)$ scaled by 2 we obtain

$$\begin{aligned}
2(\delta(\bar{\mu}, B)^T x - \delta_0(\bar{\mu}, B)) &= 2(\delta(\alpha, B)^T x - \delta_0(\alpha, B)) + |\beta|^T (\bar{B}x - \bar{b}) \\
&\quad + (1 - 2f(\alpha^T \bar{b}))\beta^T (\bar{B}x - \bar{b}).
\end{aligned}$$

Finally by noting that $|\beta| = \beta^+ + \beta^-$ and $\beta = \beta^+ - \beta^-$ we obtain

$$\begin{aligned}
2(\delta(\bar{\mu}, B)^T x - \delta_0(\bar{\mu}, B)) &= 2(\delta(\alpha, B)^T x - \delta_0(\alpha, B)) + 2f(\alpha^T \bar{b}){\beta^-}^T (\bar{B}x - \bar{b}) \\
&\quad + (2 - 2f(\alpha^T \bar{b})){\beta^+}^T (\bar{B}x - \bar{b}).
\end{aligned} \tag{17}$$

The result follows from (17) as $(2 - 2f(\alpha^T \bar{b})) \geq 0$, $f(\alpha^T \bar{b}) \geq 0$ and $\bar{B}x - \bar{b} \leq 0$ for all $x \in P(B)$.

□

Now, for any $\sigma \in \{0,1\}^r$ let $\{v^i\}_{i\in\mathcal{V}(\sigma)} \subseteq \mathcal{L}(B,\sigma)$ be a finite integral generating set of $\mathcal{L}(B,\sigma)$ (see [6] page 286). That is, a finite set $\{v^i\}_{i\in\mathcal{V}(\sigma)}$ such that

$$\mathcal{L}(B,\sigma) = \{\mu \in \mathbb{R}^r \,:\, \mu = \sum_{i\in\mathcal{V}(\sigma)} k_i v^i \quad k_i \in \mathbb{Z}_+\}$$

The existence of this set comes from using Theorem 8.1 in [6] page 289 together with an appropriate linear transformation. A detailed proof is presented in [10]. We note that the proof of the existence of this finite generating set is constructive.

For every $i \in \mathcal{V}(\sigma)$ let $m_i = \min\{m \in \mathbb{Z}_+ \setminus \{0\} \,:\, m\,\bar{b}^T v^i \in \mathbb{Z}\}$. For example if $\bar{b}^T v^i = c/d$ with $c \in \mathbb{Z}$ and $d \in \mathbb{Z}_+ \setminus \{0\}$ relatively prime we have that $m_i = d$.

Now, for every $\sigma \in \{0,1\}^r$ we define the following finite subset of $\mathcal{L}(B,\sigma)$.

$$\mathcal{L}^0(B,\sigma) := \{\mu \in \mathcal{L}(B,\sigma) \,:\, \mu = \sum_{i\in\mathcal{V}(\sigma)} r_i v^i \quad r_i \in \{0,\dots,m_i-1\}\}$$

We also define the following finite subset of $\mathcal{L}(B)$.

$$\mathcal{L}^0(B) := \bigcup_{\sigma\in\{0,1\}^r} \mathcal{L}^0(B,\sigma)$$

We now state our main result.

Theorem 1. *For any $B \in B_r^*$ we have that $SC(B)$ is a polyhedron defined by the original inequalities of $P(B)$ and the following finite set of inequalities*

$$\delta(\bar{\mu},B)^T x \le \delta_0(\bar{\mu},B) \quad \forall\, \bar{\mu} \in \mathcal{L}^0(B) \text{ s.t. } \bar{\mu}^T b \notin \mathbb{Z}.$$

Proof. Because of Proposition 5 the only thing that needs to be proved is that, for any $\bar{\mu} \in \mathcal{L}(B)$, $\delta(\bar{\mu},B)^T x \le \delta_0(\bar{\mu},B)$ is dominated by $\delta(\alpha,B)^T x \le \delta_0(\alpha,B)$ for some $\alpha \in \mathcal{L}^0(B)$.

Let $\bar{\mu} \in \mathcal{L}(B)$. Let $\sigma \in \{0,1\}^r$ be such that $\bar{\mu} \in \mathcal{L}(B,\sigma)$ and $\{k_i\}_{i\in\mathcal{V}(\sigma)} \subseteq \mathbb{Z}_+$ be such that $\bar{\mu} = \sum_{i\in\mathcal{V}(\sigma)} k_i v^i$. For all $i \in \mathcal{V}(\sigma)$ $k_i = n_i m_i + r_i$ for some $n_i, r_i \in \mathbb{Z}_+$, $0 \le r_i < m_i$. Thus

$$\sum_{i\in\mathcal{V}(\sigma)} k_i v^i = \sum_{i\in\mathcal{V}(\sigma)} r_i v^i + \sum_{i\in\mathcal{V}(\sigma)} n_i m_i v^i \tag{18}$$

but

$$\bar{b}^T\Big(\sum_{i\in\mathcal{V}(\sigma)} n_i m_i v^i\Big) = \sum_{i\in\mathcal{V}(\sigma)} n_i m_i \bar{b}^T v^i \in \mathbb{Z}. \tag{19}$$

Let

$$\alpha = \sum_{i\in\mathcal{V}(\sigma)} r_i v^i$$

and

$$\beta = \sum_{i\in\mathcal{V}(\sigma)} n_i m_i v^i$$

Because $\bar{\mu}, \alpha, \beta \in \mathcal{L}(B,\sigma)$, (18) and (19), by Lemma 1 we have that $\delta(\bar{\mu},B)^T x \le \delta_0(\bar{\mu},B)$ is dominated by $\delta(\alpha,B)^T x \le \delta_0(\alpha,B)$. The result follows by noting that $\alpha \in \mathcal{L}^0(B,\sigma) \subseteq \mathcal{L}^0(B)$. □

Combining Theorem 1 with Corollary 1 and the fact that B_r^* is a finite set we have

Corollary 2. *SC is a polyhedron.*

Note that by applying Theorem 1 to every $B \in B_r^*$ we not only prove that there *exists* a finite set of inequalities defining SC, but Theorem 1 can actually be used to develop a finite algorithm to obtain SC.

We note that the constructed set of inequalities is not minimal for the description of SC or $SC(B)$. In fact, it can be proved [10] that Theorem 1 still holds if in the definition of $\mathcal{L}^0(B,\sigma)$ we further require the r_i's to be relatively prime. We also note that finite integral generating sets are not needed in order to characterize $SC(B)$. Specifically, we have the following proposition which can be proven in a manner analogous to the proof of Theorem 1.

Proposition 7. *Let $\{w^i\}_{i\in\mathcal{W}(\sigma)} \subseteq \mathbb{R}^r$ be the extreme rays of the conic hull of $\mathcal{L}(B,\sigma)$ and let them be scaled such that they are primitive vectors of the lattice $\mathcal{L}(B)$. Also, for every $i \in \mathcal{W}(\sigma)$ let $m_i = \min\{m \in \mathbb{Z}_+ \setminus \{0\} : m\,\bar{b}^T w^i \in \mathbb{Z}\}$. Define $\tilde{\mathcal{L}}^0(B,\sigma)$ as the set of primitive vectors of the lattice $\mathcal{L}(B)$ in $\{\sum_{i\in\mathcal{W}(\sigma)} r_i w^i : 0 \leq r_i < m_i\, \forall i \in \mathcal{W}(\sigma)\}$. We then have that $\mathcal{L}^0(B,\sigma)$ can be replaced by $\tilde{\mathcal{L}}^0(B,\sigma)$ in Theorem 1 and the result still holds.*

5 Acknowledgment

This research was partially supported by NSF under grants DMI-0100020 and DMI-0121495. The author would like to thank Shabbir Ahmed and George L. Nemhauser for their very thoughtful comments regarding this work. The author would also like to thank the anonymous referee for pointing out Proposition 7 and other valuable comments.

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