Split Cuts for Convex Nonlinear Mixed Integer Programming

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joint work with

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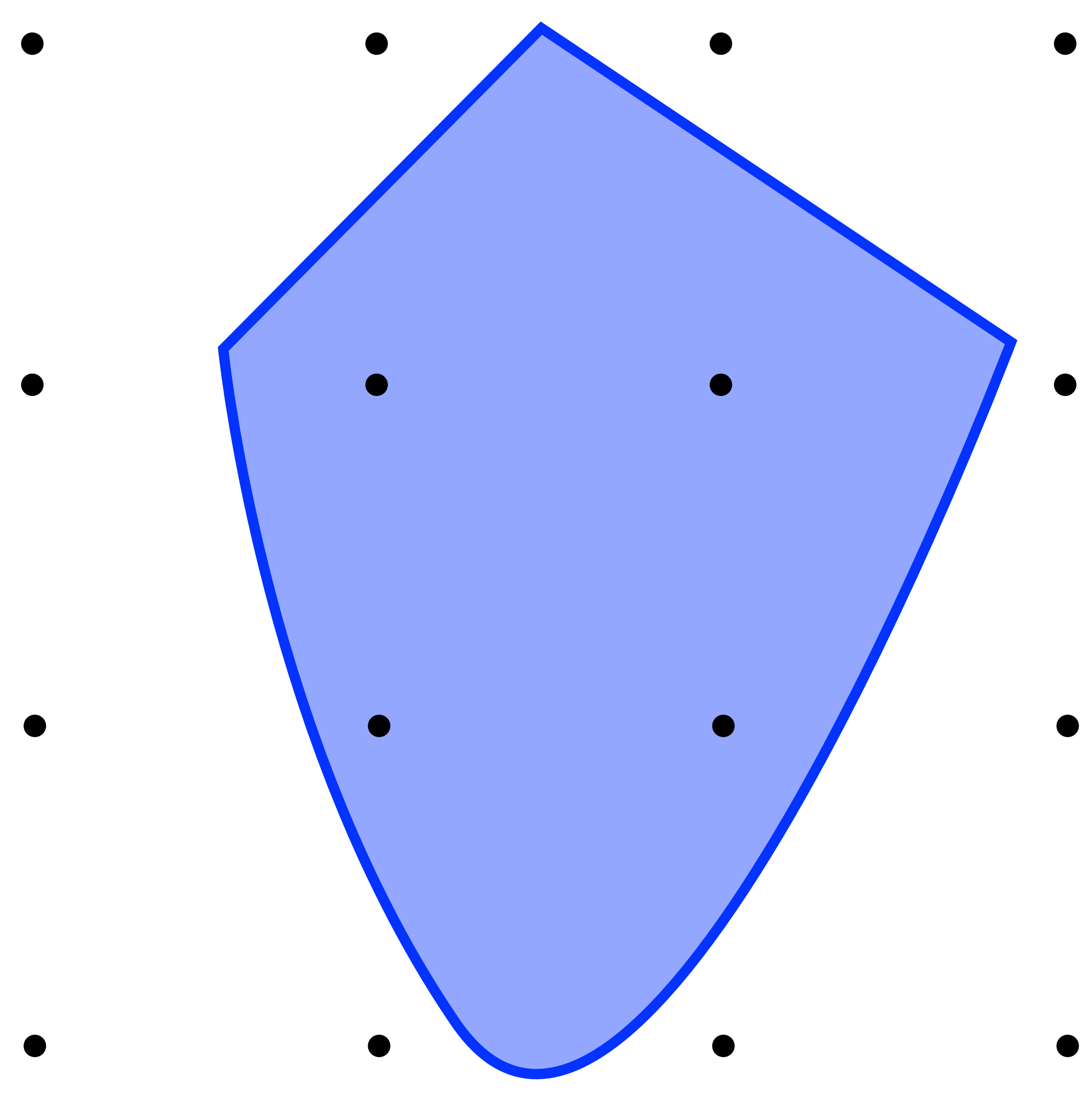
NSF CMMI-1030662 and ONR N000141110724

Integer Programming Workshop,
March 2012 – Valparaiso, Chile

Outline

- Introduction
- Split Cut Formulas
- Split Closure
- Conclusions

Split Disjunctions and Split Cuts



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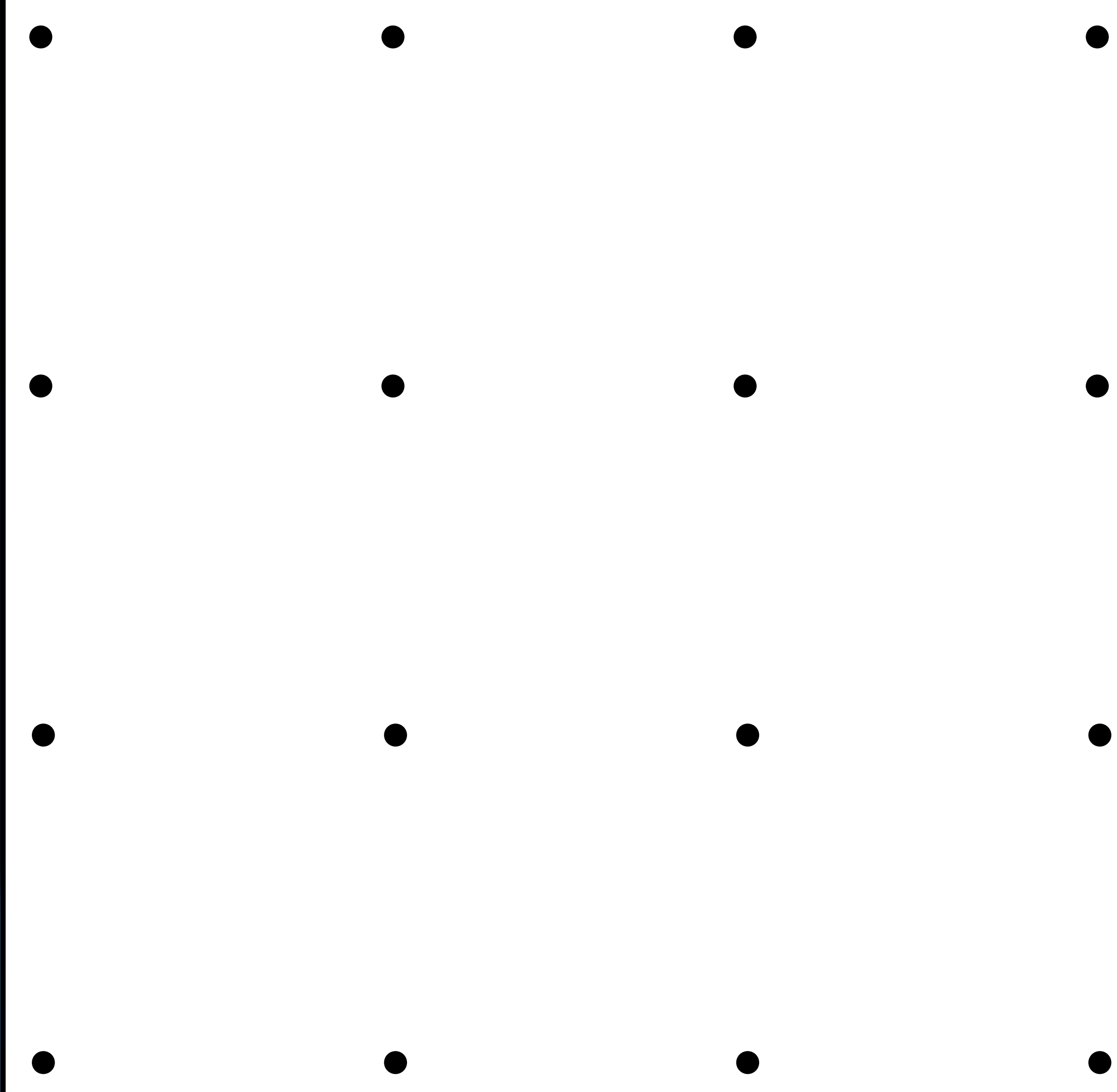
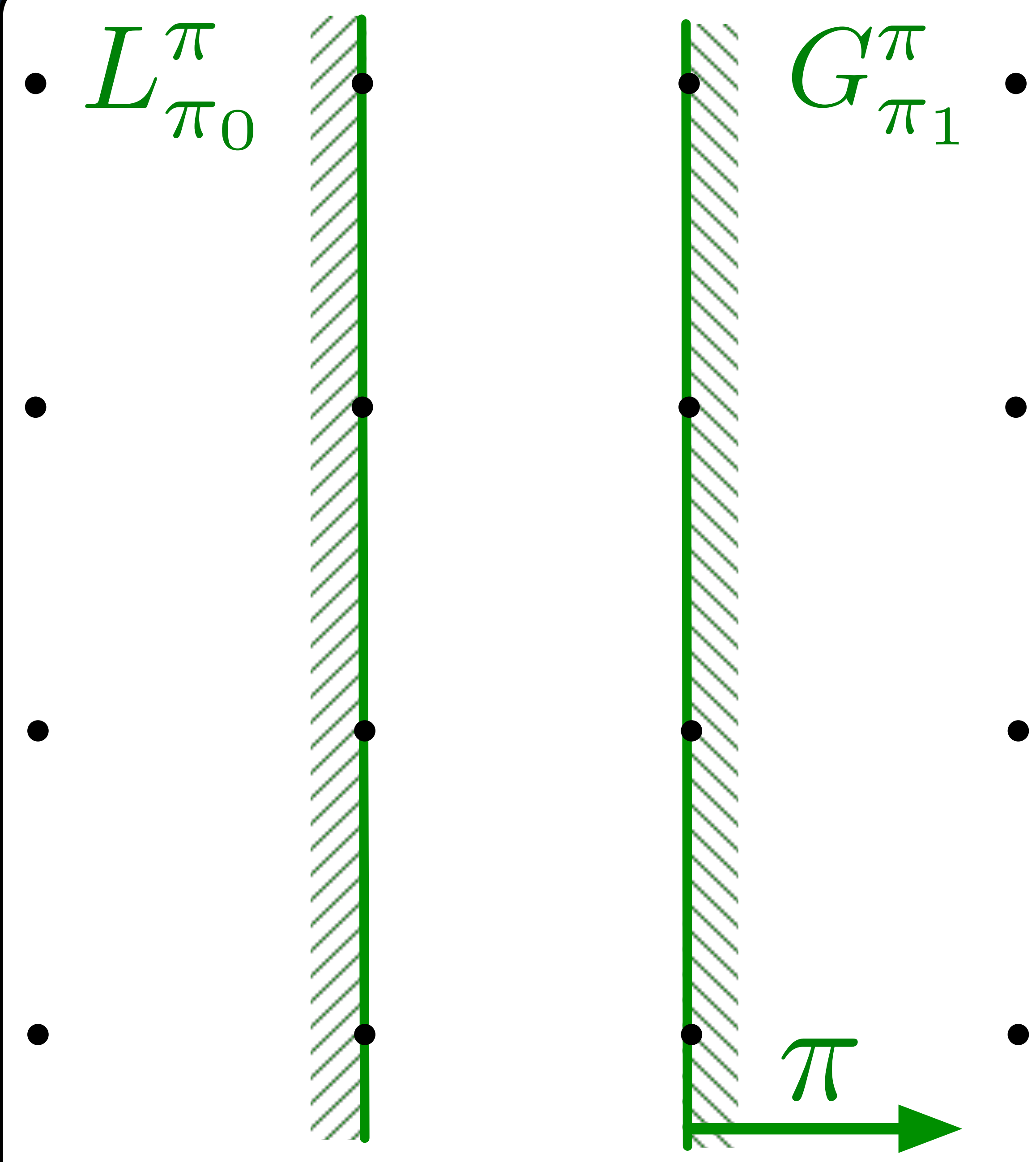


Split Disjunctions and Split Cuts

Split Disjunction

$$L^{\pi}_{\pi_0} = \{x \in \mathbb{R}^n : \langle \pi, x \rangle \le \pi_0\}$$
$$G^{\pi}_{\pi_1} = \{x \in \mathbb{R}^n : \langle \pi, x \rangle \ge \pi_1\}$$



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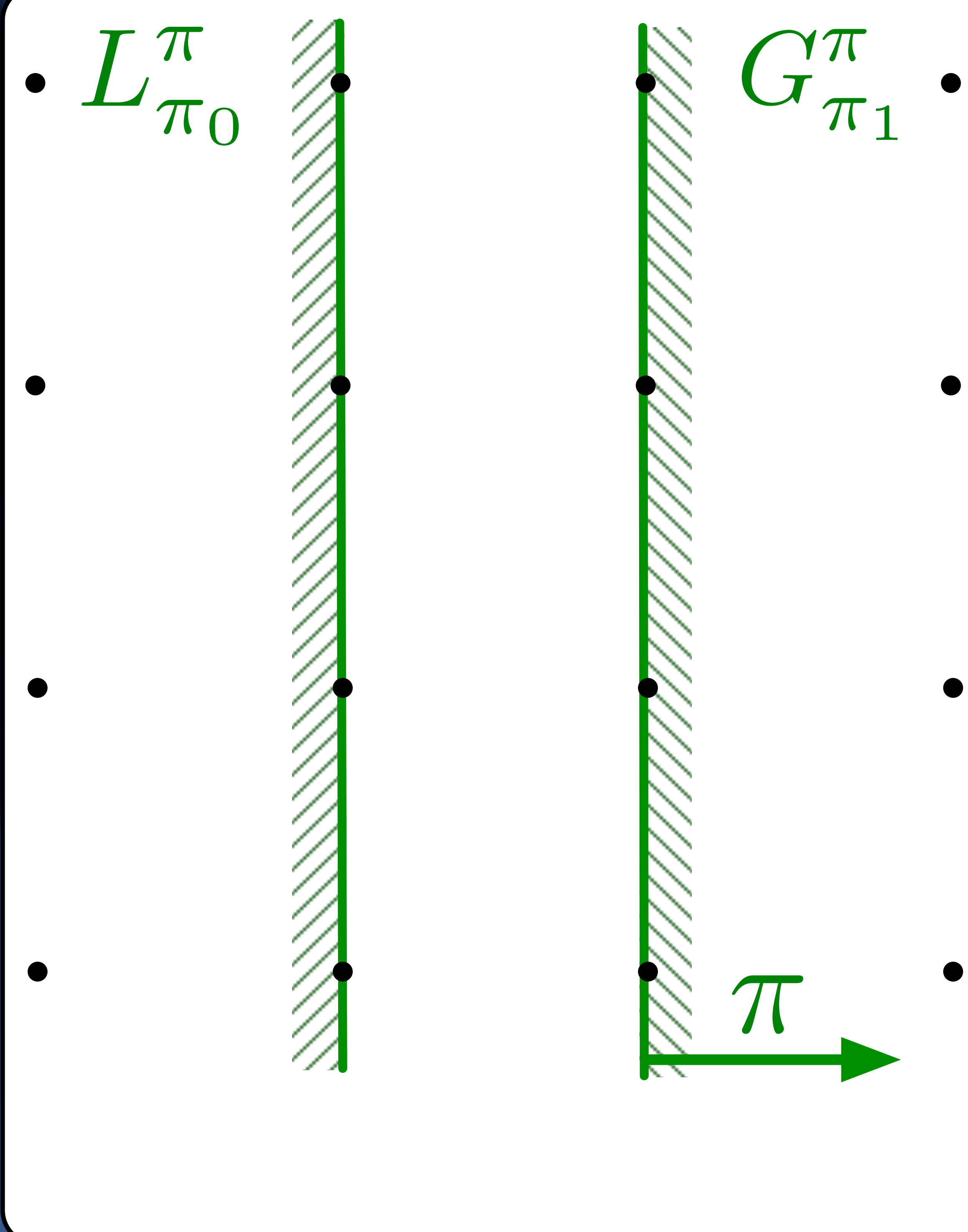
$$\pi \in \mathbb{Z}^n, \quad \pi_1 = \pi_0 + 1 \in \mathbb{Z}$$
$$\Downarrow$$
$$\mathbb{Z}^n \subseteq L^{\pi}_{\pi_0} \cup G^{\pi}_{\pi_1}$$



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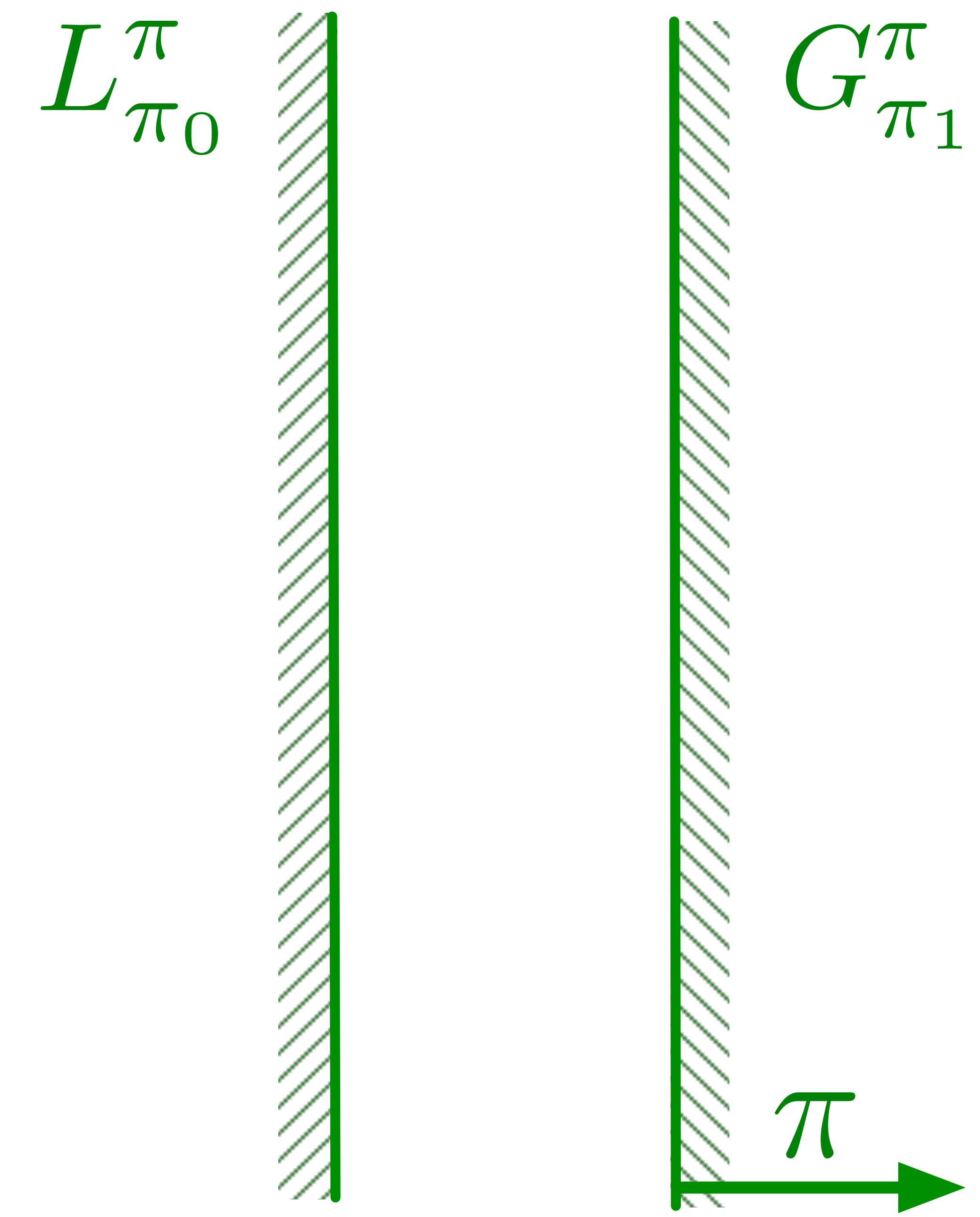

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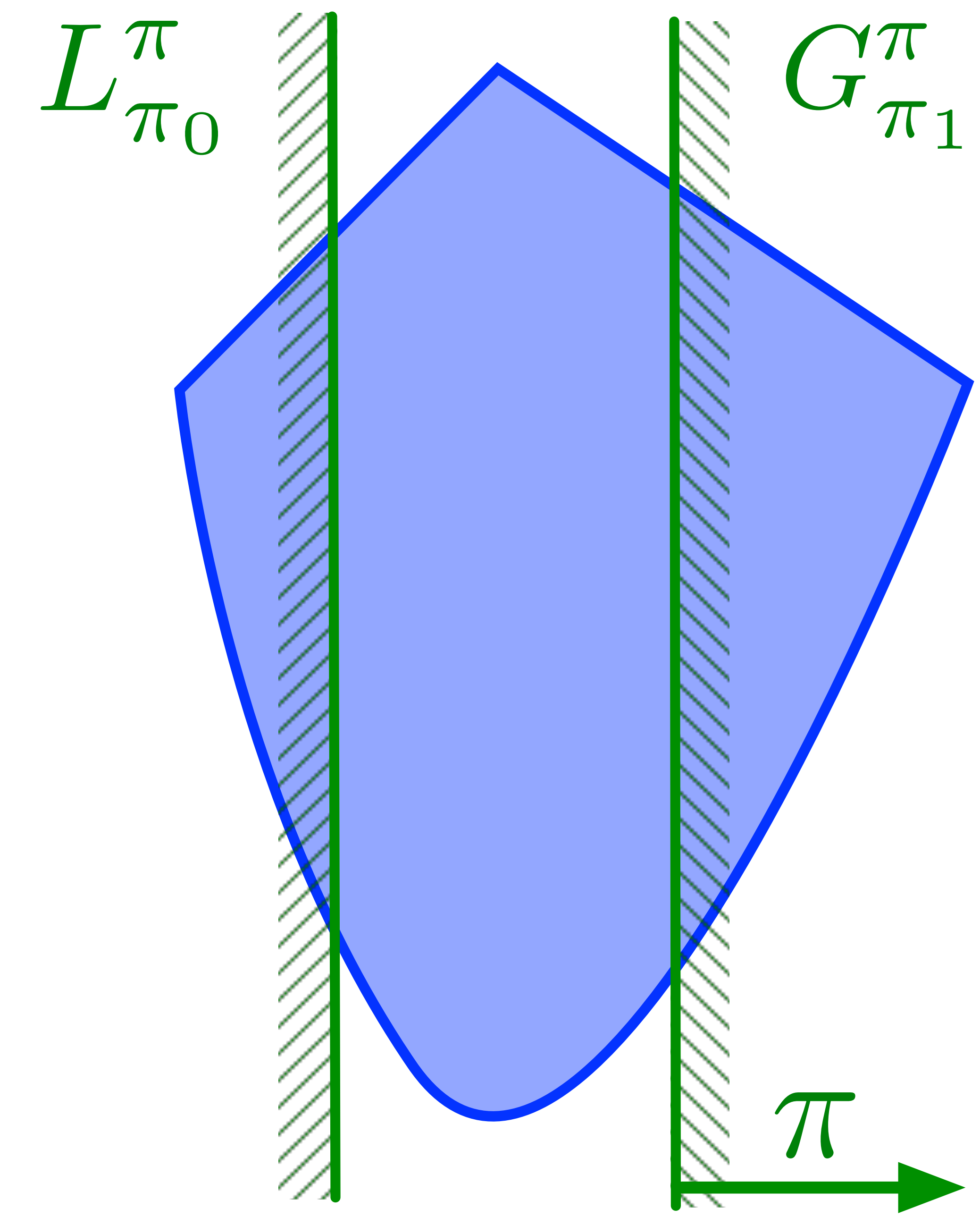

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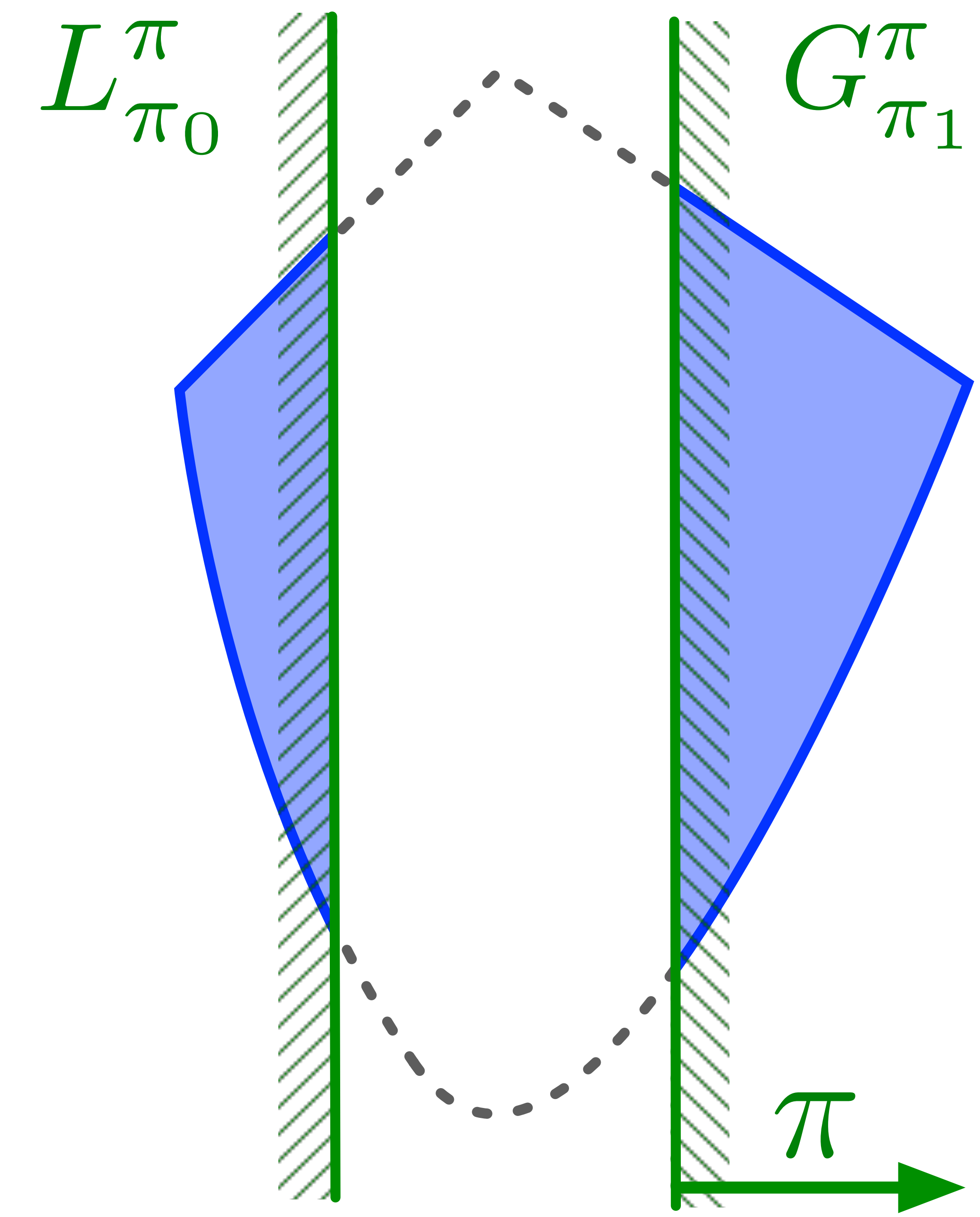

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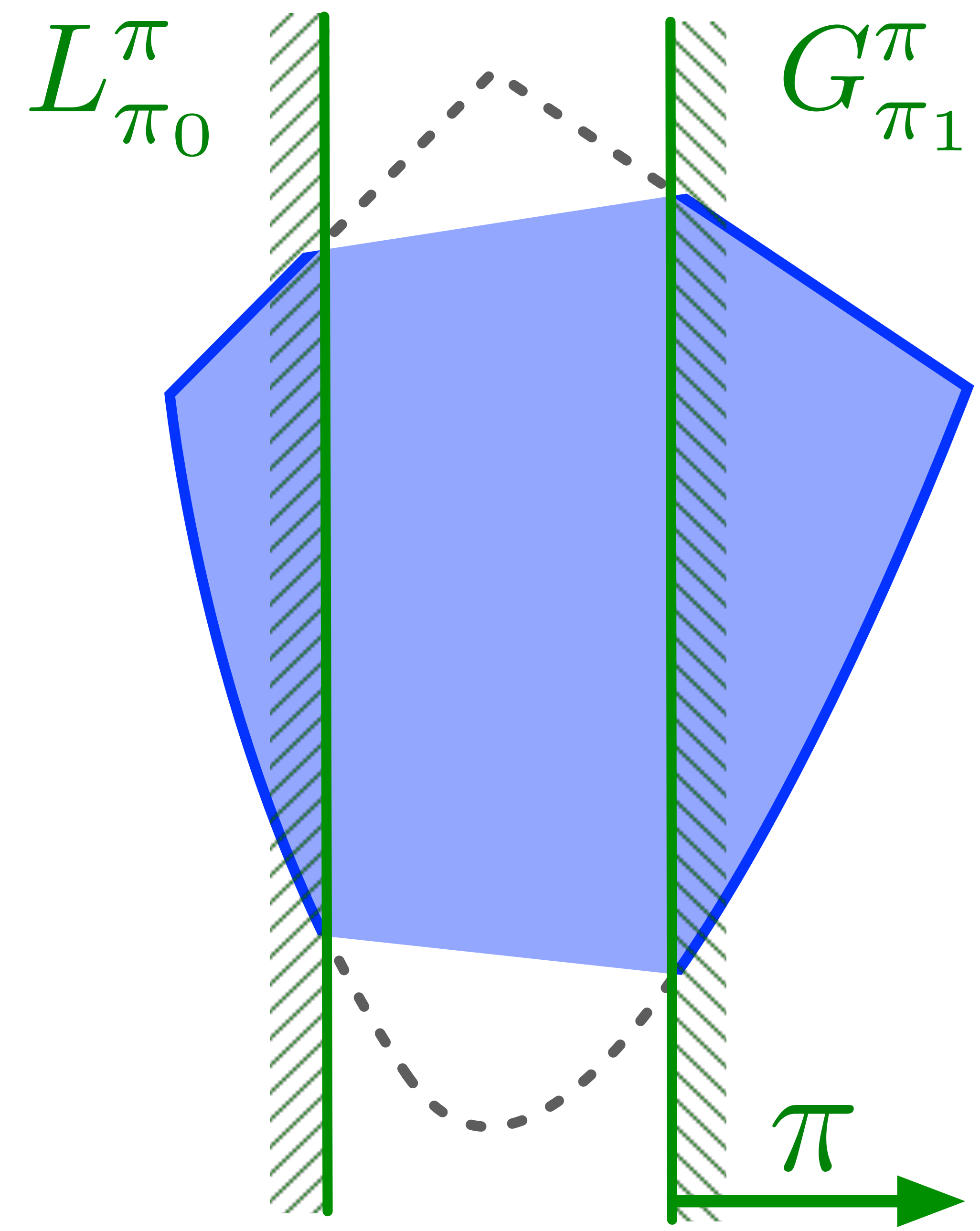

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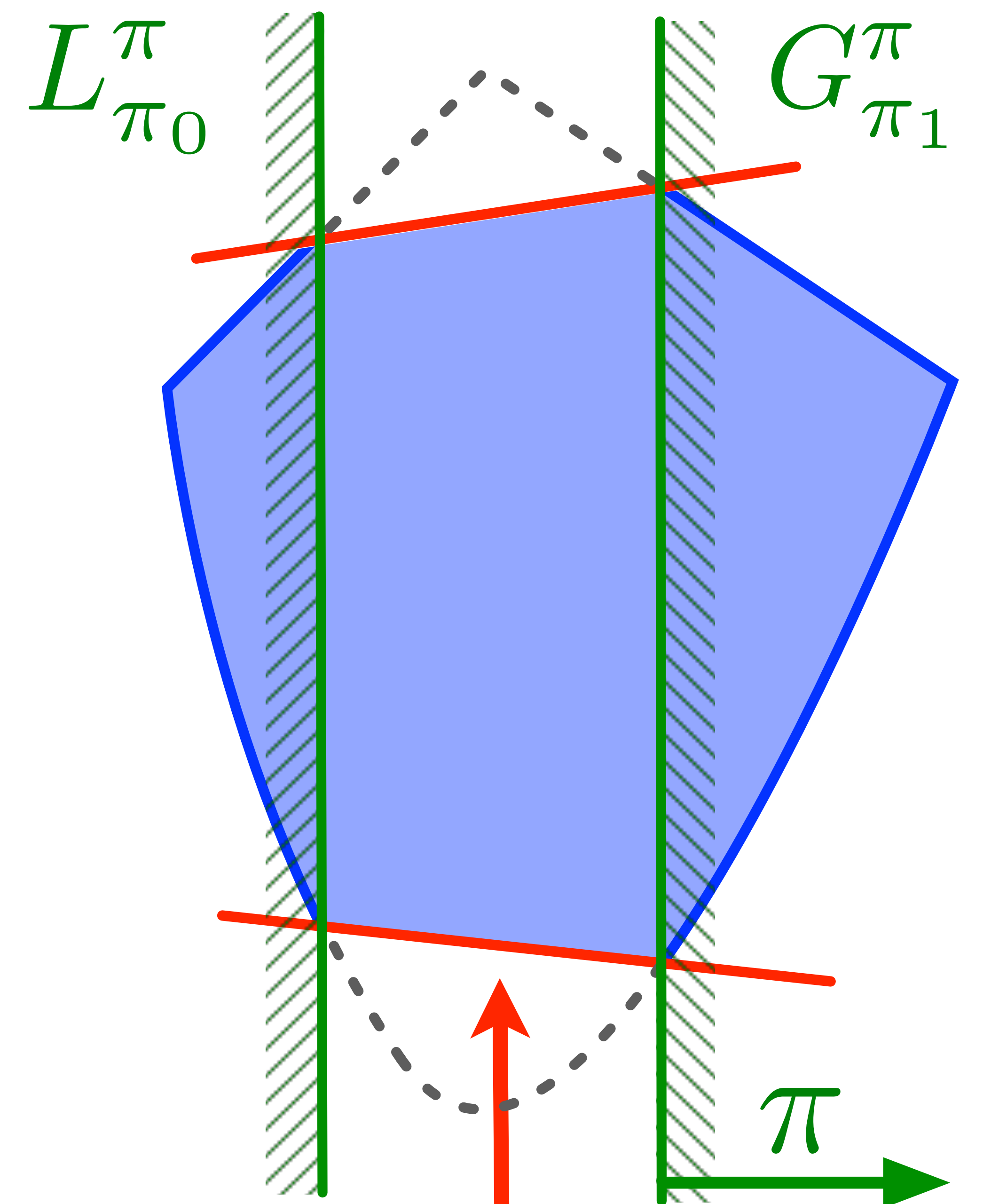
$$= \{x \ : g_i(x) \leq 0, \ i \in I, \ h_j(x) \leq 0, \ j \in J\}$$



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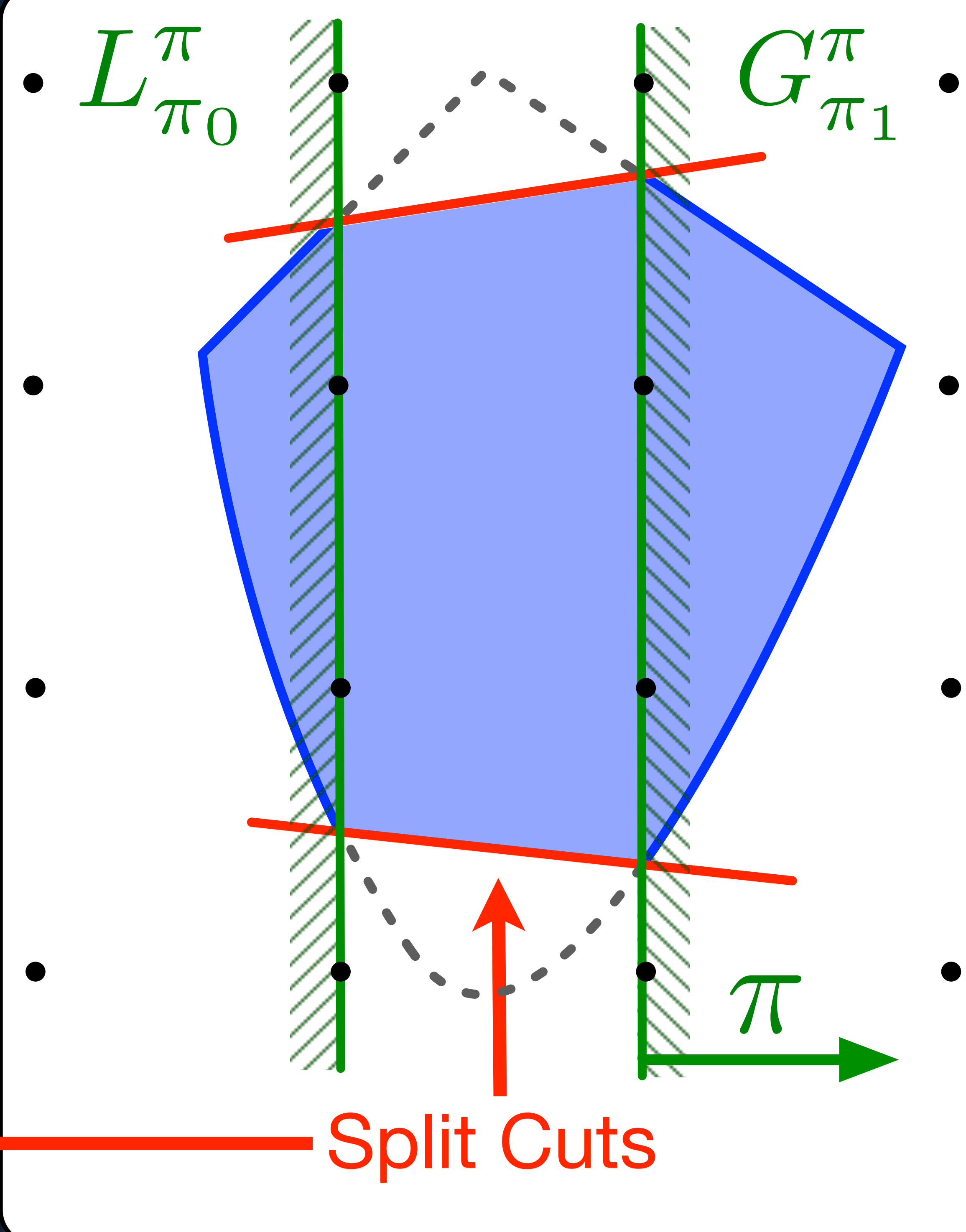
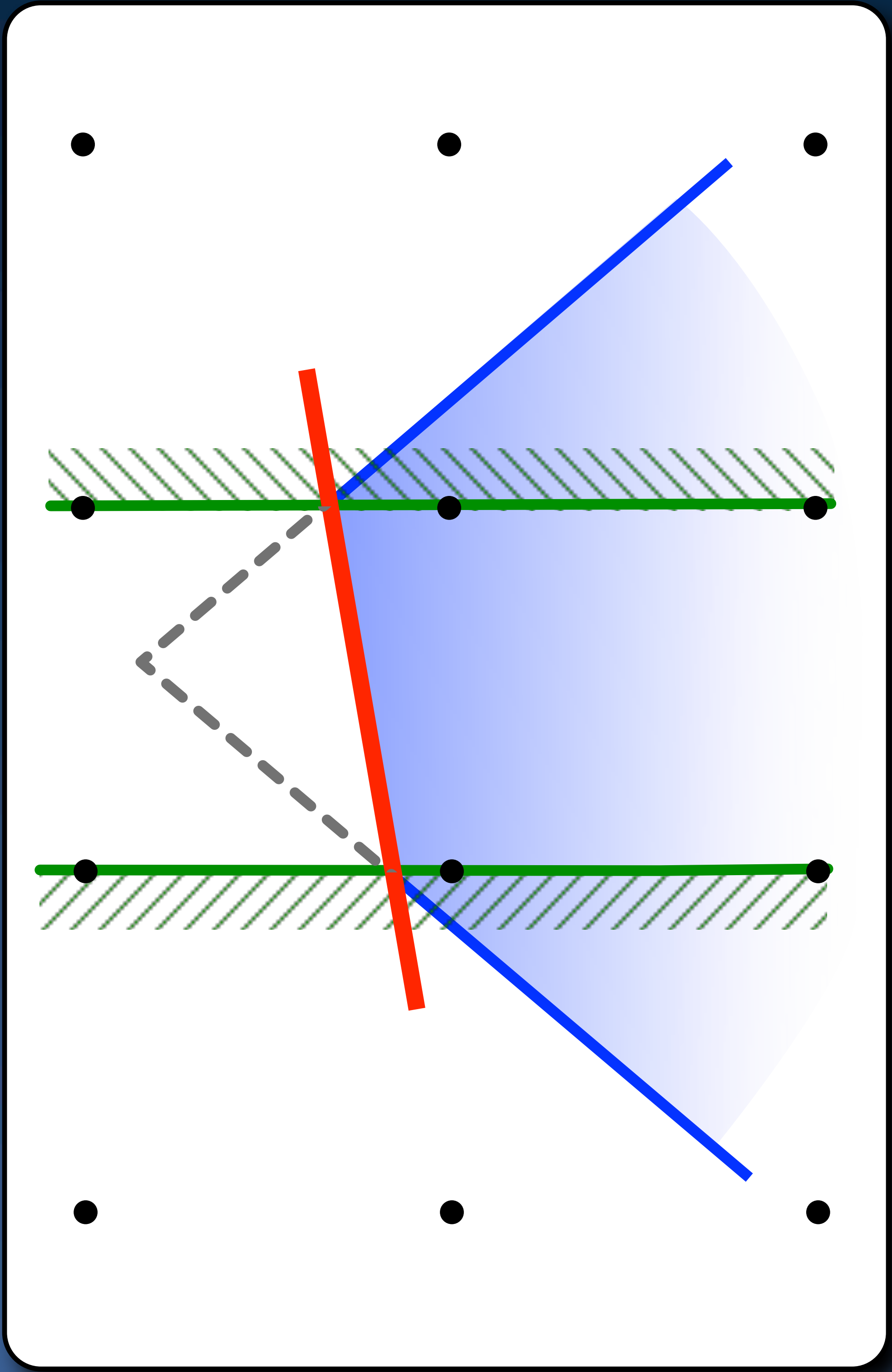


Known Facts for Rational Polyhedra

- Formulas for simplicial cones:
  - MIG (Gomory 1960) and MIR (Nemhauser and Wolsey 1988)
- Split Closure $\bigcap_{(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Z}} C^{\pi}_{\pi_0,\pi_0+1}$:
  - Rational Polyhedron (Cook, Kannan and Shrijver 1990)
  - Constructive Proofs:
    - Dash, Günlük and Lodi 2007; V. 2007



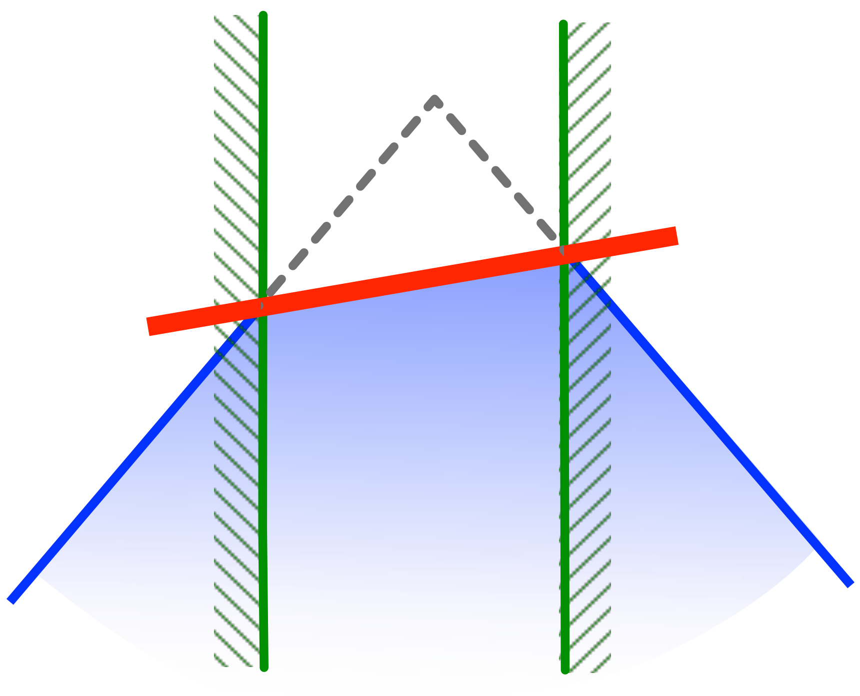
Split Cuts for Simplicial Cones

- Formulas: (MIG: Gomory 1960 and MIR: Nemhauser and Wolsey 1988)

$$C := \{x \in \mathbb{R}^n \;:\; Ax \leq b\},$$

$$\det(A) \neq 0$$

$$\pi_0 < \langle \pi, A^{-1}b \rangle < \pi_1$$



$$C^{\pi}_{\pi_0,\pi_1} := \{x \in \mathbb{R}^n \;:\; Ax \leq b, \quad \langle a, x \rangle \leq b\}$$

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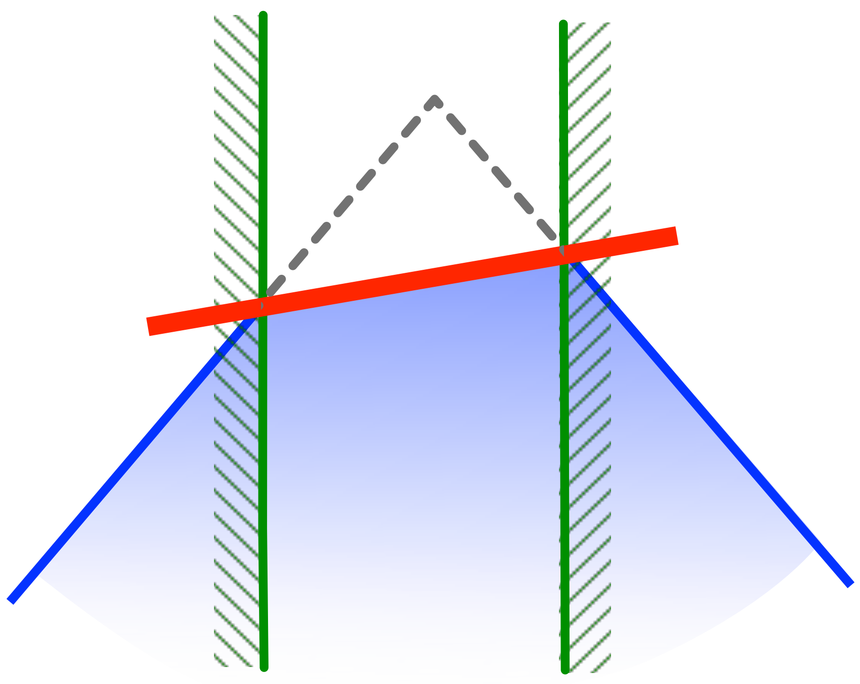


$$C^{\pi}_{\pi_0,\pi_1} := \{x \in \mathbb{R}^n \ : \quad Ax \leq b, \quad \langle a, x \rangle \leq b\}$$

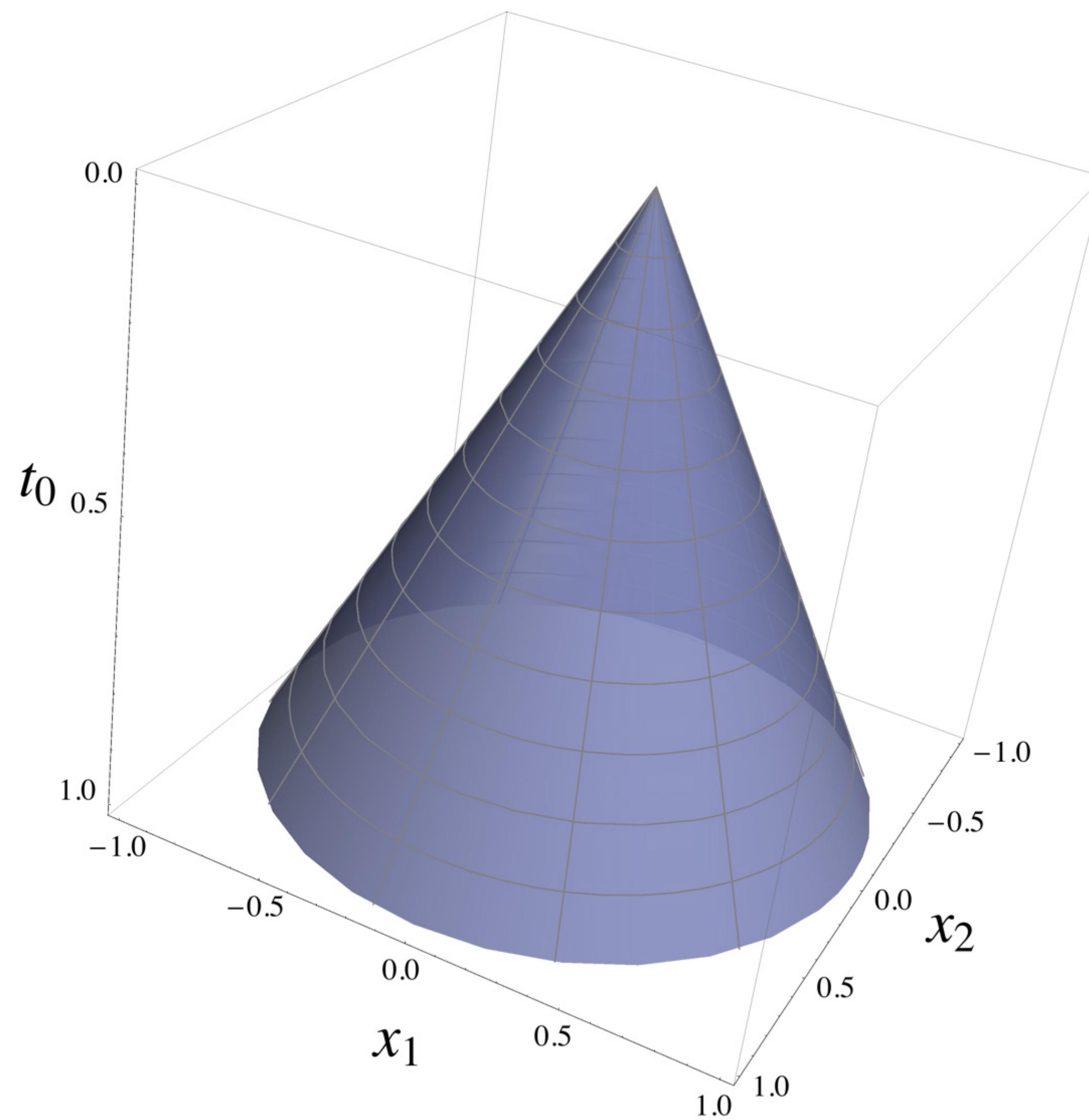
(e.g. V. 2007)

$$a := \left(2\frac{\pi_1 - \langle A^{-1}\pi, b\rangle}{\pi_1 - \pi_0} - 1\right)\pi + A^T \left|A^{-1}\pi\right|, \quad b := \left(2\frac{\pi_1 - \langle A^{-1}\pi, b\rangle}{\pi_1 - \pi_0} - 1\right)(\pi_0 + \pi_1) + \left|A^{-1}\pi\right| b + \pi_0$$

Split Cuts for Quadratic Cones

- Formulas: (Modaresi, Kılınç, V. 2011)

$$C := \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|A(x - c)\|_2 \leq t_0\}$$



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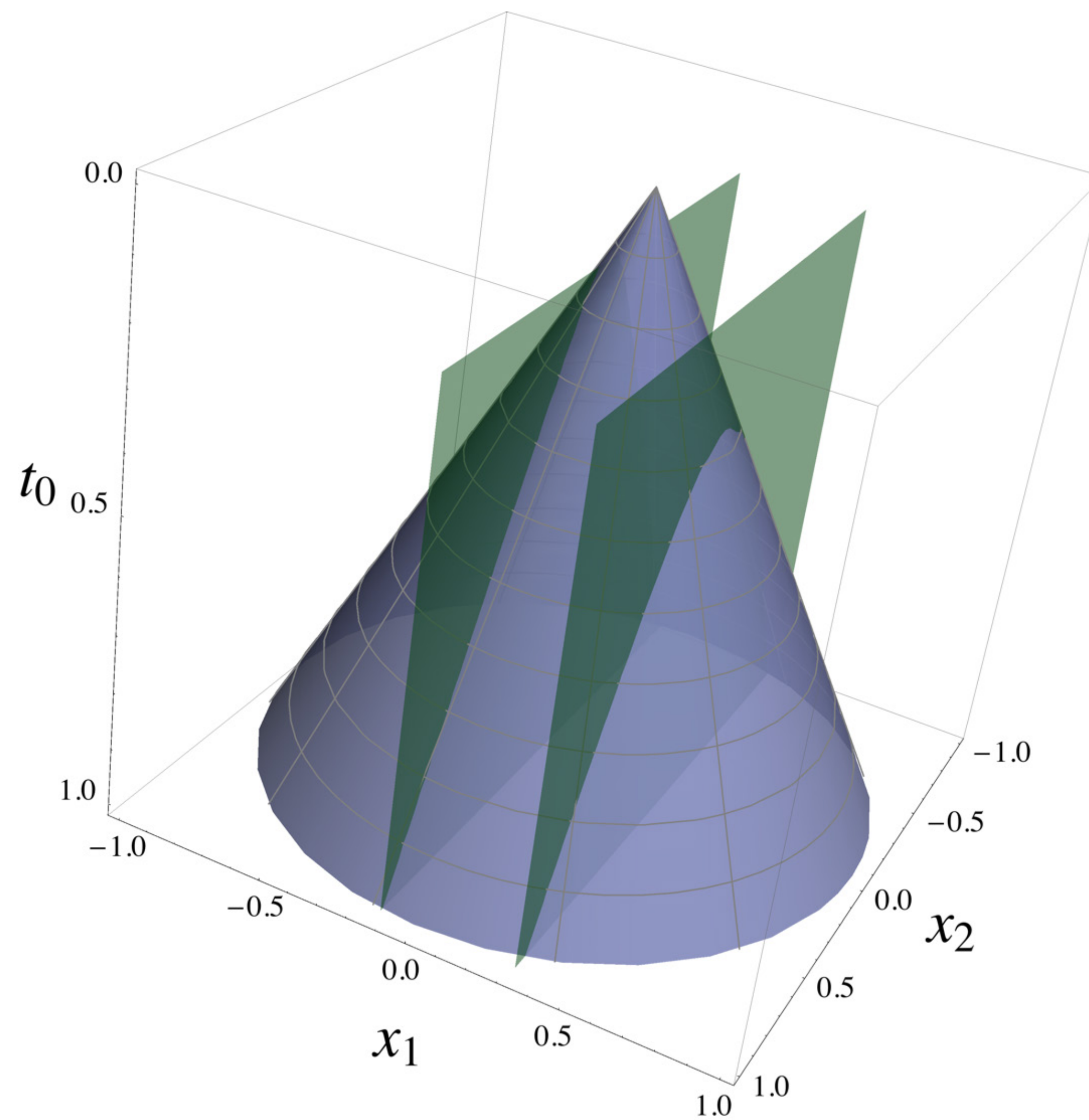

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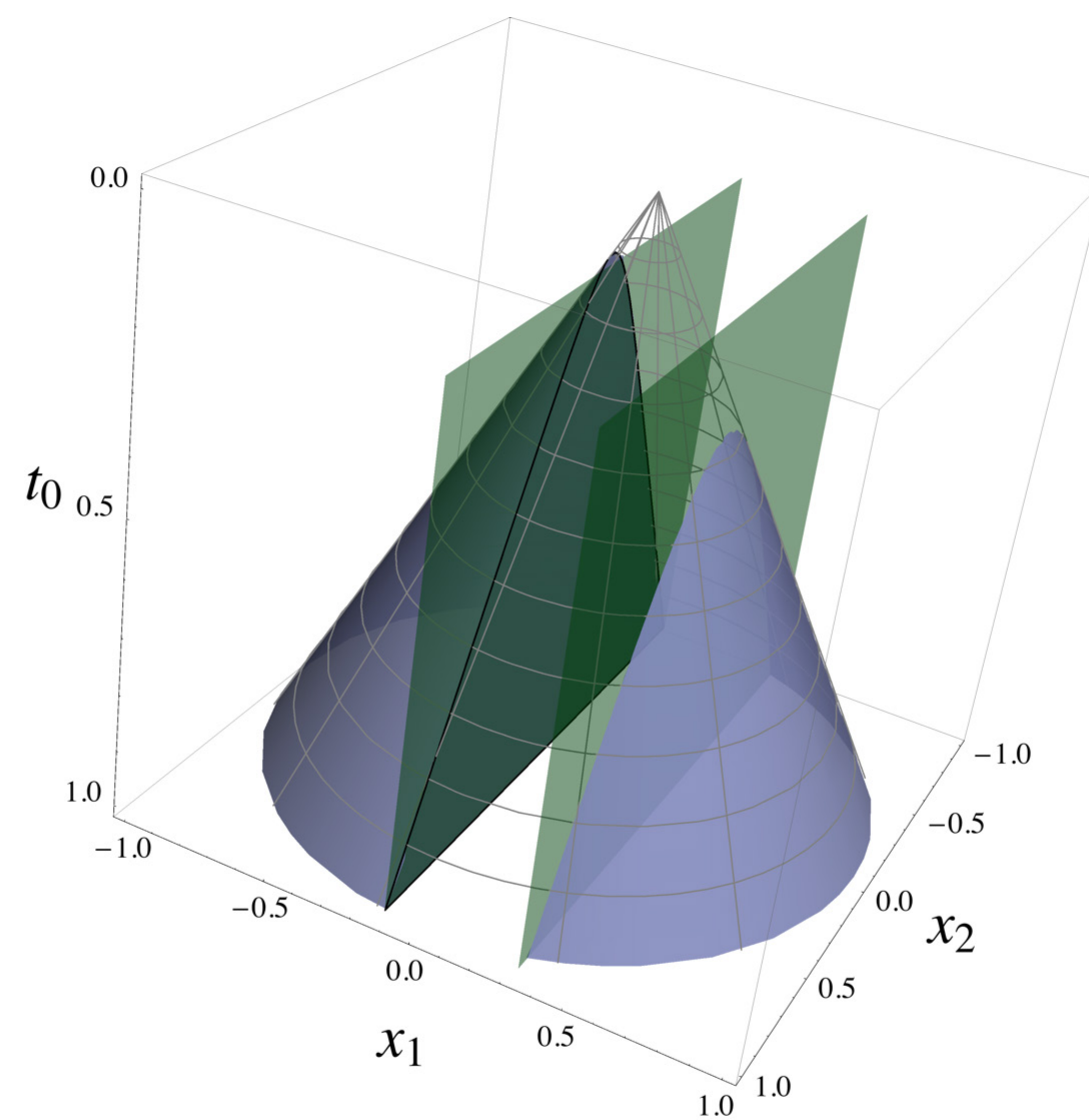

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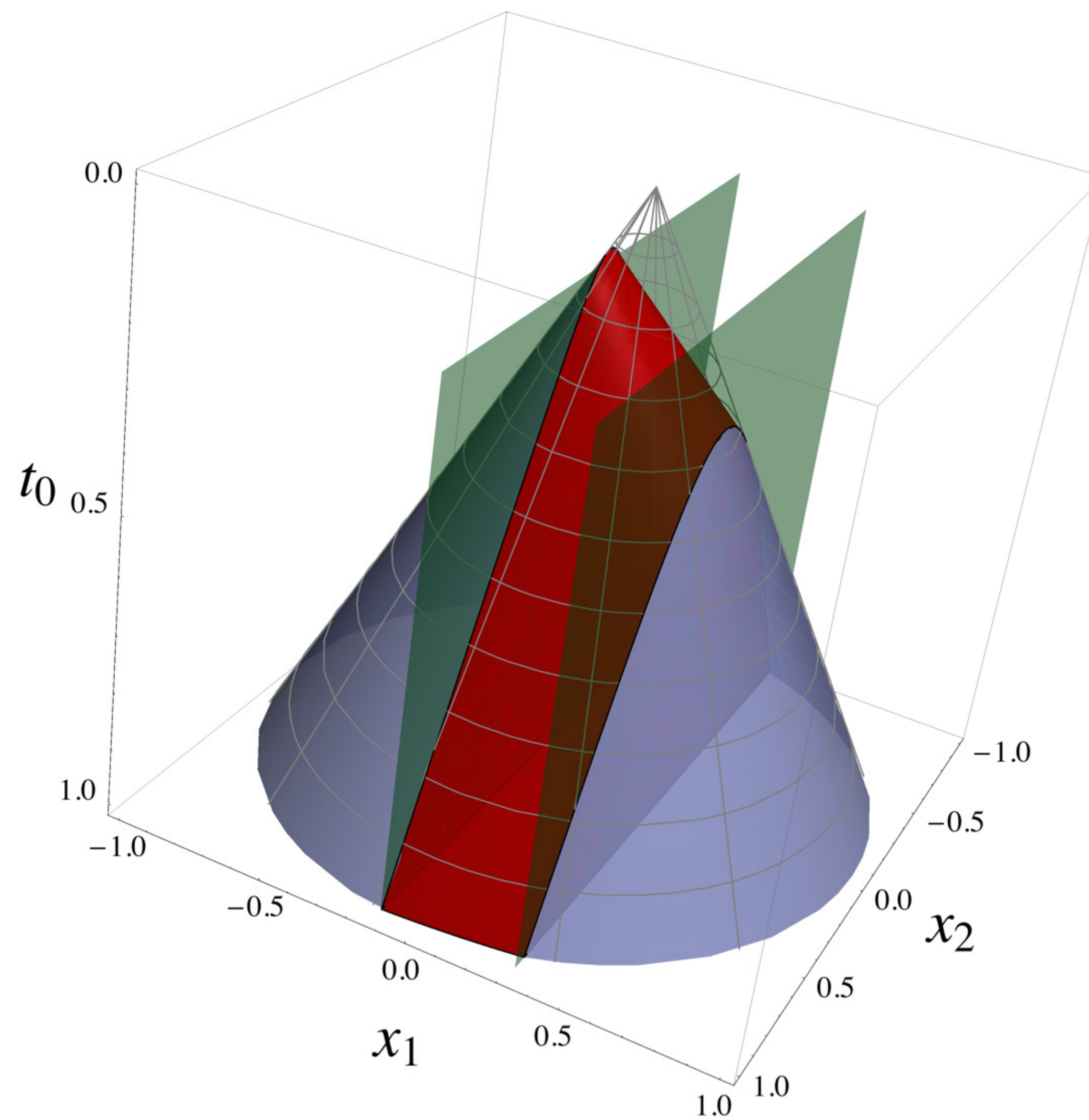

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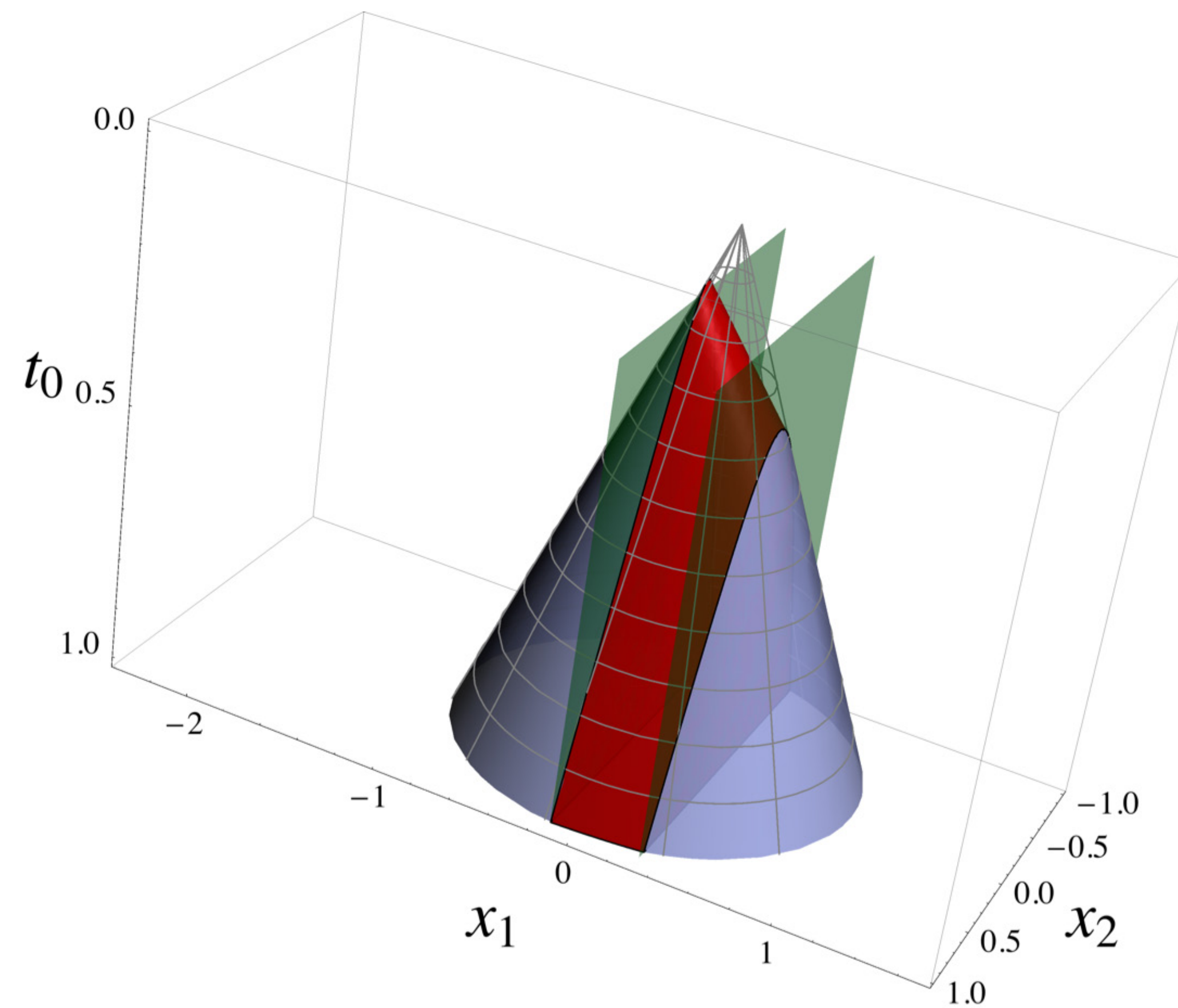

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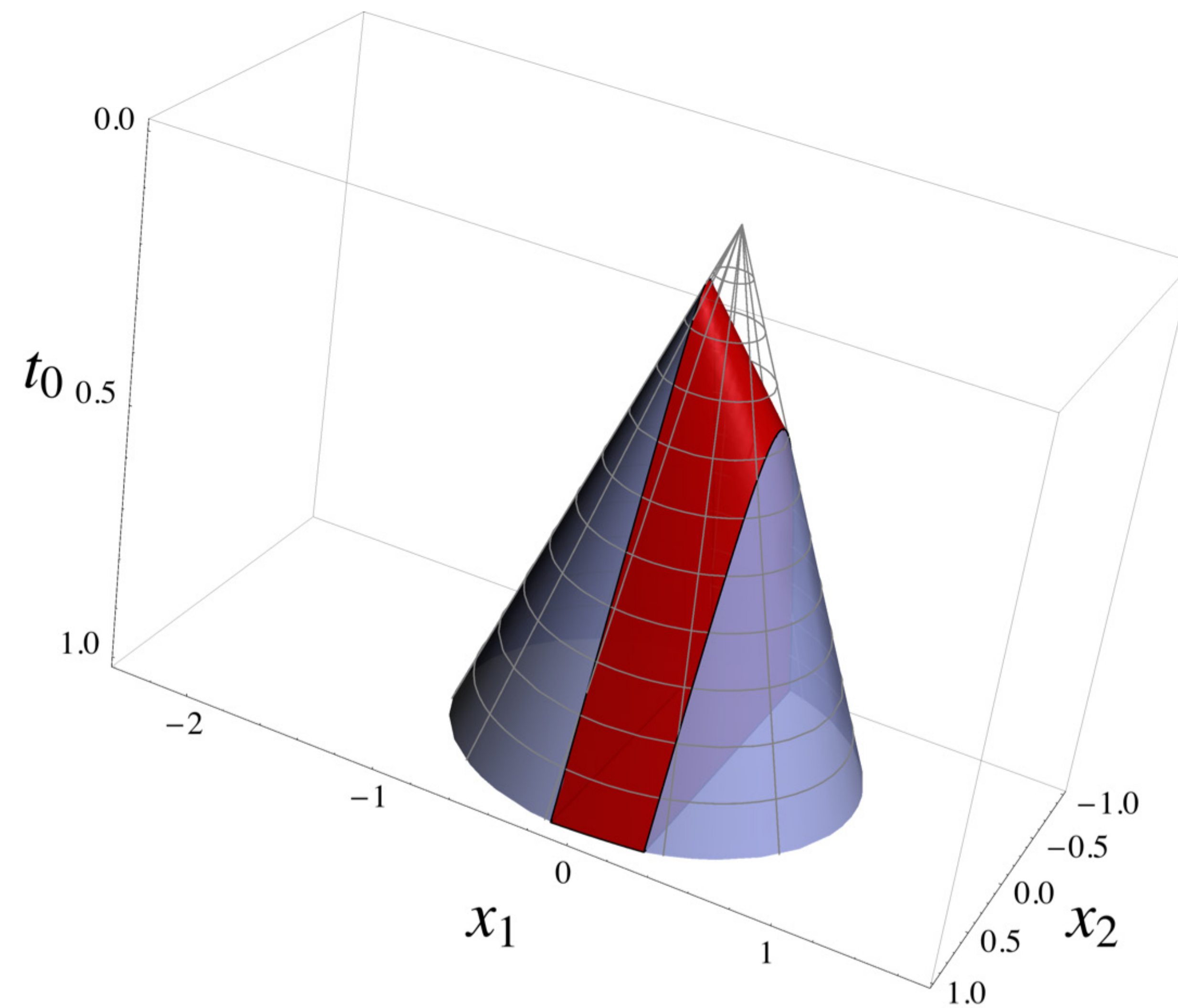

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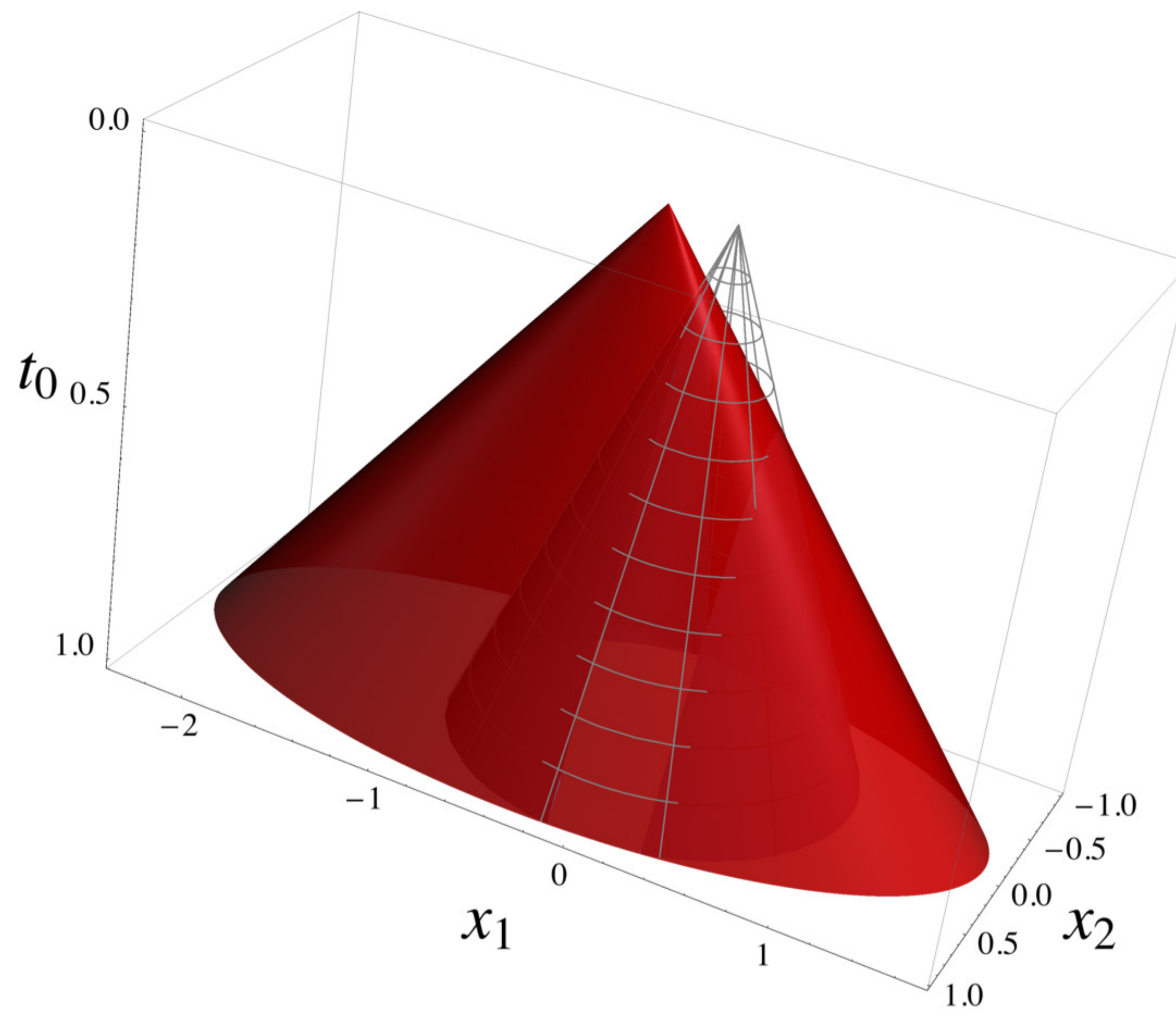

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$$C^{\pi}_{\pi_0,\pi_1} := \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|A(x - c)\|_2 \le t_0, \|Bx - d\|_2 \le t_0\}$$

(also see Atamturk and Narayanan 2010 for elementary integer splits)

Conic MIR

- Atamturk and Narayanan 2010

$$C := \left\{(x, t_0) \in \mathbb{Z}^n \times \mathbb{R} \; : \; \|A(x-c)\|_2 \leq t_0, \quad x \geq 0\right\}$$

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Extended Formulation: $(x, t, t_0) \in \mathbb{Z}^n \times \mathbb{R}^n \times \mathbb{R}_+$

$$\begin{aligned} |A(x-c)| &\leq t \\ x &\geq 0 \\ \|t\|_2 &\leq t_0 \end{aligned}$$

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$$\left.\begin{aligned} |A(x-c)| &\le t \\ x &\ge 0 \end{aligned}\right\} \longleftarrow \text{Linear Part}$$

$$\|t\|_2 \le t_0$$

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$$\|t\|_2 \le t_0 \longleftarrow \text{Nonlinear Part}$$

Aggregate: $|ax + y_0 - z_0 - b| \le s_0, \quad y_0, z_0, s_0 \ge 0, \quad x \in \mathbb{Z}^n_+$

Conic MIR

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Extended Formulation: $(x, t, t_0) \in \mathbb{Z}^n \times \mathbb{R}^n \times \mathbb{R}_+$

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$$\|t\|_2 \le t_0 \longleftarrow \text{Nonlinear Part}$$

Aggregate: $|ax + y_0 - z_0 - b| \le s_0, \quad y_0, z_0, s_0 \ge 0, \quad x \in \mathbb{Z}^n_+$

Conic MIR: $\sum_{i=1}^{n} \varphi_f(a_j)x_j - \varphi_f(b) \le s_0 + y_0 + z_0$

$$\varphi_f(a) := \lfloor a \rfloor + (a - \lfloor a \rfloor - f)^+/(1-f)$$

Conic MIR and Nonlinear Split Cut

- Modaresi, Kılınç, V. 2011

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Extended Formulation: $(x, t, t_0) \in \mathbb{Z}^n \times \mathbb{R}^n \times \mathbb{R}_+$

$|A(x-c)| \le t$ ← Linear Part

$\|t\|_2 \le t_0$ ← Nonlinear Part

Conic MIR = Split cuts for linear part ≠ Nonlinear split cut

$$(1-2f)\left(\lambda^T A x - \lfloor \lambda^T B c \rfloor\right) + f \le |\lambda|^T t$$

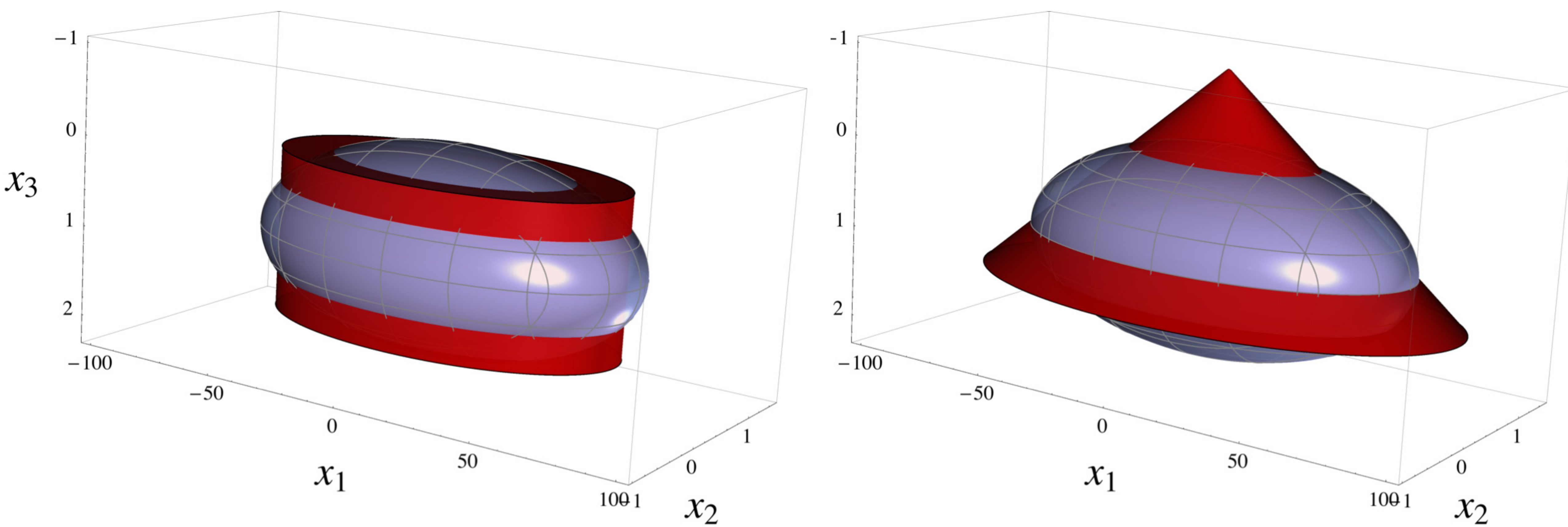
$$\lambda \in \mathbb{R}^n, \quad A^T\lambda \in \mathbb{Z}^n, \quad f := \lambda^T B c - \lfloor \lambda^T B c \rfloor$$

Split Cuts for Ellipsoids

- Formulas: (Dadush, Dey and V. 2011)

$$C := \{x \in \mathbb{R}^n \ : \ \|A(x-c)\|_2 \leq 1\}$$

$$C^{\pi}_{\pi_0,\pi_1} := \{x \in \mathbb{R}^n \ : \ \|A(x-c)\|_2 \leq 1, \quad \|Bx-d\|_2 \leq \langle a, x\rangle + b\}$$



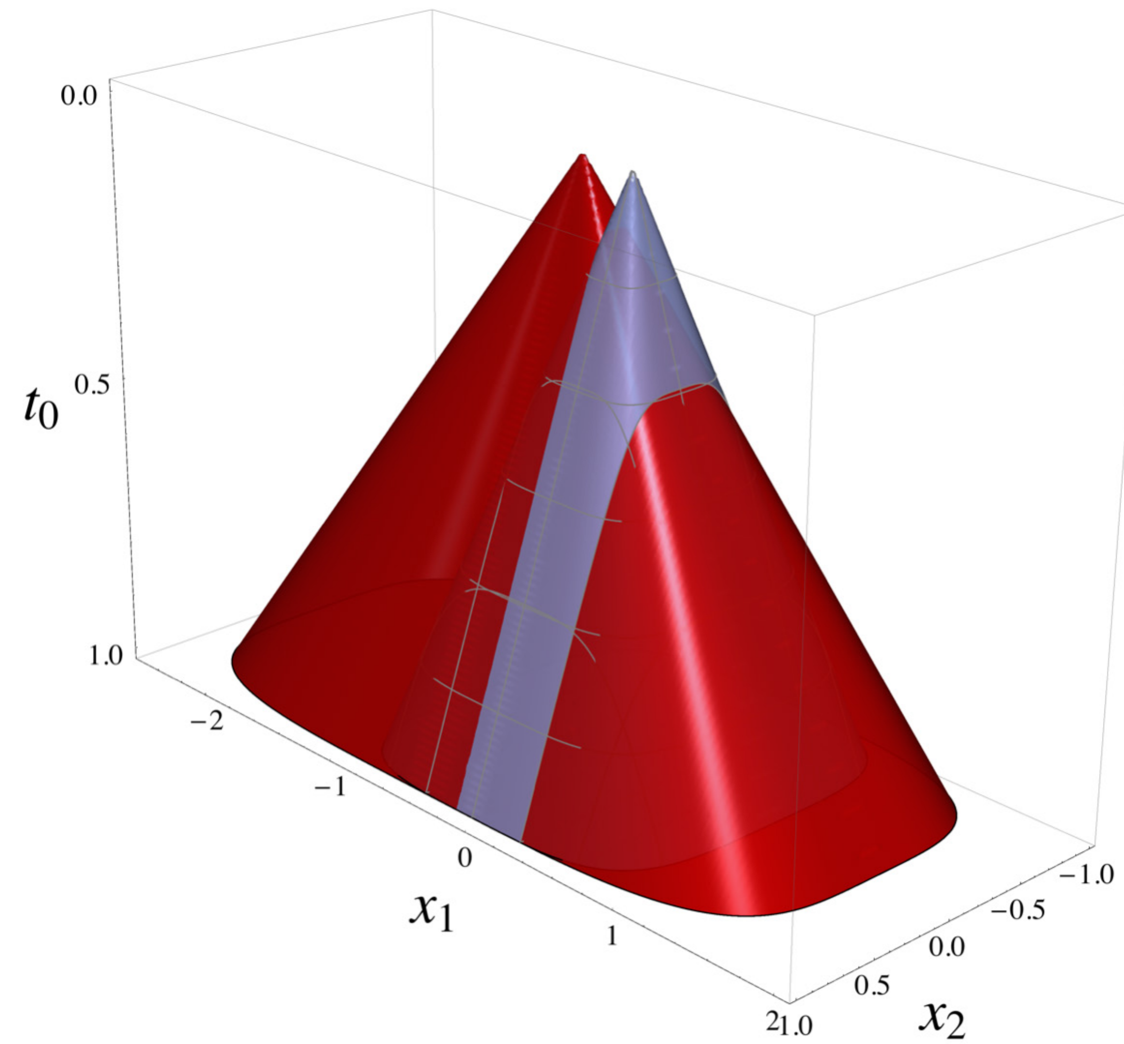
(also see Belotti, Góez, Polik, Ralphs, Terlaky 2011)

Split Cuts for P-Order Cones

- Formulas: (Modaresi, Kılınç, V. 2011)

Elementary splits: $\pi = e^i$

$$C := \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|x - c\|_p \le t_0\}$$



$$C^{\pi}_{\pi_0,\pi_1} := \Big\{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|x - c\|_2 \le t_0, \Big|(\alpha(x_1 - d_1) + \beta)^p + \sum_{i=2}^{n} (x_i - d_i)^p\Big| \le t_0^p\Big\}$$

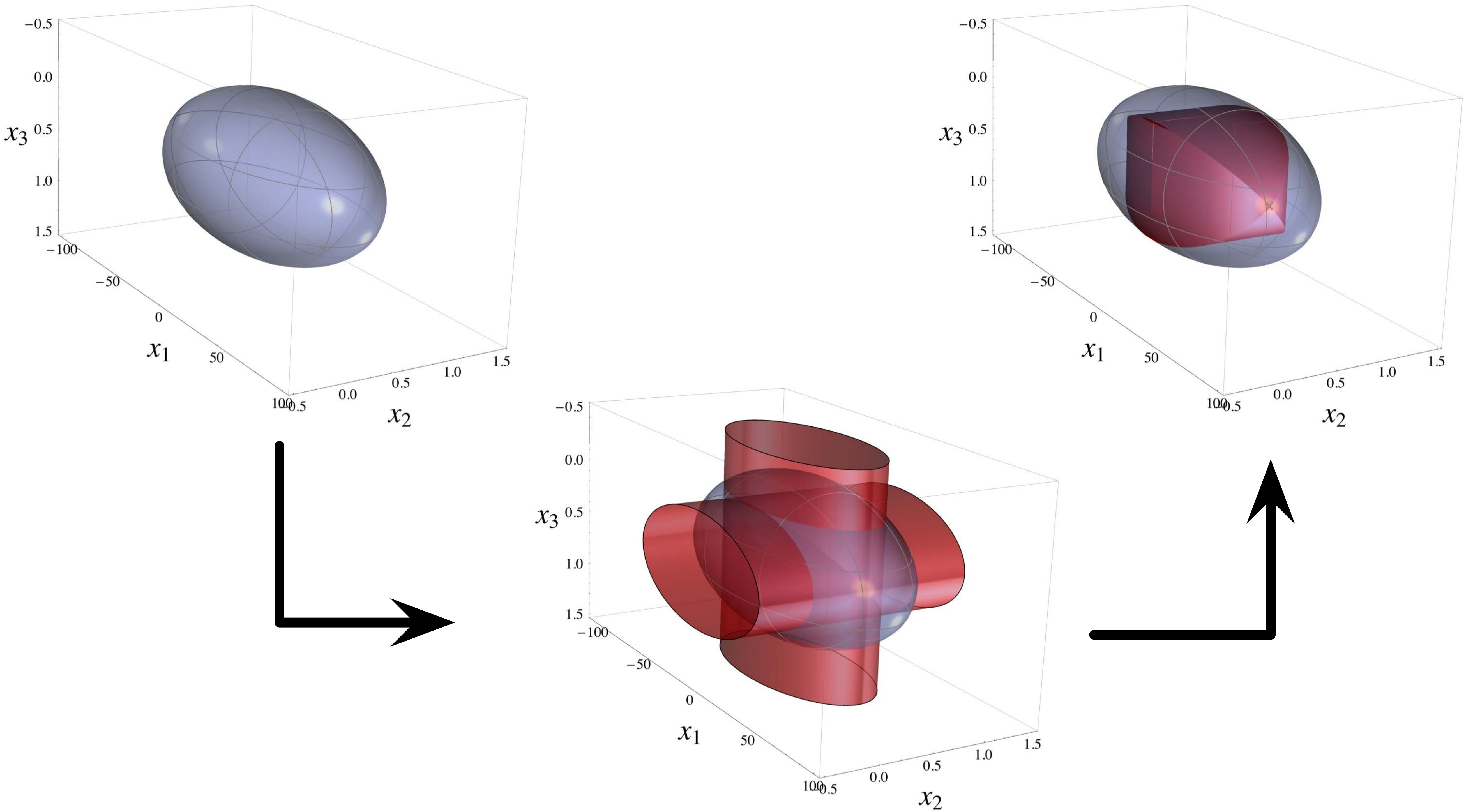
Split Closure is Finitely Generated

- Theorem (Dadush, Dey, V. 2011): If C is a strictly convex set then there exists a finite $D \subseteq \mathbb{Z}^n \times \mathbb{Z}$ such that:

$$\bigcap_{(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Z}} C^{\pi}_{\pi_0,\pi_0+1} = \bigcap_{(\pi,\pi_0)\in D} C^{\pi}_{\pi_0,\pi_0+1}$$

- Does not imply polyhedrality of split closure
- Split Closure is not stable

Split Closure Can Be Non-Polyhedral



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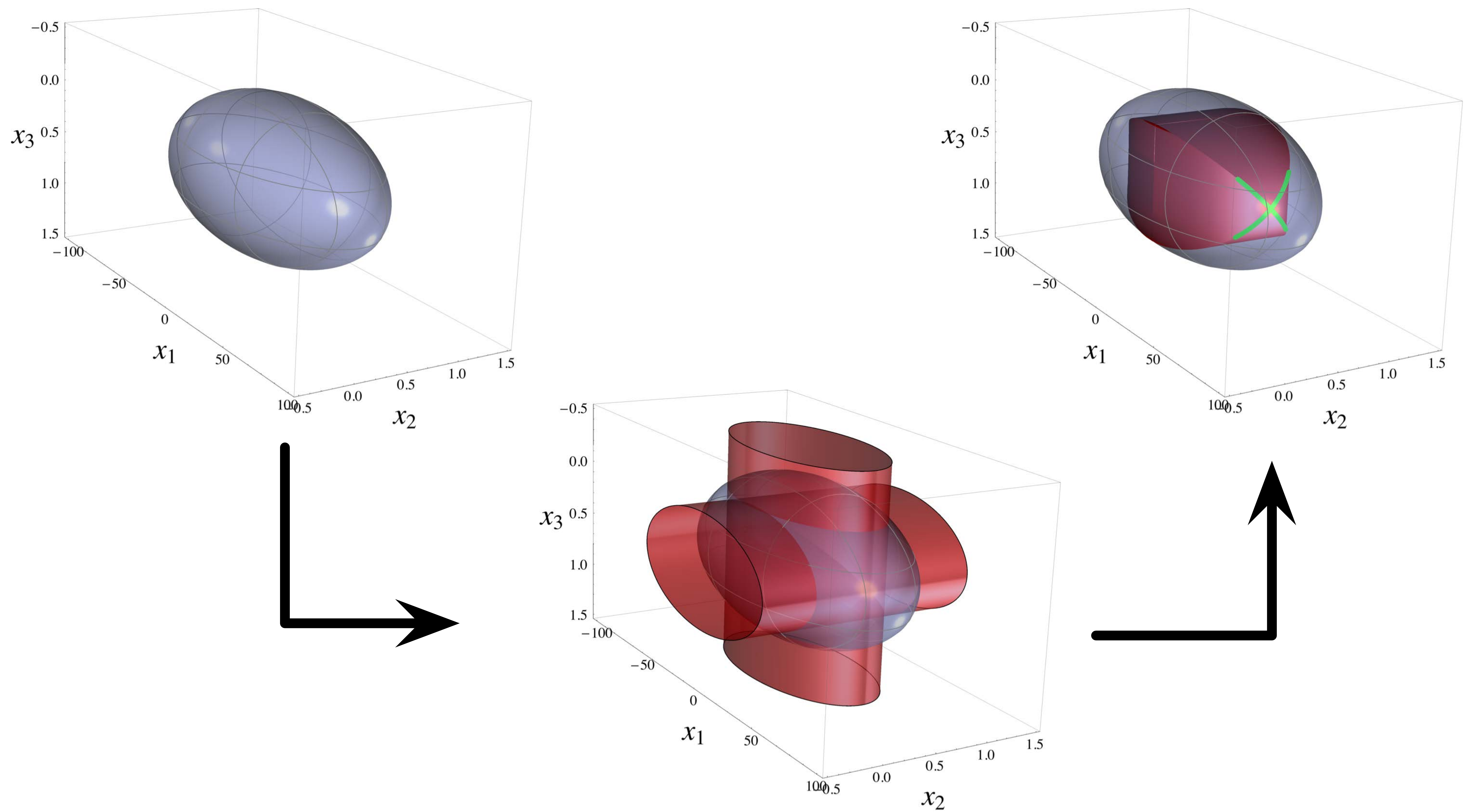

Summary and Open Questions

- Formulas for nonlinear split cuts
  - Quadratic cones, ellipsoids and others.
  - Strong ties to conic MIR
- Split closure: Finitely generated, not polyhedral
- Future:
  - More formulas
  - Computation
  - More general/constructive split closure