# A Constructive Characterization of the Split Closure of a Mixed Integer Linear Program

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October 3, 2006

# Outline

# What is the Split Closure

- Split Cuts:
  - Valid Inequalities "equivalent" to Intersection Cuts, Mixed Integer Gomory Cuts and MIR Cuts.
  - Special case of Balas's Disjunctive Cuts.
- Closure:
  - Obtained by adding all cuts in a class.
  - Class could have infinite number of cuts, so closures are not immediately polyhedrons.
  - Example: Chvátal Closure (Is a polyhedron).

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# History and Motivation

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  - Split Cuts were introduced by [Cook, et. al. 1990].
  - Split Closure is a polyhedron [Cook, et. al. 1990, Andersen, et. al. 2005]. Non-constructive proofs.
  - The Split Closure has recently been studied by [Balas and Saxena, 2005],[Dash et. al. 2005],[Vielma, 2005].
- Motivation of Constructive Characterization:
  - Algorithm to generate Split Closure? (Naive).
  - Helps understand Split Cuts better.
  - For fixed dimension. Is the number of inequalities defining the Split Closure polynomial in the size of the input? (Open even for two inequalities in $\mathbb{R}^2$).

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# Example of a Split Cut



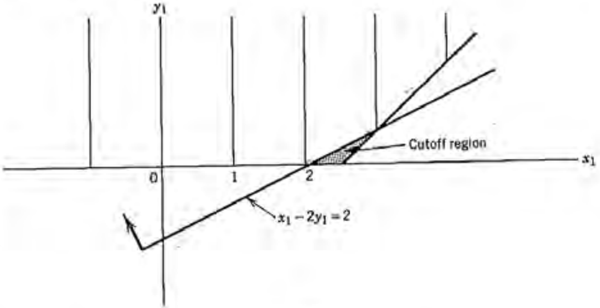


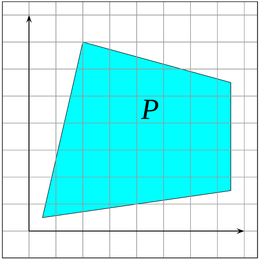

Figure 6.2

**Proposition 6.3.** *Given the two valid inequalities (6.3) for T, it follows that (6.4) is also valid for T.*

# Feasible Set of a (Mixed) Integer Linear Program and Natural Relaxations

Feasible set:

- $P := \{x \in \mathbb{R}^n : a_i^T x \leq b_i \quad \forall\, i \in M\}$
- $P_I := \{x \in P : x_j \in \mathbb{Z} \quad \forall\, j \in N_I\}$ for $N_I \subseteq \{1, \ldots, n\}$



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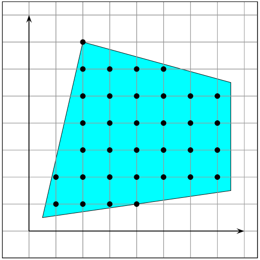

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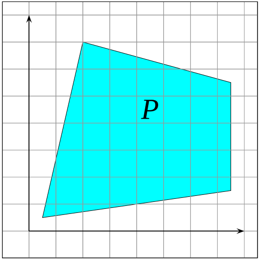
Relaxations:

- $P$, LP Relaxation
- $P(B) := \{x \in \mathbb{R}^n : a_i^T x \leq b_i \quad \forall\, i \in B\}$ for $B \in \mathcal{B} := \{B \subseteq M : |B| = n,\ \{a_i\}_{i \in B}\ l.i.\}$ *Basic or Conic Relaxation*
- $\{x \in P(B) : x_j \in \mathbb{Z} \quad \forall\, j \in N_I\}$ is a relaxation of $P_I$.
- $x(B)$ unique solution to $a_i^T x = b_i \quad \forall\, i \in B$



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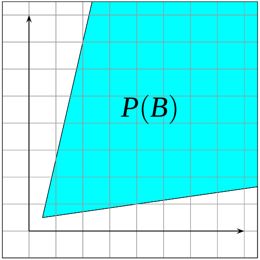

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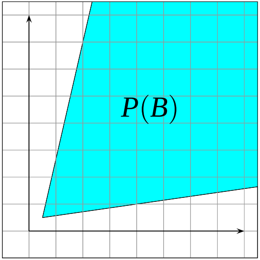

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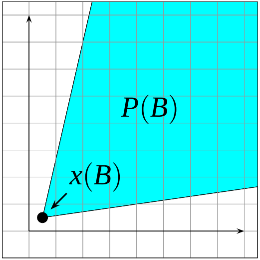

# Split Cuts are Constructed from Valid Split Disjunctions

For $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ divide $\mathbb{R}^n$ into :

- $F^l := \{x \in \mathbb{R}^n \ : \ \pi^T x \leq \pi_0\}$
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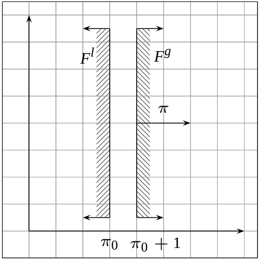

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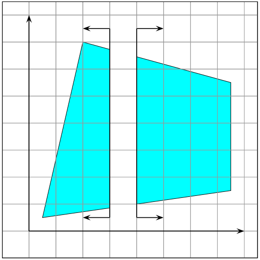

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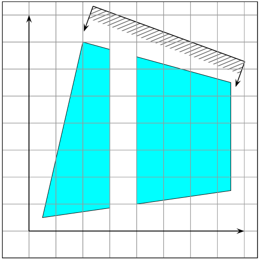
A split cut for $D(\pi, \pi_0)$ and $P$ is an inequality valid for:

- $P^l \cup P^g$
- $\text{conv}(P^l_{(\pi,\pi_0)} \cup P^g_{(\pi,\pi_0)})$



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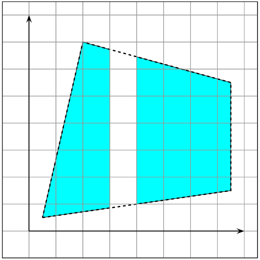

# Valid Split Disjunctions don't Cut Integer Feasible Points

For fixed $N_I$ we are interested in $(\pi, \pi_0)$ such that, for any $P$:

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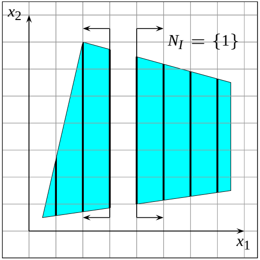

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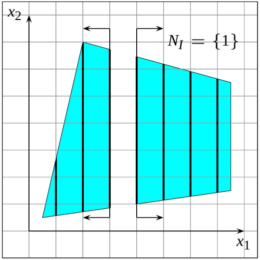
so we study

- $\Pi(N_I) := \{(\pi, \pi_0) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} : \pi_j = 0, j \notin N_I\}$



# The Split Closure is the *Polyhedron* Formed by All Split Cuts

The *split closure* [Cook, et. al. 1990] of $P_I$ is

$$SC := \bigcap_{(\pi,\pi_0)\in\Pi(N_I)} \operatorname{conv}(P^l_{(\pi,\pi_0)} \cup P^g_{(\pi,\pi_0)}).$$

**Theorem**

*[Cook, et. al. 1990] SC is a polyhedron*

# Sufficient to Study Split Cuts for Basic Relaxations

For basis $B \in \mathcal{B}$ let

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**Theorem**

*[Andersen, et. al. 2005]* $SC = \bigcap_{B\in\mathcal{B}} SC(B)$

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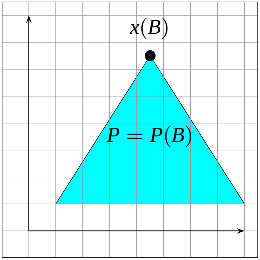
*[Andersen, et. al. 2005] $SC(B)$ is a polyhedron for all $B \in \mathcal{B}$. Hence $SC$ is a polyhedron.*

# Farkas's Lemma Can be Used to Characterize Split Cuts

- Let $P = P(B) = \{x \in \mathbb{R}^n : Bx \leq b\}$, for $B \in \mathbb{Q}^{n \times n}$, $\operatorname{rank}(B) = n$



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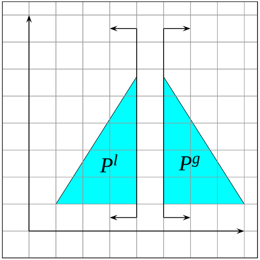

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  - F.L. for$P^l$: $\exists(\mu_0^l, \mu^l) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ s.t.
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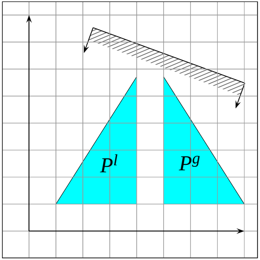

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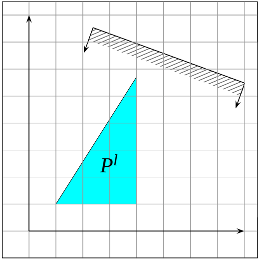

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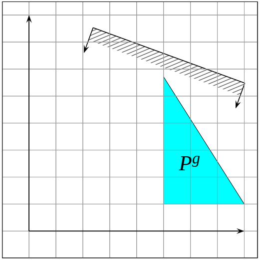

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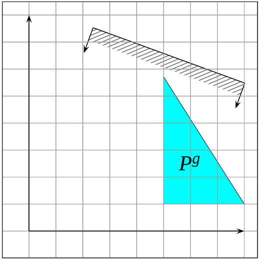

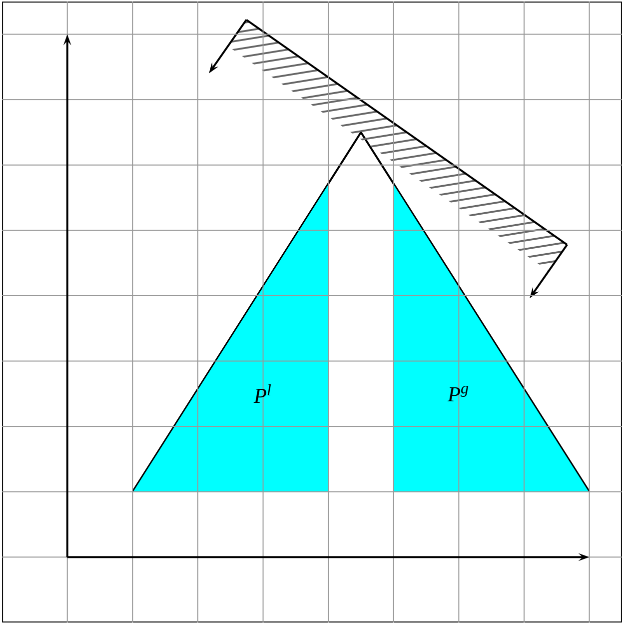
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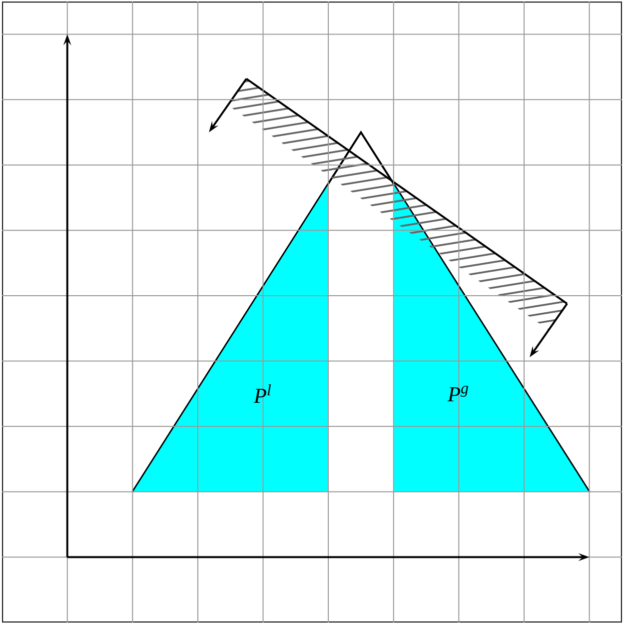
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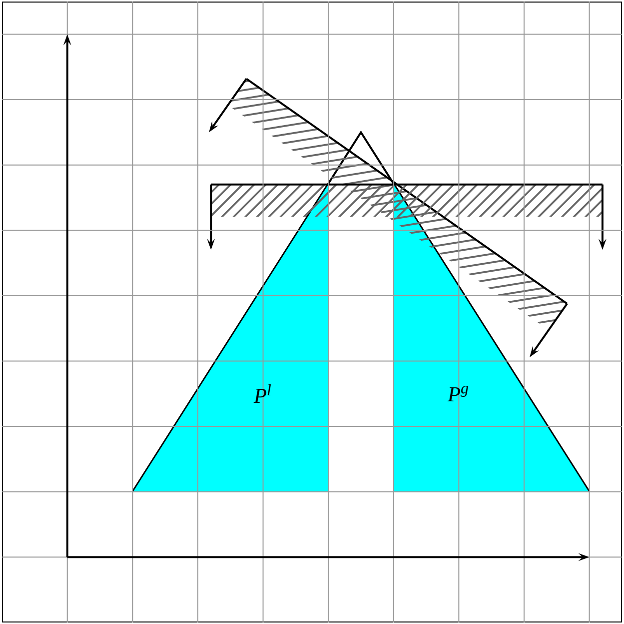
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## Proposition

*[Andersen, et. al. 2005, Balas and Perregaard, 2003, Caprara and Letchford, 2003] All non-dominated valid inequalities for* $\mathrm{conv}(P^l_{(\pi,\pi_0)} \cup P^g_{(\pi,\pi_0)})$ *are of the form* $\delta^T x \leq \delta_0$ *where*

$$\delta = B^T\mu^l + \mu_0^l \pi \ = B^T\mu^g - \mu_0^g \pi$$
$$\delta_0 = b^T\mu^l + \mu_0^l \pi_0 = b^T\mu^g - \mu_0^g(\pi_0 + 1)$$

*for* $\mu_0^l, \mu_0^g \in \mathbb{R}_+$ *and* $\mu^l, \mu^g \in \mathbb{R}^n_+$ *solutions to*

$$B^T\mu^g - B^T\mu^l = \pi$$
$$b^T\mu^g - b^T\mu^l = \pi_0 + \mu_0^g$$

$$\mu_0^l + \mu_0^g = 1, \quad \mu_0^g \in (0,1), \quad \mu_i^l \cdot \mu_i^g = 0$$

$$B^T\mu^g - B^T\mu^l = \pi$$
$$b^T\mu^g - b^T\mu^l = \pi_0 + \mu_0^g$$
$$\mu^l, \mu^g \in \mathbb{R}^n_+, \quad \mu_i^l \cdot \mu_i^g = 0$$
$$\mu_0^g \in (0,1), \quad \pi_0 \in \mathbb{Z}$$

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$$B^T\mu = \pi$$
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$$\mu_i^l = (\mu_i)^- := \max\{-\mu_i, 0\}$$

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$$Bx(B) = b$$

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$$\pi_0 < \pi^T x(B) < \pi_0 + 1$$

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$$\delta = B^T\mu^l + \mu_0^l\pi$$
$$\delta_0 = b^T\mu^l + \mu_0^l\pi_0$$

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## Proposition

$$\mathrm{conv}(P^l_{(\pi,\pi_0)} \cup P^g_{(\pi,\pi_0)}) = \{x \in P \,:\, \delta^T x \leq \delta_0\}$$

*where* $\delta(\mu)^T x \leq \delta_0(\mu)$ *is defined equivalent to*

$$(\mu^-)^T(Bx - b) + (1 - f(\mu^T b))(\mu^T Bx - \lfloor \mu^T b \rfloor) \leq 0$$

*for* $\mu$ *unique solution (if it exists) to*

$$\begin{aligned} B^T\mu &= \pi \qquad & \mu &\in \mathbb{R}^n \\ \mu^T b &\notin \mathbb{Z} \qquad & \pi_0 &= \lfloor \mu^T b \rfloor \end{aligned}$$

($y^- = \max\{-y, 0\}$ , $f(y) = y - \lfloor y \rfloor$ and operations over vectors are component wise)

# What Multipliers Induce Valid Split Disjunctions?

- We have
$\Pi(N_I) := \{(\pi, \pi_0) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} \,:\, \pi_j = 0, j \notin N_I\}$ and

$$\begin{array}{ll} B^T\mu = \pi & \mu \in \mathbb{R}^r \\ \mu^T b \notin \mathbb{Z} & \pi_0 = \lfloor \mu^T b \rfloor \end{array}$$

- Let $B = [B_I B_C]$ for $B_I \in \mathbb{R}^{n \times |N_I|}$ and $B_C \in \mathbb{R}^{n \times (n-|N_I|)}$ corresponding to the integer and continuous variables of $P_I$. Multipliers that induce valid split disjunctions are

$$\mathcal{L}(B) := \{\mu \in \mathbb{R}^n \,:\, {B_I}^T\mu \in \mathbb{Z}^{|N_I|}, \quad {B_C}^T\mu = 0\}$$

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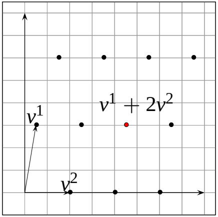
# Valid Split Disjunctions are Related to Integer Lattices

- For $\{v^i\}_{i=1}^r \subseteq \mathbb{R}^n$ l.i. a lattice is

$$\mathcal{L} := \{\mu \in \mathbb{R}^n \;:\; \mu = \sum_{i=1}^r k_i v^i \quad k_i \in \mathbb{Z}\}$$




- $\mathcal{L}(B)$ is a lattice,

$$\lceil \mu^- \rceil^T (Bx - b) + (1 - f(\mu^T b))(\mu^T Bx - \lfloor \mu^T b \rfloor) \leq 0$$

is valid for $P_I$ and cuts $x(B)$.
[Köppe and Weismantel, 2004].

- Every $\mu \in \mathcal{L}(B)$ s.t. $\mu^T b \notin \mathbb{Z}$ induces a valid split disjunction.
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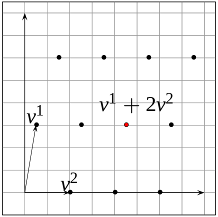


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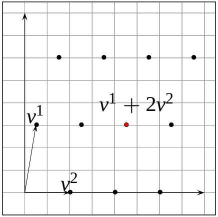


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## Proposition

$$SC(B) = \bigcap_{\substack{\mu \in \mathcal{L}(B) \\ \mu^T b \notin \mathbb{Z}}} \{x \in P(B) \;:\; \delta(\mu)^T x \leq \delta_0(\mu)\}.$$

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## Proposition

*For $\mu \in \mathcal{L}(B)$ s.t $\mu^T b \notin \mathbb{Z}$ split cut*

$$(\mu^-)^T(Bx - b) + (1 - f(\mu^T b))(\mu^T Bx - \lfloor \mu^T b \rfloor) \leq 0$$

*dominates*

$$\lceil \mu^- \rceil^T(Bx - b) + (1 - f(\mu^T b))(\mu^T Bx - \lfloor \mu^T b \rfloor) \leq 0$$

# Studying $\mathcal{L}(B)$ in Each Orthant Decomposes $SC(B)$ to the Intersection of a *Finite* Number of Sets

For $\sigma \in \{0,1\}^n$ let

$$\mathcal{L}(B,\sigma) := \{\mu \in \mathcal{L}(B) \;:\; (-1)^{\sigma_i}\mu_i \geq 0, \quad \forall\, i \in \{1,\ldots,n\}\}$$

so that

$$SC(B) = \bigcap_{\sigma\in\{0,1\}^n} SC(B,\sigma)$$

where

$$SC(B,\sigma) = \bigcap_{\substack{\mu\in\mathcal{L}(B,\sigma)\\ \mu^T b\notin\mathbb{Z}}} \{x \in P(B) \;:\; \delta(\mu)^T x \leq \delta_0(\mu)\}$$

# Studying $\mathcal{L}(B,\sigma)$ Allows Detecting Dominated Cuts

**Lemma**

*Let $\sigma \in \{0,1\}^n$ and let $\mu \in \mathcal{L}(B,\sigma)$ with $\mu = \alpha + \beta$ for $\alpha, \beta \in \mathcal{L}(B,\sigma)$ such that $\beta^T b \in \mathbb{Z}$. Then $\delta(\mu)^T x \leq \delta_0(\mu)$ is dominated by $\delta(\alpha)^T x \leq \delta_0(\alpha)$ in $P(B)$.*

**Proof.**

Uses the fact that for $\alpha, \beta$ in the same orthant $|\alpha_i + \beta_i| = |\alpha_i| + |\beta_i|$ for all $i \in \{1,\ldots,n\}$. $\square$

# A Finite Integral Generating Set (FIGS) of $\mathcal{L}(B,\sigma)$ Induces a Finite Subset of $\mathcal{L}(B,\sigma)$

- Let $\{v^i\}_{i\in\mathcal{V}(\sigma)} \subseteq \mathcal{L}(B,\sigma)$ be a (FIGS), i.e. a finite set such that

$$\mathcal{L}(B,\sigma) = \{\mu \in \mathbb{R}^r \,:\, \mu = \sum_{i\in\mathcal{V}(\sigma)} k_i v^i \quad k_i \in \mathbb{Z}_+\}$$

- We want $\mu^T b \notin \mathbb{Z}$, so for $i \in \mathcal{V}(\sigma)$ let

$$m_i = \min\{m \in \mathbb{Z}_+ \setminus \{0\} \,:\, m\, b^T v^i \in \mathbb{Z}\}$$

and define the following finite subset of $\mathcal{L}(B,\sigma)$.

$$\mathcal{L}^0(B,\sigma) := \{\mu \in \mathcal{L}(B,\sigma) \,:\, \mu = \sum_{i\in\mathcal{V}(\sigma)} r_i v^i,\, r_i \in \{0,\dots,m_i-1\}\}$$

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# Proving the Polyhedrality of $SC(B,\sigma)$ Yields the Polyhedrality of $SC$

## Theorem

*$SC(B,\sigma)$ the polyhedron given by*

$$SC(B,\sigma) = \bigcap_{\substack{\mu \in \mathcal{L}^0(B,\sigma) \\ \mu^T b \notin \mathbb{Z}}} \{x \in P(B) \,:\, \delta(\mu)^T x \le \delta_0(\mu)\}$$

## Corollary

*$SC(B)$ is a polyhedron for all $B \in \mathcal{B}$. $SC$ is a polyhedron.*

## Proof Idea.

- Goal: For $\mu \in \mathcal{L}(B,\sigma)$, $\delta(\mu)^T x \leq \delta_0(\mu)$ is dominated by $\delta(\alpha)^T x \leq \delta_0(\alpha)$ for some $\alpha \in \mathcal{L}^0(B,\sigma)$.
- How:
  - For $\mu \in \mathcal{L}(B,\sigma)$ show that $\mu = \alpha + \beta$ for $\alpha, \beta$ such that:
    - $\alpha \in \mathcal{L}^0(B,\sigma)$, $\beta \in \mathcal{L}(B,\sigma)$
    - $\beta^T b \in \mathbb{Z}$
  - Use Lemma.

□

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## Proof of Theorem.

Let $\{v^i\}_{i \in \mathcal{V}(\sigma)}$ be a FIGS for $\mathcal{L}(B, \sigma)$ and let $\{k_i\}_{i \in \mathcal{V}(\sigma)} \subseteq \mathbb{Z}_+$ be such that

$$\mu = \sum_{i \in \mathcal{V}(\sigma)} k_i v^i.$$

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Let $\{v^i\}_{i\in\mathcal{V}(\sigma)}$ be a FIGS for $\mathcal{L}(B,\sigma)$ and let $\{k_i\}_{i\in\mathcal{V}(\sigma)} \subseteq \mathbb{Z}_+$ be such that

$$\mu = \sum_{i\in\mathcal{V}(\sigma)} k_i v^i.$$

For each $i \in \mathcal{V}(\sigma)$ we have

$$k_i = n_i m_i + r_i$$

for some $n_i, r_i \in \mathbb{Z}_+$, $0 \le r_i < m_i$.Let

$$\alpha = \sum_{i\in\mathcal{V}(\sigma)} r_i v^i \quad \text{and} \quad \beta = \sum_{i\in\mathcal{V}(\sigma)} n_i m_i v^i$$

We have $\alpha \in \mathcal{L}^0(B,\sigma)$ and, as $m_i$ is such that $m_i\, b^T v^i \in \mathbb{Z}$ we have $b^T\beta \in \mathbb{Z}$. □

# Final Remarks

- The proof of the Theorem gives a way of enumerating the inequalities of $SC(B,\sigma)$, $SC(B)$ and $SC$:
  - Not practical for anything buy toy problems.
  - There is redundancy in the enumeration for $SC$ and $SC(B)$.
  - There is also redundancy in the enumeration of $SC(B,\sigma)$. In fact we can reduce $\mathcal{L}^0(B,\sigma)$ to

$$\mathcal{L}^0(B,\sigma) := \{\mu \in \mathcal{L}(B,\sigma) \,:\, \mu = \sum_{i\in\mathcal{V}(\sigma)} r_i v^i,\, r_i \in \{0,\ldots,m_i-1\}$$
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- [Dash et. al. 2005] also give a constructive characterization with similar properties.

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