Split Cuts for Convex Nonlinear Mixed Integer Programming

Juan Pablo Vielma

Massachusetts Institute of Technology

joint work with

D. Dadush and S. S. Dey
Georgia Institute of Technology

S. Modaresi and M. Kılınç
University of Pittsburgh

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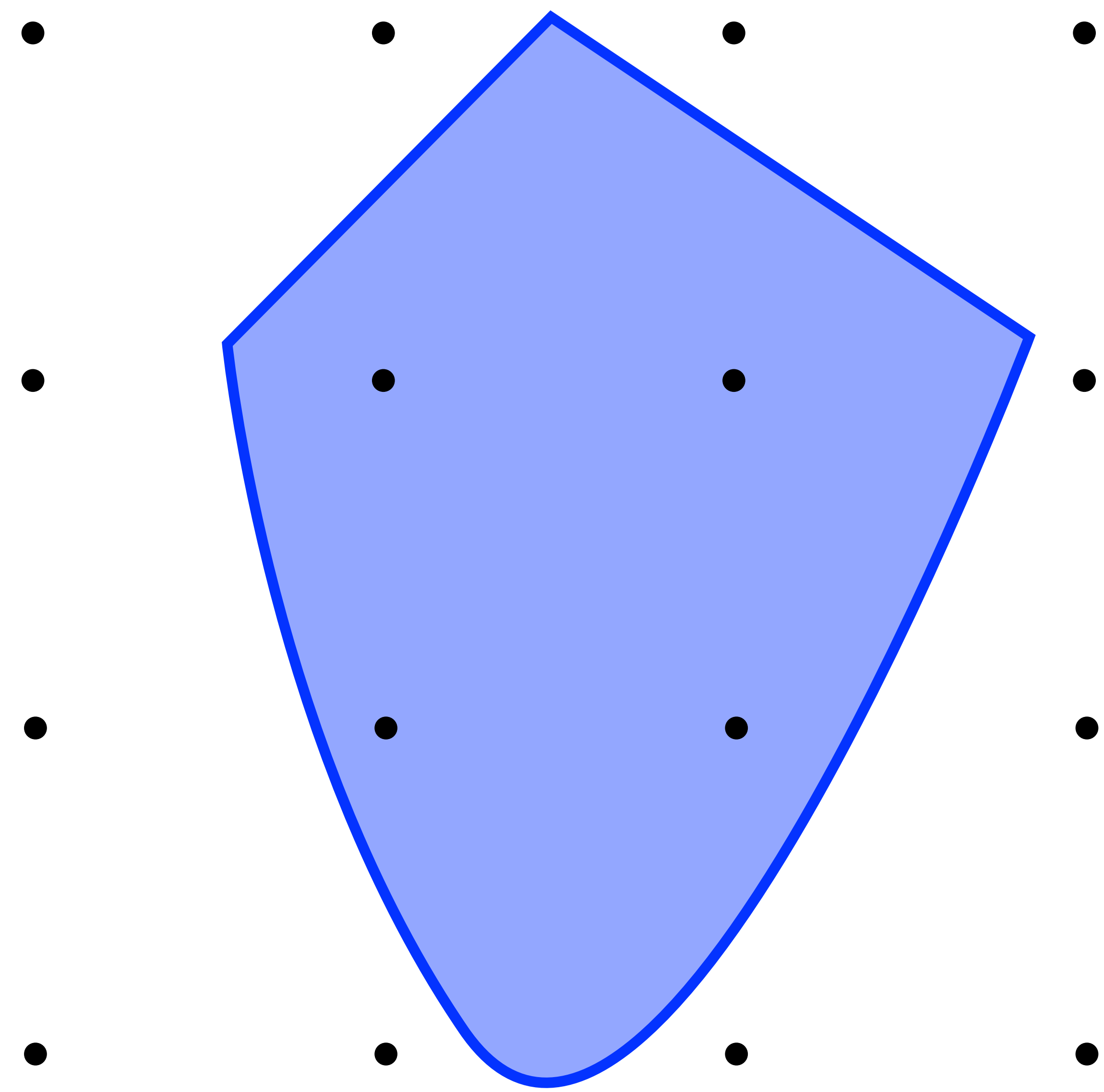
Outline

- Introduction
- Split Cut Formulas
- Conic MIR
- Summary

Split Disjunctions and Split Cuts



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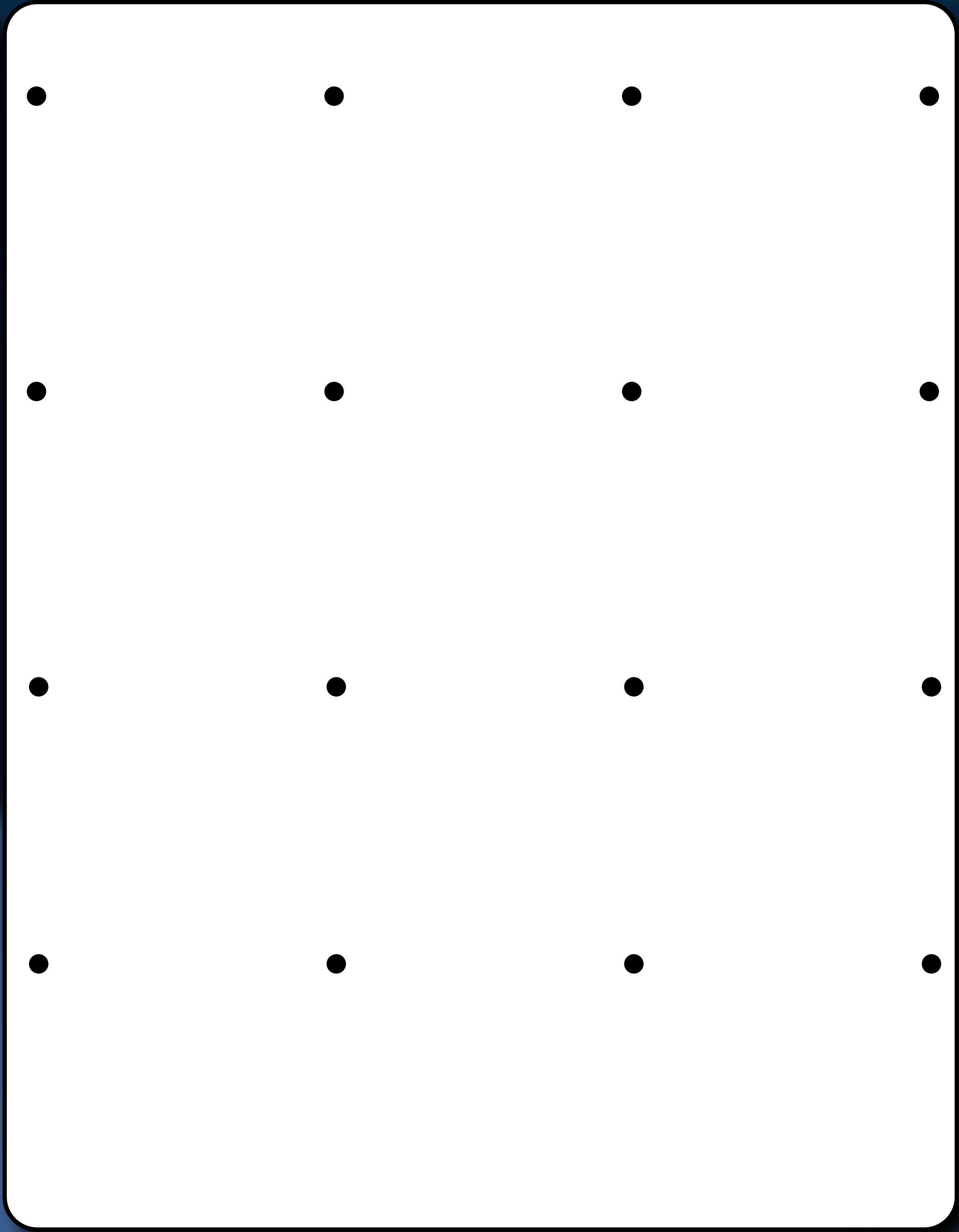

Split Disjunctions and Split Cuts

Split Disjunction

$$L^{\pi}_{\pi_0} = \{x \in \mathbb{R}^n \ : \ \langle \pi, x \rangle \leq \pi_0\}$$
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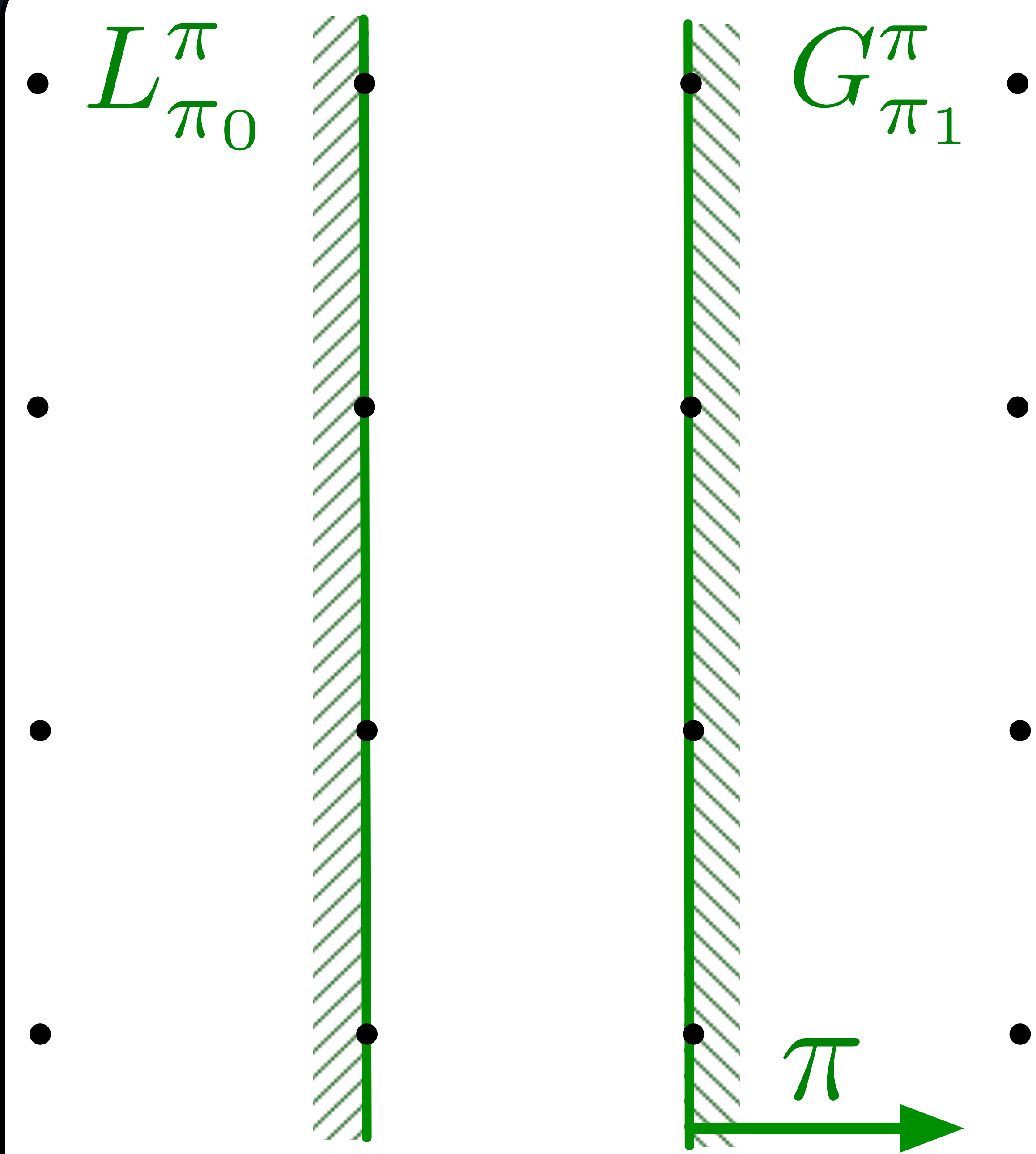

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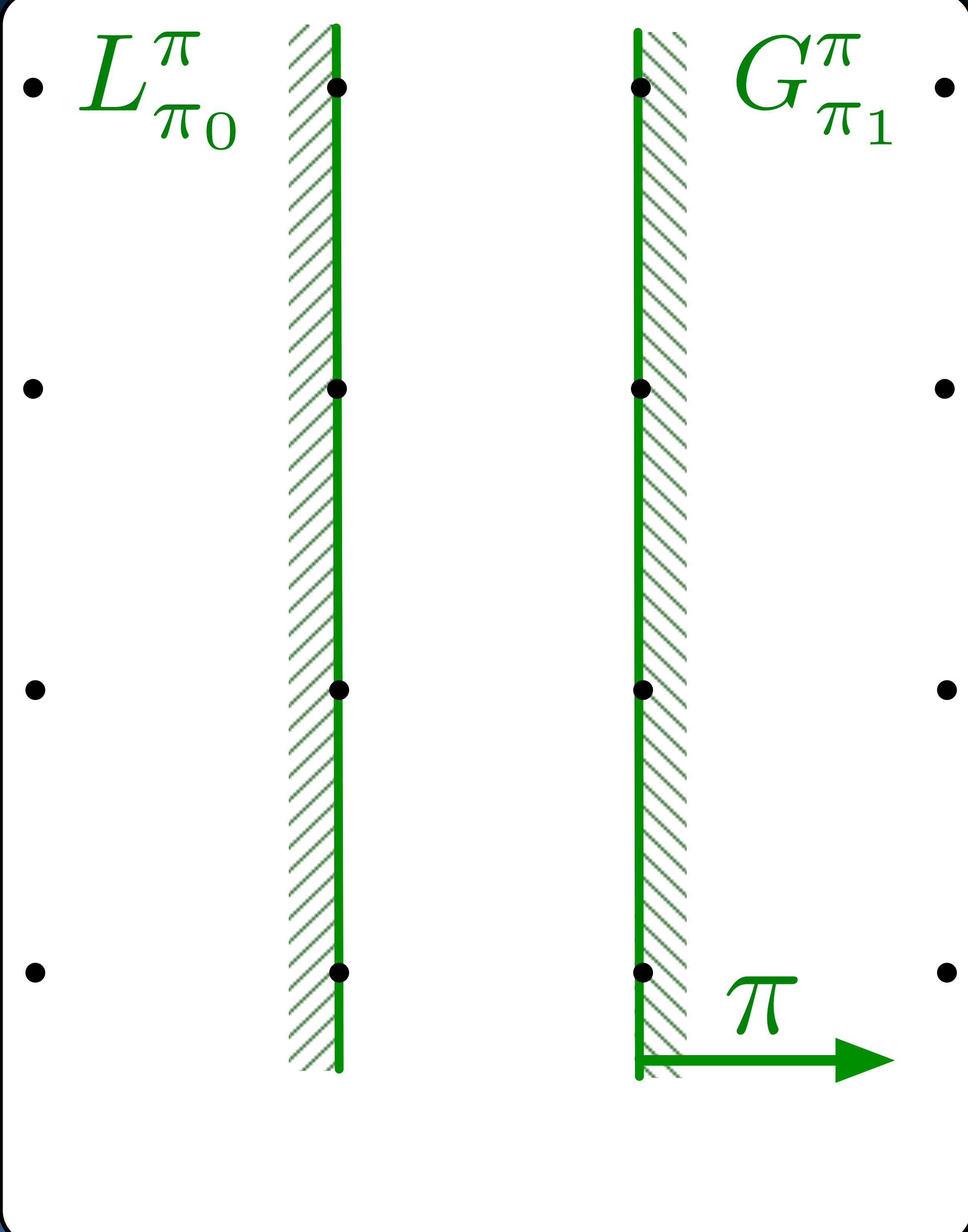
$\pi \in \mathbb{Z}^n, \quad \pi_1 = \pi_0 + 1 \in \mathbb{Z}$

$$\Downarrow$$

$$\mathbb{Z}^n \subseteq L^{\pi}_{\pi_0} \cup G^{\pi}_{\pi_1}$$



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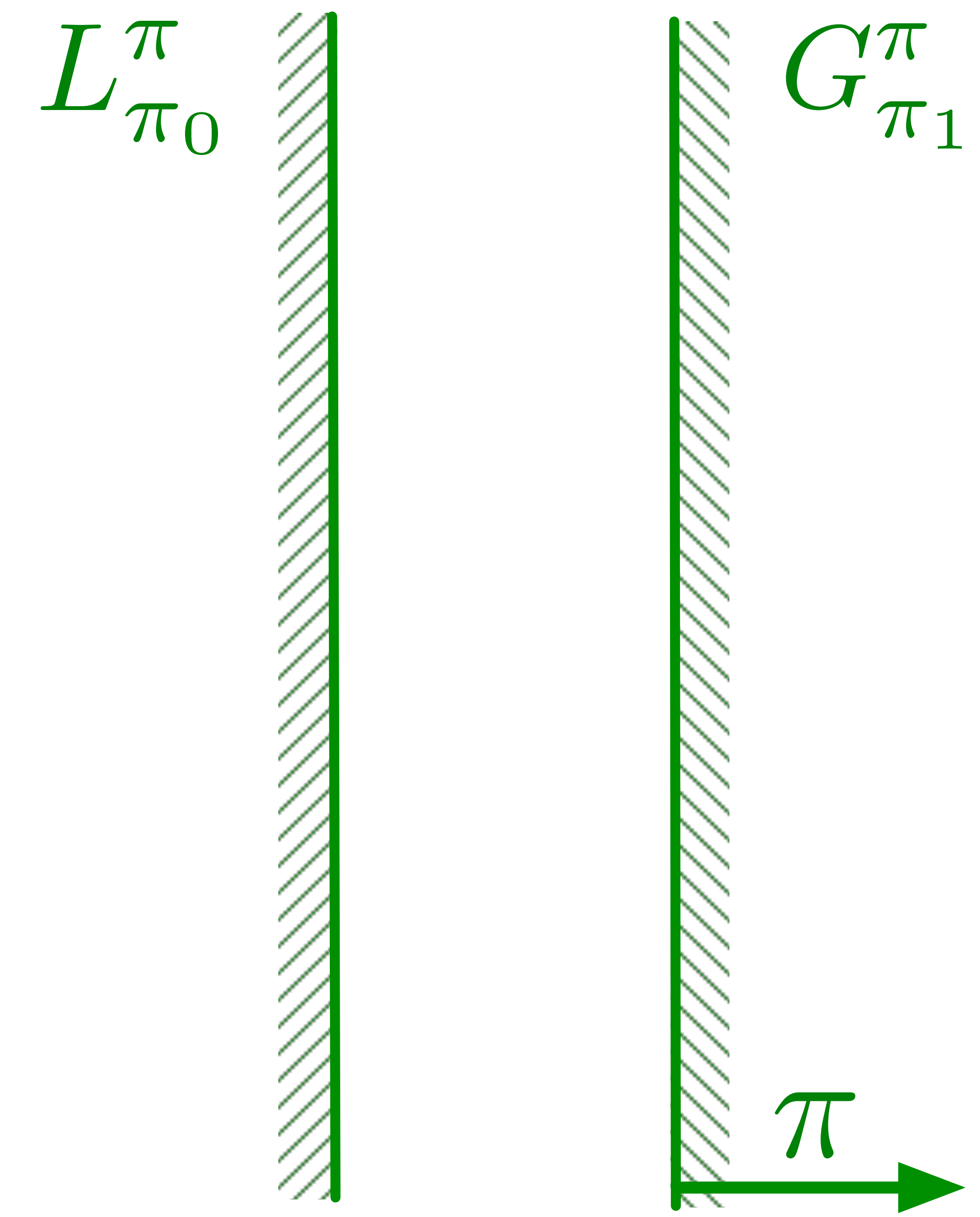

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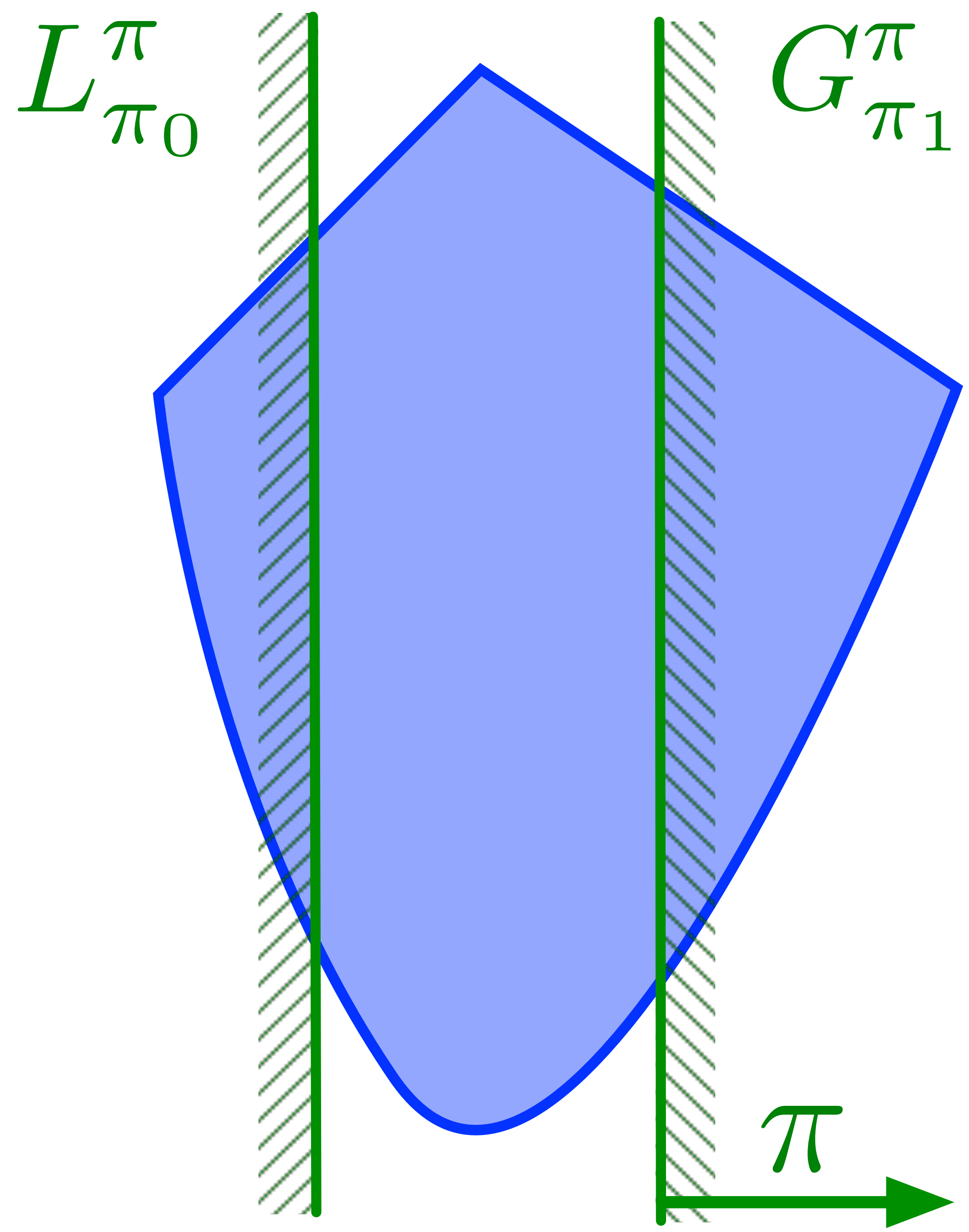

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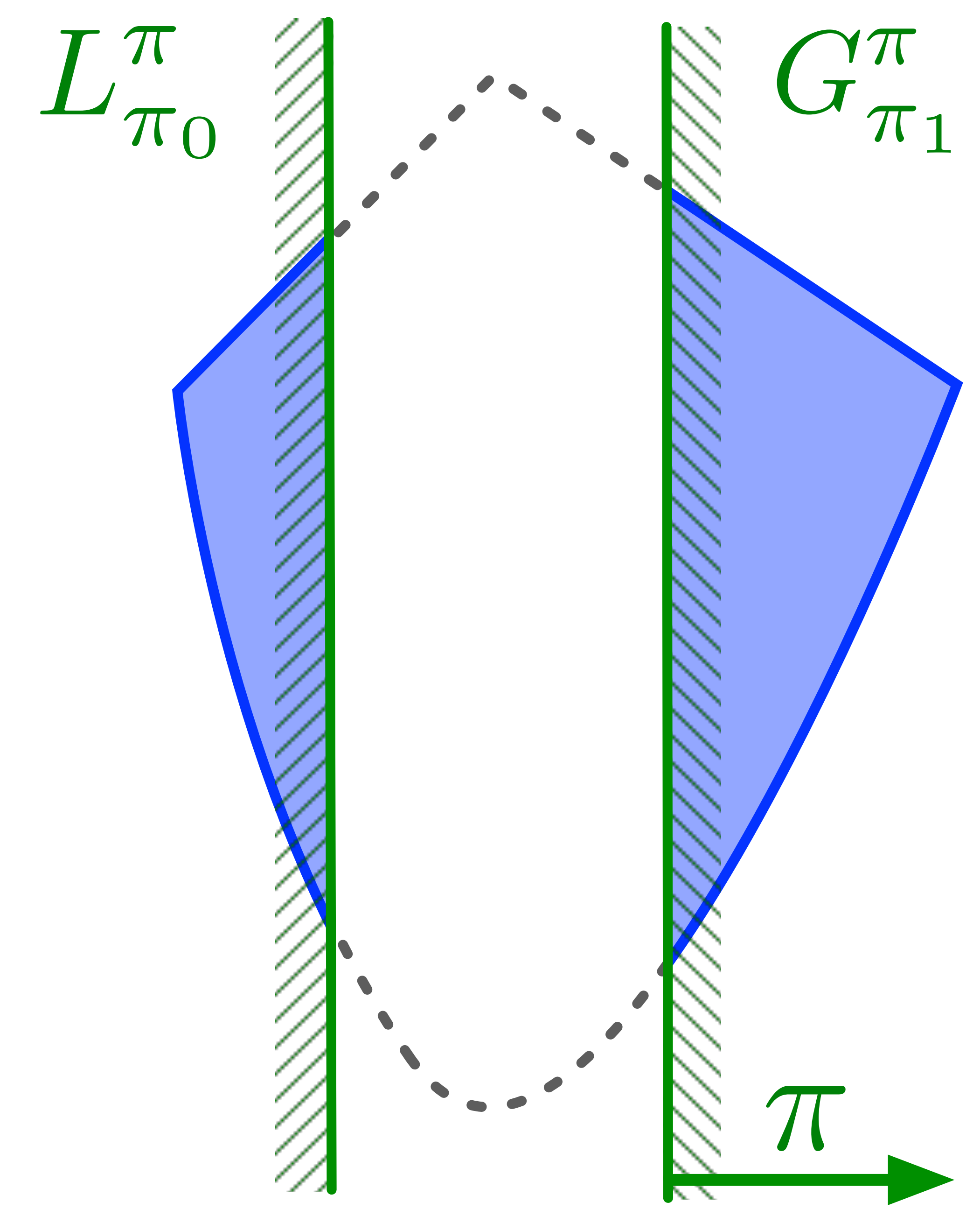

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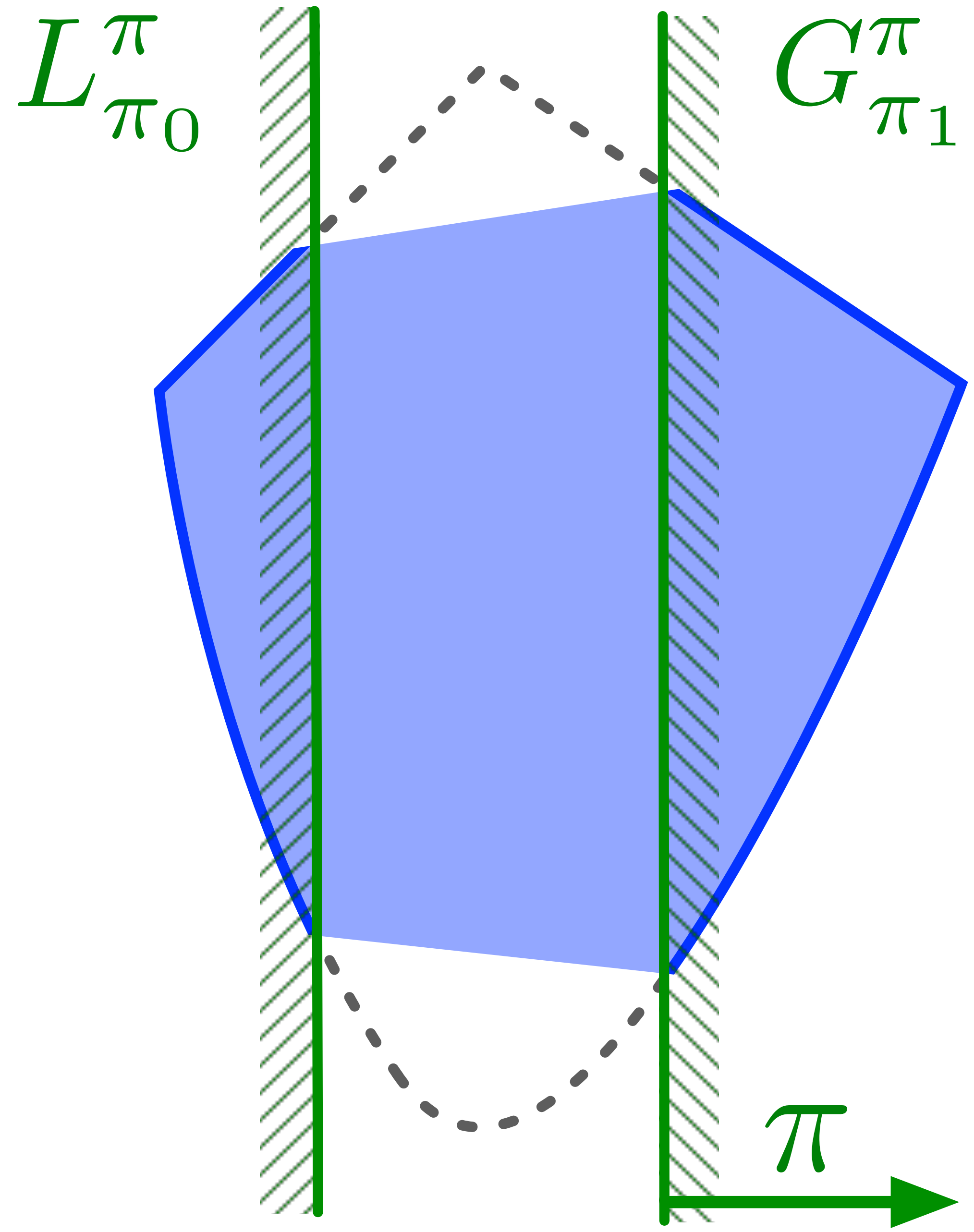

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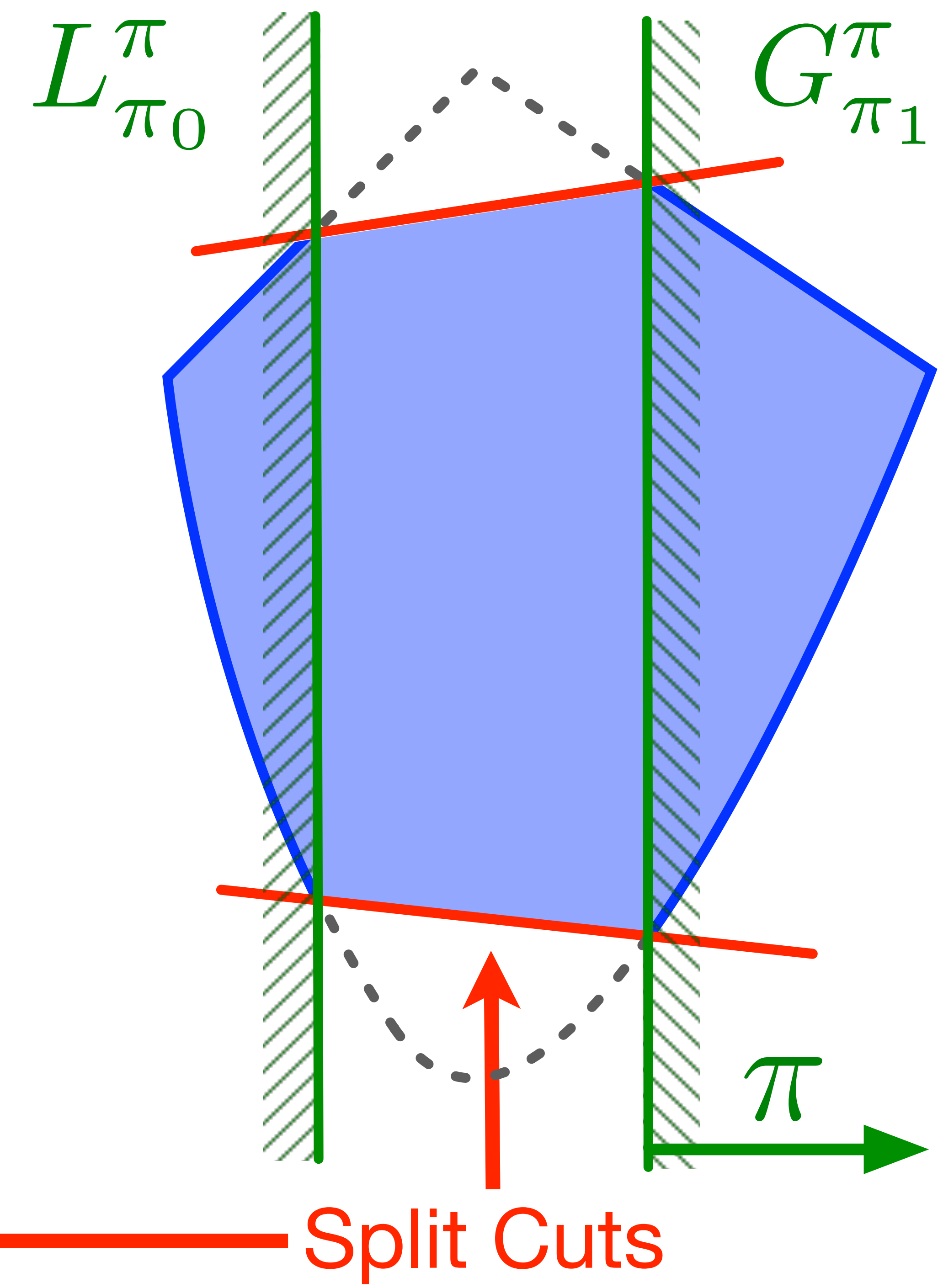

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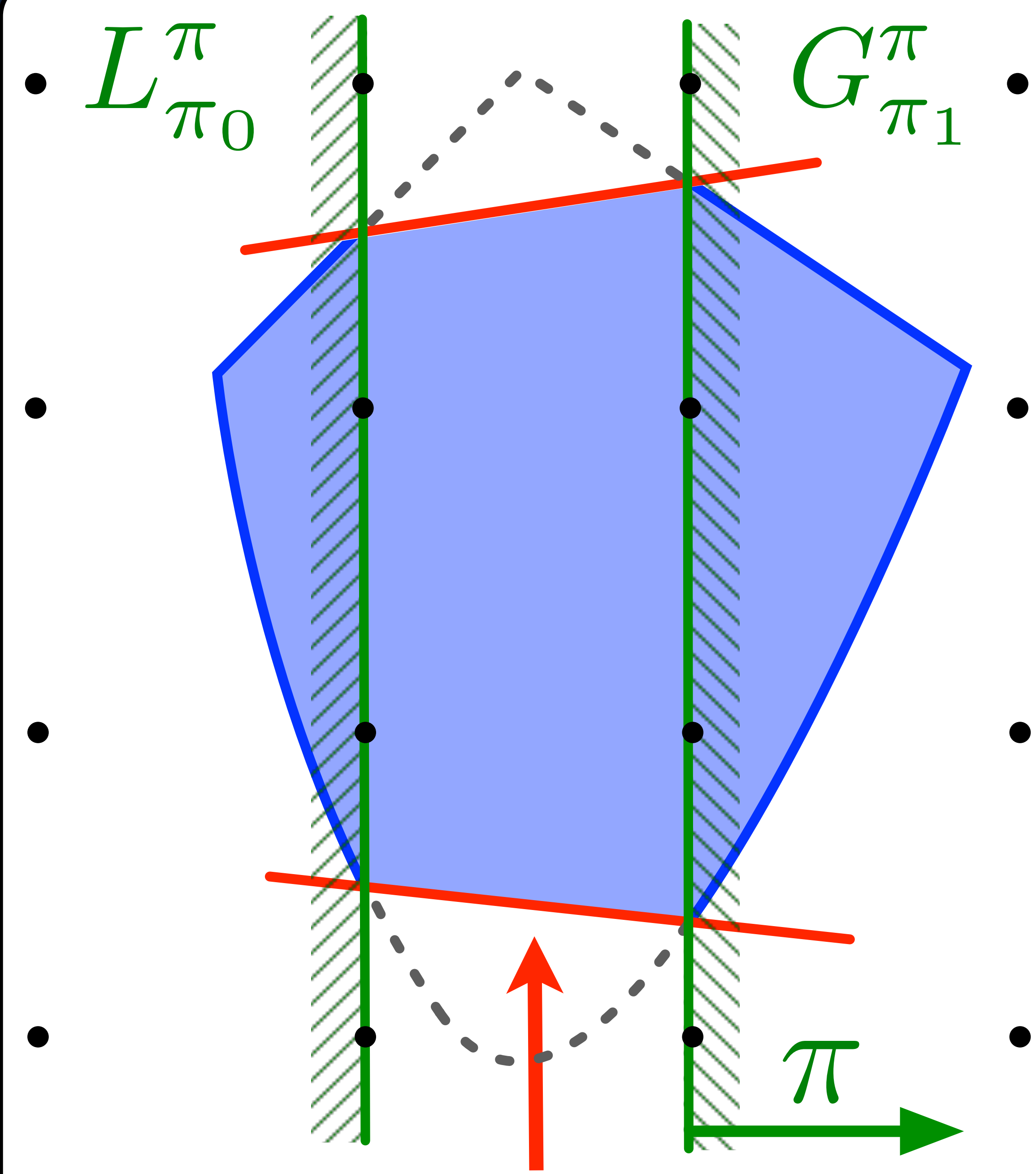


Split Cuts for Simplicial Cones

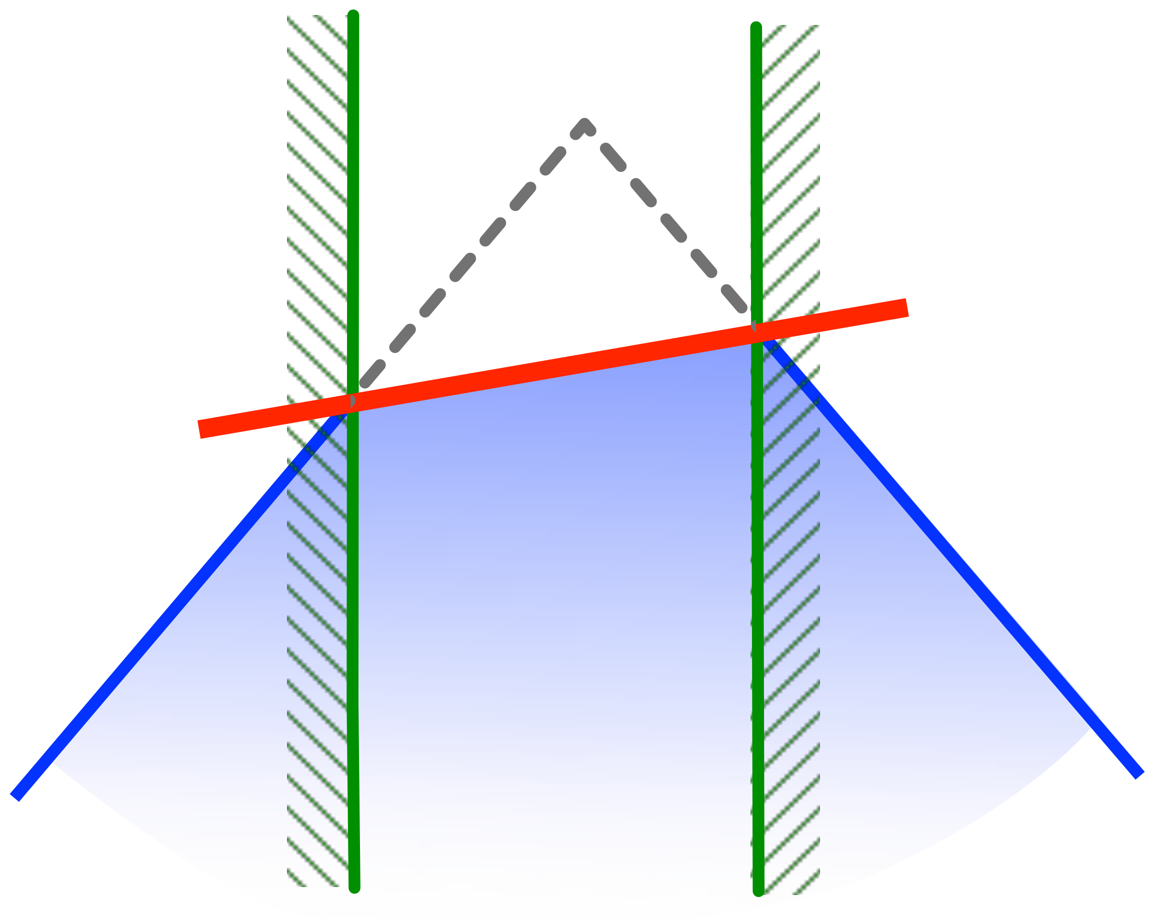
- Formulas: (MIG: Gomory 1960 and MIR: Nemhauser and Wolsey 1988)

$C := \{x \in \mathbb{R}^n \;:\; Ax \leq b\},$

$\det(A) \neq 0$

$\pi_0 < \langle \pi, A^{-1}b \rangle < \pi_1$

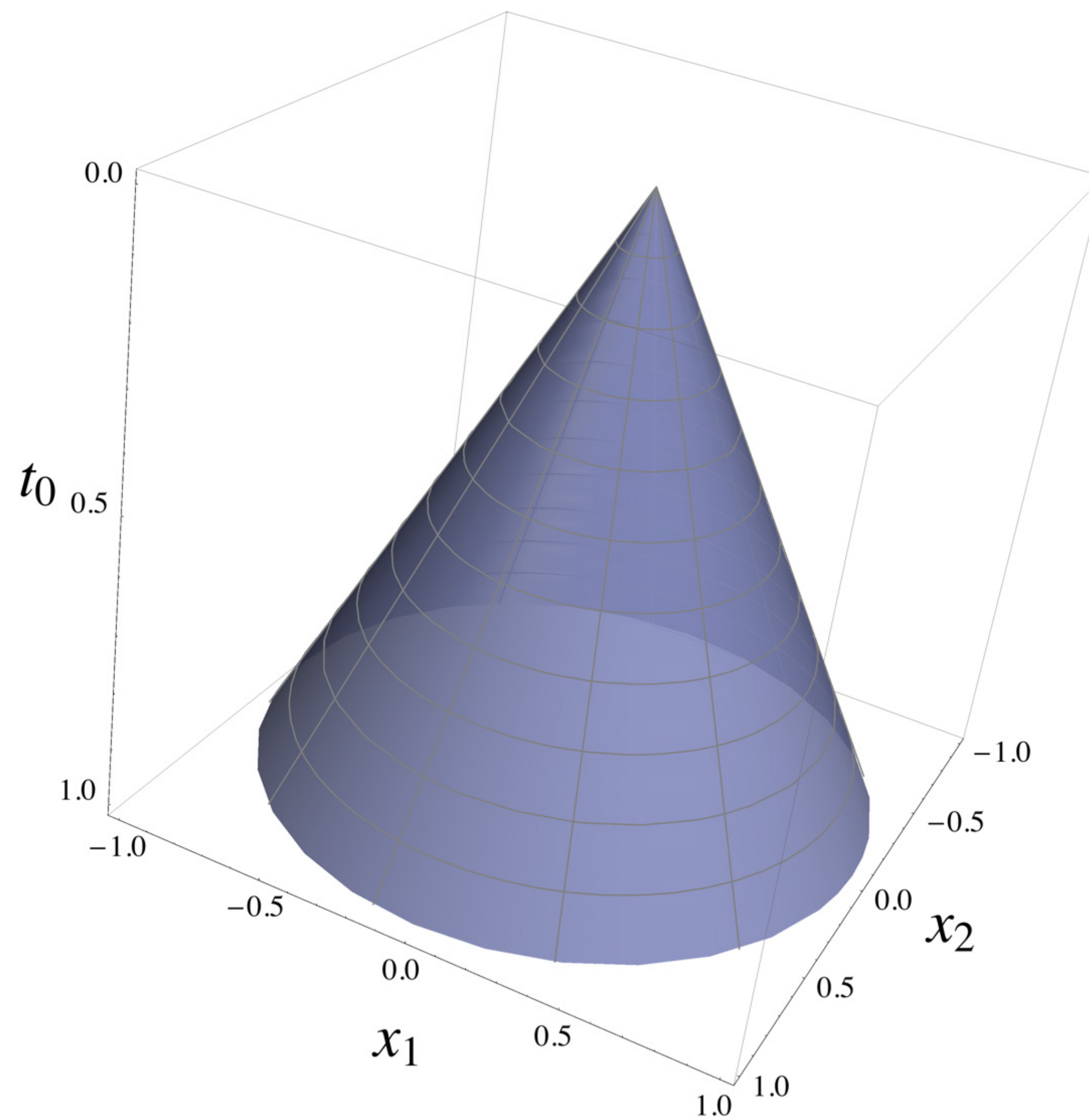
$$C^{\pi}_{\pi_0,\pi_1} := \{x \in \mathbb{R}^n \;:\; Ax \leq b,\; \langle a, x \rangle \leq b\}$$



Split Cuts for Quadratic Cones

- Formulas: (Modaresi, Kılınç, V. 2011)

$$C := \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|A(x-c)\|_2 \leq t_0\}$$



(also see Atamturk and Narayanan 2010 for elementary integer splits)

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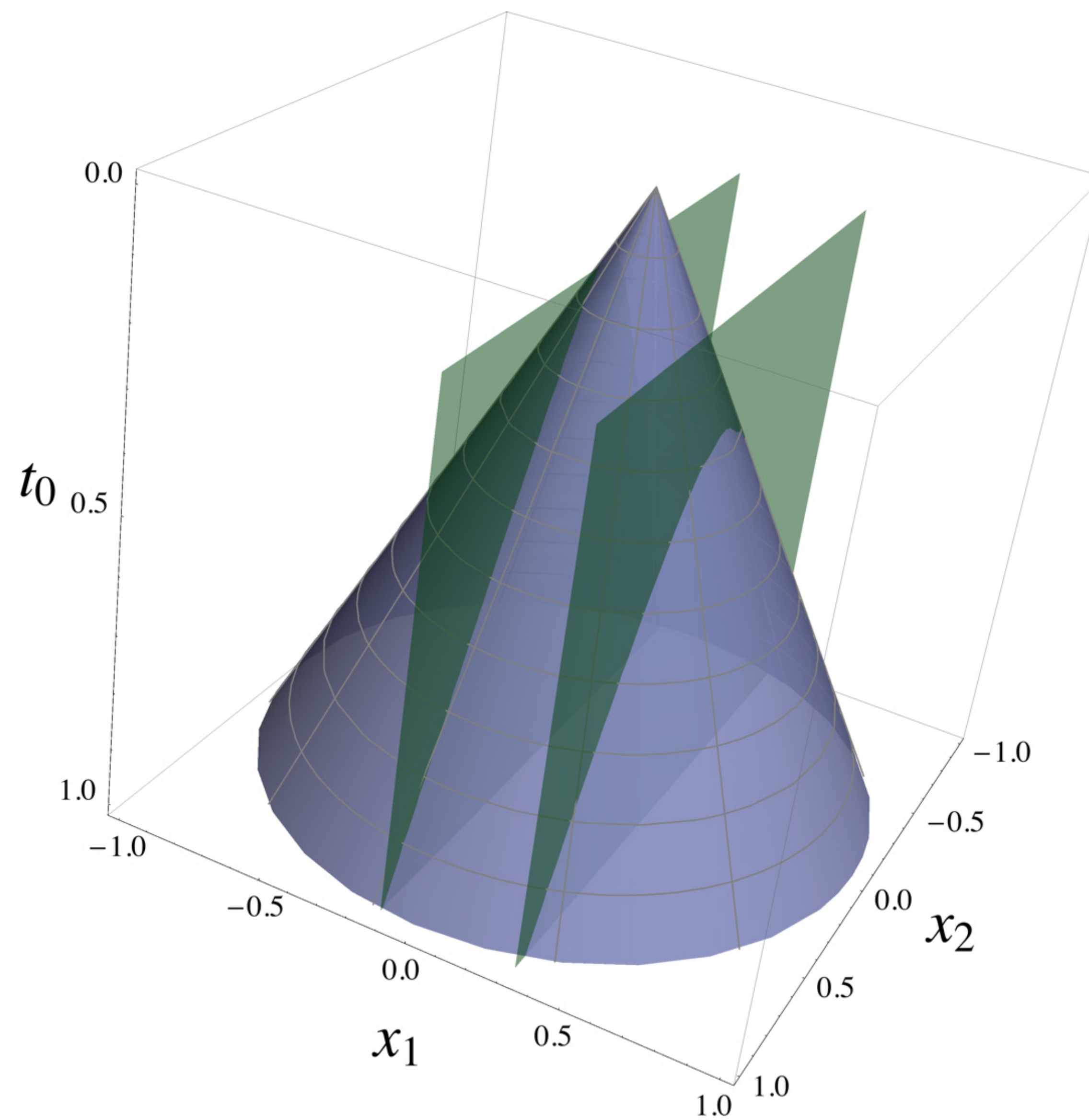

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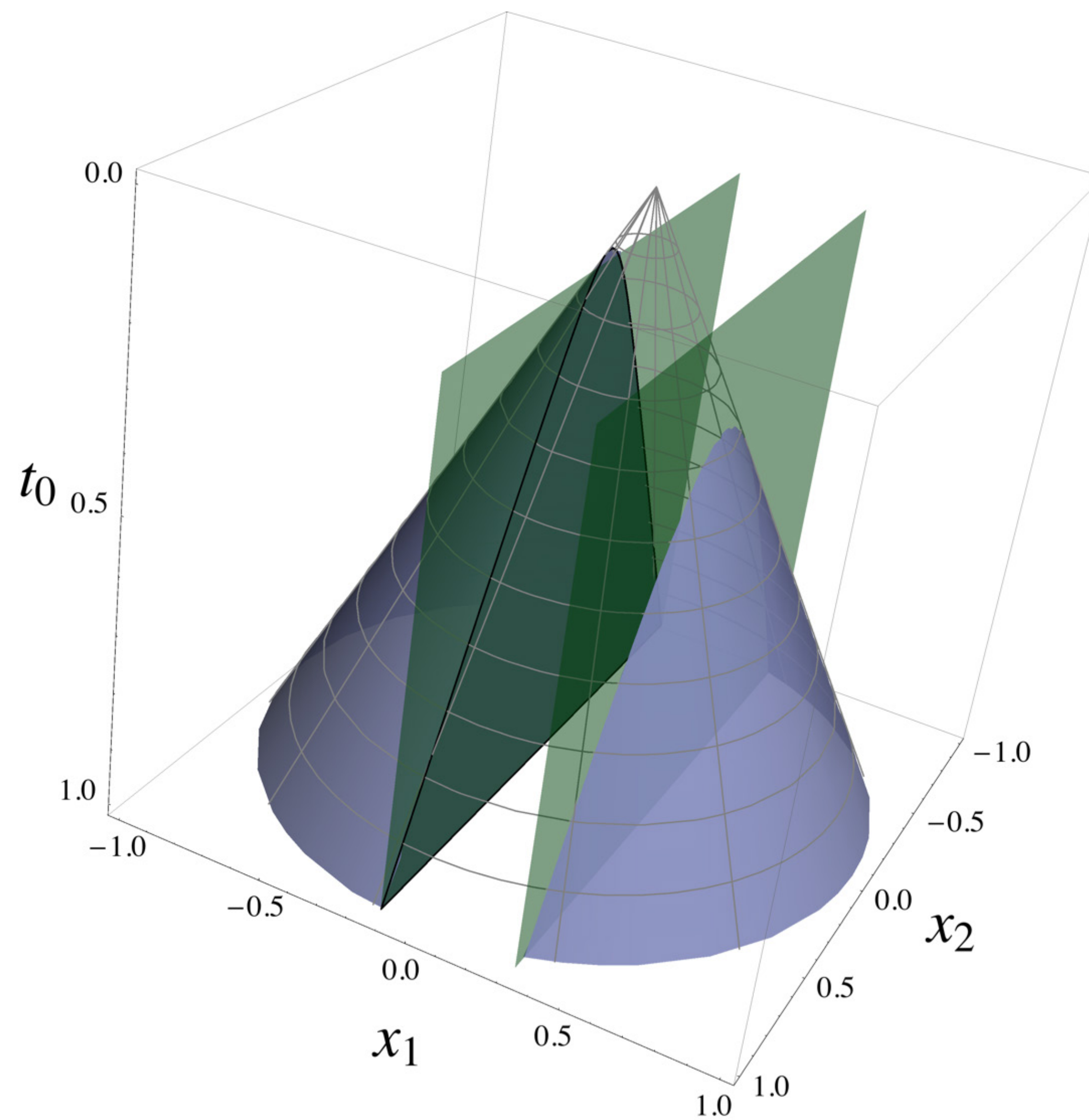

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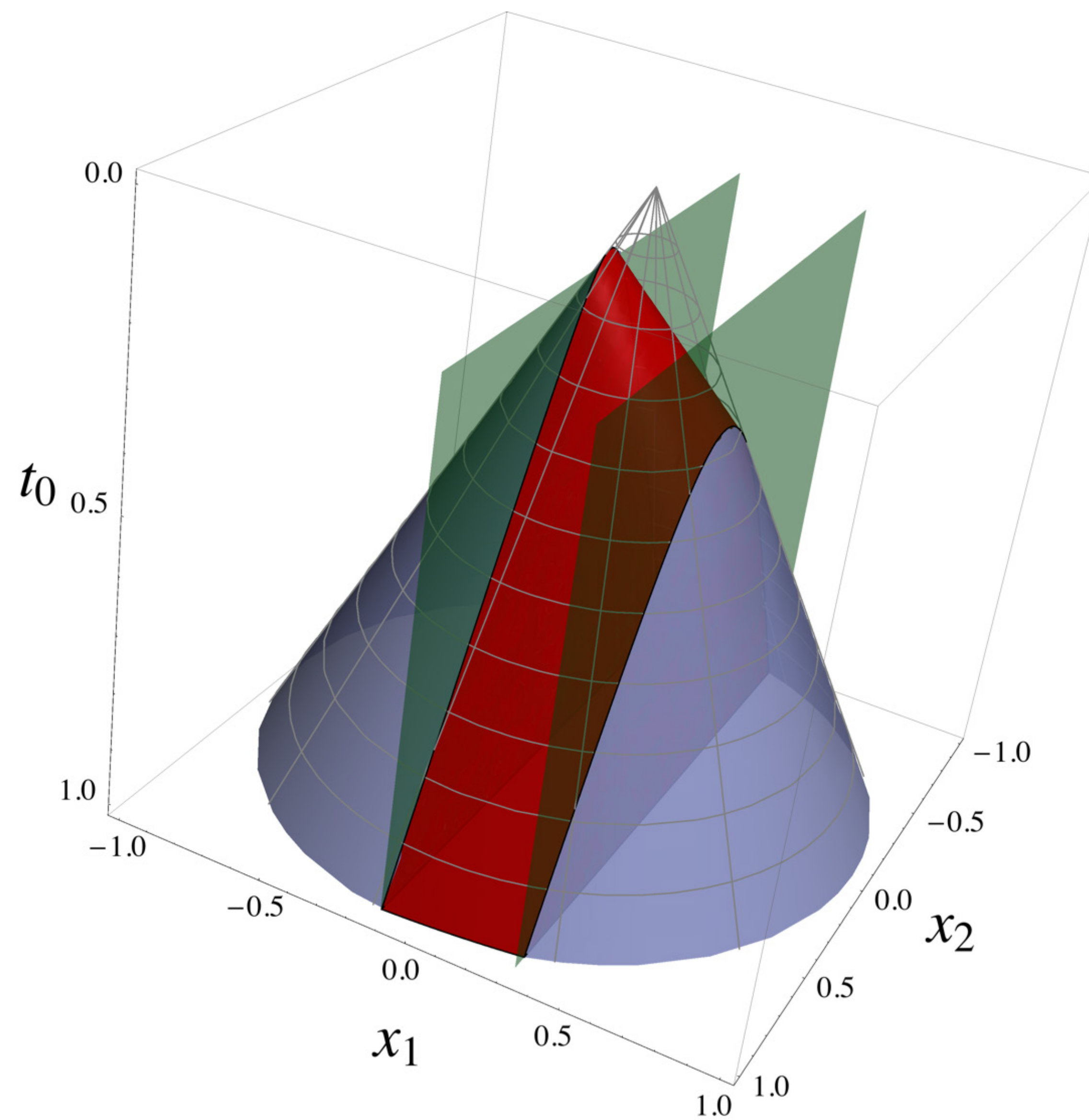

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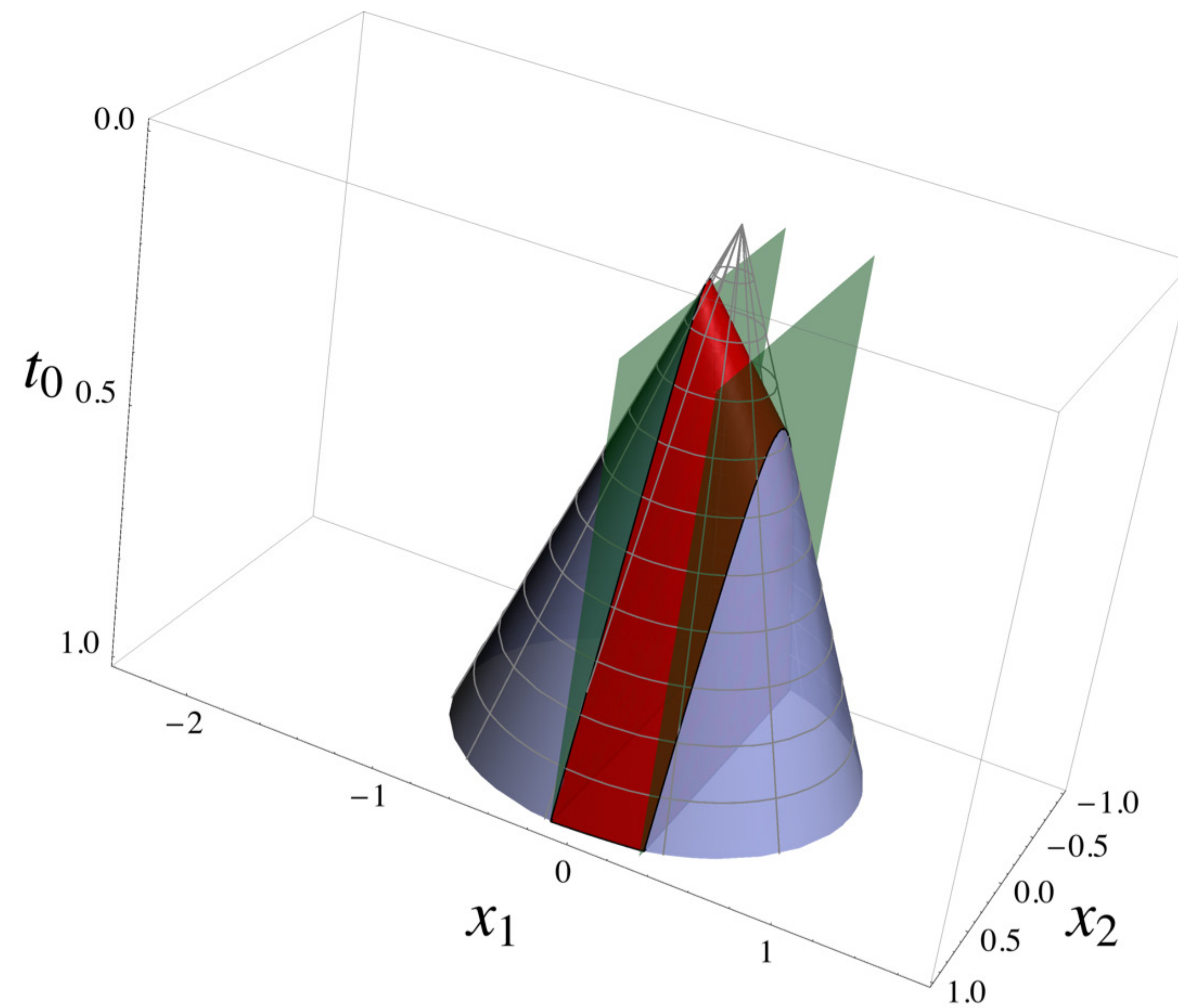

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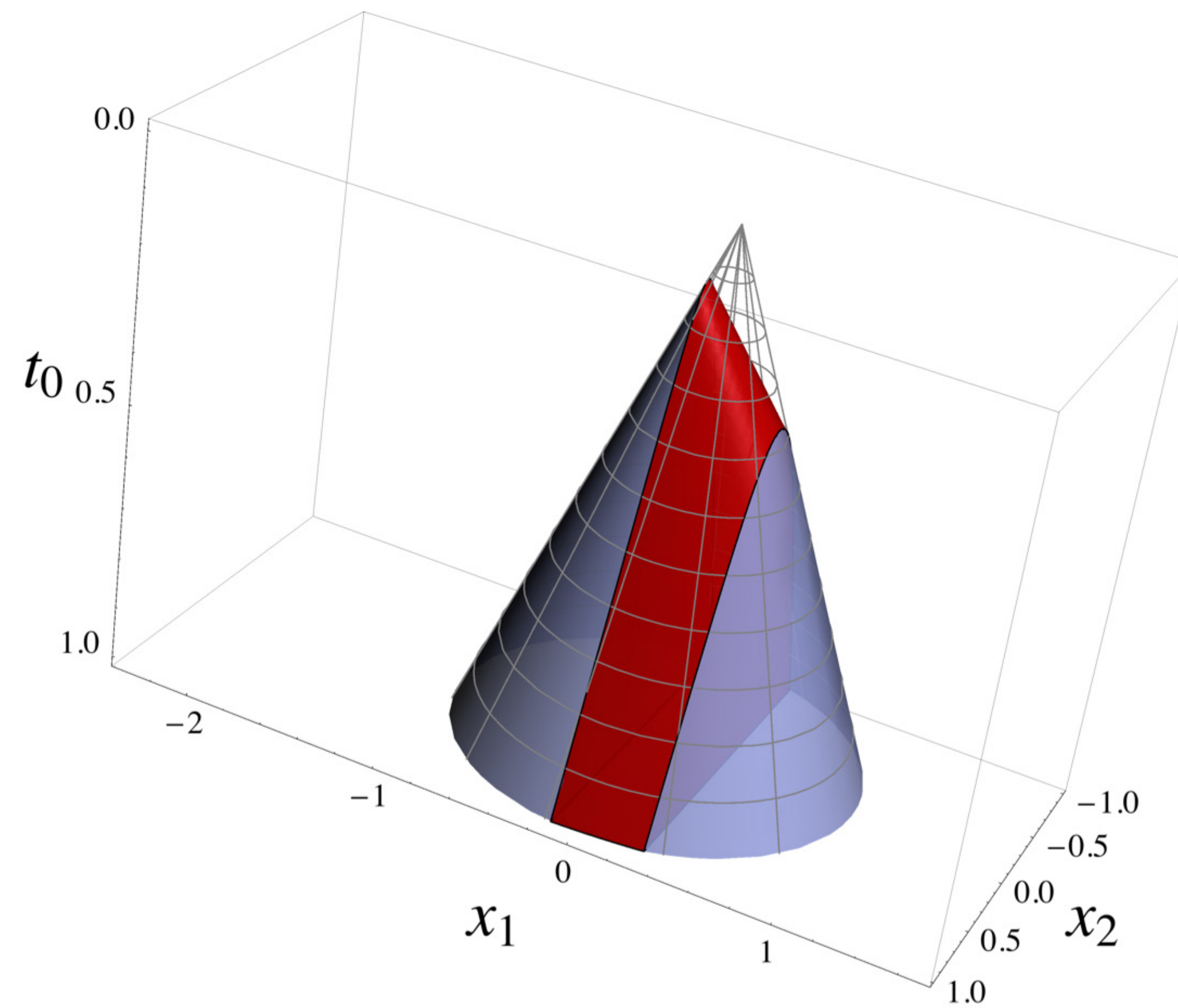

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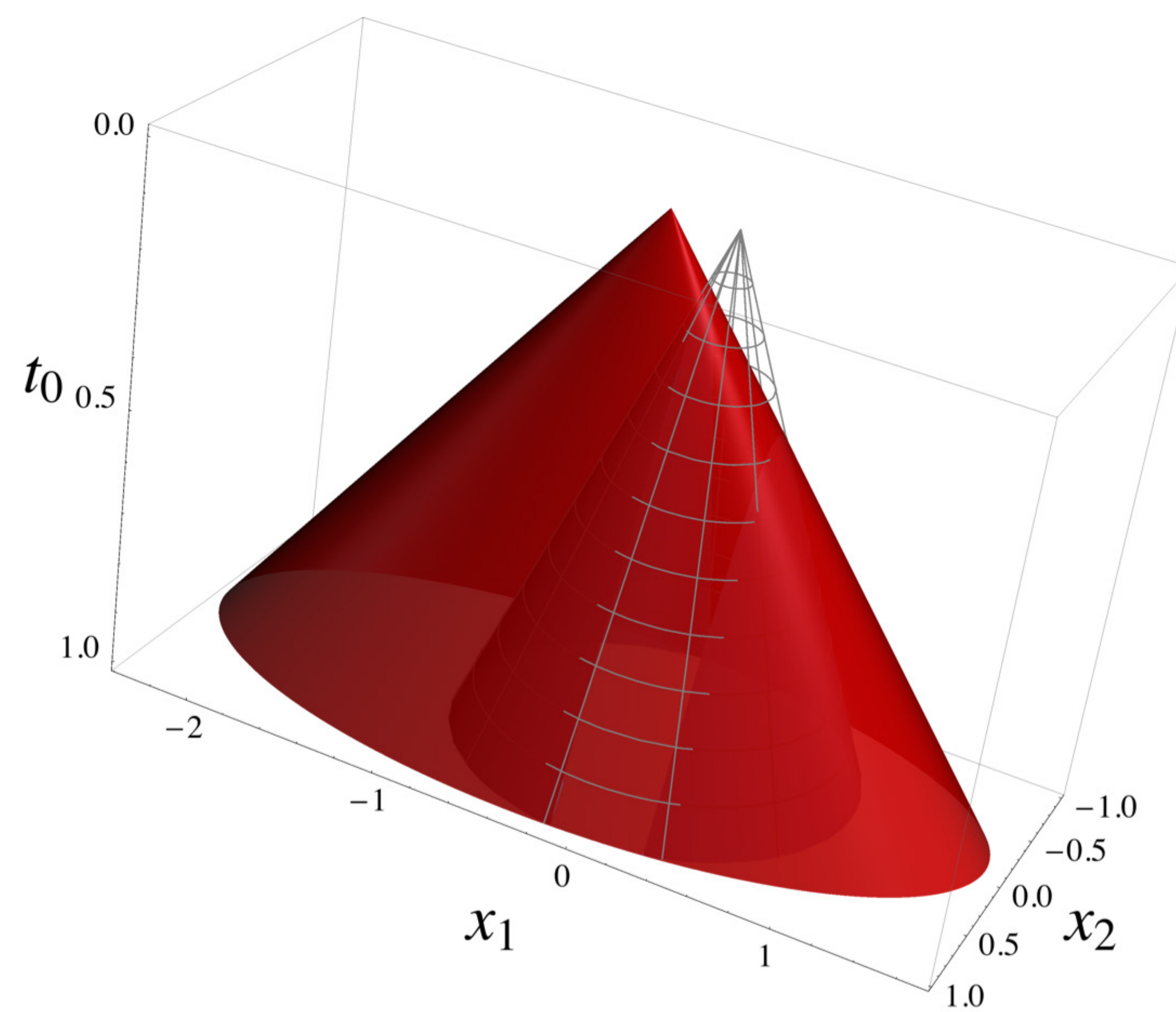

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Split Cuts for Quadratic Cones

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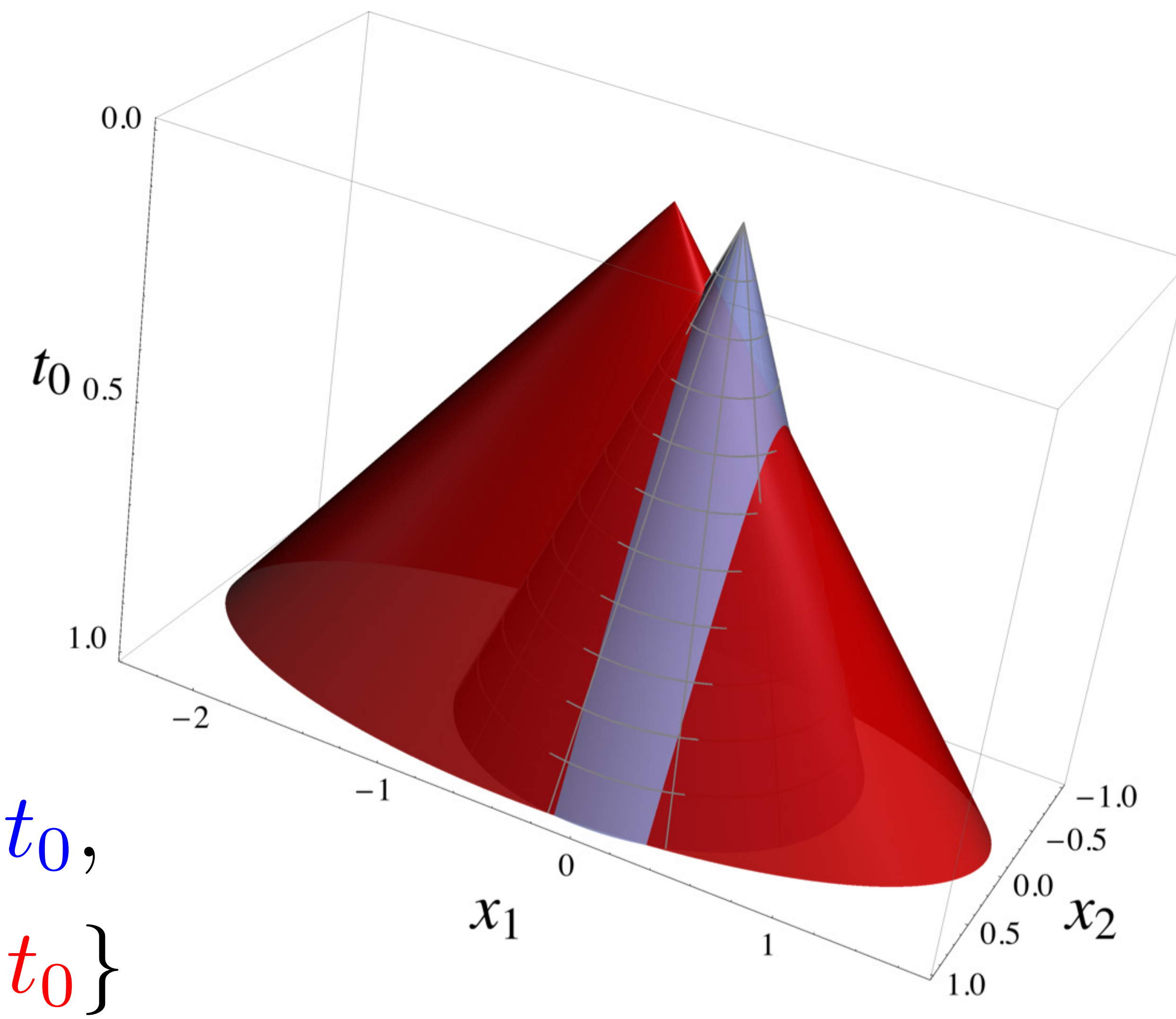
$$C := \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|A(x-c)\|_2 \le t_0\}$$

$$C^{\pi}_{\pi_0,\pi_1} := \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|A(x-c)\|_2 \le t_0, \|Bx-d\|_2 \le t_0\}$$



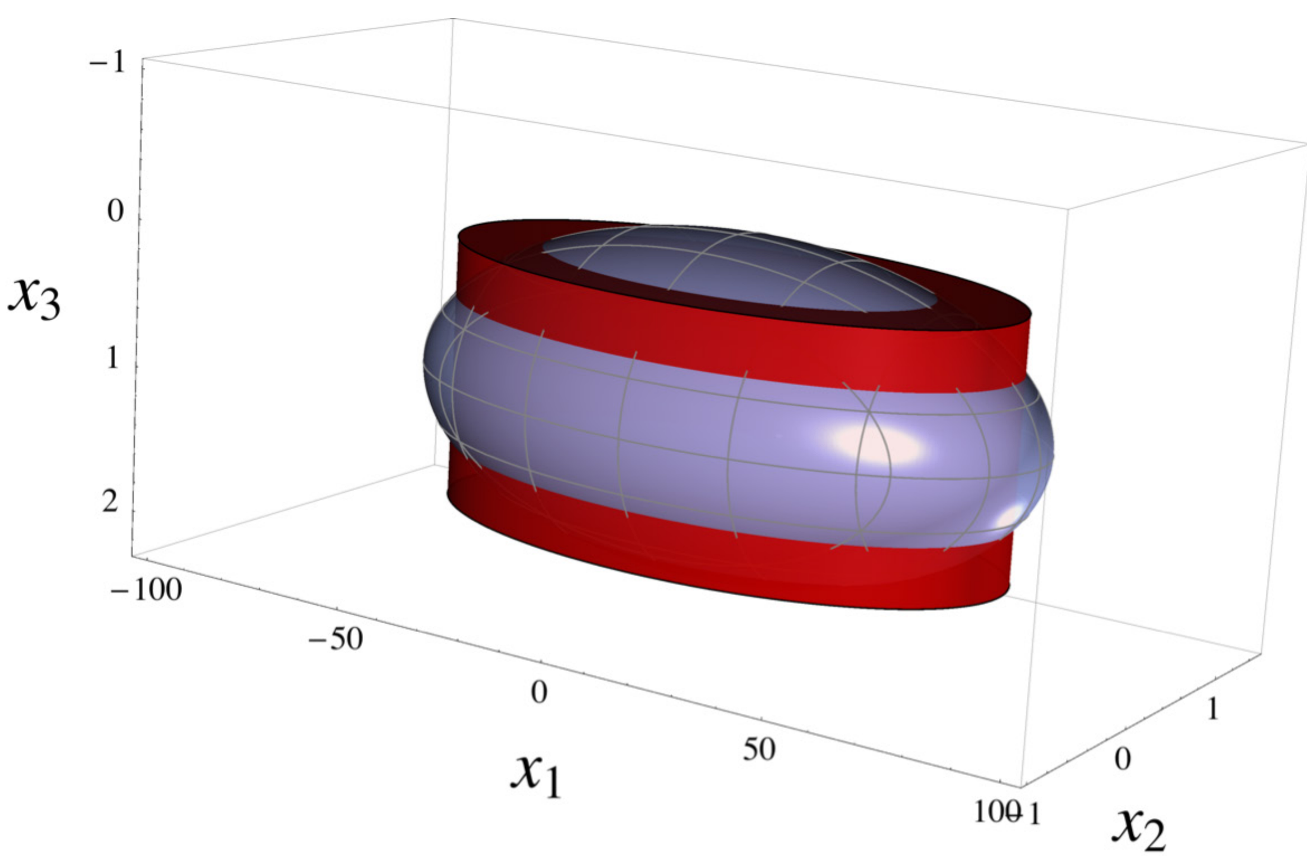
(also see Atamturk and Narayanan 2010 for elementary integer splits)

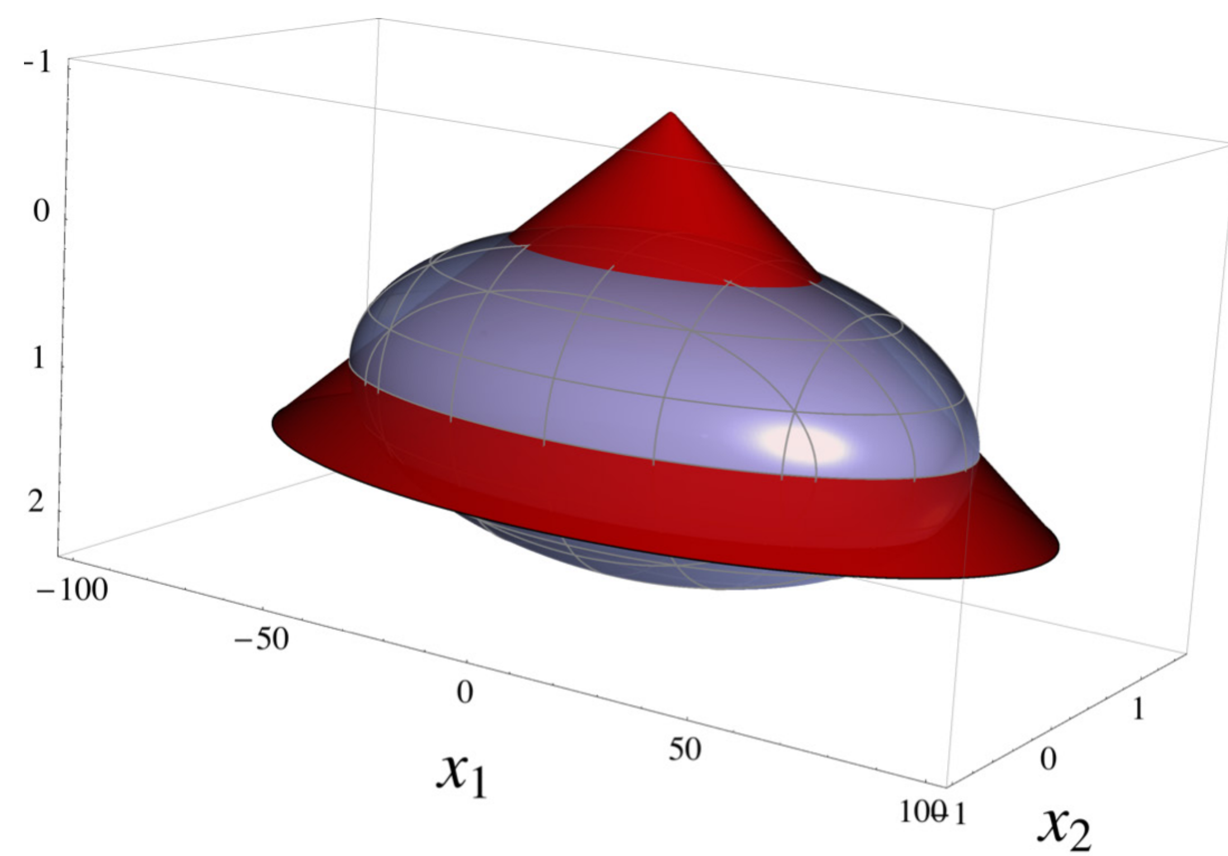
Split Cuts for Ellipsoids

- Formulas: (Dadush, Dey and V. 2011)

$$C := \{x \in \mathbb{R}^n \ : \ \|A(x-c)\|_2 \leq 1\}$$

$$C^{\pi}_{\pi_0,\pi_1} := \{x \in \mathbb{R}^n \ : \ \|A(x-c)\|_2 \leq 1, \quad \|Bx - d\|_2 \leq \langle a, x\rangle + b\}$$





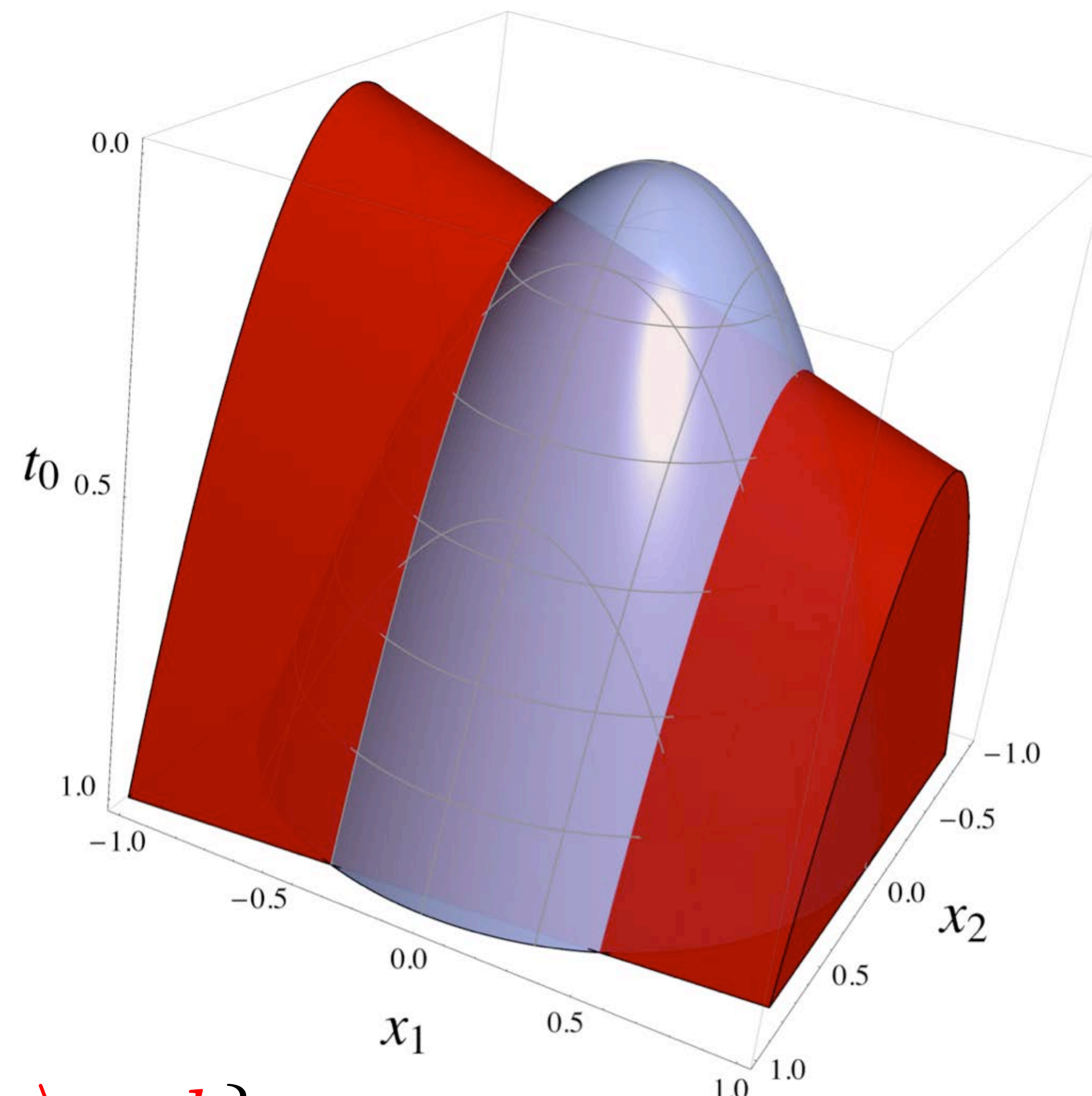
(also see Belotti, Góez, Polik, Ralphs, Terlaky 2011)

Split Cuts for Paraboloids

- Formulas: (Modaresi, Kılınç, V. 2012)

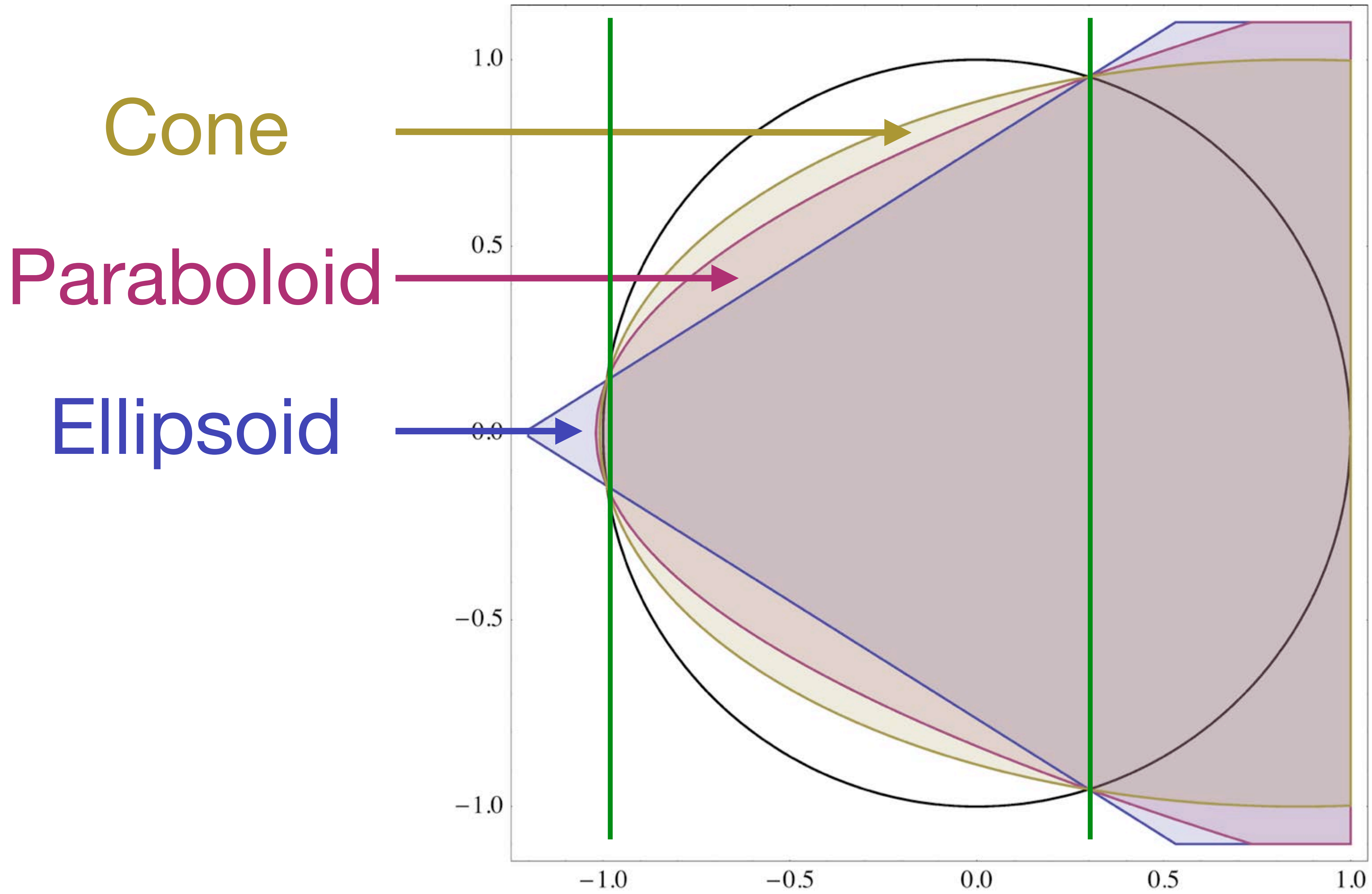
$$C := \left\{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|A(x-c)\|_2^2 \leq t_0\right\}$$

$$C^{\pi}_{\pi_0,\pi_1} := \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|A(x-c)\|_2^2 \leq t_0, \|B(x-d)\|_2^2 \leq t_0 + \langle a, x\rangle + b\}$$



(also see Belotti, Góez, Polik, Ralphs, Terlaky 2012 for bounded case)

Message: Use the Right Split Cut



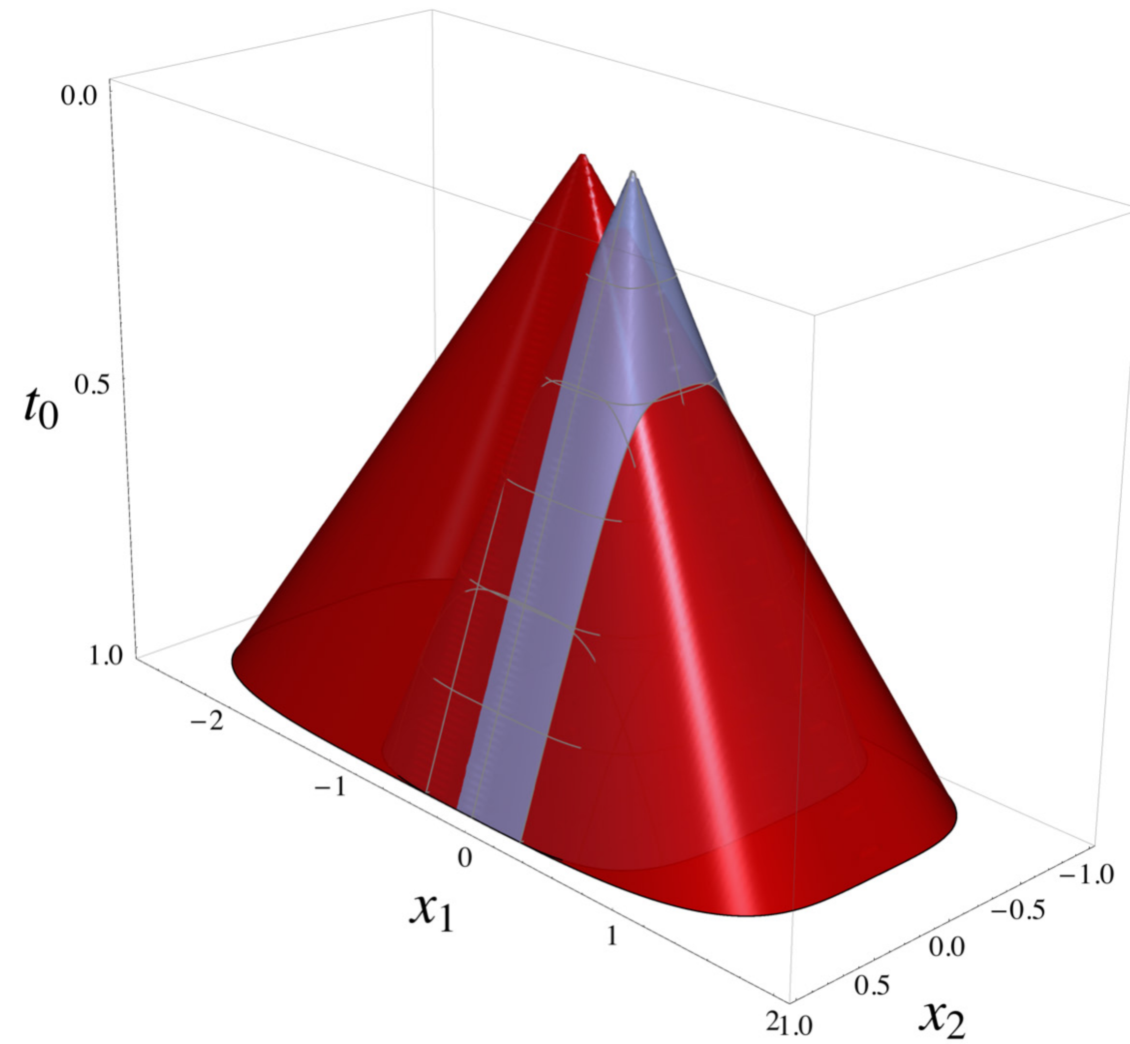
Split Cuts for P-Order Cones

- Formulas: (Modaresi, Kılınç, V. 2011)

Elementary splits: $\pi = e^i$

$$C := \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|x - c\|_p \le t_0\}$$

$$C^{\pi}_{\pi_0,\pi_1} := \Big\{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|x - c\|_2 \le t_0, \left|(\alpha(x_1 - d_1) + \beta)^p + \sum_{i=2}^{n} (x_i - d_i)^p\right| \le t_0^p\Big\}$$



Conic MIR

- Atamturk and Narayanan 2010

$$C := \left\{ (x, t_0) \in \mathbb{Z}^n \times \mathbb{R} \,:\, \|A(x-c)\|_2 \leq t_0, \quad x \geq 0 \right\}$$

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$$C := \{(x, t_0) \in \mathbb{Z}^n \times \mathbb{R} \; : \; \|A(x-c)\|_2 \leq t_0, \quad x \geq 0\}$$

Extended Formulation: $(x, t, t_0) \in \mathbb{Z}^n \times \mathbb{R}^n \times \mathbb{R}_+$

$$\begin{aligned} |A(x-c)| &\leq t \\ x &\geq 0 \\ \|t\|_2 &\leq t_0 \end{aligned}$$

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$$\left.\begin{aligned} |A(x-c)| &\leq t \\ x &\geq 0 \end{aligned}\right\} \longleftarrow \text{Linear Part}$$

$$\|t\|_2 \leq t_0$$

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$$\left.\begin{aligned} |A(x-c)| &\leq t \\ x &\geq 0 \end{aligned}\right\} \longleftarrow \text{Linear Part}$$

$$\|t\|_2 \leq t_0 \longleftarrow \text{Nonlinear Part}$$

Aggregate: $|ax + y_0 - z_0 - b| \leq s_0, \quad y_0, z_0, s_0 \geq 0, \quad x \in \mathbb{Z}_+^n$

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Aggregate: $|ax + y_0 - z_0 - b| \leq s_0, \quad y_0, z_0, s_0 \geq 0, \quad x \in \mathbb{Z}^n_+$

Conic MIR: $\sum_{i=1}^n \varphi_f(a_j)x_j - \varphi_f(b) \leq s_0 + y_0 + z_0$

$$\varphi_f(a) := \lfloor a \rfloor + (a - \lfloor a \rfloor - f)^+/(1-f)$$

Conic MIR and Nonlinear Split Cut

- Modaresi, Kılınç, V. 2011

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Extended Formulation: $(x, t, t_0) \in \mathbb{Z}^n \times \mathbb{R}^n \times \mathbb{R}_+$

$|A(x - c)| \le t$ ← Linear Part

$\|t\|_2 \le t_0$ ← Nonlinear Part

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$|A(x - c)| \leq t$ ← Linear Part

$\|t\|_2 \leq t_0$ ← Nonlinear Part

Conic MIR = Split cuts for linear part ≠ Nonlinear split cut

$$(1 - 2f)\left(\lambda^T A x - \lfloor \lambda^T B c \rfloor\right) + f \leq |\lambda|^T t$$

$$\lambda \in \mathbb{R}^n, \quad A^T \lambda \in \mathbb{Z}^n, \quad f := \lambda^T B c - \lfloor \lambda^T B c \rfloor$$

Nonlinear Split Cut v/s MIR Strength

- Single Split Cut is Stronger than MIR

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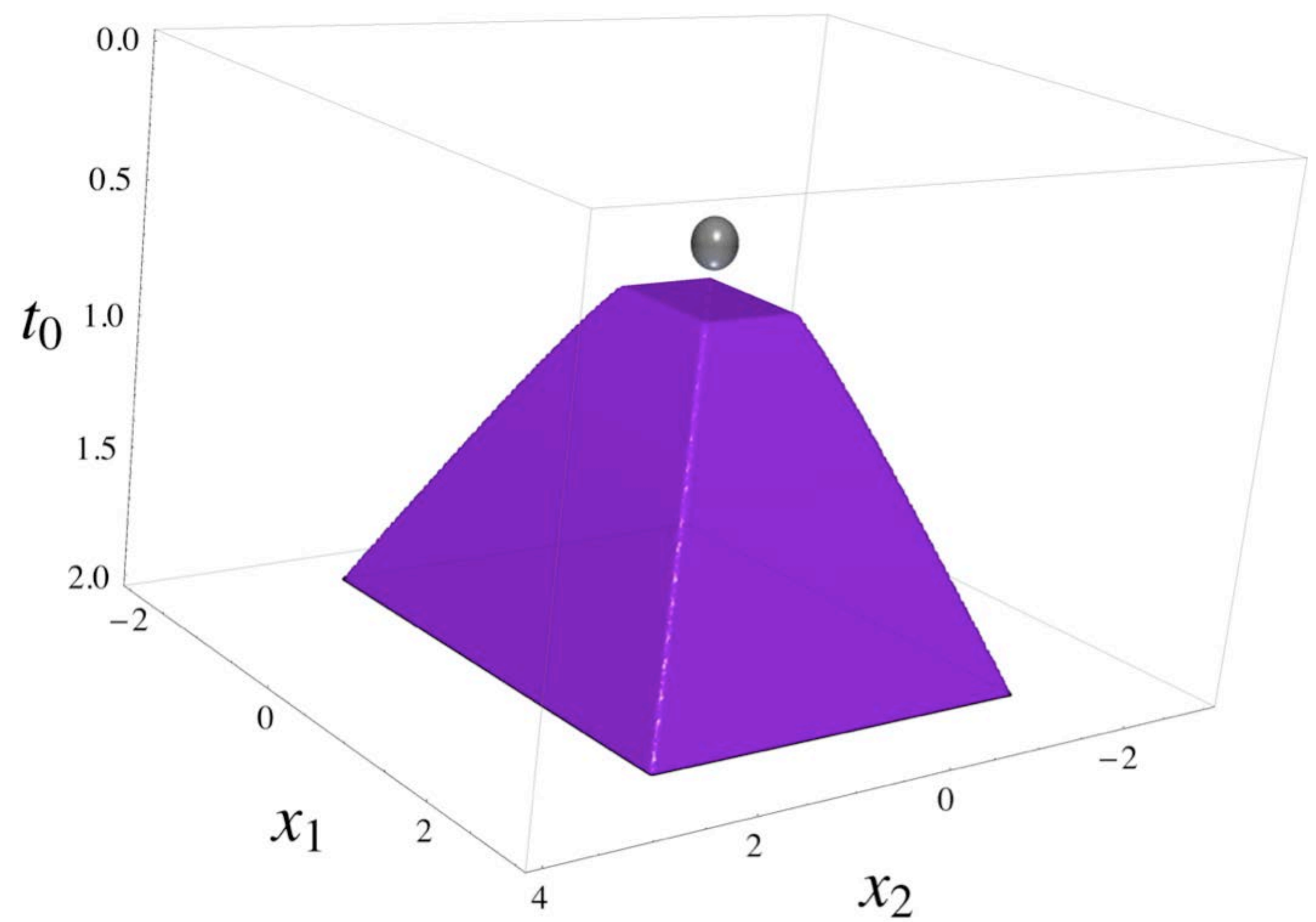
- Single Split Cut is Stronger than MIR



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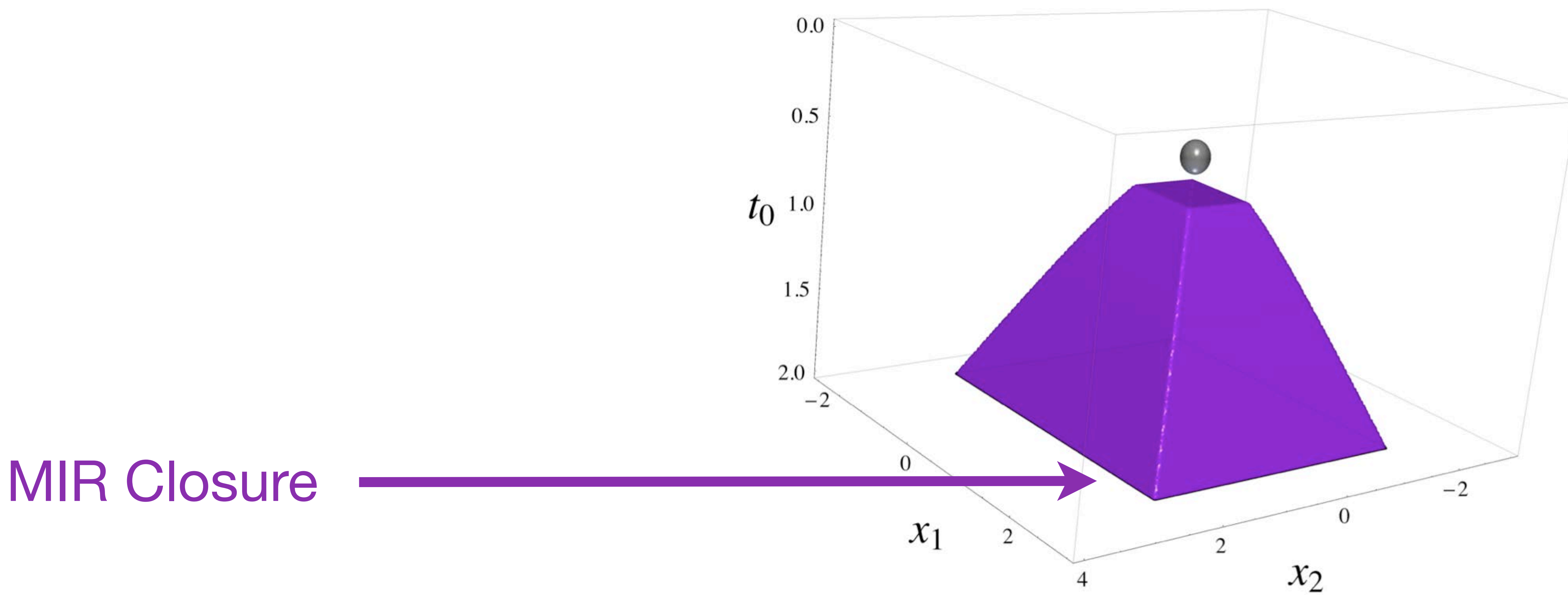

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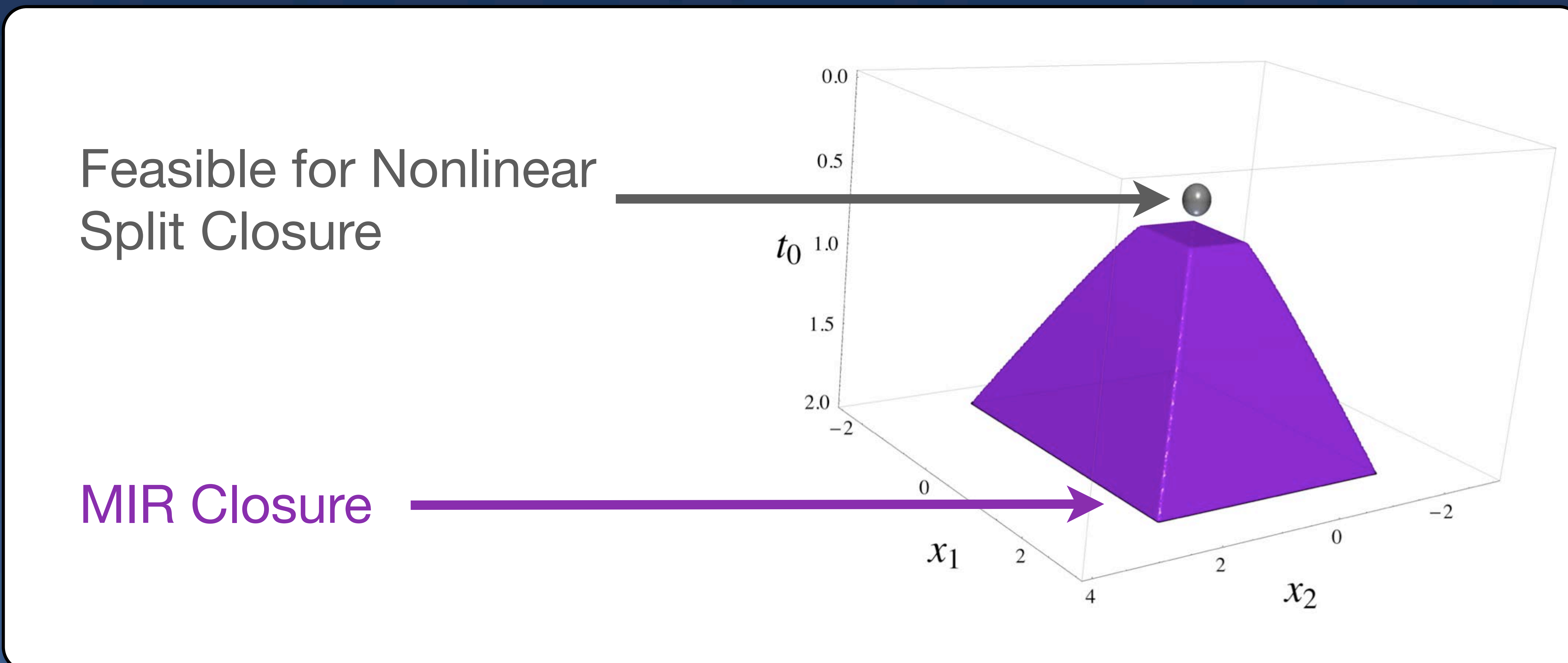


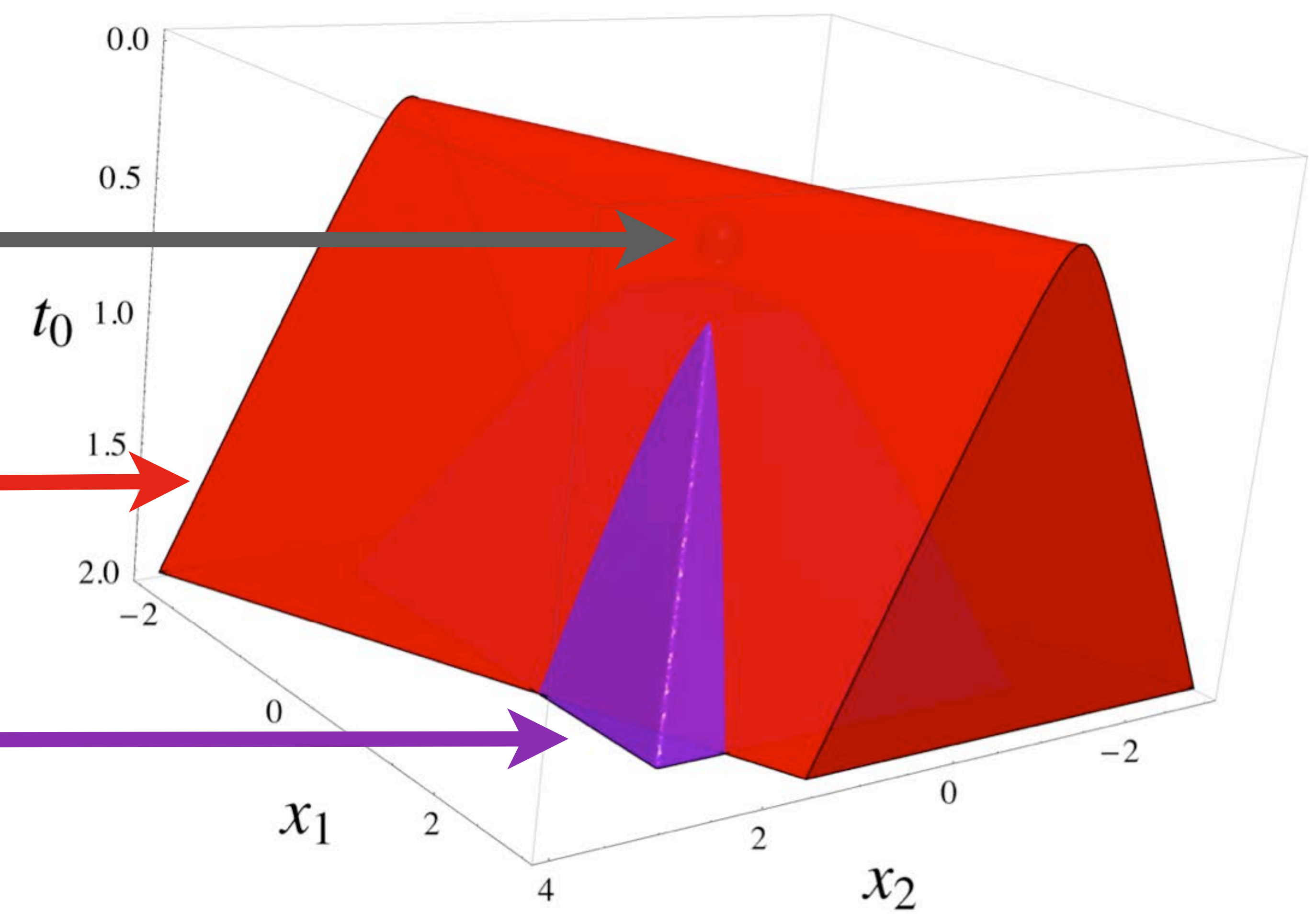
Nonlinear Split Cut v/s MIR Strength

- Single Split Cut is Stronger than MIR

Feasible for Nonlinear Split Closure

Nonlinear Split Cut

MIR Closure



Summary and Open Questions

- Formulas for nonlinear split cuts
  - Quadratic cones, ellipsoids and others.
  - Strong ties to conic MIR.
- Future:
  - Computation (INFORMS Phoenix).
  - Extended Formulations (INFORMS Phoenix).
  - More formulas.