A Constructive Characterization of the Split Closure of a Mixed Integer Linear Program. (Extended Version)

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Abstract. In 1990 Cook, Kannan and Schrijver introduced the split closure of a Mixed Integer Linear Program (MILP) and proved it to be a polyhedron. Recently, Andersen, Cornuejols and Li (2002) gave a precise relationship between split cuts and intersection cuts introduced by Balas in 1971. This allowed them to give an alternate proof of the polyhedrality of the split closure. Unfortunately, neither of these proofs is constructive. Also recently, Köppe and Weismantel (2004) introduced a family of cuts based on a mixed integer Farkas' Lemma. These cuts have been related to split cuts by Bertsimas and Weismantel in 2005. By using an algebraic characterization of split cuts by Caprara and Letchford (2003) we are able show that every cut from the family introduced by Köppe and Weismantel is dominated by the split cut to which it is related. Furthermore, by using this relationship and a result from Andersen, Cornuejols and Li (2002) we are able to give a constructive proof of the polyhedrality of the split closure.

Keywords: Mixed-integer linear programming; Lattice bases; Cutting planes; Split Cuts; Split Closure

1 Introduction

In 1990 Cook, Kannan and Schrijver [8] introduced a family of cuts for a Mixed Integer Linear Program (MILP) which they called split cuts. These cuts are a special case of Balas' disjunctive cuts [4] which arise from a particular two

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term disjunction. Split cuts are also related to intersection cuts introduced by Balas in 1971 [3]. A precise correspondence between split cuts and intersection cuts has been established for $0-1$ MILPs by Balas and Perregaard [5] and for general MILPs by Andersen, Cornuejols and Li [1, 2].

The split closure of a MILP is the intersection of all of its split cuts. Cook, Kannan and Schrijver [8] proved that the split closure of a MILP is a polyhedron. Andersen, Cornuejols and Li [1, 2] have given an alternate proof of this fact. Unfortunately, neither of these proofs is constructive in the sense that they do not provide a method for constructing the split closure for a given MILP.

Another family of cutting planes related to split cuts is the one introduced by Köppe and Weismantel in 2004 [10]. This family of cuts is based on mixed integer version of the Farkas' Lemma and they were related to split cuts by Bertsimas and Weismantel in 2005 [6].

By using an algebraic characterization of split cuts introduced by Caprara and Letchford [7] we are able to show that every cut from [10] is dominated by the split cut to which it is related. Furthermore, by using this relationship and a result from [1, 2] we are able to construct a finite set of split cuts that define the split closure, hence providing a constructive proof of its polyhedrality. The key step of this proof is using the characterization from [7] to note that every non-dominated split cut for a particular relaxation of a MILP can be associated to an element in an integer lattice introduced in [10].

To make this paper more self contained we include proofs of most of the external results. All of these proofs are slightly different or more detailed than original ones, but most of them are not different enough to be really considered alternate proofs. The exception to this last statement are the properties of intersection cuts, for which we present an alternate algebraic proof. For a geometric insight into these properties we refer to the original geometric proofs of Andersen, Cornuejols and Li [1, 2]. For this reason, we will use a notation as close as possible as the original notation used in [1, 2].

The rest of the paper is organized as follows. In section 2 we introduce some notation, the algebraic characterization of split cuts from [7] and some results from [1, 2] we will use later. Then, in section 3 we present a simplified characterization of split cuts for a particular relaxation of the MILP, present the alternate proofs of known properties of intersection cuts and show that basic split cuts are nothing more than intersection cuts. Finally, in section 4 we use this simplified characterization show that the cutting planes introduced in [10] are dominated by split cuts and develop the constructive proof of the polyhedrality of the split closure.

2 Split Cuts

We study the feasible region of a Mixed Integer Linear Programming (MILP) problem given by

$$P_I := \{x \in P \subseteq \mathbb{R}^n : x_j \in \mathbb{Z} \quad \forall\, j \in N_I\} \tag{1}$$

where $N = \{1, \dots, n\}$, $N_I \subseteq N$ and P is a rational polyhedron given by

$$P := \{x \in \mathbb{R}^n : Ax \leq b\} \tag{2}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $M = \{1, \dots, m\}$, $r = \text{rank}(A)$ and $a_{i\cdot}$ corresponds to row i of A. We will assume that $P \neq \emptyset$, but we will not assume $r = n$ allowing for P to contain a line. We also allow for P to be not full dimensional.

Now, let

$$\mathcal{B}_r^* := \{B \subseteq M \, : \, |B| = r \text{ and } \{a_{i\cdot}\}_{i \in B} \text{ are linearly independent}\}. \tag{3}$$

Then, for every $B \in \mathcal{B}_r^*$ we define the following relaxation of P

$$P(B) := \{x \in \mathbb{R}^n \, : \, a_{i\cdot}^T x \leq b_i \quad \forall \, i \in B\}. \tag{4}$$

Note that $\mathcal{B}_n^*$ corresponds to the bases of P, so for simplicity we will refer to $B \in \mathcal{B}_r^*$ as a basis even when $r < n$, noting that in this later case, feasible bases will not define extreme points of P. What will be true, even if $r < n$, is that for any basis $B \in \mathcal{B}_r^*$ we can write $P(B)$ as:

$$P(B) = \bar{x}(B) + L(B) + C(B) \tag{5}$$

where $\bar{x}(B)$ is any particular solution of the system $a_{i\cdot}^T x = b_i$, $\forall \, i \in B$, $L(B)$ is the subspace defined by the system $a_{i\cdot}^T x = 0$, $\forall \, i \in B$ and $C(B)$ is the polyhedral cone given by $\{x \in \mathbb{R}^n : a_{i\cdot}^T x \leq 0 \, \forall \, i \in B\}$. Note that $C(B)$ is a simplicial cone, i.e. its extreme rays are linearly independent. We also have that $P(B)$ is the translation of polyhedral cone $L(B) + C(B)$. We will denote this type of polyhedra as *conic polyhedra* and in particular when $L(B) = \emptyset$ we will denote them as *simple conic polyhedra*.

We will study *split disjunctions* $D(\pi, \pi_0)$ of the form $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$ where $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$. We denote the set of points satisfying split disjunction $D(\pi, \pi_0)$ as

$$F_{D(\pi,\pi_0)} := \{x \in \mathbb{R}^n \, : \, \pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1\} \tag{6}$$

and $\text{conv}(P \cap F_{D(\pi,\pi_0)})$ as the *disjunctive hull* defined by P and $D(\pi, \pi_0)$. Similarly for $B \in \mathcal{B}_r^*$ we define the *basic disjunctive hull* defined by B and $D(\pi, \pi_0)$ as $\text{conv}(P(B) \cap F_{D(\pi,\pi_0)})$.

We say that a disjunction $D(\pi, \pi_0)$ is valid for P_I if $P_I \subseteq F_{D(\pi,\pi_0)} \subsetneq \mathbb{R}^n$. We are interested in the following set of disjunctions, which are always valid for P_I.

$$\Pi_0^n(N_I) := \{(\pi, \pi_0) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} \, : \, \pi_j = 0, \, j \notin N_I\} \tag{7}$$

We also define the projection of $\Pi_0^n(N_I)$ onto the π variables as

$$\Pi^n(N_I) := \{\pi \in \mathbb{Z}^n \setminus \{0\} \, : \, \pi_j = 0, \, j \notin N_I\}. \tag{8}$$

With this, the *split closure* of P_I is defined as

$$SC := \bigcap_{(\pi,\pi_0)\in\Pi_0^n(N_I)} \operatorname{conv}(P \cap F_{D(\pi,\pi_0)}). \tag{9}$$

Similarly, for $B \in \mathcal{B}_k^*$ we define the *basic split closure* as

$$SC(B) := \bigcap_{(\pi,\pi_0)\in\Pi_0^n(N_I)} \operatorname{conv}(P(B) \cap F_{D(\pi,\pi_0)}). \tag{10}$$

A *split cut* is an inequality valid for SC and hence valid for P_I. Similarly a *basic split cut* is an inequality valid for $SC(B)$ for some $B \in \mathcal{B}_r^*$. It is known that basic split cuts are exactly the same as intersection cuts (see, for example, [1, 2]).

If $\delta^T x \leq \delta_0$ and $\{{\gamma^l}^T x \leq \gamma_0^l\}_{l=1}^q$ are inequalities valid for SC, we will say that $\delta^T x \leq \delta_0$ is *dominated* by $\{{\gamma^l}^T x \leq \gamma_0^l\}_{l=1}^q$ if and only if

$$\{x \in P \,:\, {\gamma^i}^T x \leq \gamma_0^i \quad \forall i \in \{1,\ldots,q\}\} \subseteq \{x \in P \,:\, \delta^T x \leq \delta_0\}. \tag{11}$$

Similarly, if the inequalities are valid for $SC(B)$ for some $B \in \mathcal{B}_r^*$, we will say that $\delta^T x \leq \delta_0$ is *dominated* by $\{{\gamma^l}^T x \leq \gamma_0^l\}_{l=1}^q$ if and only if

$$\{x \in P(B) \,:\, {\gamma^i}^T x \leq \gamma_0^i \quad \forall i \in \{1,\ldots,q\}\} \subseteq \{x \in P(B) \,:\, \delta^T x \leq \delta_0\}. \tag{12}$$

In particular, we will say that a split cut or basic split cut $\delta^T x \leq \delta_0$ is *non-trivial* if and only if it is not dominated by the trivial inequality $\mathbf{0}^T x \leq 1$. In other words, non-trivial split cuts and non-trivial basic split cuts are cuts that are not already valid for P and $P(B)$ respectively.

Any split cut $\delta^T x \leq \delta_0$ is a valid inequality for $\operatorname{conv}(P \cap F_{D(\pi,\pi_0)})$ for some $(\pi,\pi_0) \in \Pi_0^n(N_I)$, so we will concentrate on characterizing split cuts for a fixed, but arbitrary $(\pi,\pi_0) \in \Pi_0^n(N_I)$. We will refer to these split cuts for a fixed $(\pi,\pi_0) \in \Pi_0^n(N_I)$ as a split cut for $D(\pi,\pi_0)$. Similarly we will also talk about basic split cuts for $D(\pi,\pi_0)$ and B for some basis $B \in \mathcal{B}_r^*$.

For any $(\pi,\pi_0) \in \Pi_0^n(N_I)$ we have that $P \cap F_{D(\pi,\pi_0)}$ is the disjoint union of sets $P_1(\pi,\pi_0)$ and $P_2(\pi,\pi_0)$ given by

$$P_1(\pi,\pi_0) := \{x \in \mathbb{R}^n : Ax \leq b, \quad \pi^T x \leq \pi_0\} \tag{13}$$

$$P_2(\pi,\pi_0) := \{x \in \mathbb{R}^n : Ax \leq b, \quad -\pi^T x \leq -\pi_0 - 1\} \tag{14}$$

either of which could be empty. Note that $P_1(\pi,\pi_0)$ and $P_2(\pi,\pi_0)$ respectively correspond to the *Left* and *Right* polyhedra of [7].

The following characterization is constructed by using Farkas' Lemma and the fact that an inequality is valid for $\operatorname{conv}(P \cap F_{D(\pi,\pi_0)})$ if and only if it is valid for $P_1(\pi,\pi_0)$ and $P_2(\pi,\pi_0)$. This approach was first proposed by Cook, Kannan and Schrijver [8], but was not used to give a characterization of split cuts. This characterization can also be seen as a slight simplification of the reverse polar characterization of Balas [4] for the particular case of split disjunctions. It has been implicitly used for $0-1$ MILPs by Balas and Perregaard [5] and for general MILPs by Andersen, Cornuejols and Li [1, 2], but to the best of our knowledge it was explicitly introduced for the first time by Caprara and Letchford [7].

Proposition 1. *Let* $(\pi, \pi_0) \in \Pi_0^n(N_I)$*. Then for any* $\mu_0^1, \mu_0^2 \in \mathbb{R}_+$ *and* $\mu^1, \mu^2 \in \mathbb{R}_+^m$ *which are solutions solutions to the system*

$$\sum_{i \in M} \mu_i^2 a_{i\cdot} - \sum_{i \in M} \mu_i^1 a_{i\cdot} = \pi \tag{15}$$

$$\sum_{i \in M} \mu_i^2 b_i - \sum_{i \in M} \mu_i^1 b_i - \mu_0^2 = \pi_0 \tag{16}$$

$$\mu_0^1 + \mu_0^2 = 1 \tag{17}$$

we have that $\delta(\mu, \pi, \pi_0)^T x \le \delta_0(\mu, \pi, \pi_0)$ *given by*

$$\delta(\mu, \pi, \pi_0) := \mu_0^1 \pi + \sum_{i \in M} \mu_i^1 a_{i\cdot} = -\mu_0^2 \pi + \sum_{i \in M} \mu_i^2 a_{i\cdot} \tag{18}$$

$$\delta_0(\mu, \pi, \pi_0) := \mu_0^1 \pi_0 + \sum_{i \in M} \mu_i^1 b_i = -\mu_0^2 (\pi_0 + 1) + \sum_{i \in M} \mu_i^2 b_i \tag{19}$$

is a split cut for $D(\pi, \pi_0)$*. Conversely, any non-trivial split cut for* $D(\pi, \pi_0)$ *is dominated by* $\delta(\mu, \pi, \pi_0)^T x \le \delta_0(\mu, \pi, \pi_0)$ *for some* $\mu_0^1, \mu_0^2 \in \mathbb{R}_+$ *and* $\mu^1, \mu^2 \in \mathbb{R}_+^m$ *which are solutions to* (15)-(17) *that also satisfy*

$$\mu_0^2 \in (0, 1). \tag{20}$$

The validity of this characterization can be established from [4], in which a higher dimensional characterization of $\text{conv}(P \cap F_{D(\pi,\pi_0)})$ is introduced. It can also be proved without using this higher dimensional characterization [7].

To rigorously prove the validity of the characterization in Proposition 1 we will use the following version of the Inhomogeneous Farkas' Lemma.

Proposition 2 (Farkas' Lemma). *Let* $\gamma \in \mathbb{R}^n$*,* $\gamma_0 \in \mathbb{R}$ *and* $Q = \{x \in \mathbb{R}^n : Ex \le f\}$ *with* $E \in \mathbb{R}^{m \times n}$ *and* $f \in \mathbb{R}^m$*. Then* $\gamma^T x \le \gamma_0$ *is a valid inequality for* Q *if and only if there exists* $\mu \in \mathbb{R}_+^m$ *such that one of the following conditions hold*

1. $\gamma = E^T \mu$ *and* $\gamma_0 > \mu^T f$.
2. $\gamma = E^T \mu$ *and* $\gamma_0 = \mu^T f$.
3. $0 = E^T \mu$ *and* $0 > \mu^T f$.

Furthermore, $\{x \in Q : \gamma^T x = \gamma_0\} \ne \emptyset$ *if and only if alternative 2 holds and alternative 3 is equivalent to* $Q = \emptyset$.

This version of Farkas' Lemma can easily be proved using linear programming duality. For a discussion of different versions of Farkas' Lemma we refer to [12] page 39. In particular Proposition 2 is a slight specialization of *Farkas Lemma III* in page 41 of [12].

To determine the appropriate alternative of Farkas' Lemma we first need to study the faces induced by split cut $\delta^T x \leq \delta_0$. These faces are the ones induced by $\delta^T x \leq \delta_0$ on $\text{conv}(P \cap F_{D(\pi,\pi_0)})$, $P_1(\pi, \pi_0)$ and $P_1(\pi, \pi_0)$, which are given by

$$F(\delta, \delta_0) := \{x \in \text{conv}(P \cap F_{D(\pi,\pi_0)}) : \delta^T x = \delta_0\} \tag{21}$$

$$F_k(\delta, \delta_0) := \{x \in P_k(\pi, \pi_0) : \delta^T x = \delta_0\} \quad k \in \{1, 2\}. \tag{22}$$

From these definitions we have that $F(\delta, \delta_0) = \text{conv}(F_1(\delta, \delta_0) \cup F_2(\delta, \delta_0))$. Furthermore, we the have that

Lemma 1. *Let $(\pi, \pi_0) \in \Pi_0^n(N_I)$ and $\delta^T x \leq \delta$ be a non-trivial split cut for $D(\pi, \pi_0)$. If $P_k(\pi, \pi_0) \neq \emptyset$ for $k \in \{1, 2\}$ then there exists a split cut $\gamma^T x \leq \gamma_0$ for $D(\pi, \pi_0)$ such that*

1. *$\delta^T x \leq \delta$ is dominated by $\gamma^T x \leq \gamma_0$.*
2. *$F_k(\gamma, \gamma_0) \neq \emptyset$ for $k \in \{1, 2\}$.*

Proof. If both $F_k(\delta, \delta_0)$'s are empty then $F(\delta, \delta_0) = \emptyset$ by changing δ_0 to $\max\{\delta^T x : x \in \text{conv}(P \cap F_{D(\pi,\pi_0)})\} < \delta_0$ we can reduce to the case when only one of the $F_k(\delta, \delta_0)$'s is empty.

For the case where one $F_k(\delta, \delta_0)$ is empty we will assume that $F_1(\delta, \delta_0) \neq \emptyset$ and $F_2(\delta, \delta_0) = \emptyset$ as the other case is analogous.

If $F_2(\delta, \delta_0) = \emptyset$ then $\delta^T x \leq \delta_0$ does not *touch* $P_2(\pi, \pi_0)$. To obtain $\gamma^T x \leq \gamma_0$ we will *tilt* $\delta^T x \leq \delta_0$ until it does *touch* $P_2(\pi, \pi_0)$.

Applying Farkas' Lemma to $P_1(\pi, \pi_0)$ and $P_2(\pi, \pi_0)$ we get alternatives 2 and 1 respectively. By combining them we get that there exist $\mu^1, \mu^2 \in \mathbb{R}^m_+$, $\mu^1_0, \mu^2_0 \in \mathbb{R}_+$ such that

$$\sum_{i \in M} \mu^1_i a_{i\cdot} + \mu^1_0 \pi = \delta = \sum_{i \in M} \mu^2_i a_{i\cdot} - \mu^2_0 \pi \tag{23}$$

$$\sum_{i \in M} \mu^1_i b_i + \mu^1_0 \pi_0 = \delta_0 > \sum_{i \in M} \mu^2_i b_i - \mu^2_0 (\pi_0 + 1). \tag{24}$$

By eliminating δ and δ_0 from (23)-(24) and reordering we get

$$(\mu^1_0 + \mu^2_0)\pi = \sum_{i \in M} \mu^2_i a_{i\cdot} - \sum_{i \in M} \mu^1_i a_{i\cdot} \tag{25}$$

$$(\mu^1_0 + \mu^2_0)\pi_0 + \mu^2_0 > \sum_{i \in M} \mu^2_i b_i - \sum_{i \in M} \mu^1_i b_i. \tag{26}$$

Given that $\delta^T x \leq \delta_0$ is a non-trivial split cut and hence by Farkas' Lemma it is not equivalent to or dominated by a non-negative linear combination of the original constraints we have that $\mu_0^k > 0$ for $k \in \{1,2\}$. Then, if we keep $\mu_0^1 + \mu_0^2$ constant while decreasing μ_0^2 until (26) is complied at equality or μ_0^2 reaches zero we will get $\bar{\mu}_0^1, \bar{\mu}_0^2 \in \mathbb{R}_+$ such that

$$(\bar{\mu}_0^1 + \bar{\mu}_0^2)\pi = \sum_{i\in M} \mu_i^2 a_{i\cdot} - \sum_{i\in M} \mu_i^1 a_{i\cdot}. \tag{27}$$

and either $\bar{\mu}_0^2 = 0$ or

$$(\bar{\mu}_0^1 + \bar{\mu}_0^2)\pi_0 + \bar{\mu}_0^2 = \sum_{i\in M} \mu_i^2 b_i - \sum_{i\in M} \mu_i^1 b_i. \tag{28}$$

Let γ, γ_0 be such that

$$\gamma = \sum_{i\in M} \mu_i^1 a_{i\cdot} + \bar{\mu}_0^1 \pi \tag{29}$$

$$\gamma_0 = \sum_{i\in M} \mu_i^1 b_i + \bar{\mu}_0^1 \pi_0. \tag{30}$$

Then if $\bar{\mu}_0^2 = 0$ we have

$$\gamma = \sum_{i\in M} \mu_i^2 a_{i\cdot} \tag{31}$$

$$\gamma_0 \geq \sum_{i\in M} \mu_i^2 b_i. \tag{32}$$

and if not

$$\gamma = \sum_{i\in M} \mu_i^2 a_{i\cdot} - \bar{\mu}_0^2 \pi \tag{33}$$

$$\gamma_0 = \sum_{i\in M} \mu_i^2 b_i - \bar{\mu}_0^2(\pi_0 + 1). \tag{34}$$

In both cases by using Farkas' lemma we see that $\gamma^T x \leq \gamma_0$ is valid for $P_1(\pi, \pi_0)$ and $P_2(\pi, \pi_0)$ so it is a split cut. Now, as $\bar{\mu}_0^1 > \mu_0^1$, from (23)-(24) and (29)-(30) we have that

$$\{x \in \mathbb{R}^n \,:\, \pi^T x \geq \pi_0,\, \gamma^T x \leq \gamma_0\} \subseteq \{x \in \mathbb{R}^n \,:\, \pi^T x \geq \pi_0,\, \delta^T x \leq \delta_0\}. \tag{35}$$

We also have that $\delta^T x \leq \delta_0$ is valid for $P_1(\pi, \pi_0)$ so

$$\{x \in P \,:\, \pi^T x \leq \pi_0\} \subseteq \{x \in \mathbb{R}^n \,:\, \delta^T x \leq \delta_0\}. \tag{36}$$

By combining (35)-(36) we get

$$\{x \in P \,:\, \gamma^T x \leq \gamma_0\} \subseteq \{x \in \mathbb{R}^n \,:\, \delta^T x \leq \delta_0\} \tag{37}$$

and hence $\{x \in P \,:\, \gamma^T x \leq \gamma_0\} \subseteq \{x \in P \,:\, \delta^T x \leq \delta_0\}$.

As $\delta^T x \leq \delta_0$ is non-trivial and dominated by $\gamma^T x \leq \gamma_0$ we have that $\bar{\mu}_0^2 \neq 0$ and then $F_k(\gamma, \gamma_0) \neq \emptyset$ for $k \in \{1,2\}$.

□

With this we can now prove Proposition 1.

Proof of Proposition 1. Using Farkas' Lemma and (18)-(19) we have that $\delta(\mu,\pi,\pi_0)^T x \le \delta_0(\mu,\pi,\pi_0)$ is valid for $P_k(\pi,\pi_0)$ for $k \in \{1,2\}$ and hence it is valid for $\text{conv}(P \cap F_{D(\pi,\pi_0)})$.

For the converse, lets suppose that $\delta^T x \le \delta_0$ is a non-trivial split cut for $D(\pi,\pi_0)$ and lets distinguish between the cases $P_k(\pi,\pi_0) \ne \emptyset$ for $k \in \{1,2\}$ and $P_k(\pi,\pi_0) = \emptyset$ for exactly one $k \in \{1,2\}$.

If $P_k(\pi,\pi_0) \ne \emptyset$ for $k \in \{1,2\}$ from Lemma 1 $\delta^T x \le \delta_0$ is dominated by $\gamma^T x \le \gamma_0$ where $\gamma^T x \le \gamma_0$ is a valid inequality for $P_k(\pi,\pi_0)$ for $k \in \{1,2\}$ that induces non-empty faces in both these polyhedra. Then by Farkas' Lemma alternative 2 we have that there exist $\mu_0^1, \mu_0^2 \in \mathbb{R}_+$ and $\mu^1, \mu^2 \in \mathbb{R}_+^m$ such that

$$\gamma = \mu_0^1 \pi + \sum_{i \in M} \mu_i^1 a_{i\cdot} = -\mu_0^2 \pi + \sum_{i \in M} \mu_i^2 a_{i\cdot} \qquad = \delta(\mu,\pi,\pi_0) \tag{38}$$

$$\gamma_0 = \mu_0^1 \pi_0 + \sum_{i \in M} \mu_i^1 b_i = -\mu_0^2 (\pi_0 + 1) + \sum_{i \in M} \mu_i^2 b_i = \delta_0(\mu,\pi,\pi_0). \tag{39}$$

As $\delta^T x \le \delta_0$ is a non-trivial split cut for $D(\pi,\pi_0)$ we have that $\gamma^T x \le \gamma_0$ is a non-trivial split cut for $D(\pi,\pi_0)$. Then, by Farkas' Lemma $\gamma^T x \le \gamma_0$ is not a non-negative linear combination of the original constraints and hence $\mu_0^k > 0$ for $k \in \{1,2\}$. Then by scaling (38)-(39) by $\lambda = 1/(\mu_0^L + \mu_0^R)$ and renaming the variables we obtain a solution to

$$\lambda\gamma = \mu_0^1 \pi + \sum_{i \in M} \mu_i^1 a_{i\cdot} = -\mu_0^2 \pi + \sum_{i \in M} \mu_i^2 a_{i\cdot} \qquad = \delta(\mu,\pi,\pi_0) \tag{40}$$

$$\lambda\gamma_0 = \mu_0^1 \pi_0 + \sum_{i \in M} \mu_i^1 b_i = -\mu_0^2 (\pi_0 + 1) + \sum_{i \in M} \mu_i^2 b_i = \delta_0(\mu,\pi,\pi_0) \tag{41}$$

that complies with (17) and (20).

By eliminating $\lambda\gamma$ and $\lambda\gamma_0$ from (40)-(41) and reordering we get that the multipliers also comply with (15)-(16). Finally from (40)-(41) we get that $\gamma^T x \le \gamma_0$ is equivalent to $\delta(\mu,\pi,\pi_0)^T x \le \delta_0(\mu,\pi,\pi_0)$. Then $\delta^T x \le \delta_0$ is dominated by $\delta(\mu,\pi,\pi_0)^T x \le \delta_0(\mu,\pi,\pi_0)$.

For the remaining case we will assume $P_2(\pi,\pi_0) = \emptyset$ as the other case is analogous. In this case $\pi^T x \le \pi_0$ is a split cut for $D(\pi,\pi_0)$ as it is trivially valid for $P_k(\pi,\pi_0)$ for $k \in \{1,2\}$. We may assume that $\pi^T x \le \pi_0$ is not valid for P or else we would have $\text{conv}(P \cap F_{D(\pi,\pi_0)}) = P$ in which case there would be no non-trivial split cuts for $D(\pi,\pi_0)$. On the other hand, we also have that $\text{conv}(P \cap F_{D(\pi,\pi_0)}) = P_1(\pi,\pi_0)$ so $\delta^T x \le \delta_0$ is valid for $P_1(\pi,\pi_0)$. Then $\delta^T x \le \delta_0$ is dominated by $\pi^T x \le \pi_0$.

Now as $P_2(\pi,\pi_0) = \emptyset$ we have alternative 3 of Farkas' Lemma for $P_2(\pi,\pi_0)$ and hence there exists $\mu_0^2 \in \mathbb{R}_+$ and $\mu^2 \in \mathbb{R}_+^m$ such that

$$0 = -\mu_0^2 \pi + \sum_{i \in M} \mu_i^2 a_{i\cdot} \tag{42}$$

$$0 > -\mu_0^2 (\pi_0 + 1) + \sum_{i \in M} \mu_i^2 b_i. \tag{43}$$

As $P \neq \emptyset$ we do not have alternative 3 of Farkas' Lemma for P and hence $\mu_0^2 > 0$. Then dividing (42)-(43) by μ_0^2, reordering terms and renaming μ^2 we get $\mu^2 \in \mathbb{R}_+^m$ such that

$$\pi = \sum_{i \in M} \mu_i^2 a_{i\cdot} \tag{44}$$

$$\pi_0 + 1 > \sum_{i \in M} \mu_i^2 b_i. \tag{45}$$

Now, as $\pi^T x \leq \pi_0$ is not valid for P we have that

$$\sum_{i \in M} \mu_i^2 b_i > \pi_0. \tag{46}$$

Hence there exists $\mu_0^2 \in (0, 1)$ such that

$$\sum_{i \in M} \mu_i^2 b_i - \mu_0^2 = \pi_0. \tag{47}$$

By letting $\mu_0^1 = 1 - \mu_0^2$ and $\mu^1 = \mathbf{0}$ we have that $\pi^T x \leq \pi_0$ is equivalent to $\delta(\mu, \pi, \pi_0)^T x \leq \delta_0(\mu, \pi, \pi_0)$ and the result follows.

□

We will also use the following results from [1, 2].

Proposition 3. *Let* $(\pi, \pi_0) \in \Pi_0^n(N_I)$. *Then*

$$\operatorname{conv}(P \cap F_{D(\pi,\pi_0)}) = \bigcap_{B \in \mathcal{B}_r^*} \operatorname{conv}(P(B) \cap F_{D(\pi,\pi_0)}). \tag{48}$$

Corollary 1. $SC = \bigcap_{B \in \mathcal{B}_r^*} SC(B)$.

We will prove Proposition 3 by refining Proposition 1 as follows.

Proposition 4. *Let* $(\pi, \pi_0) \in \Pi_0^n(N_I)$. *Any non-trivial split cut for* $D(\pi, \pi_0)$ *is dominated by* $\{\delta(\mu(l), \pi, \pi_0)^T x \leq \delta_0(\mu(l), \pi, \pi_0)\}_{l=1}^q$ *for some for some* $q \in \mathbb{Z}_+ \setminus \{0\}$ *and* $\mu(l) = (\mu_0^1(l), \mu_0^2(l), \mu^1(l), \mu^2(l)) \in \mathbb{R}_+^{2m+2}$ *solutions to* (15)-(17),(20) *that further comply with*

$$\mu_i^1(l) \cdot \mu_i^2(l) = 0 \quad \forall i \in M \tag{49}$$

$$\{a_{i\cdot}\}_{\{i\,:\,\exists\, k \in \{0,1\} \text{ such that } \mu_i^k(l) > 0\}} \text{ are linearly independent} \tag{50}$$

$$|\{i \,:\, \exists\, k \in \{0,1\} \text{ such that } \mu_i^k(l) > 0\}| \leq \operatorname{rank}(A). \tag{51}$$

Proof. Let $\delta^T x \leq \delta_0$ be a non-trivial split cut for $D(\pi, \pi_0)$ and let μ be the solution to (15)-(17),(20) from Proposition 1 such that $\delta^T x \leq \delta_0$ is dominated by $\delta(\mu, \pi, \pi_0)^T x \leq \delta_0(\mu, \pi, \pi_0)$.

Now, system (15)-(17) plus the non-negativity constraints is a standard form linear problem over μ which we will denote by (SFP). After possibly removing redundant constraints we have that basic solutions to (SFP) have at most $\text{rank}(A) + 2$ non-zero components. Then, when (20) holds, extra requirements (49)-(51) are equivalent to requiring μ to be a basic solution to (SFP). Then if our particular μ is a basic solution for (SFP) we are done. If not, as μ is feasible for (SFP), we have that μ is a convex combination of basic feasible solution $\{\mu(l)\}_{l=1}^{q}$ for some $q \in \mathbb{Z}_+ \setminus \{0, 1\}$. Let $\alpha \in \mathbb{R}_+^q$ be such that

$$\sum_{l=1}^{q} \alpha_l = 1 \tag{52}$$

$$\mu = \sum_{l=1}^{q} \alpha_l \mu(l). \tag{53}$$

Given that $(\delta(\mu, \pi, \pi_0), \delta_0(\mu, \pi, \pi_0))$ is linear in μ, we have that

$$(\delta(\mu, \pi, \pi_0), \delta_0(\mu, \pi, \pi_0)) = \sum_{l=1}^{q} \alpha_l (\delta(\mu(l), \pi, \pi_0), \delta_0(\mu(l), \pi, \pi_0)). \tag{54}$$

Then, by Farkas' Lemma we have that $\delta^T x \leq \delta_0$ is dominated by $\{\delta(\mu(l), \pi, \pi_0)^T x \leq \delta_0(\mu(l), \pi, \pi_0)\}_{l=1}^{q}$.

□

Proposition 4 shows us that, besides the original constraints of P, the only inequalities needed to describe $\text{conv}(P \cap F_{D(\pi,\pi_0)})$ are split cuts constructed from at most $\text{rank}(A)$ linearly independent constraints, hence Proposition 3 holds. We note that the proof of Proposition 4 is similar to the algebraic proof of Proposition 3 presented in [1, 2]. Furthermore, the fact that any non-trivial split cut for $D(\pi, \pi_0)$ is dominated by $\delta(\mu, \pi, \pi_0)^T x \leq \delta_0(\mu, \pi, \pi_0)$ for some $\mu \in \mathbb{R}_+^{2m+2}$ solution (15)-(17),(20),(49) was directly proved in [7].

Given that $\mathcal{B}_r^*$ is finite, from Corollary 1 we have that proving that SC is a polyhedron reduces to proving that $SC(B)$ is a polyhedron for all $B \in \mathcal{B}_r^*$. This was exactly the approach used in [1, 2] to prove the polyhedrality of SC. In this same spirit we now study the characterization of basic split cuts to give a constructive proof of the polyhedrality of $SC(B)$.

3 Basic Split Cuts

In this section we study basic split cuts for a particular basis $B \in \mathcal{B}_r^*$. We will use Proposition 1 to give a family of inequalities that contain all non-dominated basic split cuts for B and does not explicitly depend on (π, π_0). We will also show that a basic split cut for $D(\pi, \pi_0)$ and basis B coincides with with the intersection cut for $D(\pi, \pi_0)$ and basis B and present algebraic proof of some known properties of intersection cuts.

For a particular basis $B \in \mathcal{B}_r^*$ we denote by $\bar{B}$ the $r \times n$ submatrix of A defined by this basis and by $\bar{b}$ the corresponding r-dimensional righthand side.

By defining $y^- = \max\{-y, 0\}$, $y^+ = \max\{y, 0\}$ and $f(y) = y - \lfloor y \rfloor$ for any $y \in \mathbb{R}$ and assuming that these operations together with $|y|$ are applied component wise for $y \in \mathbb{R}^r$ we have that

Proposition 5. *For $B \in \mathcal{B}_r^*$ and $\bar{\mu} \in \mathbb{R}^r$ define the following inequalities:*

1. $\delta^1(\bar{\mu}, B)^T x \leq \delta_0^1(\bar{\mu}, B)$ *given by*

$$(\bar{\mu}^-)^T(\bar{B}x - \bar{b}) + (1 - f(\bar{\mu}^T\bar{b}))(\bar{\mu}^T\bar{B}x - \lfloor \bar{\mu}^T\bar{b} \rfloor) \leq 0 \tag{55}$$

2. $\delta^2(\bar{\mu}, B)^T x \leq \delta_0^2(\bar{\mu}, B)$ *given by*

$$(\bar{\mu}^+)^T(\bar{B}x - \bar{b}) - f(\bar{\mu}^T\bar{b})(\bar{\mu}^T\bar{B}x - \lfloor \bar{\mu}^T\bar{b} \rfloor) + f(\bar{\mu}^T\bar{b}) \leq 0 \tag{56}$$

3. $\delta^3(\bar{\mu}, B)^T x \leq \delta_0^3(\bar{\mu}, B)$ *given by*

$$\frac{1}{2}\Big(|\bar{\mu}|^T(\bar{B}x - \bar{b}) + (1 - 2f(\bar{\mu}^T\bar{b}))(\bar{\mu}^T\bar{B}x - \lfloor \bar{\mu}^T\bar{b} \rfloor) + f(\bar{\mu}^T\bar{b})\Big) \leq 0 \tag{57}$$

Also, for $\pi \in \Pi^n(N_I)$ define the following linear system over $\bar{\mu} \in \mathbb{R}^r$

$$\bar{B}^T\bar{\mu} = \pi. \tag{58}$$

Then

1. *If $\bar{\mu}$ is a solution to* (58) *then, for all $k \in \{1, 2, 3\}$ we have that $\delta^k(\bar{\mu}, B)^T x \leq \delta_0^k(\bar{\mu}, B)$ is valid for $\operatorname{conv}(P(B) \cap F_{D(\pi, \lfloor \bar{\mu}^T\bar{b} \rfloor)})$. Furthermore, $\delta^1(\bar{\mu}, B) = \delta^2(\bar{\mu}, B) = \delta^3(\bar{\mu}, B)$ and $\delta_0^1(\bar{\mu}, B) = \delta_0^2(\bar{\mu}, B) = \delta_0^3(\bar{\mu}, B)$.*

2. *If $\bar{\mu}$ is the unique solution to* (58) *and $\bar{\mu}^T\bar{b} \notin \mathbb{Z}$ then for any $k \in \{1, 2, 3\}$*

$$\begin{aligned}\operatorname{conv}(P(B) \cap F_{D(\pi, \lfloor \bar{\mu}^T\bar{b} \rfloor)}) &= \{x \in P(B) : \delta^k(\bar{\mu}, B)x \leq \delta_0^k(\bar{\mu}, B)\} \\ &\subsetneq P(B).\end{aligned} \tag{59}$$

3. *If* (58) *is infeasible or the unique solution $\bar{\mu}$ to* (58) *is such that $\bar{\mu}^T\bar{b} \in \mathbb{Z}$ then*

$$\operatorname{conv}(P(B) \cap F_{D(\pi, \lfloor \bar{\mu}^T\bar{b} \rfloor)}) = P(B). \tag{60}$$

4. $\operatorname{conv}(P(B) \cap F_{D(\pi, \pi_0)}) = P(B)$ *for all $\pi_0 \neq \lfloor \bar{\mu}^T\bar{b} \rfloor$.*

Proof. To prove this proposition we will use Proposition 1 for the special case $P = P(B)$ for the given $B \in \mathcal{B}_r^*$. For this case, condition (49) of Proposition 1 for (π, π_0) allow us to combine μ^1 and μ^2 into $\bar{\mu} \in \mathbb{R}^r$. We can then write conditions (15)-(17),(20) of Proposition 1 in variables $\bar{\mu}$ and the original μ_0^1, μ_0^2 as:

$$\bar{B}^T \bar{\mu} = \sum_{i \in B} \bar{\mu}_i a_{i \cdot} = \pi \quad (61)$$

$$\bar{\mu}^T b = \sum_{i \in B} \bar{\mu}_i b_i = \pi_0 + \mu_0^2 \quad (62)$$

$$\mu_0^1 + \mu_0^2 = 1 \quad (63)$$

$$\mu_0^2 \in (0,1) \quad (64)$$

Then, from (61),(62),(64) and Proposition 1 we have that for $P = P(B)$ there can be a non-trivial split cut for $D(\pi, \pi_0)$ only if (58) is feasible and the unique solution $\bar{\mu}$ to (58) is such that $\bar{\mu}^T \bar{b} \notin \mathbb{Z}$ and $\pi_0 = \lfloor \bar{\mu}^T \bar{b} \rfloor$. This proves parts 3 and 4 by noting that, besides the original constraints of $P(B)$, the only necessary inequalities for the description of $\text{conv}(P(B) \cap F_{D(\pi,\pi_0)})$ are non-trivial split cuts.

Now, if $\bar{\mu}$ is a solution to (58), by the identities $\mu_i^1 = \bar{\mu}_i^-$, $\mu_i^2 = \bar{\mu}_i^+$, $\mu_0^2 = f(\bar{\mu}^T \bar{b})$, $\mu_0^1 = 1 - f(\bar{\mu}^T \bar{b})$ and $\pi_0 = \lfloor \bar{\mu}^T \bar{b} \rfloor$ we have that $\delta^k(\bar{\mu}, B) = \delta(\mu, \pi, \pi_0)$ and $\delta_0^k(\bar{\mu}, B) = \delta_0(\mu, \pi, \pi_0)$ for $k \in \{1,2\}$. Hence for $k \in \{1,2\}$ we have that $\delta^k(\bar{\mu}, B)^T x \leq \delta_0^k(\bar{\mu}, B)$ is valid for $\text{conv}(P(B) \cap F_{D(\pi,\pi_0)})$. By noting that (57) is the average of (55) and (56) we have part 1.

Finally, we already know that under the conditions of part 2 the first equality of (59) holds. To prove that the strict containment holds we will show that for any $k \in \{1,2,3\}$, $\delta^k(\bar{\mu}) \bar{x}(B) > \delta_0^k(\bar{\mu})$. First, by multiplying $\bar{B} \bar{x}(B) = \bar{b}$ by $\bar{\mu}^T$ and using $\bar{\mu}^T \bar{b} \notin \mathbb{Z}$ we have that $\bar{\mu}^T \bar{B} \bar{x}(B) - \lfloor \bar{\mu}^T \bar{b} \rfloor > 0$. By multiplying this last strict inequality by $(1 - f(\mu^T \bar{b})) > 0$ and $\bar{B} \bar{x}(B) = \bar{b}$ by $\mu^- \geq 0$ and adding them together we have that $\delta^1(\bar{\mu}) \bar{x}(B) > \delta_0^1(\bar{\mu})$. The result follows from the equivalence of $\delta^k(\bar{\mu}, B)^T x \leq \delta_0^k(\bar{\mu}, B)$ for $k \in \{1,2,3\}$.

□

With this characterization we can prove the following properties of basic split cuts which correspond to known properties of intersection cuts for split disjunctions.

Proposition 6. $\text{conv}(P(B) \cap F_{D(\pi,\pi_0)}) \neq P(B)$ *if and only if any of the following equivalent conditions hold*

1. $\pi^T x(B) \in (\pi_0, \pi_0 + 1)$ *and* π *is in the linear span of* $\{a_{i\cdot}\}_{i \in B}$.
2. $\pi^T x(B) \in (\pi_0, \pi_0 + 1)$ *and* $\pi^T x$ *is constant over* $x \in x(B) + L(B)$.
3. $\pi^T x(B) \in (\pi_0, \pi_0 + 1)$ *and* $\pi^T y = 0$ *for all* $y \in L(B)$.
4. $\pi^T x \in (\pi_0, \pi_0 + 1)$ *for all* $x \in x(B) + L(B)$.

Proof. Direct from linear algebra and Proposition 5 by noting that when (58) has a solution $\bar{\mu}$ then $\pi^T x(B) = \bar{\mu}^T \bar{b}$. □

Propositions 5 and 6 suggest the following definitions

Definition 1. *For any $B \in \mathcal{B}_r^*$ and $\bar{\mu} \in \mathbb{R}^r$ such that $\bar{\mu}$ is the unique solution for* (58) *for some $\pi \in \Pi^n(N_I)$ we define $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ to be the inequality defined by $\delta^k(\bar{\mu}, B)^T x \leq \delta_0^k(\bar{\mu}, B)$ for any $k \in \{1, 2, 3\}$.*

And for any $B \in \mathcal{B}_r^*$ and $(\pi, \pi_0) \in \Pi_0^n(N_I)$ such that (π, π_0) complies with any of conditions 1-4 of Proposition 6 we define $\delta(\pi, \pi_0, B)^T x \leq \delta_0(\pi, \pi_0, B)$ to be $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ for $\bar{\mu}$ unique solution to (58) *for B and π.*

Corollary 2. *For any $B \in \mathcal{B}_r^*$ and $(\pi, \pi_0) \in \Pi_0^n(N_I)$ such that (π, π_0) complies with any of conditions 1-4 of Proposition 6 and $\bar{\mu}$ unique solution to* (58) *we have that*

1. *$\delta(\pi, \pi_0, B)^T x \leq \delta_0(\pi, \pi_0, B)$ is the unique non-trivial basic split cut for B and $D(\pi, \pi_0)$ and $\delta(\pi, \pi_0, B)^T x > \delta_0(\pi, \pi_0, B)$ for all $x \in x(B) + L(B)$.*

2. *If $P_1(\pi, \pi_0, B) \neq \emptyset$ then*

$$F_1 = \{x \in P_1(\pi, \pi_0, B) \,:\, \pi^T x = \pi_0,\, a_{i\cdot}^T x = b_i \,\forall i \in B \text{ s.t. } \bar{\mu}_i < 0\} \tag{65}$$

and if $P_2(\pi, \pi_0, B) \neq \emptyset$ then

$$F_2 = \{x \in P_2(\pi, \pi_0, B) \,:\, \pi^T x = \pi_0 + 1,\, a_{i\cdot}^T x = b_i \,\forall i \in B \text{ s.t. } \bar{\mu}_i > 0\} \tag{66}$$

where $F_k = \{x \in P_k(\pi, \pi_0, B) \,:\, \delta(\pi, \pi_0, B)^T x = \delta_0(\pi, \pi_0, B)\}$

3. *If $P_k(\pi, \pi_0, B) \neq \emptyset$ for $k \in \{1, 2\}$ then*

$$\{a_{i\cdot}\}_{\{i \in B \,:\, \bar{\mu}_i < 0\}} \cup \{\pi\} \text{ are linearly independent} \tag{67}$$
$$\{a_{i\cdot}\}_{\{i \in B \,:\, \bar{\mu}_i > 0\}} \cup \{\pi\} \text{ are linearly independent.} \tag{68}$$

Proof. 1 is direct from Propositions 5 and 6. 2 comes from using characterizations $\delta^k(\bar{\mu})^T x \leq \delta_0^k(\bar{\mu})$ of $\delta(\pi, \pi_0, B)^T x \leq \delta_0(\pi, \pi_0, B)$ for $k \in \{1, 2\}$ and using (58). 3 comes direct from (58), the fact that $\{a_{i\cdot}\}_{i \in B}$ are linearly independent and that if $P_k(\pi, \pi_0, B) \neq \emptyset$ for $k \in \{1, 2\}$ then there exists $i, j \in B$ such that $\bar{\mu}_i < 0$ and $\bar{\mu}_j > 0$. □

We now show that basic split cuts are in fact intersection cuts.

In general, an intersection cut [3] is a cut based on the intersection of the extreme rays of $P(B)$ for some $B \in \mathcal{B}_r^*$ with a convex set $\mathcal{C}$ that contains $\bar{x}(B) + L(B)$ in its interior but does not contain any integer feasible solutions in its interior. For a description of this general intersection cut see [3] or [9] page 203.

In particular, given a valid split disjunction $D(\pi, \pi_0)$ such that $\pi^T x \in (\pi_0, \pi_0 + 1)$ for all $x \in \bar{x}(B) + L(B)$ we have that $\mathcal{C} = \{x \in \mathbb{R}^n : \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$ complies with the requirements of an intersection cut.

We then have the following definition of an intersection cut for a split disjunction from [1, 2].

Definition 2. *Lets chose the extreme rays of $P(B)$ to be $\{r^i\}_{i\in B}$ such that*

$$a_{i\cdot}^T r^k = \begin{cases} -1 & \text{if } k{=}i \\ 0 & o.w. \end{cases} \tag{69}$$

If $D(\pi,\pi_0)$ is such that $\pi_0 < \pi^T x < \pi_0 + 1$ for all $x \in x(B) + L(B)$ then the unique intersection cut for $P(B)$ and $D(\pi,\pi_0)$ is

$$\sum_{i\in B} \beta_i (b_i - a_{i\cdot}^T x) \geq 1 \tag{70}$$

where

$$\beta_i := \begin{cases} -(\pi^T r^i)/\epsilon^1 & \text{if } \pi^T r^i < 0 \\ (\pi^T r^i)/\epsilon^2 & \text{if } \pi^T r^i > 0 \\ 0 & o.w. \end{cases} \tag{71}$$

with $\epsilon^1 := \pi^T \bar{x}(B) - \lfloor \pi^T \bar{x}(B) \rfloor$ and $\epsilon^2 := \lfloor \pi^T \bar{x}(B) \rfloor + 1 - \pi^T \bar{x}(B)$.

Note that the requirements for the existence of an intersection cut for $P(B)$ and $D(\pi,\pi_0)$ are the same as the requirements for the existence of a non-trivial basic split cut for B and $D(\pi,\pi_0)$. We in fact have that

Proposition 7. *Under the requirements for the existence of an intersection cut for $P(B)$ and $D(\pi,\pi_0)$ we have that intersection cut* (70) *is equivalent to basic split cut $\delta(\pi,\pi_0,B)^T x \leq \delta_0(\pi,\pi_0,B)$.*

Proof. Let $\bar{\mu}$ be the unique solution to (58) such that $\bar{\mu}^T \bar{b} \notin \mathbb{Z}$; which exists under the requirements for the existence of an intersection cut for $P(B)$ and $D(\pi,\pi_0)$. Multiplying (58) by r^i and using (69) we get:

$$\pi^T r^i = -\bar{\mu}_i \tag{72}$$

and by using (58) and the fact that $\bar{B}\bar{x}(B) = \bar{b}$ we have that

$$\epsilon^1 = f(\bar{\mu}^T \bar{b}) \text{ and } \epsilon^2 = 1 - f(\bar{\mu}^T \bar{b}). \tag{73}$$

By multiplying (70) by $-\epsilon^1\epsilon^2$,using these identities and reordering we obtain that (70) is equivalent to

$$\sum_{\substack{i\in B\\ \bar{\mu}_i>0}} \bar{\mu}_i (a_{i\cdot}^T x - b_i) - f(\bar{\mu}^T \bar{b}) \sum_{i\in B} \bar{\mu}_i (a_{i\cdot}^T x - b_i) \leq -f(\bar{\mu}^T \bar{b}) + f(\bar{\mu}^T \bar{b})^2 \tag{74}$$

and by using the definition of $\bar{B}$ and reordering again we get that (70) is equivalent to

$$(\bar{\mu}^+)^T (\bar{B}x - \bar{b}) - f(\bar{\mu}^T \bar{b})(\bar{\mu}^T \bar{B} x - \lfloor \bar{\mu}^T \bar{b} \rfloor) + f(\bar{\mu}^T \bar{b}) \leq 0 \tag{75}$$

which is equal to (56) and hence equivalent to $\delta(\pi,\pi_0,B)^T x \leq \delta_0(\pi,\pi_0,B)$. □

4 Mixed Integer Lattices and Polyhedrality of the Split Closure

In this section we show that every non-dominated basic split cut for a particular basis $B \in \mathcal{B}_r^*$ can be associated to an element of the integer lattice used by [10]. Then we construct a finite set of inequalities defining $SC(B)$.

We start by summarizing the results from [10] in Proposition 8. For this we let $\bar{B}_I \in \mathbb{R}^{r\times|N_I|}$ and $\bar{B}_C \in \mathbb{R}^{r\times(n-|N_I|)}$ be the submatrices of $\bar{B}$ corresponding to the integer and the continuous variables of P_I respectively and we use the following definition of an integer lattice.

Definition 3. *Let $\{v^i\}_{i\in\mathcal{V}} \subseteq \mathbb{R}^r$ be a finite set of linear independent vectors. The* integer lattice *generated by $\{v^i\}_{i\in\mathcal{V}}$ is $\mathcal{L} := \{\mu \in \mathbb{R}^r \,:\, \mu = \sum_{i\in\mathcal{V}} k_i v^i \quad k_i \in \mathbb{Z}\}$. The set $\{v^i\}_{i\in\mathcal{V}}$ is called a* basis *of $\mathcal{L}$. (See chapter 6 in [6] or chapter I.7 in [11]).*

Proposition 8. *For every $B \in \mathcal{B}_r^*$*

1. *$\mathcal{L}(B) := \{\bar{\mu} \in \mathbb{R}^r \,:\, \bar{B}_I{}^T\bar{\mu} \in \mathbb{Z}^{|N_I|}, \quad \bar{B}_C{}^T\bar{\mu} = 0\}$ is an integer lattice*
2. *If $\bar{\mu} \in \mathcal{L}(B)$ is such that $\bar{\mu}^T b \notin \mathbb{Z}$ then the inequality defined by*

$$\lceil\bar{\mu}^-\rceil^T(\bar{B}x - \bar{b}) + (1 - f(\bar{\mu}^T\bar{b}))(\bar{\mu}^T\bar{B}x - \lfloor\bar{\mu}^T\bar{b}\rfloor) \leq 0 \tag{76}$$

is valid for $\{x \in P(B) \,:\, x_j \in \mathbb{Z}\,\forall\, j \in N_I\}$. Furthermore this inequality is not satisfied by $\bar{x}(B)$.

Proof. See Proposition 2 and Theorem 3 of [10] for the case $r = n$. The case $r < n$ is analogous. □

Bertsimas and Weismantel [6] related split cuts to inequality (76) by showing that every $\bar{\mu} \in \mathcal{L}(B)$ such that $\bar{\mu}^T b \notin \mathbb{Z}$ induces a valid split disjunction for P_I. We will now see that in fact, the only split disjunctions necessary for the description of $SC(B)$ are the ones induced by elements of $\mathcal{L}(B)$.

We have that $\mathcal{L}(B)$ precisely corresponds to all the $\bar{\mu} \in \mathbb{R}^r$ such that $\bar{\mu}$ is the unique solution to (58) for B and some $\pi \in \Pi^n(N_I)$. Hence, every non-dominated basic split cuts for B is associated to an element in $\mathcal{L}(B)$. Furthermore, from Proposition 5 we get the following characterization of $SC(B)$.

Proposition 9. *For every $B \in \mathcal{B}_r^*$ and $\bar{\mu} \in \mathcal{L}(B)$ we have that $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ is a valid inequality for $SC(B)$. Furthermore we have that*

$$SC(B) = \bigcap_{\substack{\bar{\mu}\in\mathcal{L}(B)\\ \bar{\mu}^T\bar{b}\notin\mathbb{Z}}} \{x \in P(B) \,:\, \delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)\}. \tag{77}$$

We then have that every element of $\mathcal{L}(B)$ that is associated to a cut given by (76) is also associated to a non-dominated basic split cuts for B. Furthermore we have that

Proposition 10. *Let $\bar{\mu} \in \mathcal{L}(B)$ be such that $\bar{\mu}^T b \notin \mathbb{Z}$ then cut (76) for $\bar{\mu}$ is dominated by the basic split cut $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$.*

Proof. The result follows directly from representation (55) of $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ and the facts that $\bar{B}x - \bar{b} \leq 0$ for all $x \in P(B)$ and that $\lceil \bar{\mu}^- \rceil \geq \bar{\mu}^-$. □

We will now construct a finite subset of $\mathcal{L}(B)$ that suffices to characterize $SC(B)$. To do this we will need to study the intersection of $\mathcal{L}(B)$ with each orthant separately. For any $\sigma \in \{0,1\}^r$ let

$$\mathcal{L}(B, \sigma) := \{\mu \in \mathcal{L}(B) \,:\, (-1)^{\sigma_i} \mu_i \geq 0, \quad \forall\, i \in \{1, \ldots, r\}\} \tag{78}$$

be the intersection of $\mathcal{L}(B)$ with the orthant defined by σ. We then have that

$$\mathcal{L}(B) = \bigcup_{\sigma \in \{0,1\}^r} \mathcal{L}(B, \sigma) \tag{79}$$

Now, for each $\sigma \in \{0,1\}^r$, we construct a finite subset of $\mathcal{L}(B, \sigma)$, such that the finite union of these sets suffices to characterize $SC(B)$. To do this we will need the following lemma

Lemma 2. *Let $\sigma \in \{0,1\}^r$ and let $\bar{\mu} \in \mathcal{L}(B, \sigma)$ with $\bar{\mu} = \alpha + \beta$ for $\alpha, \beta \in \mathcal{L}(B, \sigma)$ such that $\beta^T b \in \mathbb{Z}$. Then $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ is dominated by $\delta(\alpha, B)^T x \leq \delta_0(\alpha, B)$.*

Proof. We will use representation (57) of $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ and $\delta(\alpha, B)^T x \leq \delta_0(\alpha, B)$. First note that

$$\lfloor \bar{\mu}^T b \rfloor = \lfloor \alpha^T b \rfloor + \beta^T b \tag{80}$$

$$f(\bar{\mu}^T b) = f(\alpha^T b). \tag{81}$$

Then

$$\begin{aligned}
2(\delta(\bar{\mu}, B)^T x - \delta_0(\bar{\mu}, B)) &= |\bar{\mu}|^T (\bar{B}x - \bar{b}) + (1 - 2f(\bar{\mu}^T \bar{b}))(\bar{\mu}^T \bar{B}x - \lfloor \bar{\mu}^T \bar{b} \rfloor) \\
&\quad + f(\bar{\mu}^T \bar{b}) && (82) \\
&= |\alpha + \beta|^T (\bar{B}x - \bar{b}) \\
&\quad + (1 - 2f(\alpha^T \bar{b}))(\alpha^T \bar{B}x - \lfloor \alpha^T \bar{b} \rfloor + \beta^T \bar{B}x - \beta^T \bar{b}) \\
&\quad + f(\alpha^T \bar{b}) && (83) \\
&= |\alpha|^T (\bar{B}x - \bar{b}) + (1 - 2f(\alpha^T \bar{b}))(\alpha^T \bar{B}x - \lfloor \alpha^T \bar{b} \rfloor) \\
&\quad + f(\alpha^T \bar{b}) + |\beta|^T (\bar{B}x - \bar{b}) \\
&\quad + (1 - 2f(\alpha^T \bar{b}))(\beta^T \bar{B}x - \beta^T \bar{b}) && (84)
\end{aligned}$$

where the first equality follows from using representation (57) of $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ scaled by 2, the second one follows from $\bar{\mu} = \alpha + \beta$ and (80)-(81) and the last equality follows from the fact when α and β are on the same orthant $|\alpha + \beta| = |\alpha| + |\beta|$.

Then, by using representation (57) of $\delta(\alpha, B)^T x \leq \delta_0(\alpha, B)$ scaled by 2 we obtain

$$2(\delta(\bar{\mu}, B)^T x - \delta_0(\bar{\mu}, B)) = 2(\delta(\alpha, B)^T x - \delta_0(\alpha, B)) + |\beta|^T(\bar{B}x - \bar{b}) + (1 - 2f(\alpha^T\bar{b}))\beta^T(\bar{B}x - \bar{b}). \tag{85}$$

Finally by noting that $|\beta| = \beta^+ + \beta^-$ and $\beta = \beta^+ - \beta^-$ we obtain

$$2(\delta(\bar{\mu}, B)^T x - \delta_0(\bar{\mu}, B)) = 2(\delta(\alpha, B)^T x - \delta_0(\alpha, B)) + 2f(\alpha^T\bar{b}){\beta^-}^T(\bar{B}x - \bar{b}) + (2 - 2f(\alpha^T\bar{b})){\beta^+}^T(\bar{B}x - \bar{b}). \tag{86}$$

The result follows from (86) as $(2 - 2f(\alpha^T\bar{b})) \geq 0$, $f(\alpha^T\bar{b}) \geq 0$ and $\bar{B}x - \bar{b} \leq 0$ for all $x \in P(B)$.

□

We also need the following proposition.

Proposition 11. *For any $\sigma \in \{0,1\}^r$ there exists a finite integral generating set for $\mathcal{L}(B, \sigma)$. That is, a finite set $\{v^i\}_{i \in \mathcal{V}(\sigma)} \subseteq \mathcal{L}(B, \sigma)$ such that*

$$\mathcal{L}(B, \sigma) = \{\mu \in \mathbb{R}^r \,:\, \mu = \sum_{i \in \mathcal{V}(\sigma)} k_i v^i \quad k_i \in \mathbb{Z}_+\} \tag{87}$$

Moreover this set can be constructed by a finite algorithm.

Proof. Let $V \subseteq \mathbb{R}^n$ be the subspace of $\mathbb{R}^n$ generated by the columns of $\bar{B}^T$ and let

$$V^0 := \{x \in V \,:\, x_j = 0 \,\forall \notin N_I\} \tag{88}$$

Let $Q \in \mathbb{Z}^{l \times n}$ the matrix such that subspace V^0 is equal to $\{x \in \mathbb{R}^n \,:\, Qx = 0\}$. Then

$$\bar{B}^T \mathcal{L}(B) = \{x \in \mathbb{Z}^n \,:\, Qx = 0\} \tag{89}$$

By Theorem 6.4 in page 207 of [6] we have that $\bar{B}^T \mathcal{L}(B)$ is a lattice whose basis $\{z^i\}_{i=1}^W \subseteq \mathbb{R}^n$ can be constructed using the integral normal form of Q. Let $\{w^i\}_{i=1}^W \subseteq \mathbb{R}^r$ be such that each w^i is the unique solution to

$$\bar{B}^T w^i = z^i \tag{90}$$

Then $\{w^i\}_{i=1}^W \subseteq \mathbb{R}^r$ is a basis of $\mathcal{L}(B)$. Lets define the following linear operator from $\mathbb{R}^W$ to $\mathbb{R}^r$.

$$L(x) := \sum_{i=1}^W x_i w^i \tag{91}$$

Let $\sigma \in \{0,1\}^r$, we that have that $\mathcal{L}(B, \sigma)$ is the image under L of

$$C_\sigma := \{x \in \mathbb{Z}^W \,:\, (-1)^{\sigma_i} \left(\sum_{i=1}^W x_i {w^i}^T e^j \right) \geq 0 \,\forall j \in \{1, \ldots, r\}\} \tag{92}$$

where e^j is now the j-th unit vector in $\mathbb{R}^r$. C_σ is the set of all integer points in a rational polyhedral cone, so by Theorem 8.1 in page 289 of [6] there exists a finite set $\{u^i\}_{i\in\mathcal{U}(\sigma)} \subseteq \mathbb{R}^W$ such that

$$C_\sigma = \{x \in \mathbb{R}^W \,:\, x = \sum_{i\in\mathcal{U}(\sigma)} k_i u^i \quad k_i \in \mathbb{Z}_+\} \tag{93}$$

Then by letting $\mathcal{V}(\sigma) = \mathcal{U}(\sigma)$ and $v^i = L(u^i)$ for all $i \in \mathcal{V}(\sigma)$ we have a finite set $\{v^i\}_{i\in\mathcal{V}(\sigma)}$ such that (87) holds.

The finite algorithm for constructing the finite integral generating set can be extracted from this proof and the proof of Theorem 8.1 of [6] which is also constructive. □

Now, for any $\sigma \in \{0,1\}^r$ let $\{v^i\}_{i\in\mathcal{V}(\sigma)}$ be a finite integral generating set of $\mathcal{L}(B,\sigma)$.

For every $i \in \mathcal{V}(\sigma)$ let $m_i = \min\{m \in \mathbb{Z}_+ \setminus \{0\} \,:\, m\,\bar{b}^T v^i \in \mathbb{Z}\}$. For example if $\bar{b}^T v^i = c/d$ with $c \in \mathbb{Z}$ and $d \in \mathbb{Z}_+ \setminus \{0\}$ relatively prime we have that $m_i = d$.

Now, for every $\sigma \in \{0,1\}^r$ we define the following finite subset of $\mathcal{L}(B,\sigma)$.

$$\mathcal{L}^0(B,\sigma) := \{\mu \in \mathcal{L}(B,\sigma) \,:\, \mu = \sum_{i\in\mathcal{V}(\sigma)} r_i v^i \quad r_i \in \{0,\ldots,m_i-1\}\} \tag{94}$$

We also define the following finite subset of $\mathcal{L}(B)$.

$$\mathcal{L}^0(B) := \bigcup_{\sigma\in\{0,1\}^r} \mathcal{L}^0(B,\sigma) \tag{95}$$

We now state our main result.

Theorem 1. *For any $B \in B_r^*$ we have that $SC(B)$ is a polyhedron defined by the original inequalities of $P(B)$ and the following finite set of inequalities*

$$\delta(\bar{\mu},B)^T x \le \delta_0(\bar{\mu},B) \quad \forall\, \bar{\mu} \in \mathcal{L}^0(B) \text{ s.t. } \bar{\mu}^T b \notin \mathbb{Z}. \tag{96}$$

Proof. Because of Proposition 9 the only thing that needs to be proved is that, for any $\bar{\mu} \in \mathcal{L}(B)$, $\delta(\bar{\mu},B)^T x \le \delta_0(\bar{\mu},B)$ is dominated by $\delta(\alpha,B)^T x \le \delta_0(\alpha,B)$ for some $\alpha \in \mathcal{L}^0(B)$.

Let $\bar{\mu} \in \mathcal{L}(B)$. Let $\sigma \in \{0,1\}^r$ be such that $\bar{\mu} \in \mathcal{L}(B,\sigma)$ and $\{k_i\}_{i\in\mathcal{V}(\sigma)} \subseteq \mathbb{Z}_+$ be such that $\bar{\mu} = \sum_{i\in\mathcal{V}(\sigma)} k_i v^i$. For all $i \in \mathcal{V}(\sigma)$ $k_i = n_i m_i + r_i$ for some $n_i, r_i \in \mathbb{Z}_+$, $0 \le r_i < m_i$. Thus

$$\sum_{i\in\mathcal{V}(\sigma)} k_i v^i = \sum_{i\in\mathcal{V}(\sigma)} r_i v^i + \sum_{i\in\mathcal{V}(\sigma)} n_i m_i v^i \tag{97}$$

but

$$\bar{b}^T(\sum_{i\in\mathcal{V}(\sigma)} n_i m_i v^i) = \sum_{i\in\mathcal{V}(\sigma)} n_i m_i \bar{b}^T v^i \in \mathbb{Z}. \tag{98}$$

Let

$$\alpha = \sum_{i \in \mathcal{V}(\sigma)} r_i v^i \tag{99}$$

and

$$\beta = \sum_{i \in \mathcal{V}(\sigma)} n_i m_i v^i \tag{100}$$

Because $\bar{\mu}, \alpha, \beta \in \mathcal{L}(B, \sigma)$, (97) and (98), by Lemma 2 we have that $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ is dominated by $\delta(\alpha, B)^T x \leq \delta_0(\alpha, B)$. The result follows by noting that $\alpha \in \mathcal{L}^0(B, \sigma) \subseteq \mathcal{L}^0(B)$. □

Combining Theorem 1 with Corollary 1 and the fact that B_r^* is a finite set we have

Corollary 3. *SC is a polyhedron.*

Note that by applying Theorem 1 to every $B \in B_r^*$ we not only prove that there *exists* a finite set of inequalities defining SC, but Theorem 1 can actually be used to develop a finite algorithm to obtain SC.

We note that the constructed set of inequalities is not minimal for the description of SC or $SC(B)$. In fact we have that

Proposition 12. *Let*

$$\bar{\mathcal{L}}^0(B, \sigma) = \{\mu \in \mathcal{L}(B, \sigma) \, : \mu = \sum_{i \in \mathcal{V}(\sigma)} r_i v^i \quad r_i \in \{0, \ldots, m_i - 1\} \text{ and } \{r_i\}_{i \in \mathcal{V}} \text{ are relatively prime } \} \tag{101}$$

and

$$\bar{\mathcal{L}}^0(B) := \bigcup_{\sigma \in \{0,1\}^r} \bar{\mathcal{L}}^0(B, \sigma) \tag{102}$$

then

$$SC(B) = \bigcap_{\substack{\bar{\mu} \in \bar{\mathcal{L}}^0(B) \\ \bar{\mu}^T b \notin \mathbb{Z}}} \{x \in P(B) \, : \, \delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)\} \tag{103}$$

To prove this Proposition we will need the following lemma.

Lemma 3. *Let $\bar{\mu} \in \mathcal{L}(B, \sigma)$ with $\bar{\mu} = k\alpha$ for $\alpha \in \mathcal{L}(B, \sigma)$ and $k \in \mathbb{Z}_+$. Then $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ is dominated by $\delta(\alpha, B)^T x \leq \delta_0(\alpha, B)$.*

Proof. The result true if $\bar{\mu}^T \bar{b} \in \mathbb{Z}$ as then $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ is trivial, so we will assume that $\bar{\mu}^T \bar{b} \notin \mathbb{Z}$.

Let $\pi = \bar{\mu}^T \bar{B}$, $\pi_0 = \bar{\mu}^T \bar{b}$, $\tilde{\pi} = \alpha^T \bar{B}$ and $\tilde{\pi}_0 = \bar{\alpha}^T \bar{b}$. From the proof of Proposition 7 we have that $\delta(\bar{\mu}, B)^T x \leq \delta_0(\bar{\mu}, B)$ is equivalent to the intersection cut for $D(\pi, \pi_0)$ and B and that $\delta(\alpha, B)^T x \leq \delta_0(\alpha, B)$ is equivalent to the intersection cut for $D(\tilde{\pi}, \tilde{\pi}_0)$ and B. Then it suffices to prove that the intersection cut for $D(\pi, \pi_0)$ and B is dominated by intersection cut for $D(\tilde{\pi}, \tilde{\pi}_0)$ and B. By using identifications (72) and (73) we have that the intersection cut for $D(\pi, \pi_0)$ and B is given by

$$\sum_{i \in B} \bar{\beta}_i (b_i - a_{i\cdot}^T x) \geq 1 \tag{104}$$

with

$$\bar{\beta}_i := \begin{cases} \bar{\mu}_i / f(\bar{\mu}^T \bar{b}) & \text{if } \bar{\mu}_i > 0 \\ (-\bar{\mu}_i)/(1 - \bar{\mu}^T \bar{b}) & \text{if } \bar{\mu}_i < 0 \ . \\ 0 & \text{o.w.} \end{cases} \tag{105}$$

Similarly the intersection cut for $D(\tilde{\pi}, \tilde{\pi}_0)$ and B is given by

$$\sum_{i \in B} \tilde{\beta}_i (b_i - a_{i\cdot}^T x) \geq 1 \tag{106}$$

with

$$\tilde{\beta}_i := \begin{cases} \alpha_i / f(\alpha^T \bar{b}) & \text{if } \alpha_i > 0 \\ (-\alpha_i)/(1 - \alpha^T \bar{b}) & \text{if } \alpha_i < 0 \ . \\ 0 & \text{o.w.} \end{cases} \tag{107}$$

By using the identity $\lfloor \bar{\mu}^T \bar{b} \rfloor = k \lfloor \alpha^T \bar{b} \rfloor + \lfloor k f(\alpha^T \bar{b}) \rfloor$ we have that

$$k \lfloor \alpha^T \bar{b} \rfloor \leq \lfloor \bar{\mu}^T \bar{b} \rfloor \leq k \lfloor \alpha^T \bar{b} \rfloor + k - 1 \tag{108}$$

From which we obtain that $f(\bar{\mu}^T \bar{b}) \leq k f(\alpha^T \bar{b})$ and $1 - f(\bar{\mu}^T \bar{b}) \leq k(1 - f(\alpha^T \bar{b}))$. By noting that $\bar{\mu}_i = k \alpha_i$ we have

$$\tilde{\beta}_i \leq \bar{\beta}_i \quad \forall i \in B. \tag{109}$$

Then any $x \in P(B)$ that complies with (106) also complies with (104) and hence (104) is dominated by (106). □

With this we can prove Proposition 12.

Proof of Proposition 12. Direct from Theorem 1, Lemma 3 and noting that if $\mu = \sum_{i \in \mathcal{V}(\sigma)} r_i v^i \in \mathcal{L}^0(B, \sigma)$ is such that $\{r_i\}_{i \in \mathcal{V}(\sigma)}$ are not relatively prime we have that $\mu = k\tilde{\mu}$ for $k \in \mathbb{Z}_+ \setminus \{0, 1\}$ and some $\tilde{\mu} \in \mathcal{L}^0(B, \sigma)$. □

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