Split Cuts for Convex Nonlinear Mixed Integer Programming

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joint work with

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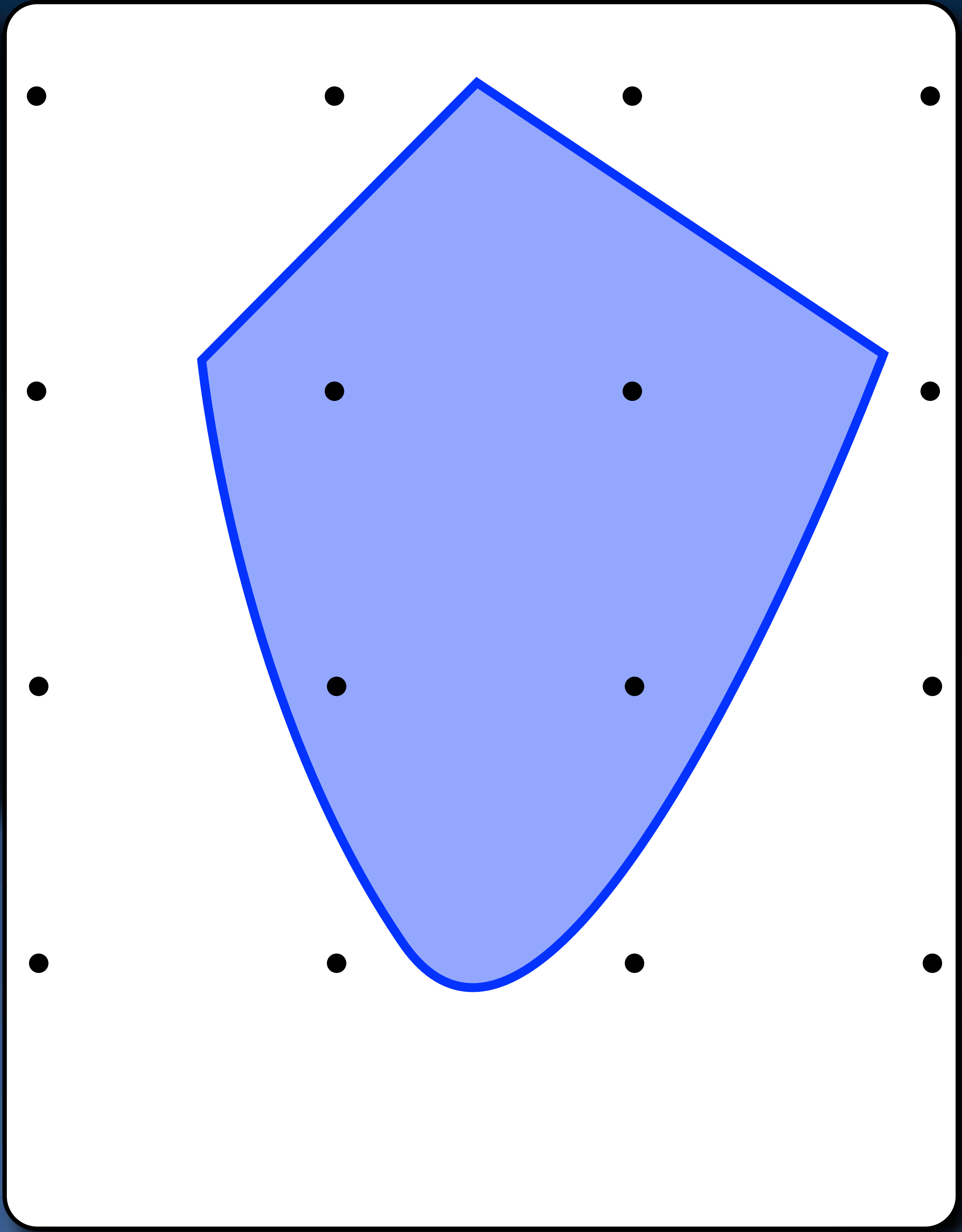
Outline

- Introduction
- Split Cut Formulas
- Split Closure
- Conclusions

Split Disjunctions and Split Cuts



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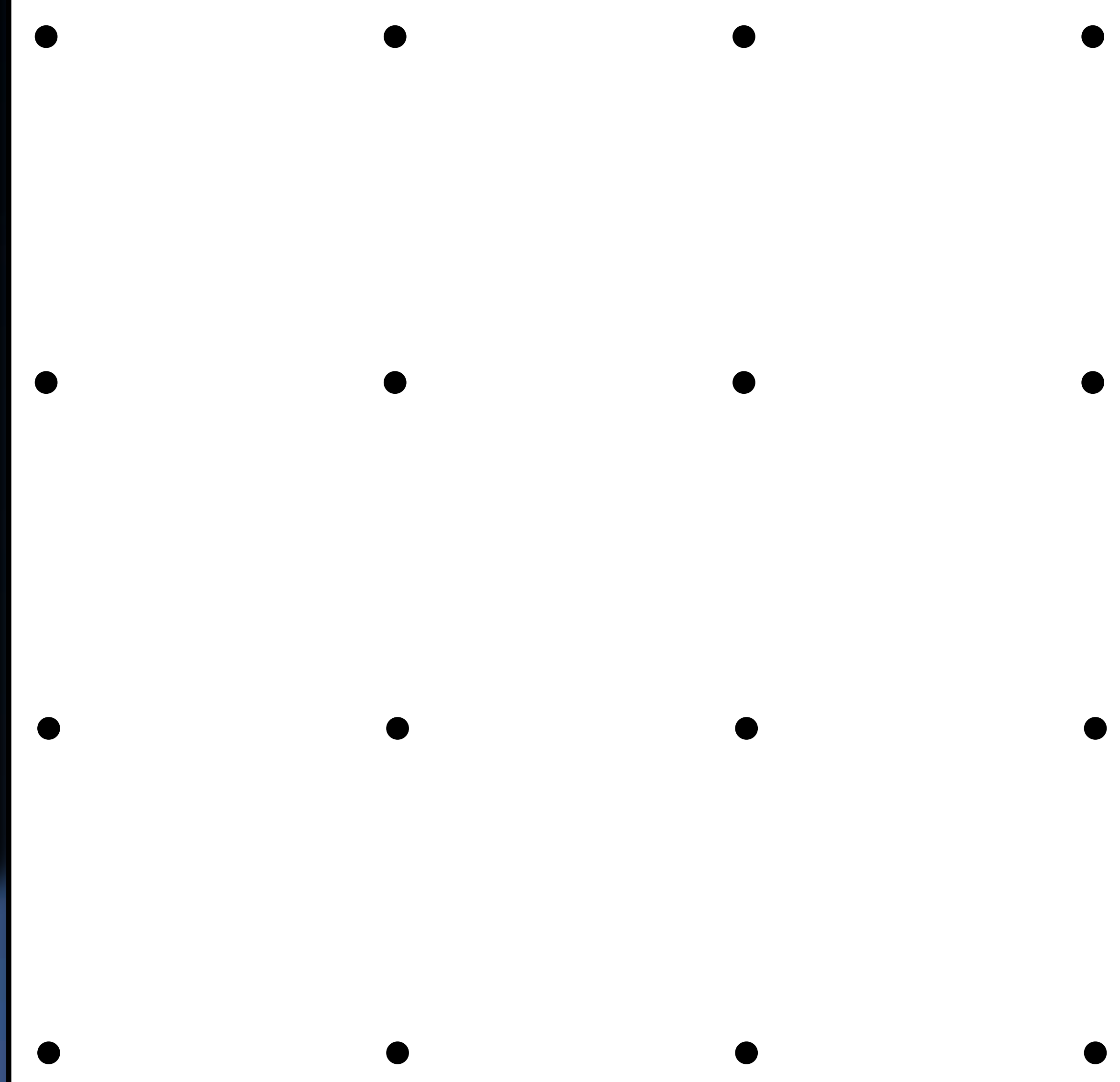

Split Disjunctions and Split Cuts

Split Disjunction

$$L^{\pi}_{\pi_0} = \{x \in \mathbb{R}^n \ : \ \langle \pi, x \rangle \leq \pi_0\}$$
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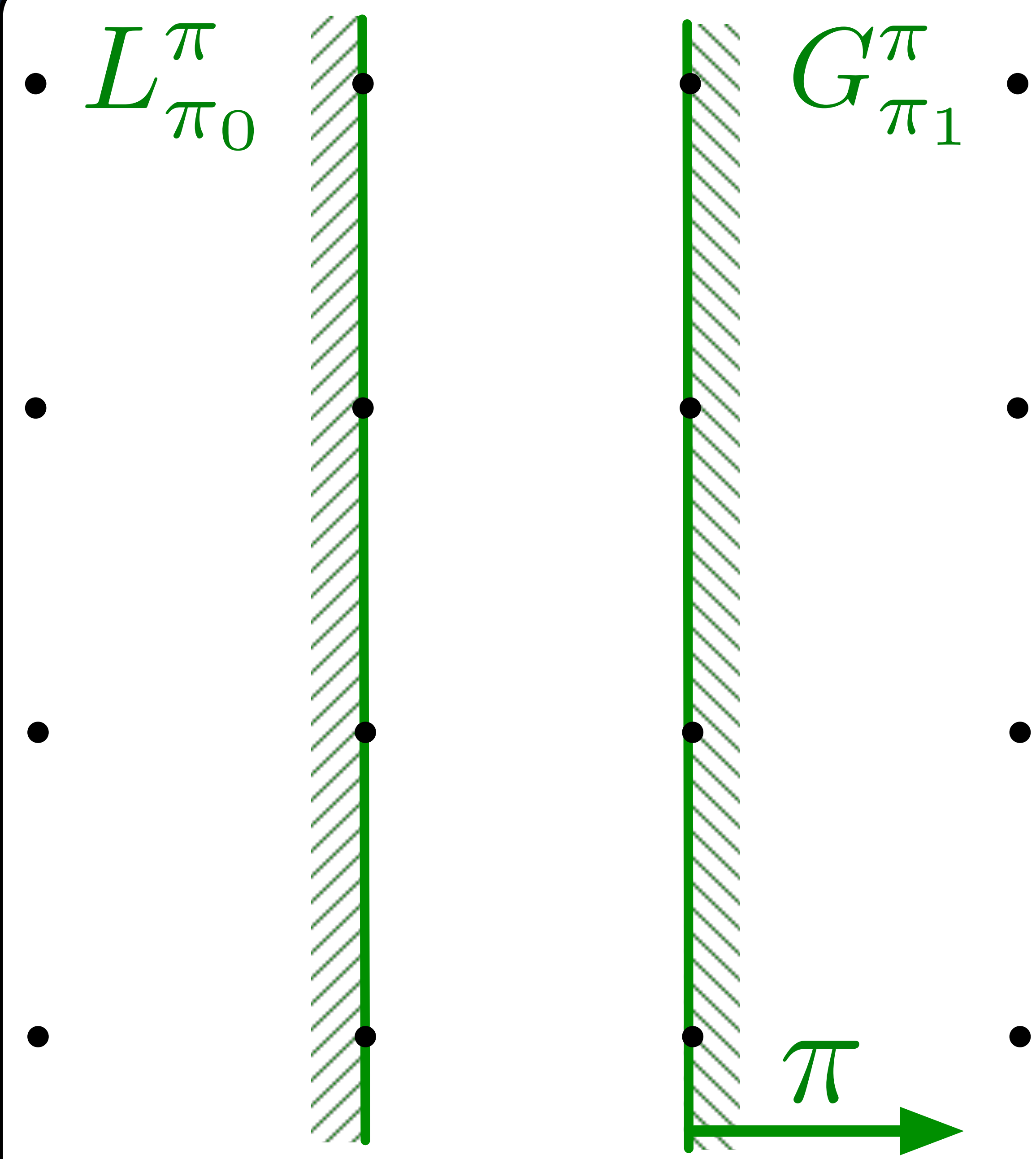

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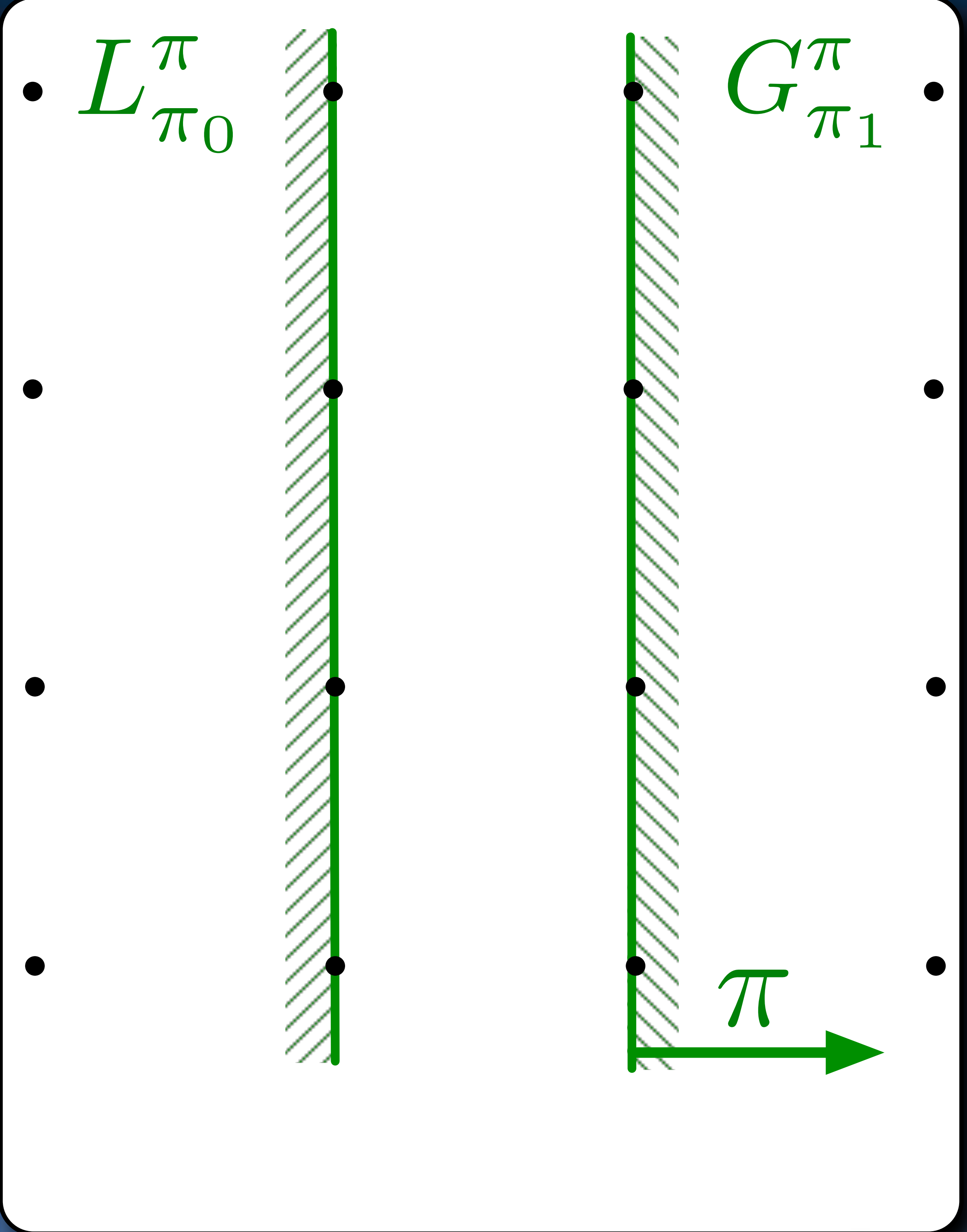

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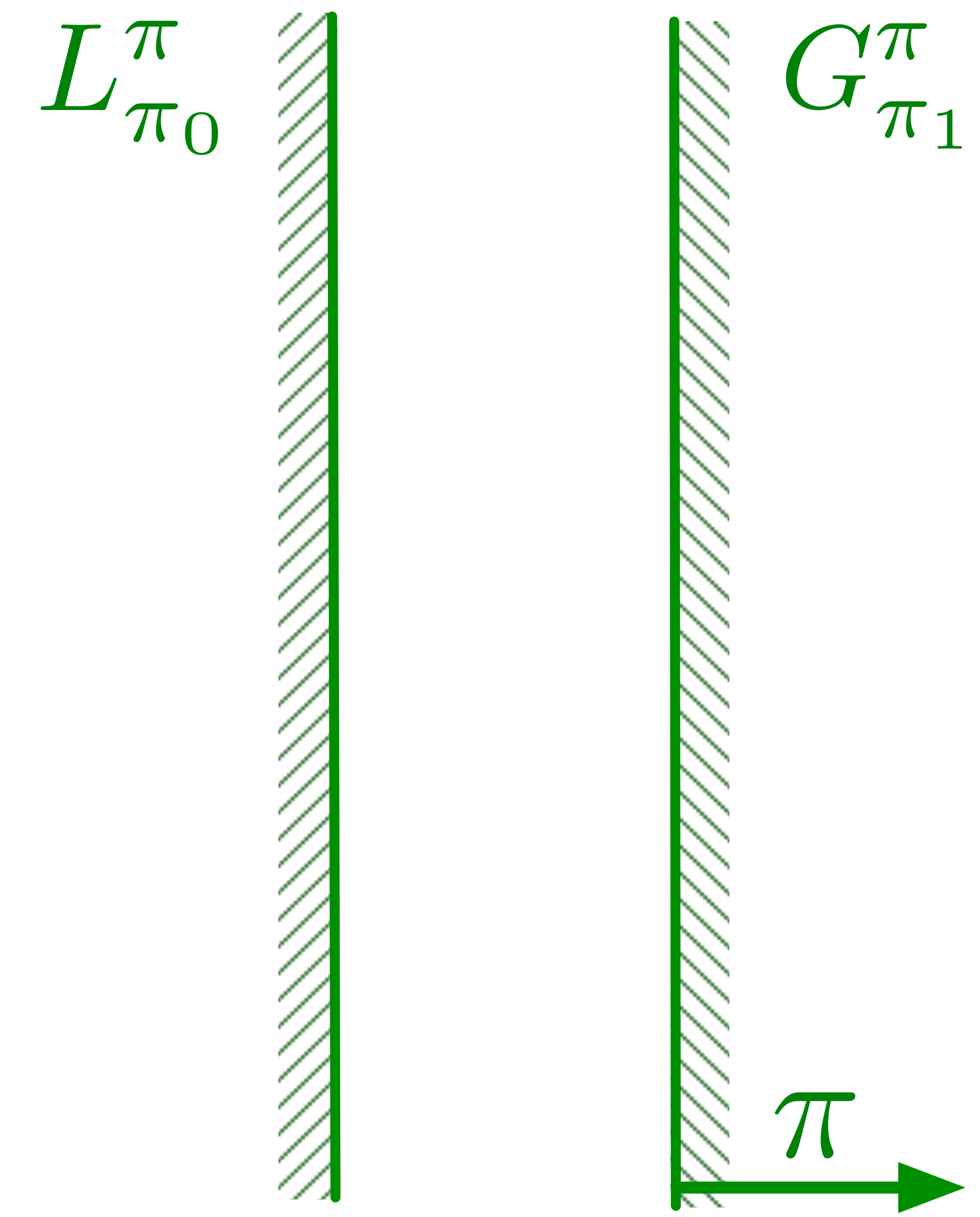

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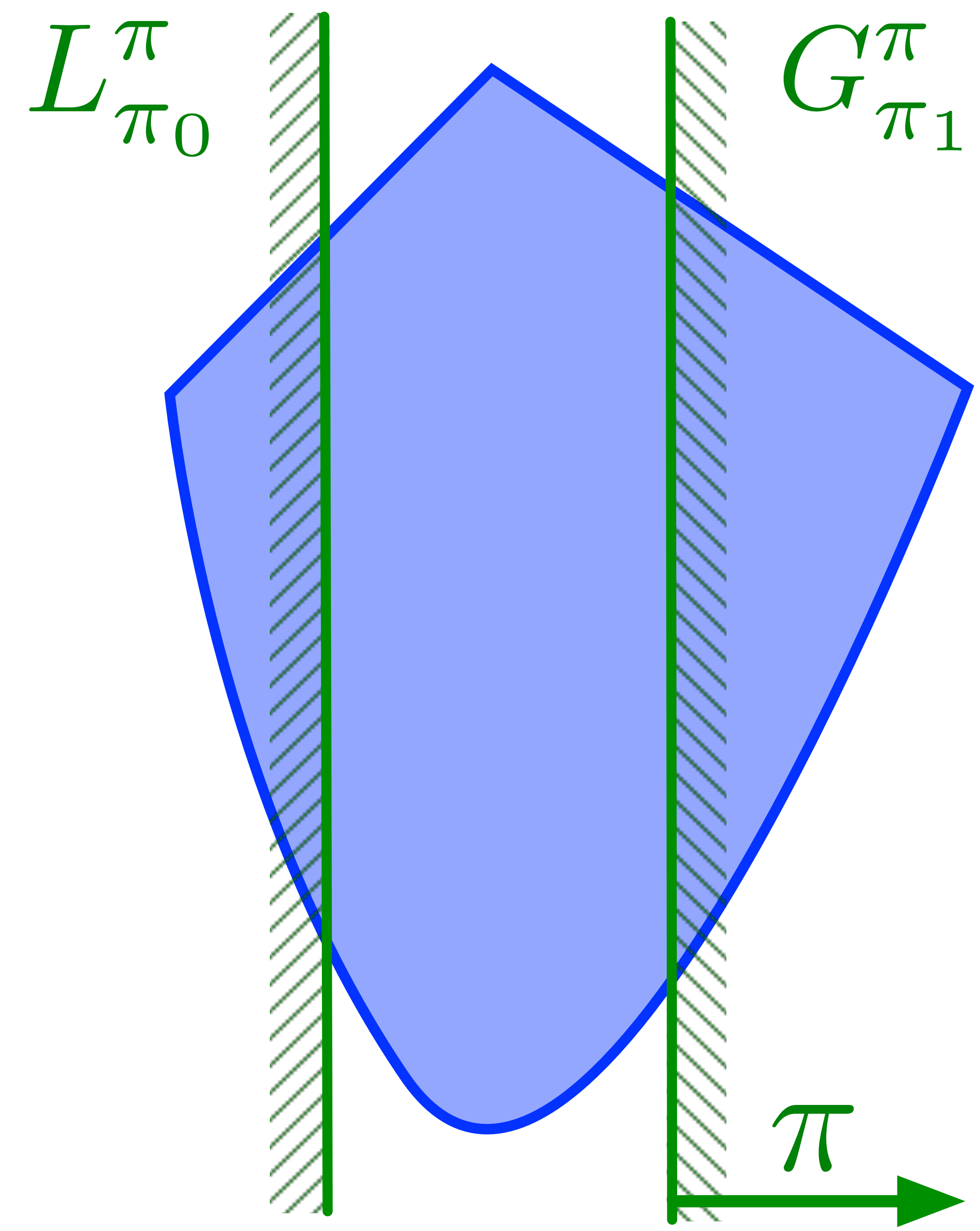

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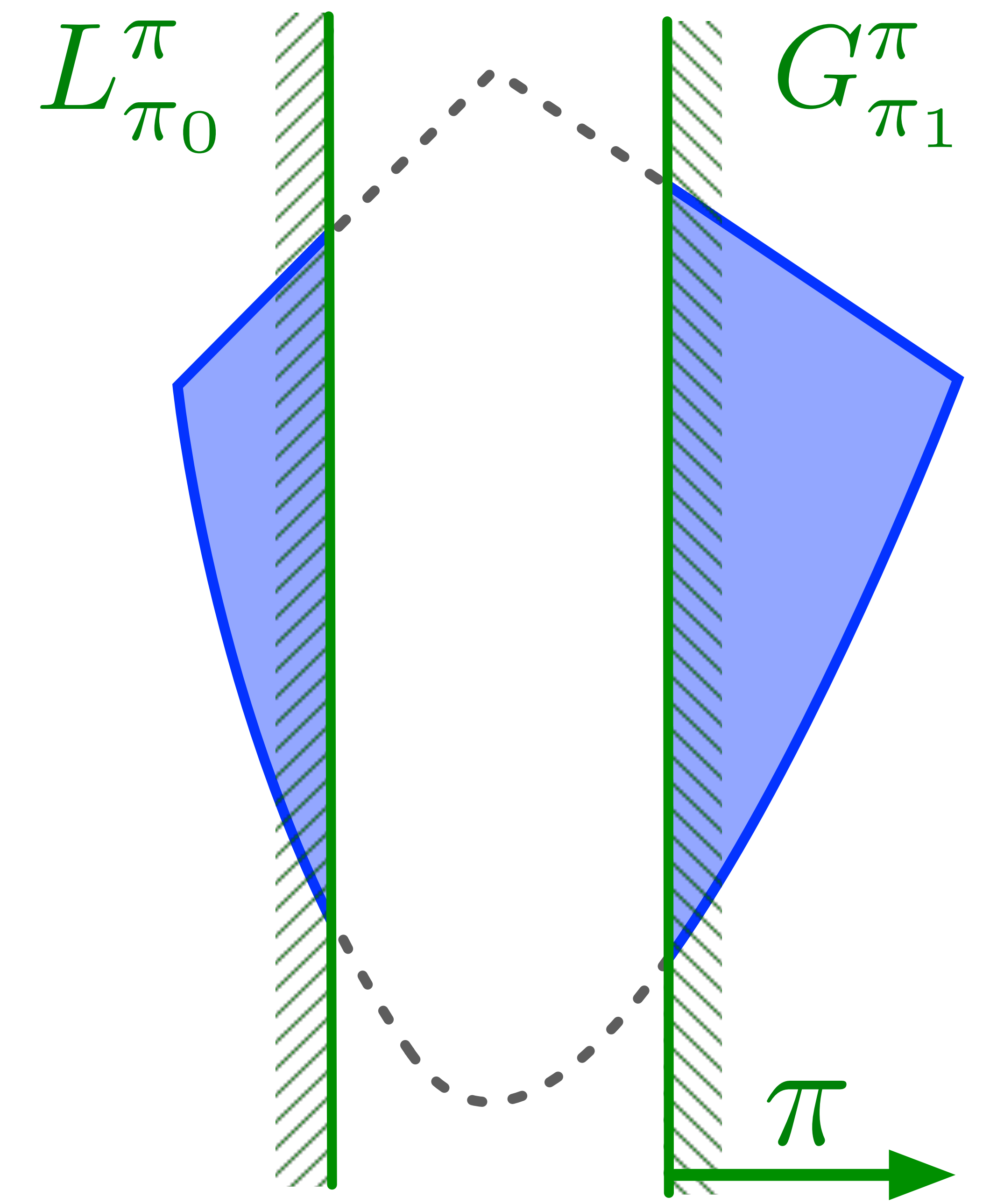

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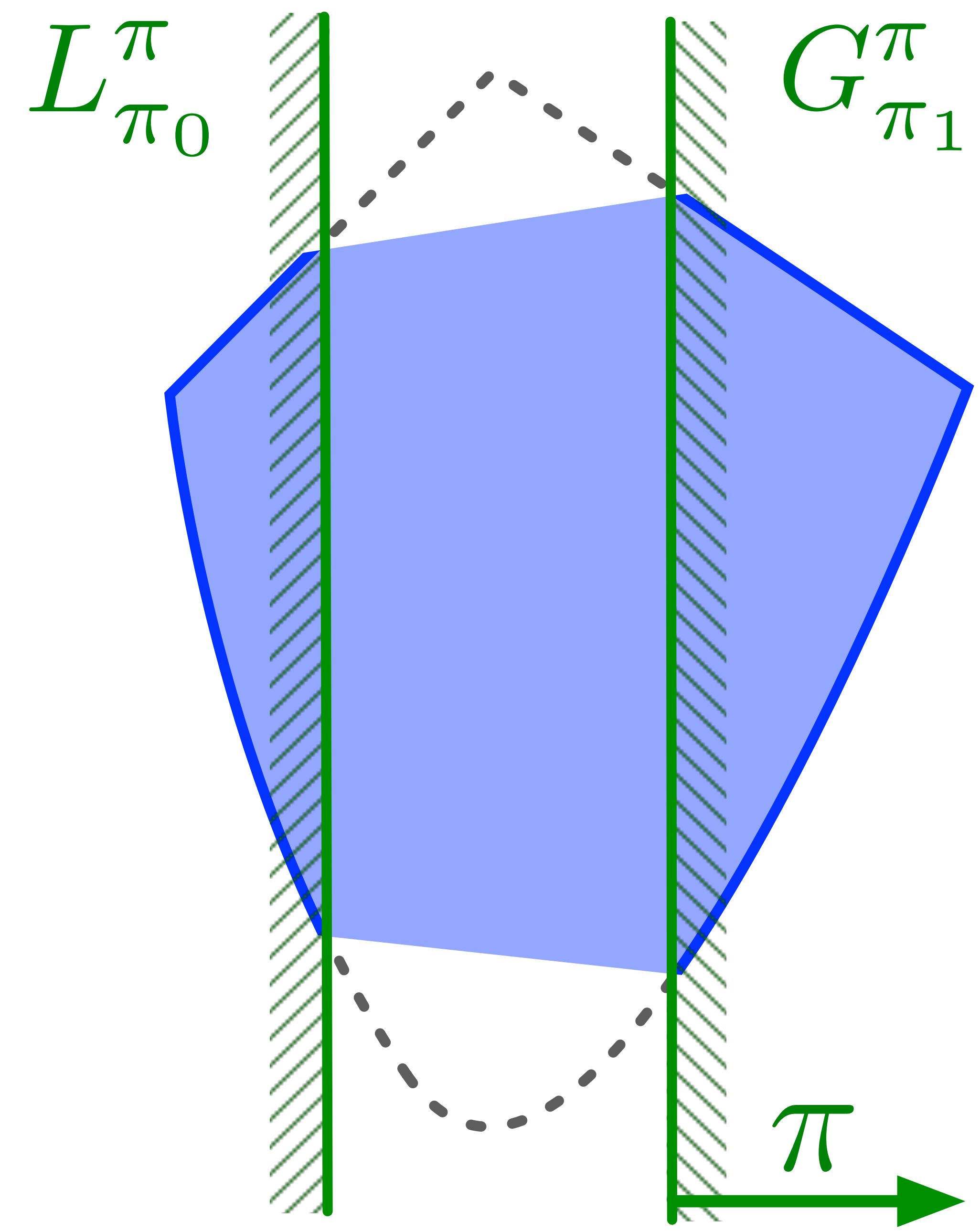

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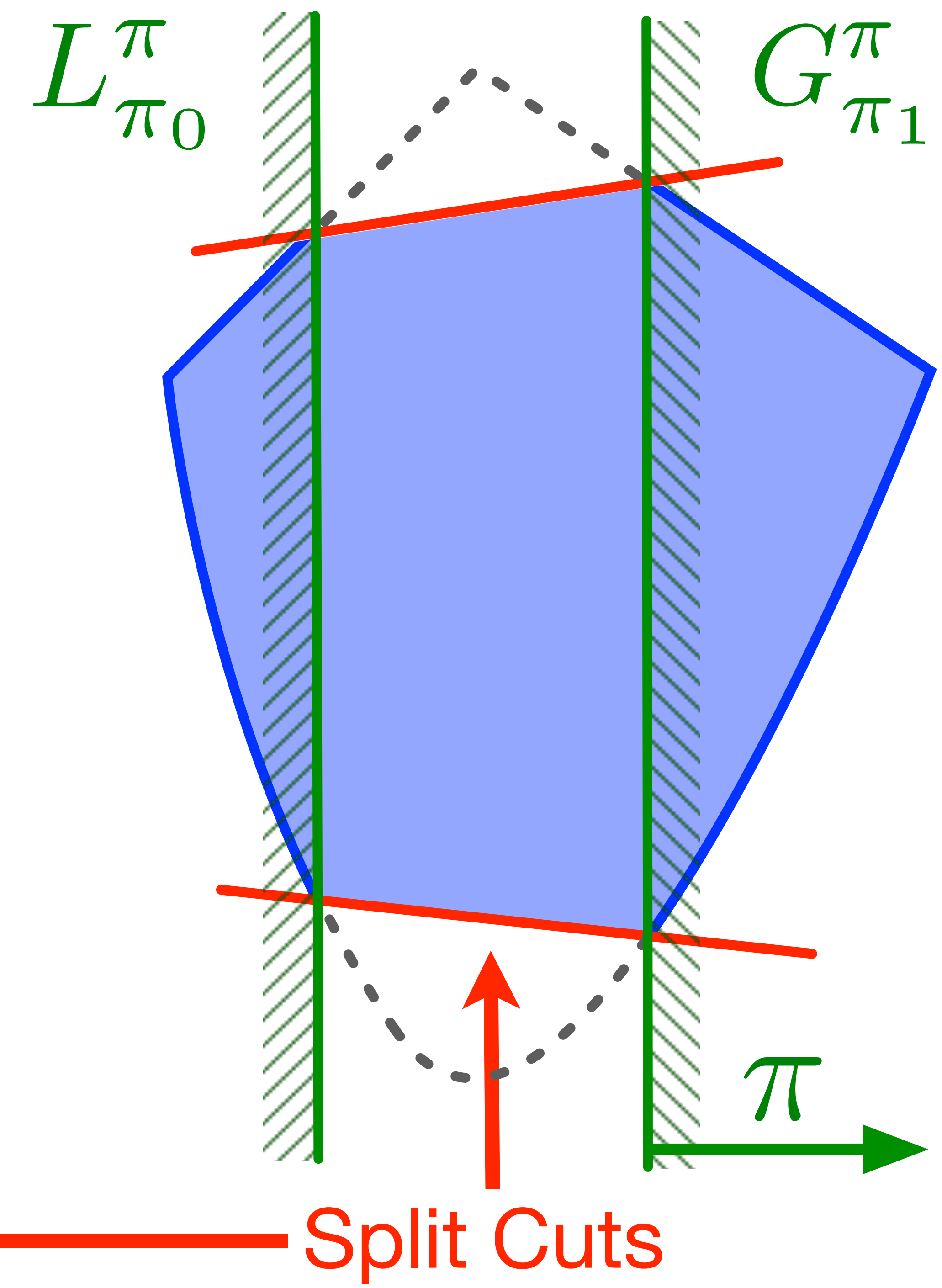

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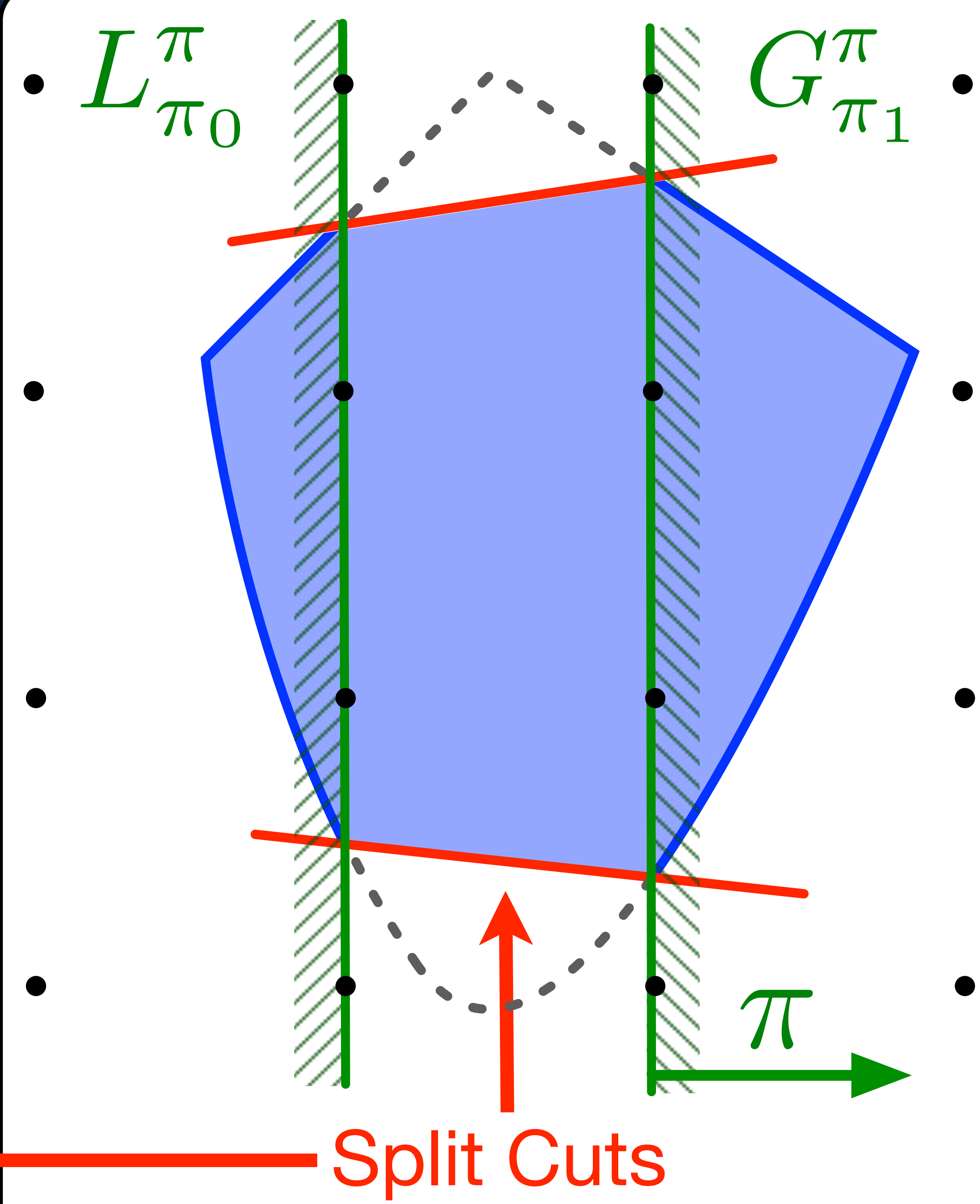
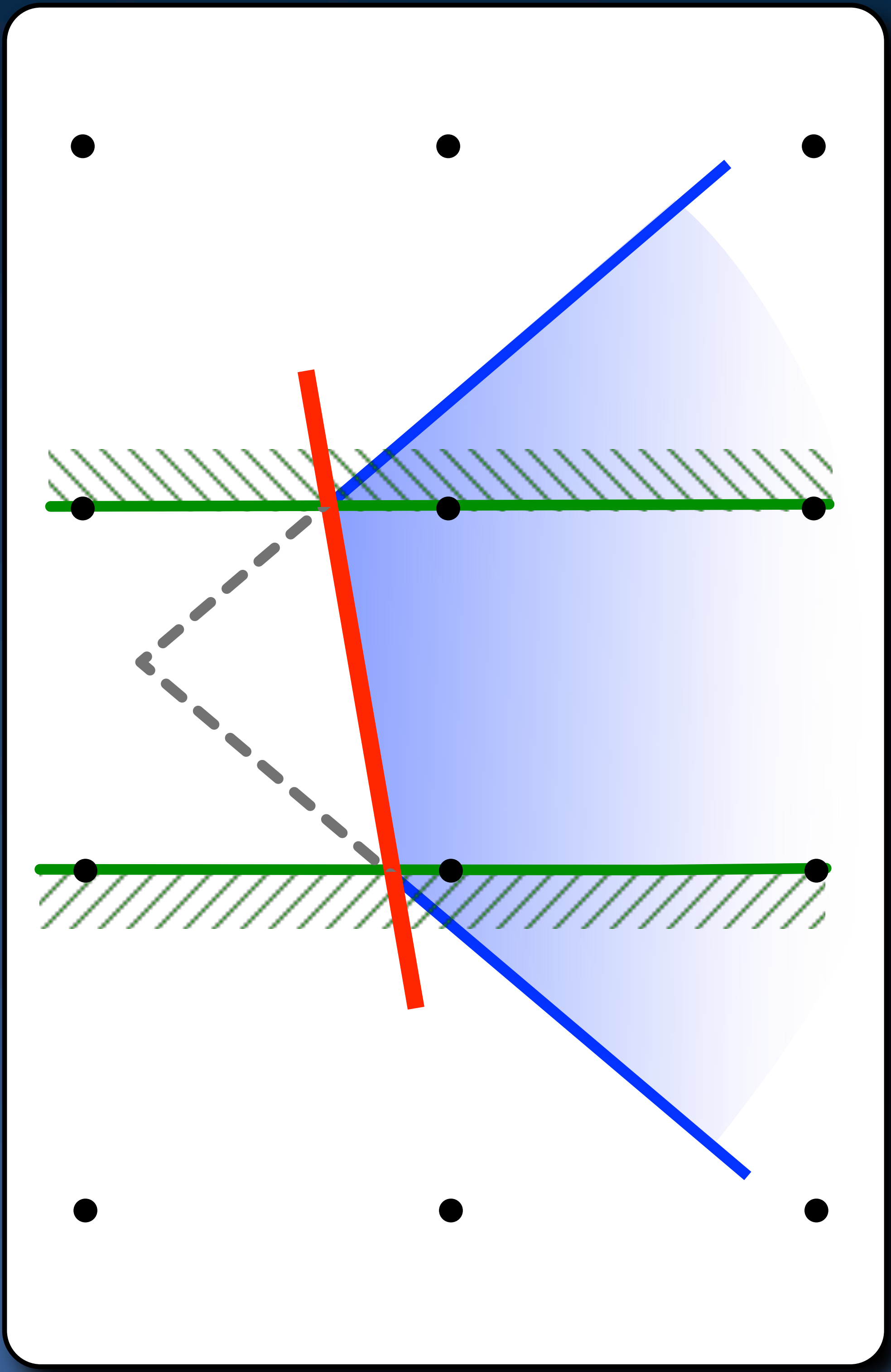


Known Facts for Rational Polyhedra

- Formulas for simplicial cones:
  - MIG (Gomory 1960) and MIR (Nemhauser and Wolsey 1988)
- Split Closure $\bigcap_{(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Z}} C^{\pi}_{\pi_0,\pi_0+1}$:
  - Rational Polyhedron (Cook, Kannan and Shrijver 1990)
  - Constructive Proofs:
    - Dash, Günlük and Lodi 2007; V. 2007



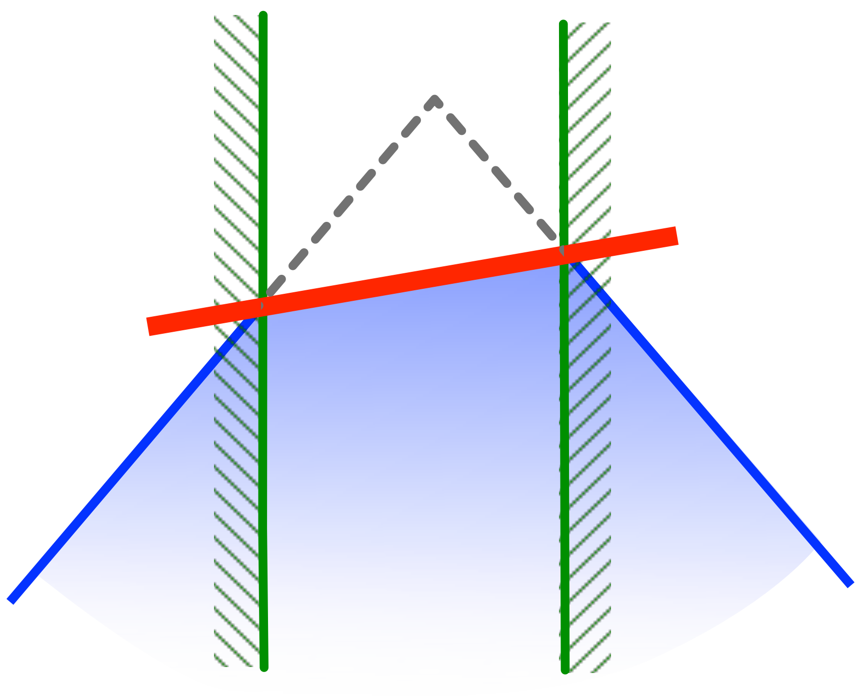
Split Cuts for Simplicial Cones

- Formulas: (MIG: Gomory 1960 and MIR: Nemhauser and Wolsey 1988)

$$C := \{x \in \mathbb{R}^n \ : \ Ax \le b\},$$

$$\det(A) \neq 0$$

$$\pi_0 < \langle \pi, A^{-1}b \rangle < \pi_1$$



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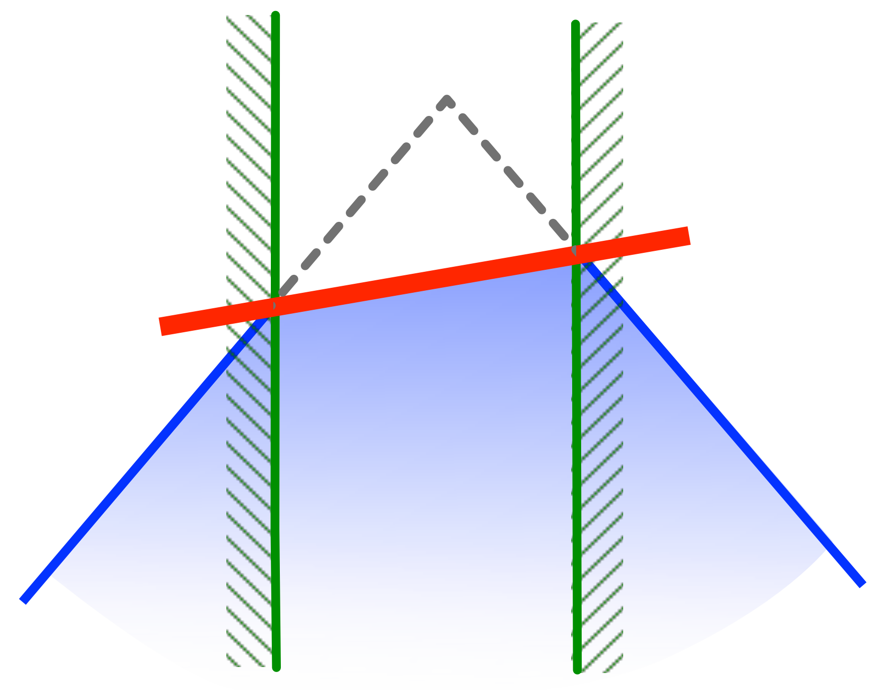


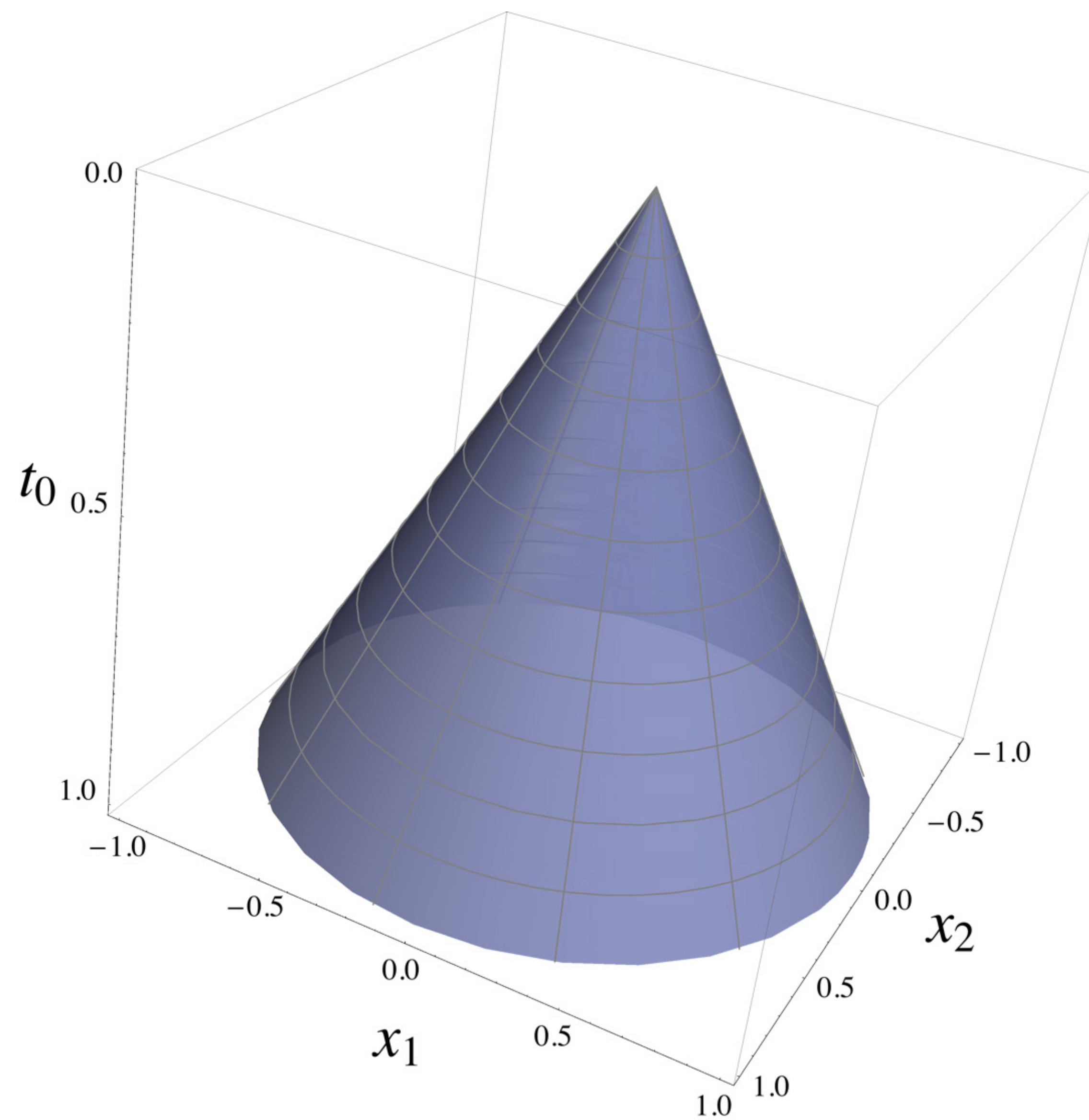
(e.g. V. 2007)

$$a := \left(2\frac{\pi_1 - \langle A^{-1}\pi, b\rangle}{\pi_1 - \pi_0} - 1\right)\pi + A^T\left|A^{-1}\pi\right|, \quad b := \left(2\frac{\pi_1 - \langle A^{-1}\pi, b\rangle}{\pi_1 - \pi_0} - 1\right)(\pi_0 + \pi_1) + \left|A^{-1}\pi\right| b + \pi_0$$

Split Cuts for Quadratic Cones

- Formulas: (Modaresi, Kılınç, V. 2011)

$$C := \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|A(x - c)\|_2 \leq t_0\}$$



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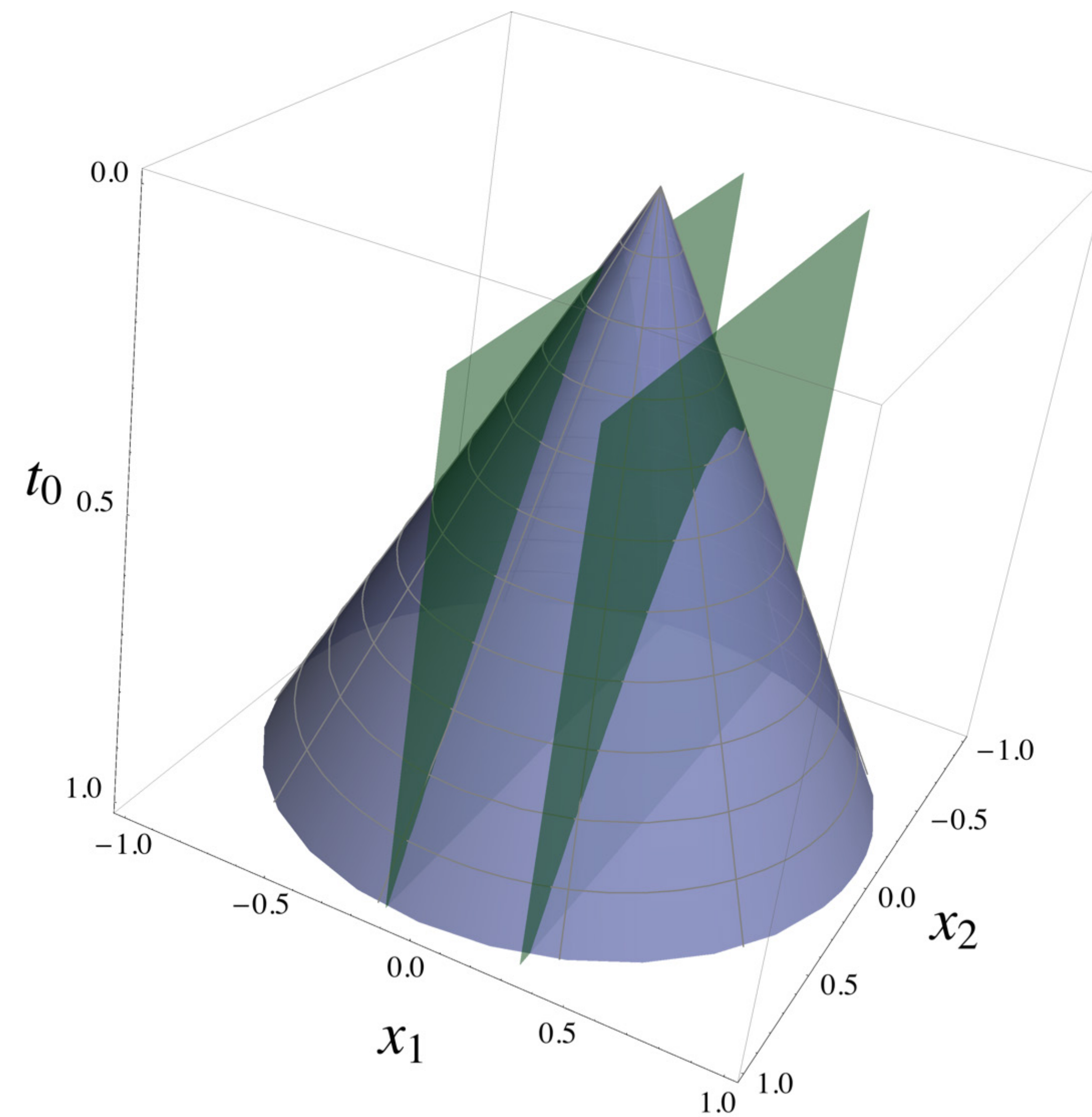

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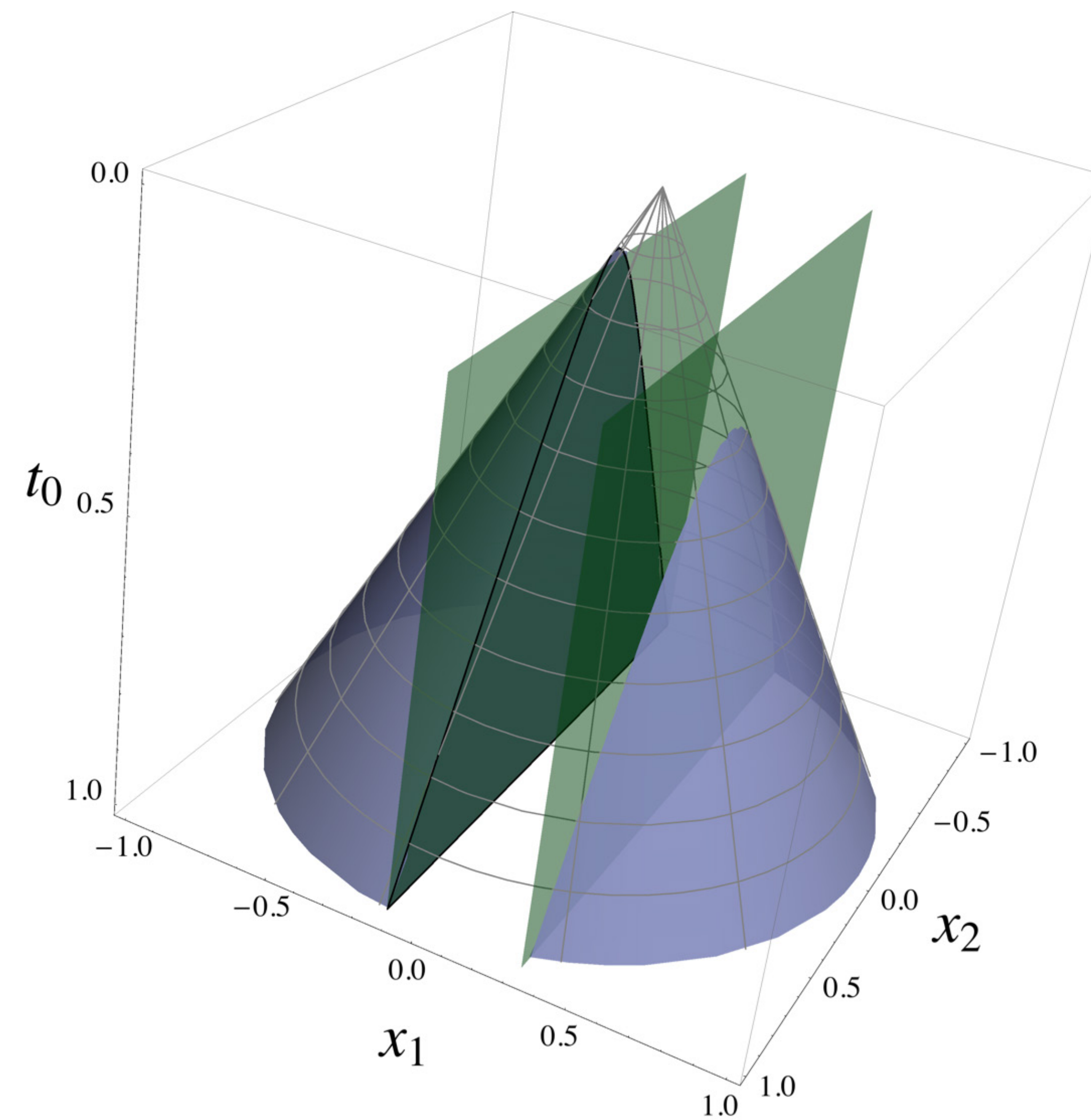

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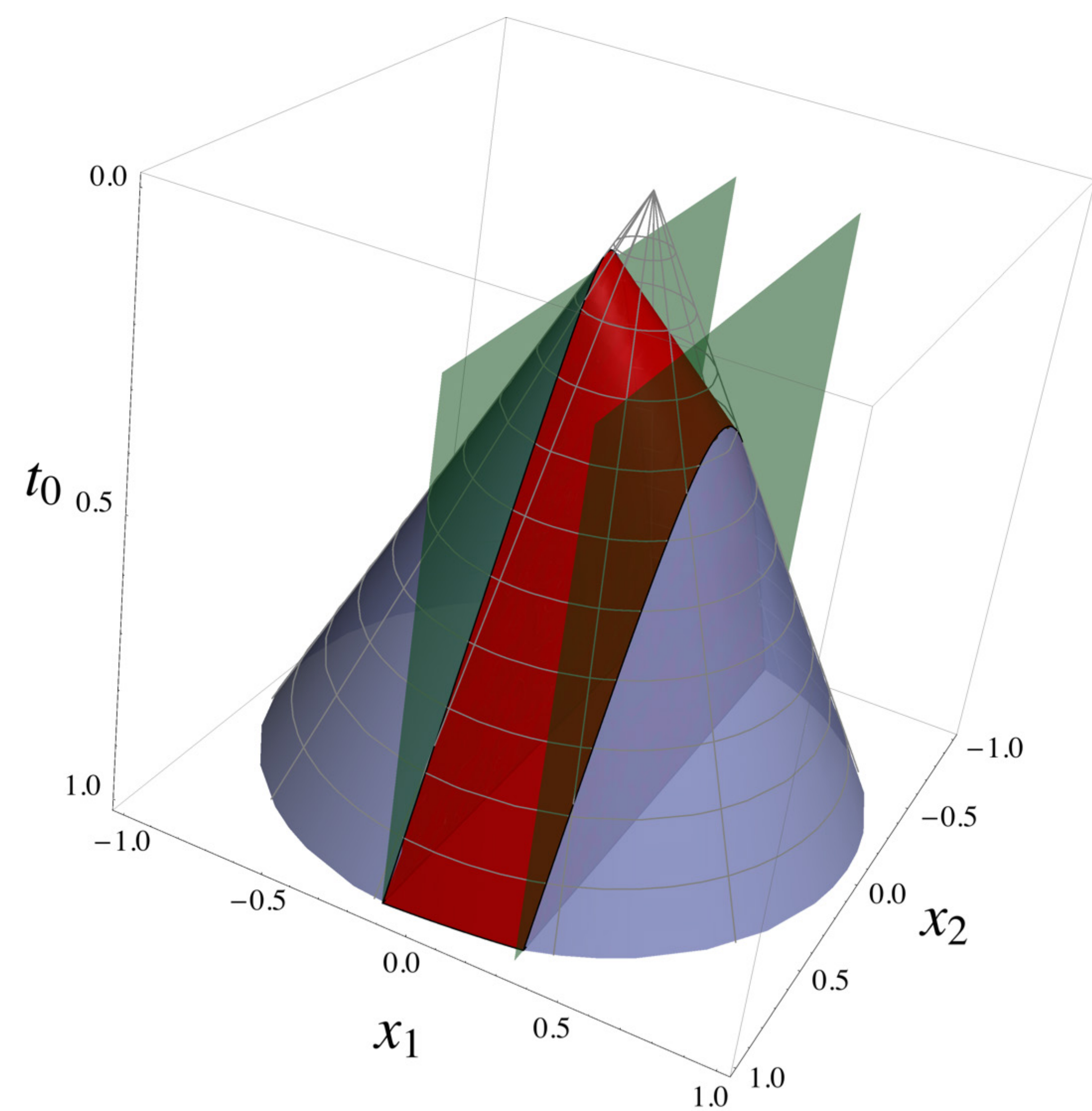

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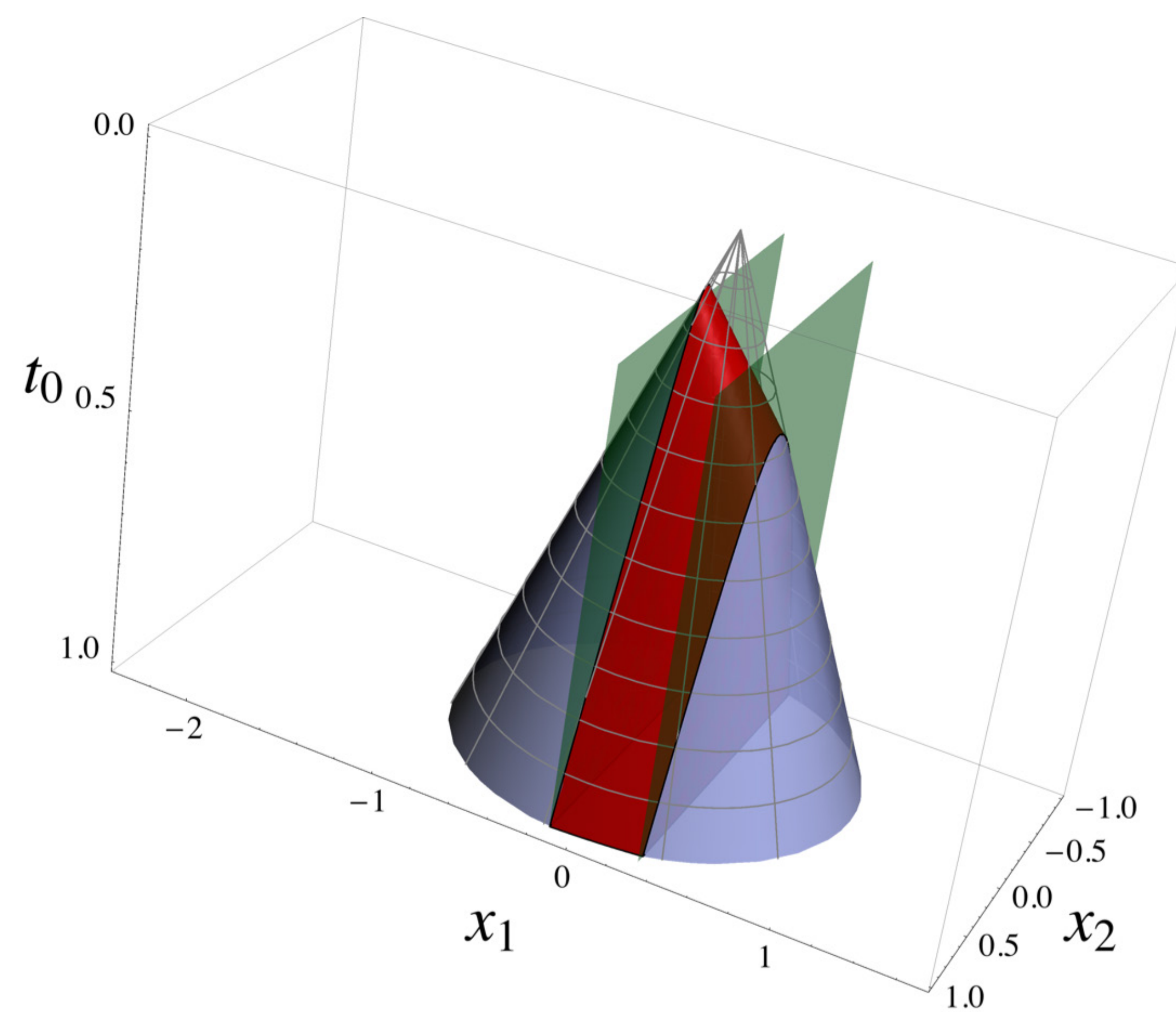

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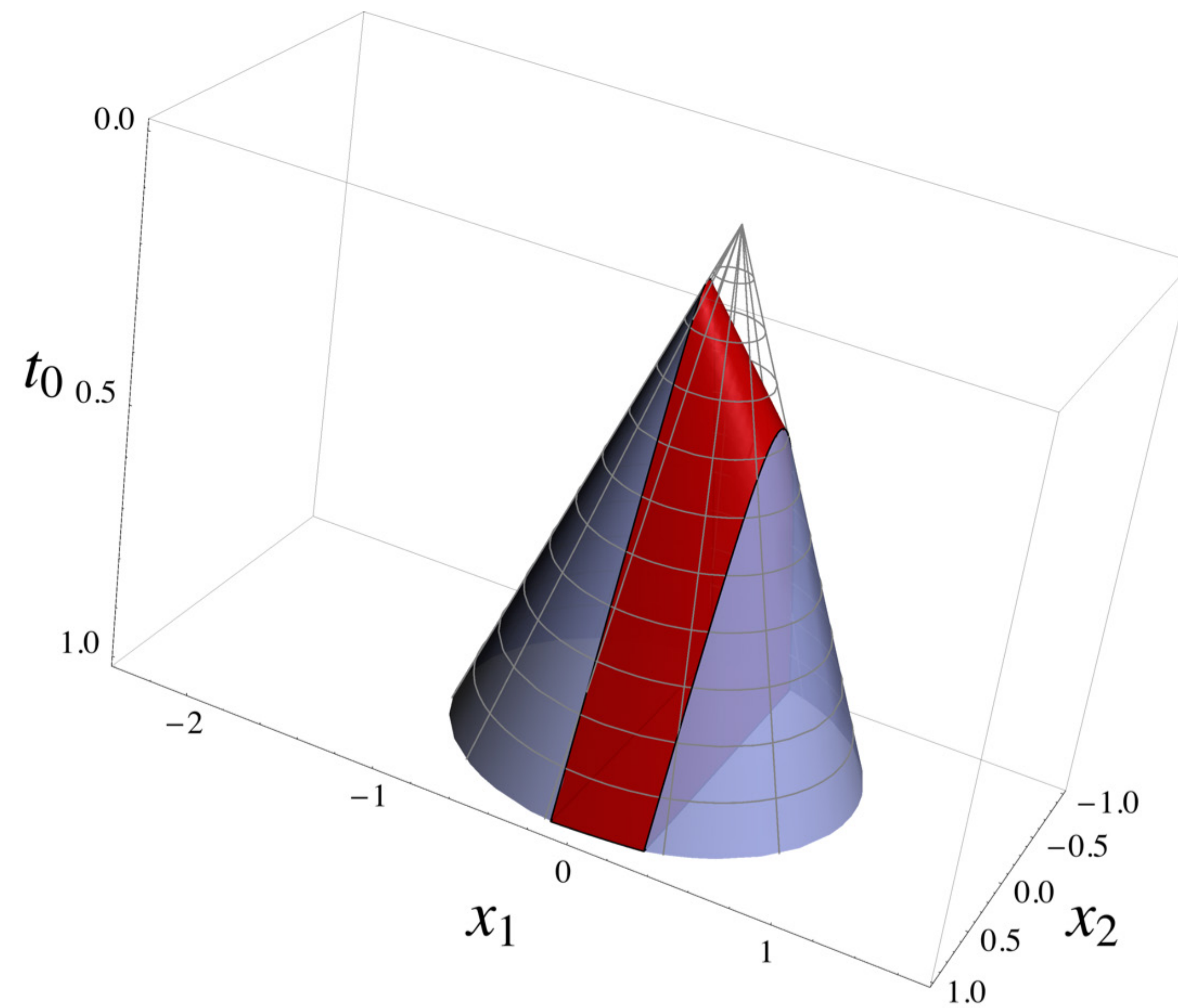

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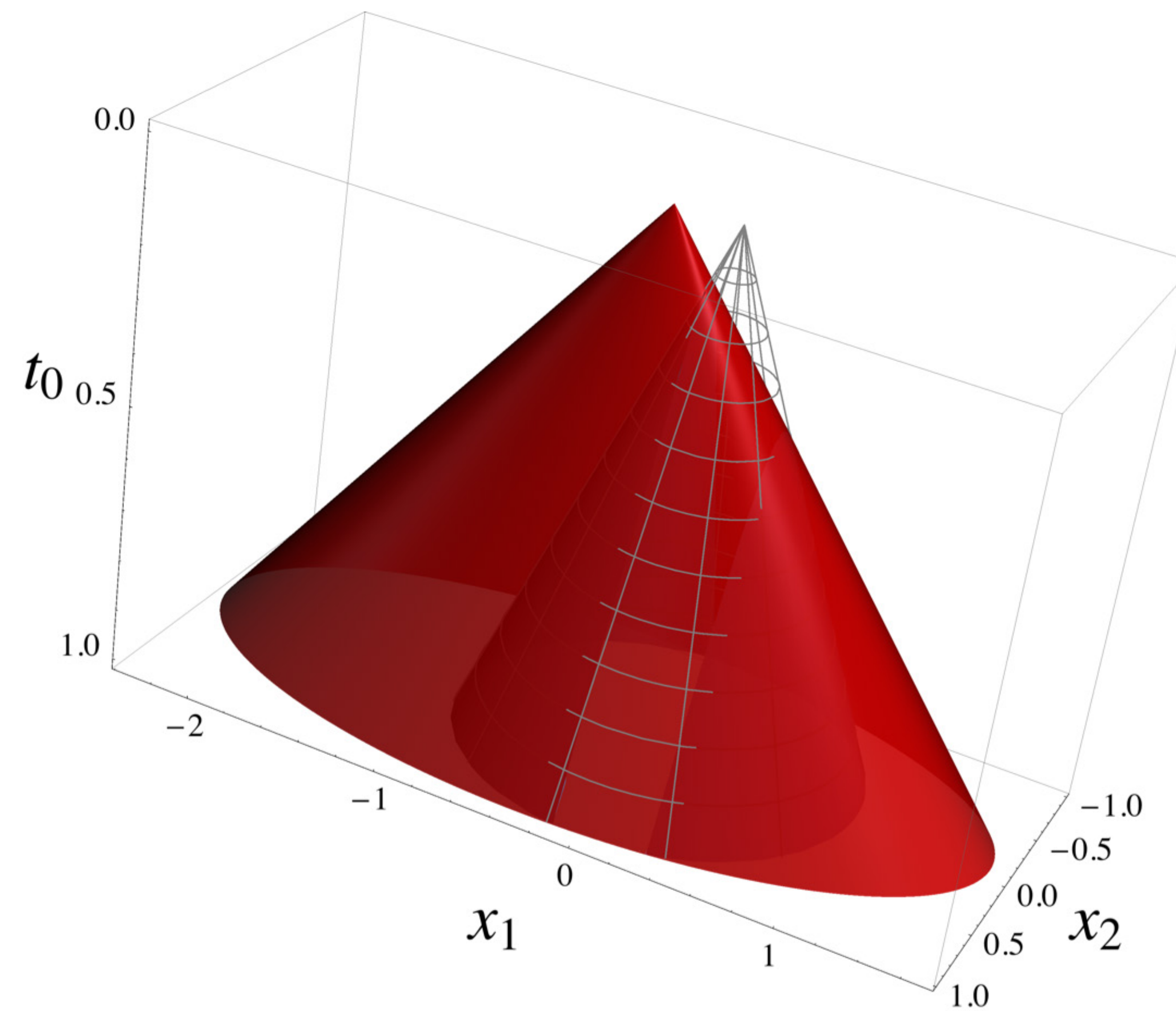

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Conic MIR and Nonlinear Split Cut

$$C := \left\{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} \,:\, \|A(x-c)\|_2 \leq t_0\right\}$$

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Extended Formulation: $(x, t, t_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$

$$\begin{aligned} |A(x-c)| &\le t \\ \|t\|_2 &\le t_0 \end{aligned}$$

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$$|A(x-c)| \leq t \quad \longleftarrow \text{Linear Part}$$

$$\|t\|_2 \leq t_0$$

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$\|t\|_2 \le t_0$ ← Nonlinear Part

Conic MIR = Split cuts for linear part ≠ Nonlinear split cut

$$(1-2f)\left(\lambda^T Ax - \lfloor \lambda^T Bc \rfloor\right) + f \le |\lambda|^T t$$

$$\lambda \in \mathbb{R}^n, \quad A^T\lambda \in \mathbb{Z}^n, \quad f := \lambda^T Bc - \lfloor \lambda^T Bc \rfloor$$

Split Cuts for Ellipsoids

- Formulas: (Dadush, Dey and V. 2011)

$$C := \{x \in \mathbb{R}^n \ : \ \|A(x-c)\|_2 \leq 1\}$$

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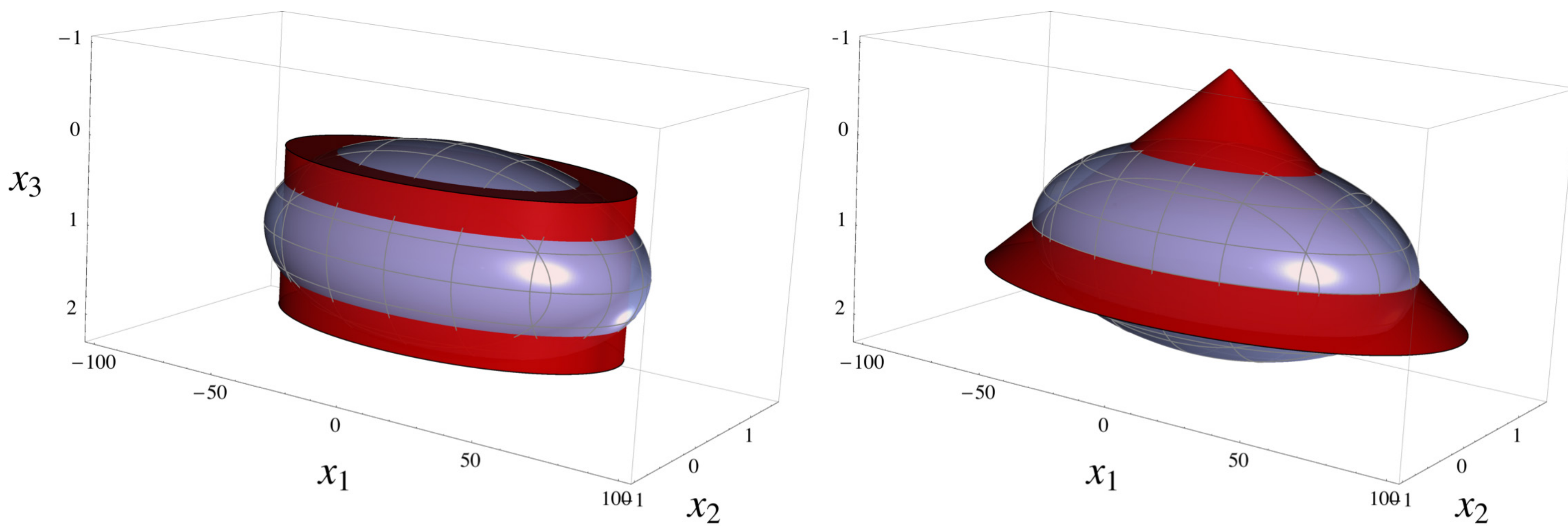


(also see Belotti, Góez, Polik, Ralphs, Terlaky 2011)

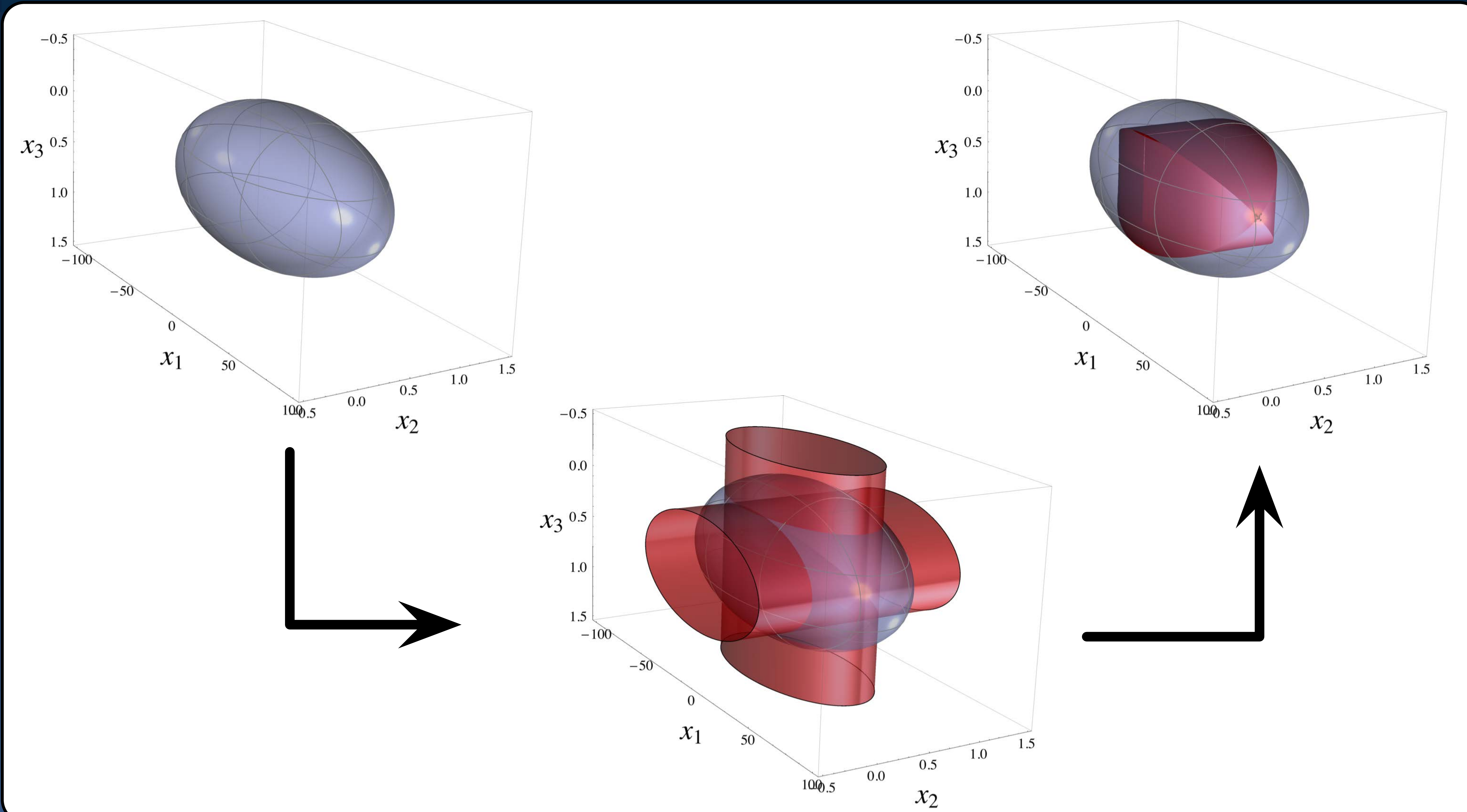
Split Closure is Finitely Generated

- Theorem (Dadush, Dey, V. 2011): If C is a strictly convex set then there exists a finite $D \subseteq \mathbb{Z}^n \times \mathbb{Z}$ such that:

$$\bigcap_{(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Z}} C^{\pi}_{\pi_0,\pi_0+1} = \bigcap_{(\pi,\pi_0)\in D} C^{\pi}_{\pi_0,\pi_0+1}$$

- Does <u>not</u> imply polyhedrality of split closure
- Split Closure is <u>not</u> stable

Split Closure Can Be Non-Polyhedral



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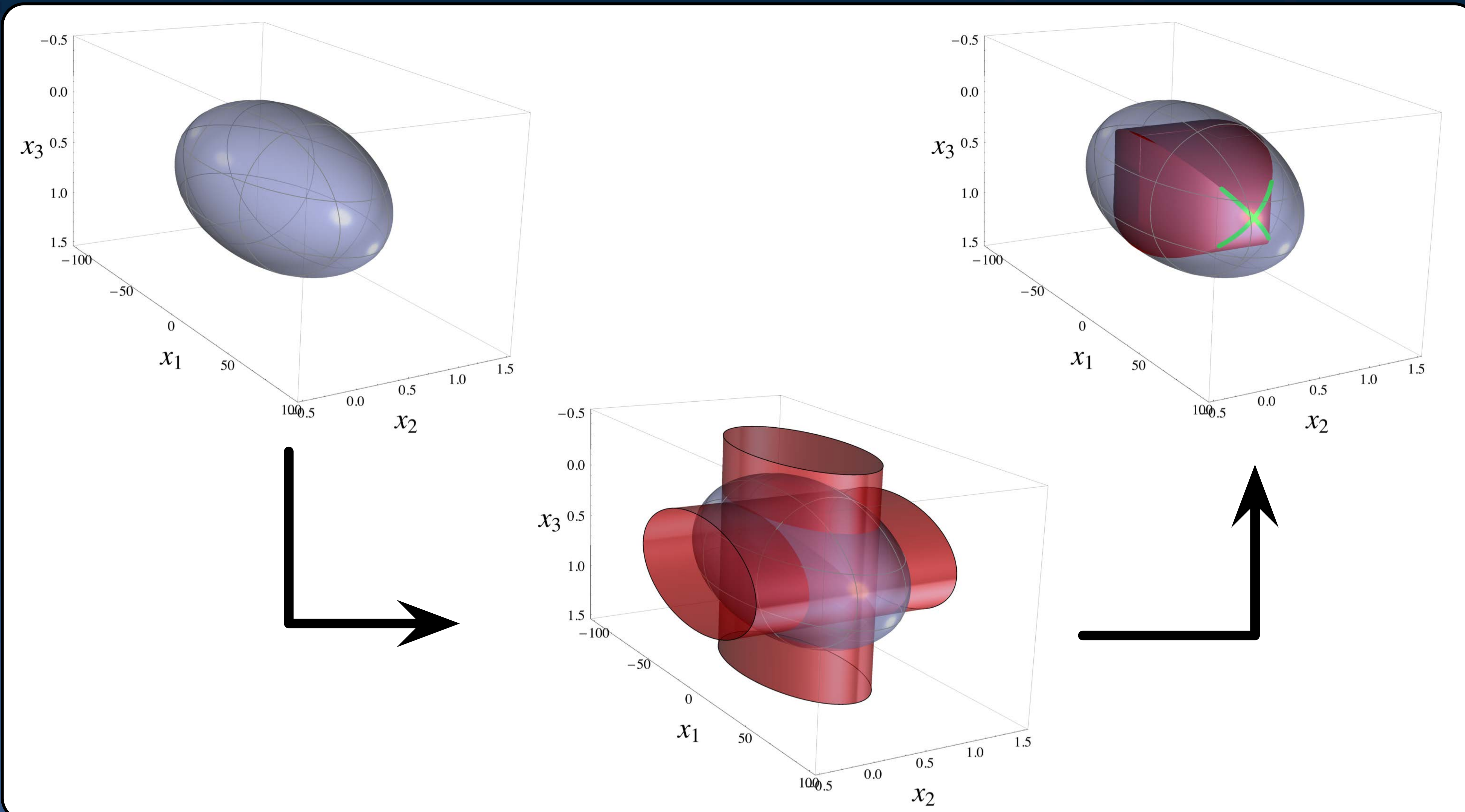

Other Results and Open Questions

- Formulas for nonlinear split cuts
  - Quadratic cones and ellipsoids
  - Strong ties to conic MIR
- Split closure: Finitely generated, not polyhedral
- Future:
  - More formulas
  - Computation
  - More general/constructive split closure