Mixed Integer Programming Models for Non-Separable Piecewise Linear Cost Functions

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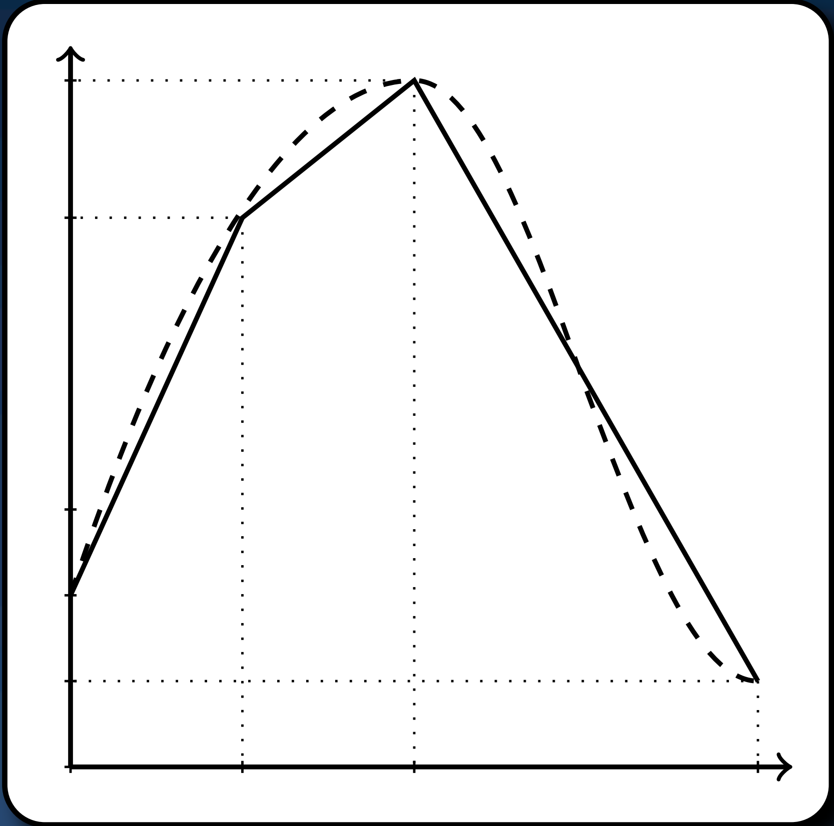
Joint work with Shabbir Ahmed and George Nemhauser.

University of Pittsburgh, 2008 – Pittsburgh, PA

Piecewise Linear Optimization

$$\begin{aligned} \min \quad & f_0(x) \\ s.t. \quad & \\ & f_i(x) \leq 0 \quad \forall i \in I \\ & x \in X \subset \mathbb{R}^n \end{aligned}$$



- $\forall i \in \{0\} \cup I \quad f_i(x) : D \longrightarrow \mathbb{R}$ is a piecewise linear function (PLF) and X is any compact set.
- Convex = Linear Programming. Non-Convex = NP Hard.
- Specialized algorithms (Tomlin 1981, ..., de Farias et al. 2008) or Mixed Integer Programming Models (12+ papers).

Mixed Integer Models for PLFs

- Existing studies are for separable functions:

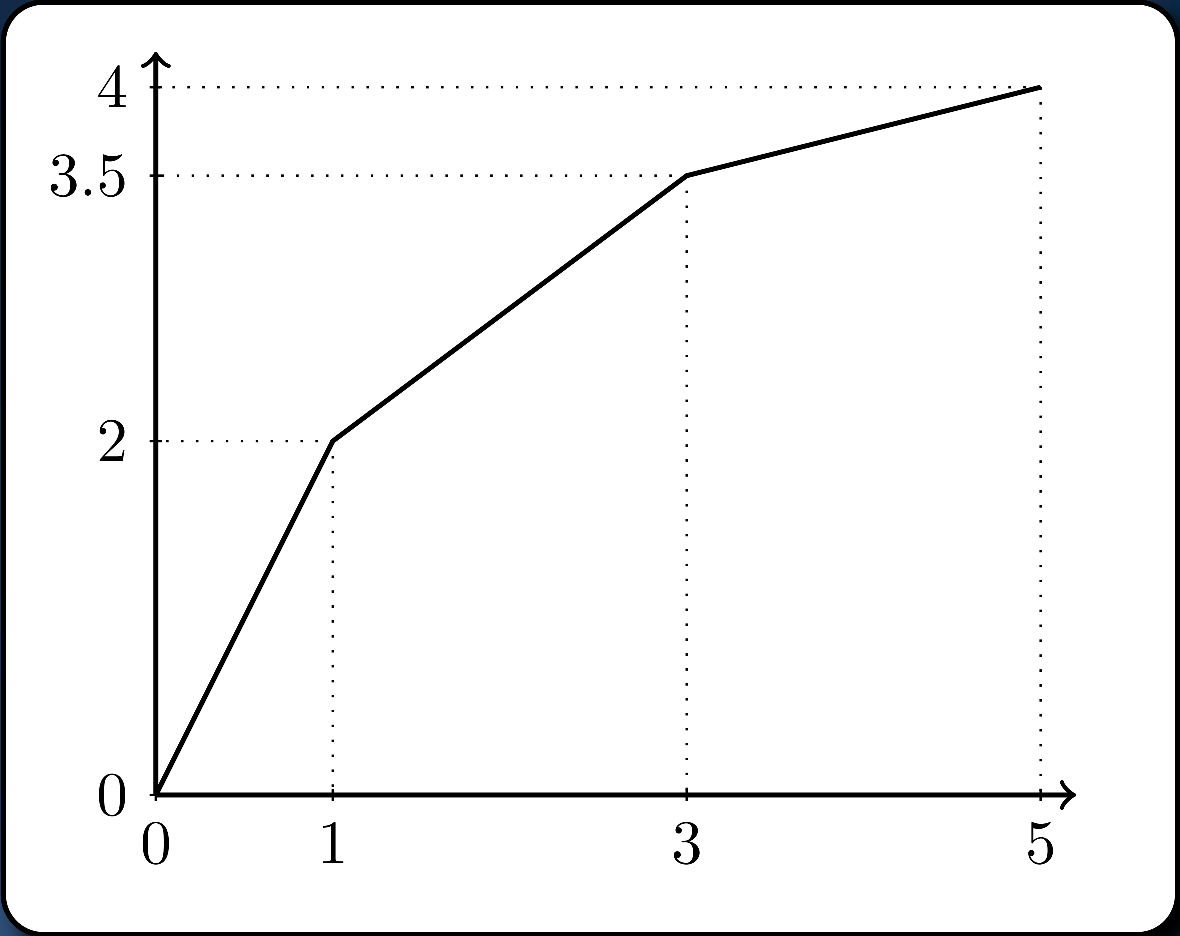
$$f(x) = \sum_{j=1}^{n} f_j(x_j) \text{ for } f_j(x_j) : \mathbb{R} \to \mathbb{R}$$

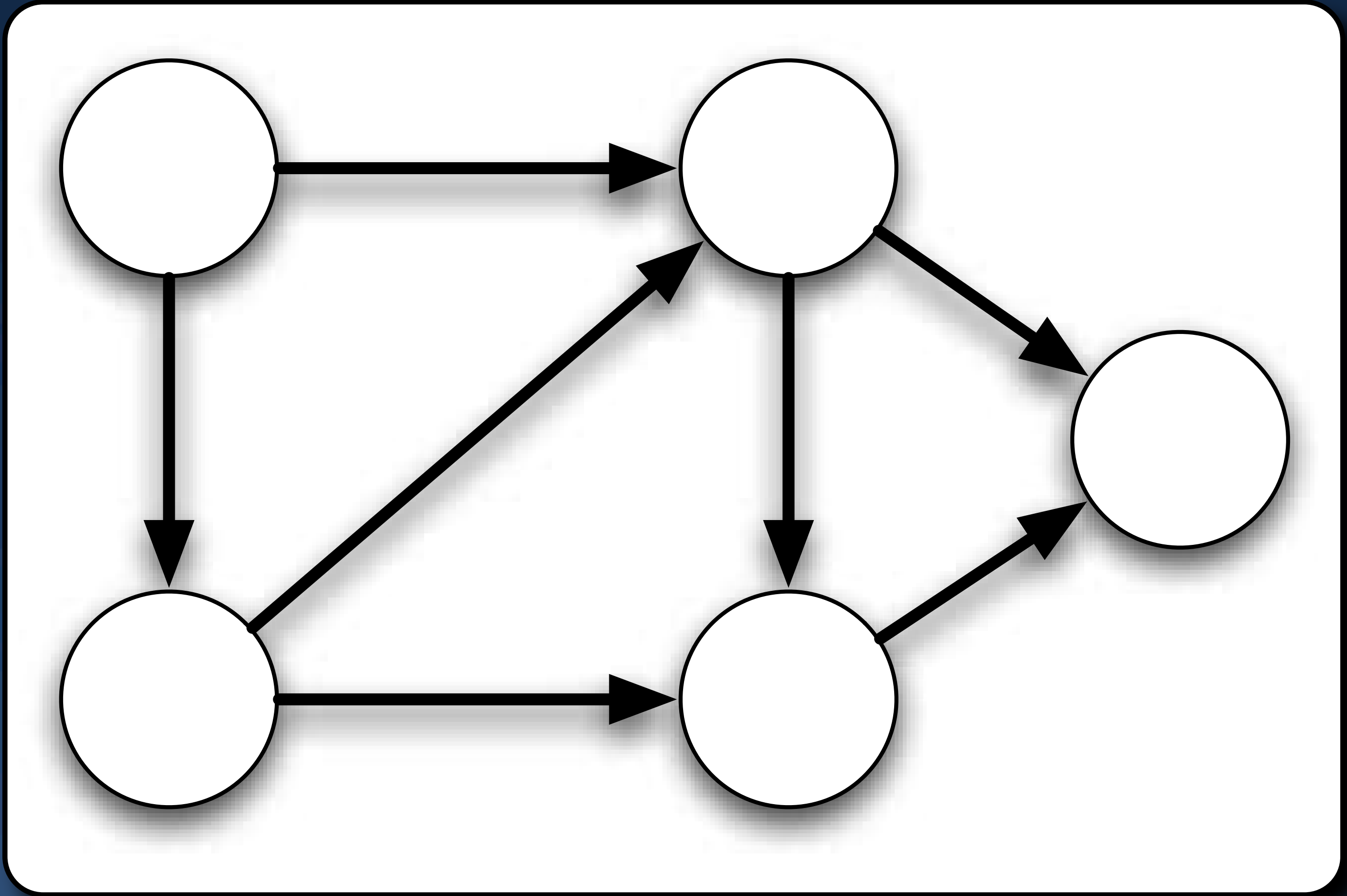
- Contributions (Vielma et al. 2008a,b):
  - First models with a logarithmic # of binary variables.
  - Theoretical and computational comparison: multivariate (non-separable) and lower semicontinuous functions in a unifying framework.

Outline

- Applications of Piecewise Linear Functions.
- Modeling Piecewise Linear Functions.
- Logarithmic Formulations.
- Comparison of Formulations.
- Extension to Lower Semicontinuous Functions.
- Final Remarks.

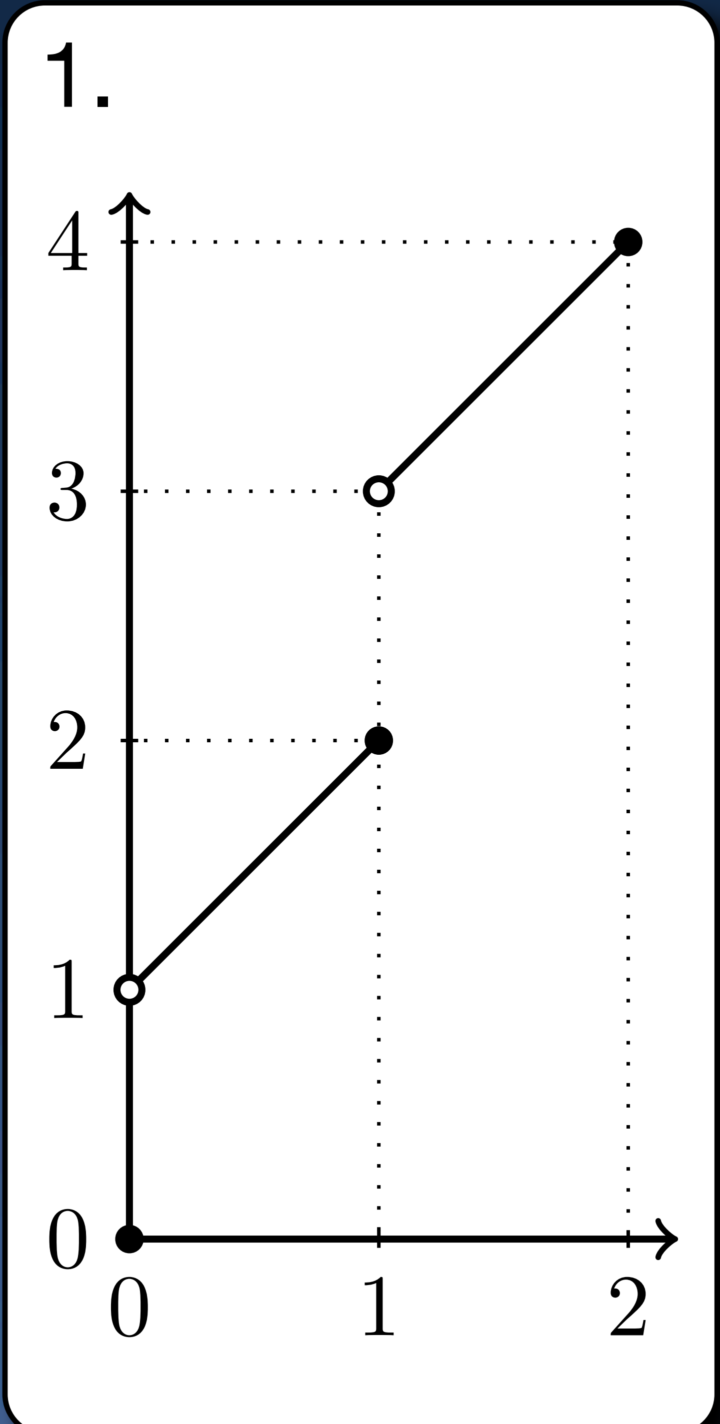
Economies of Scale: Concave

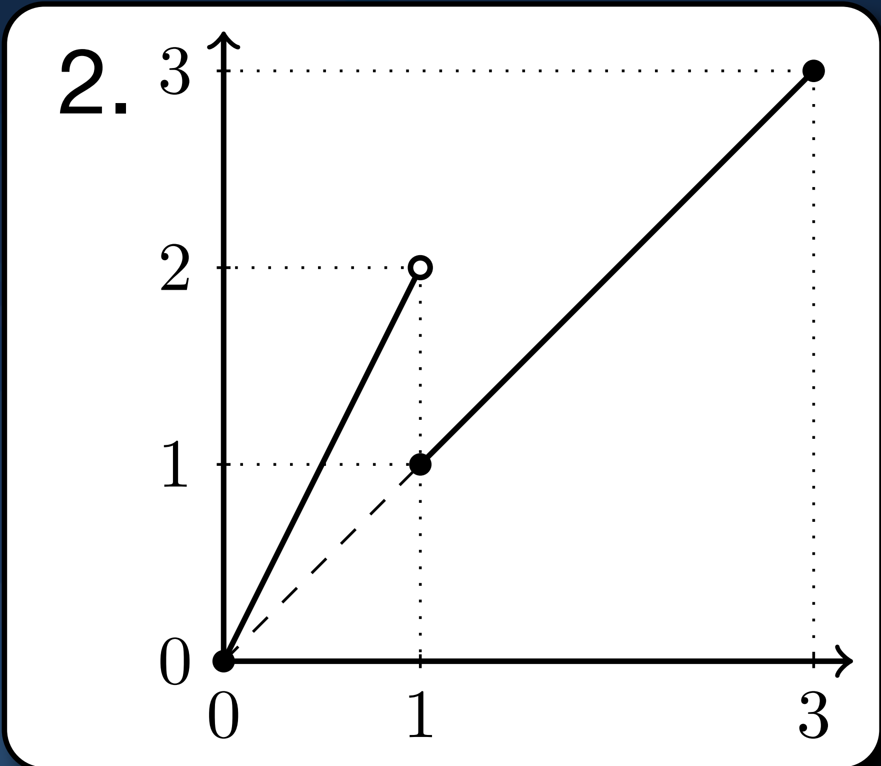


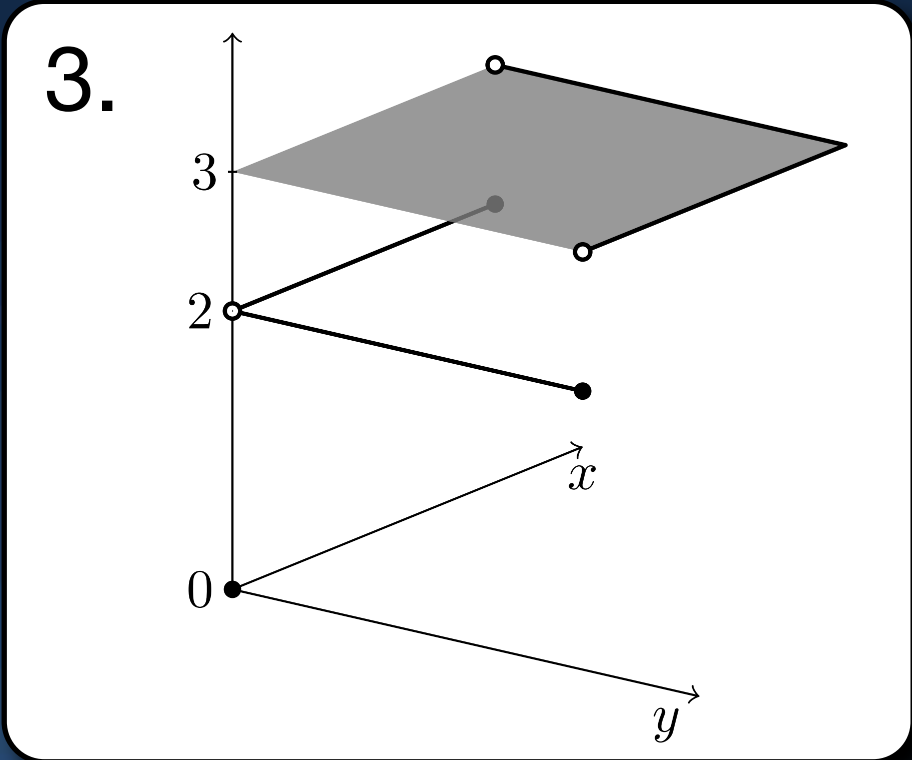


- Single and multi-commodity network flow.
- Applications in telecommunications, transportation, and logistics.
- (Balakrishnan and Graves 1989, ..., Croxton, et al. 2007).

Fixed Charges and Discounts

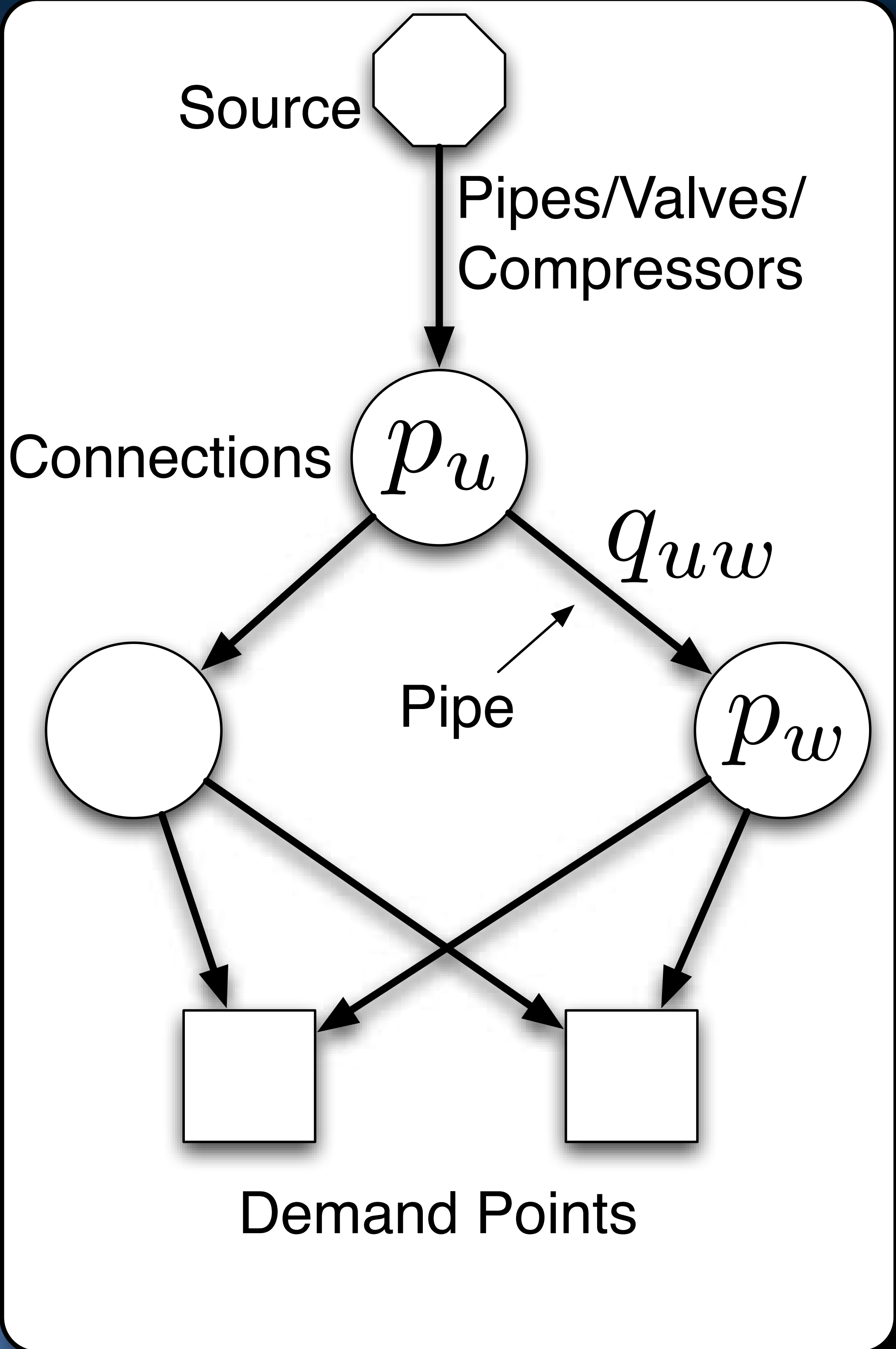






1. Fixed Costs in Logistics.
2. Discounts (e.g. Auctions: Sandholm, et al. 2006, CombineNet).
3. Discounts in fixed charges (Lowe 1984).

Non-Linear and PDE Constraints



$$p(x,t) = \text{gas pressure}$$

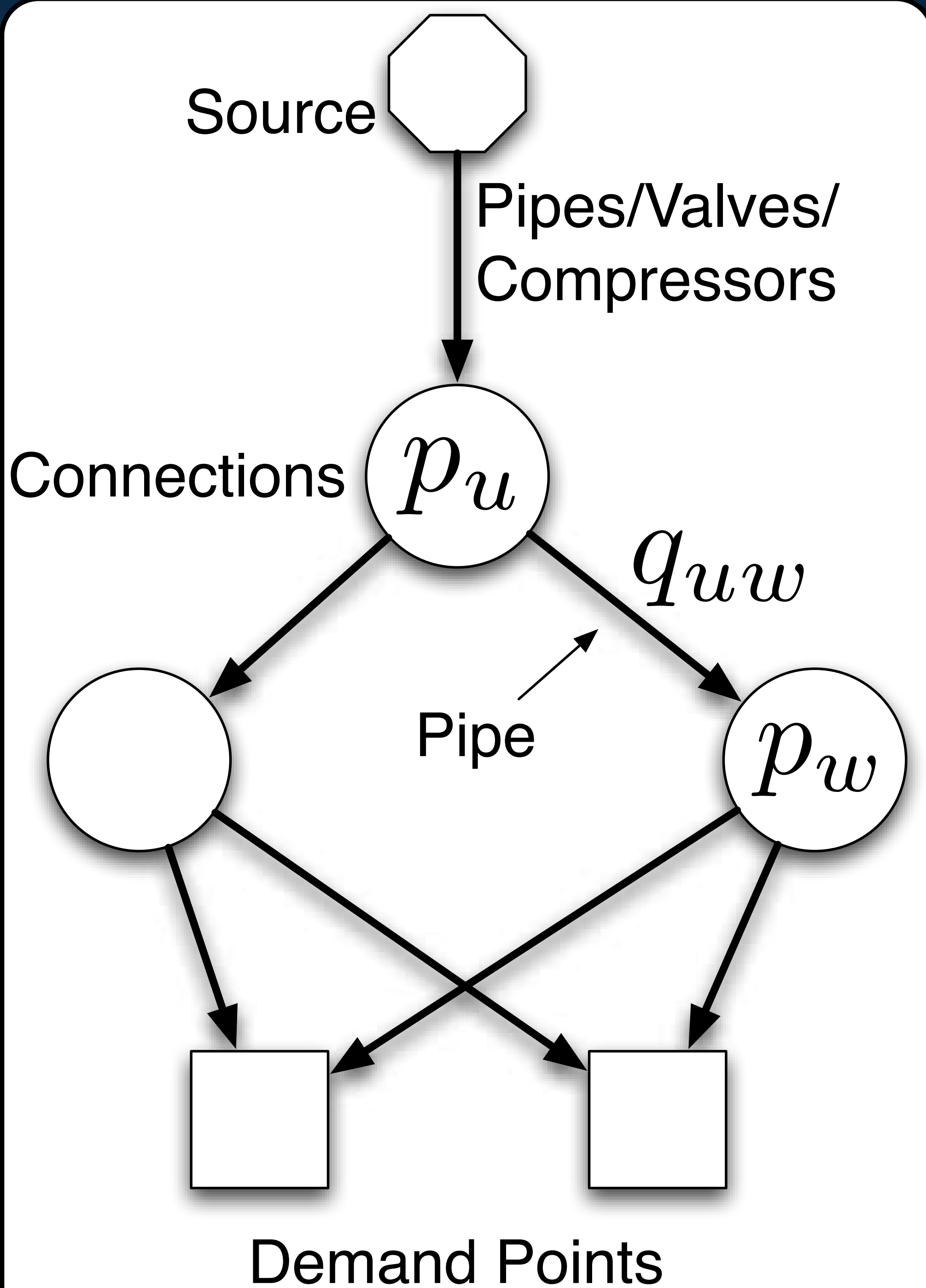
$$q(x,t) = \text{gas volume flow}$$

$$A\,\frac{\partial \rho}{\partial t} + \rho_0\,\frac{\partial q}{\partial x} = 0,$$

$$\frac{\partial p}{\partial x} = -\lambda\frac{|v|v}{2D}\,\rho\,.$$

- Gas Network Optimization (Martin et al. 2006).

Non-Linear and PDE Constraints



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Discretize non-linear stationary solution $p_v = g(p_u, q_{uv})$

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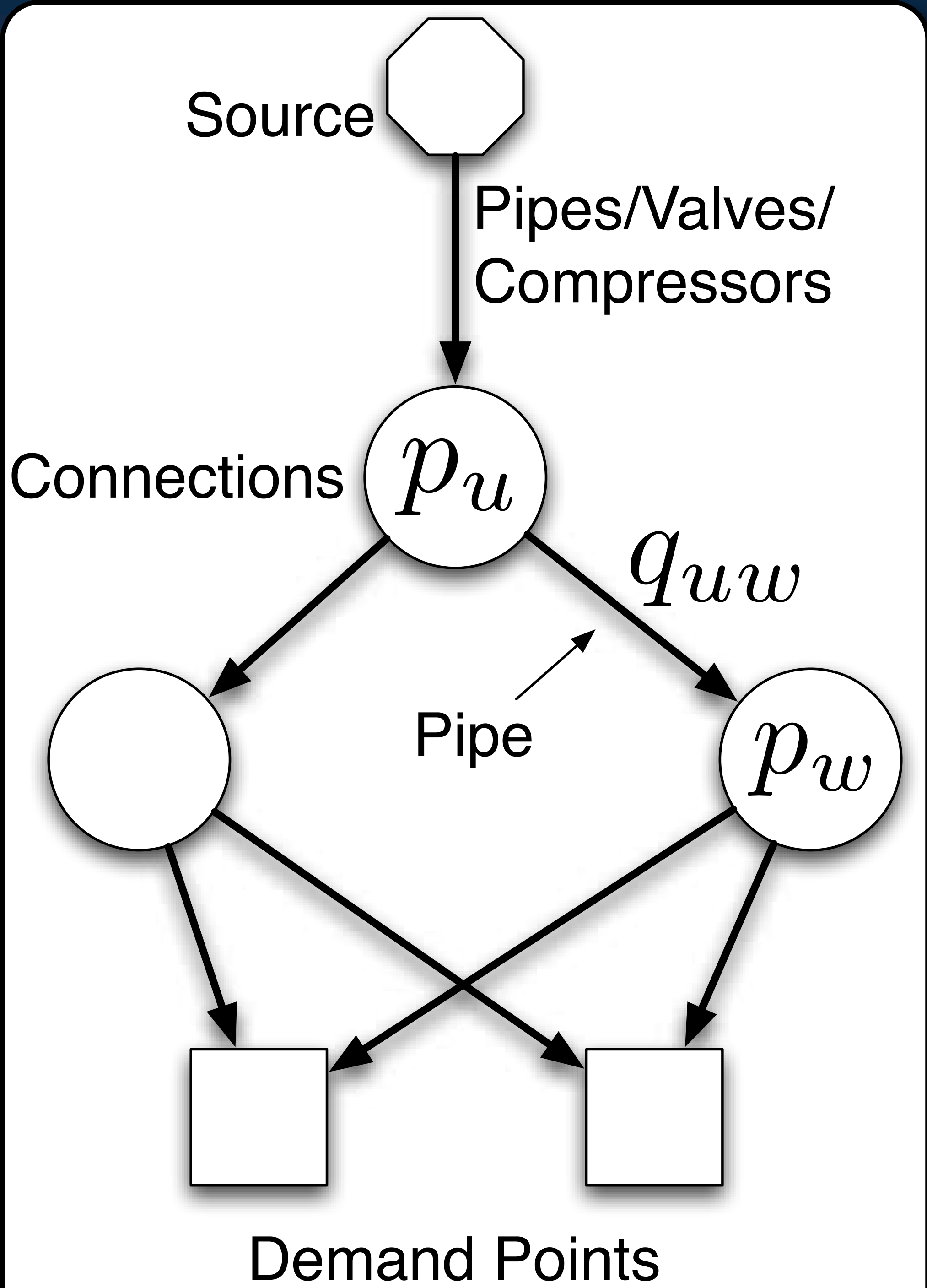

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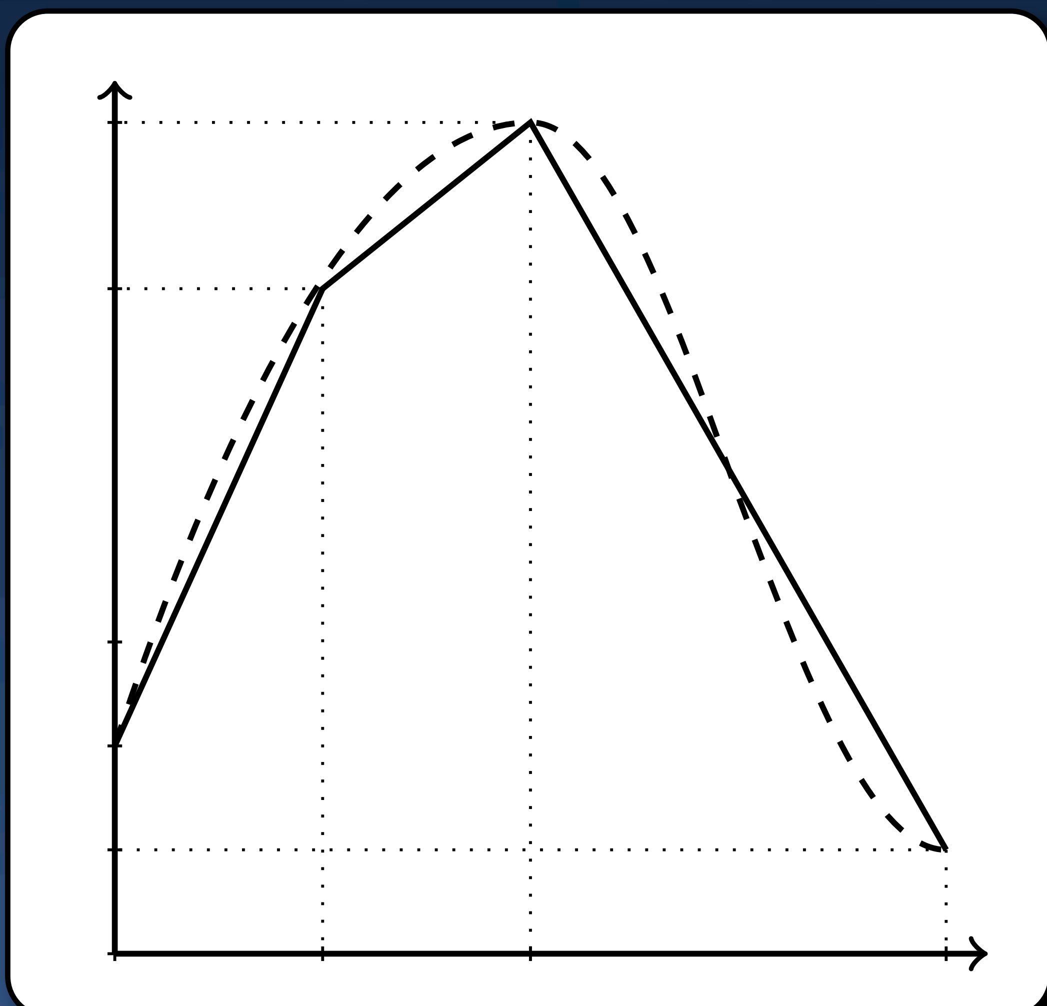
Discretize PDE

(Fügenschuh, et al. 2008)

- Gas Network Optimization (Martin et al. 2006).

Numerically Exact Global Optimization



- Process engineering (Bergamini et al. 2005, 2008, Computers and Chemical Eng.)
- Wetland restoration (Stralberg et al. 2009).

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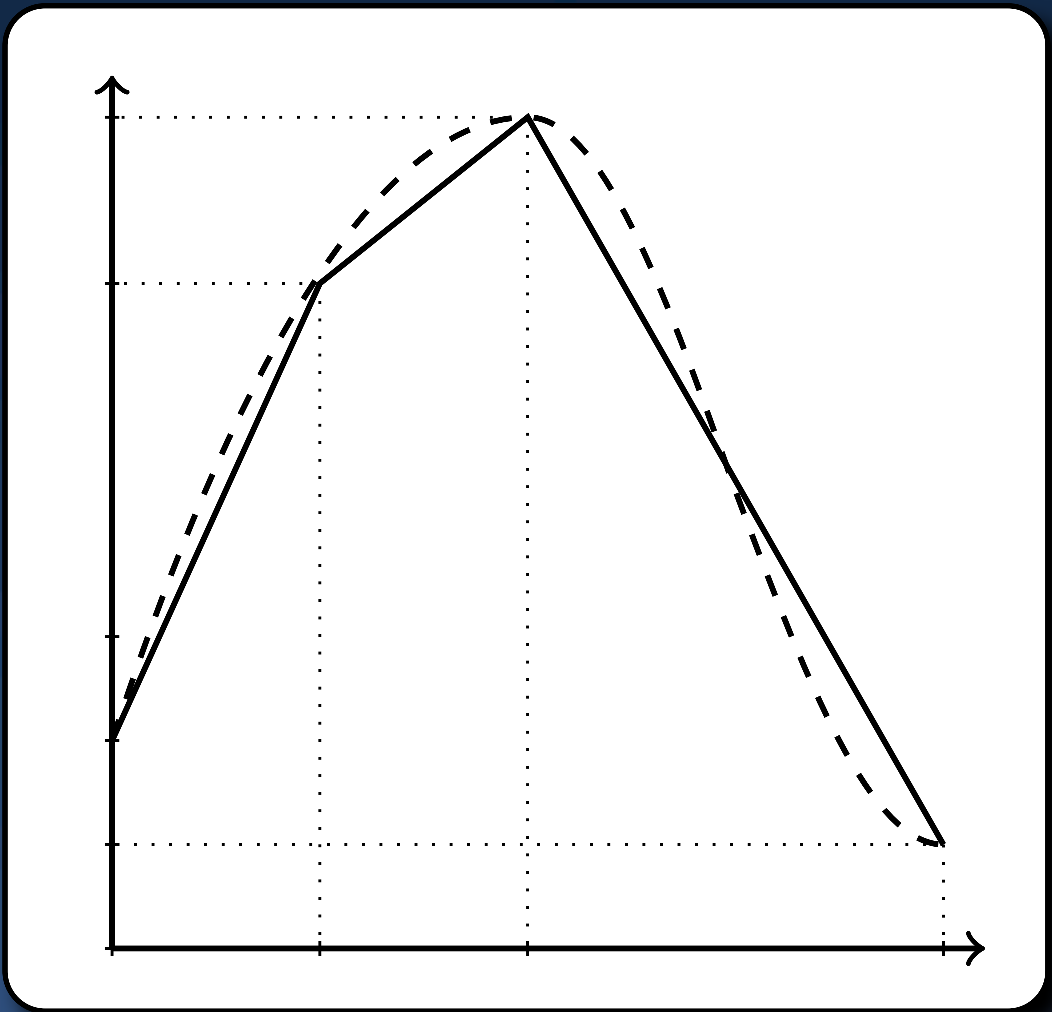

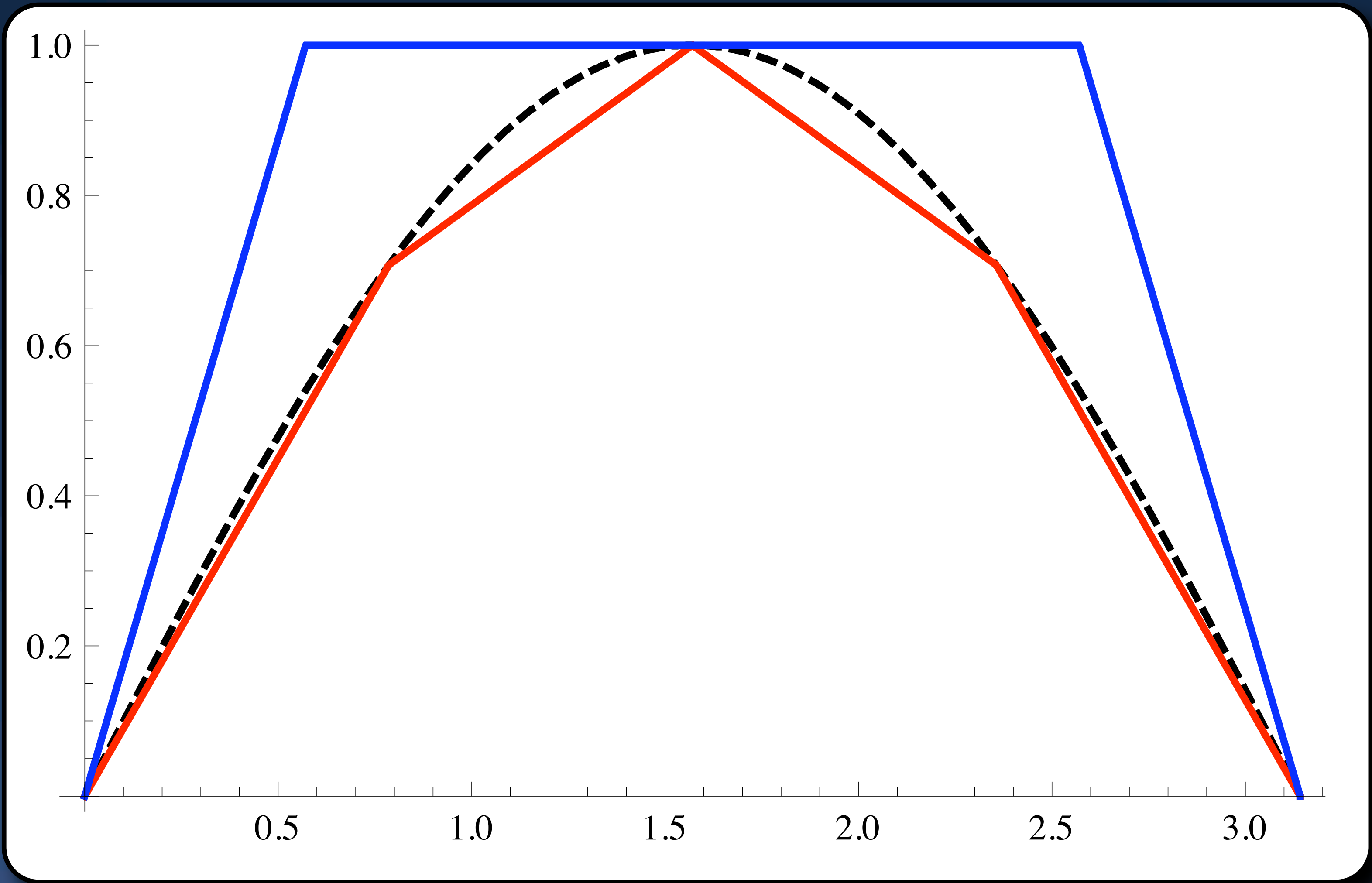

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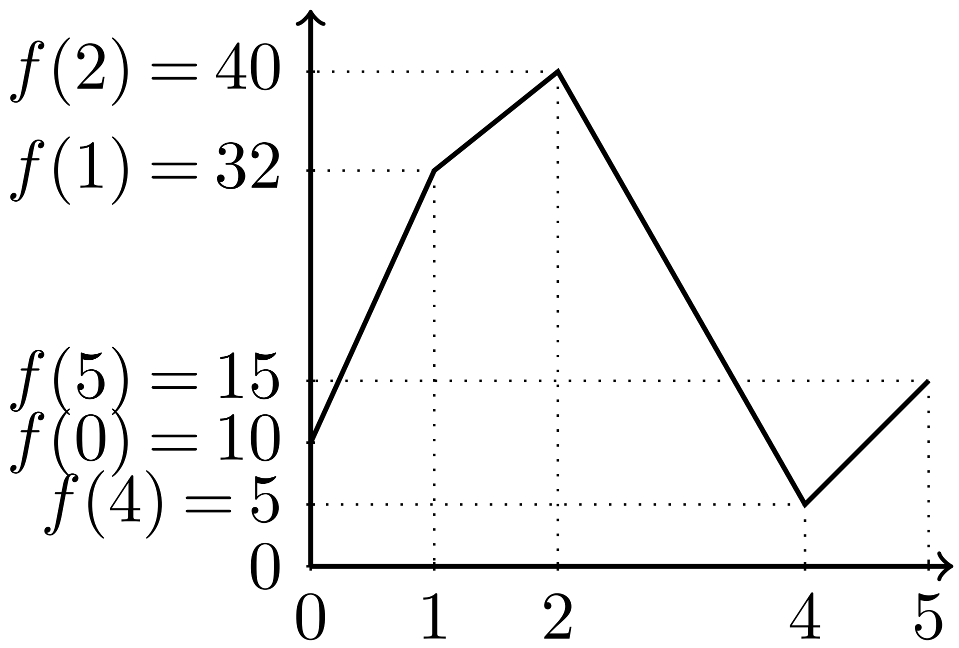
Piecewise Linear Functions: Definition



$$f(x) := \begin{cases} 22x + 10 & x \in [0,1] \\ 8x + 24 & x \in [1,2] \\ -17.5x + 75 & x \in [2,4] \\ 10x - 35 & x \in [4,5] \end{cases}$$

Definition 1. Piecewise Linear $f : D \subset \mathbb{R}^n \to \mathbb{R}$:

$$f(x) := \left\{ m_P x + c_P \quad x \in P \quad \forall P \in \mathcal{P}. \right.$$

for finite family of polytopes $\mathcal{P}$ such that $D = \bigcup_{P \in \mathcal{P}} P$

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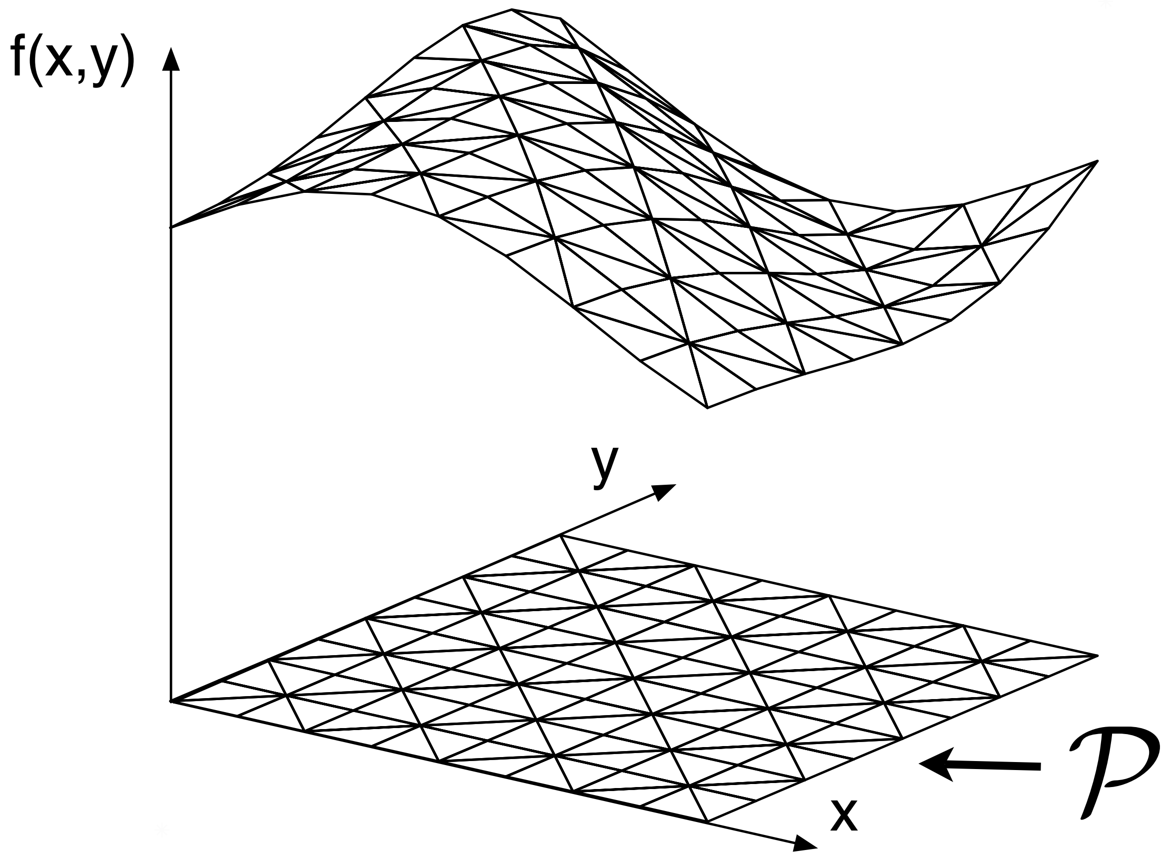

$$f(x,y) := \begin{cases} 0.48x + 0.03y + 6 & (x,y) \in P_1 \\ & \vdots \\ -0.4x - 0.04y + 8.45 & (x,y) \in P_{128} \end{cases}$$

$$P_1 := \{(x,y) \in \mathbb{R} : y \geq 0, x \leq 1, y - x \leq 0\}$$

$$\vdots$$

$$P_{128} := \{(x,y) \in \mathbb{R} : y \geq 0, x \geq 7, x + y \leq 8\}$$

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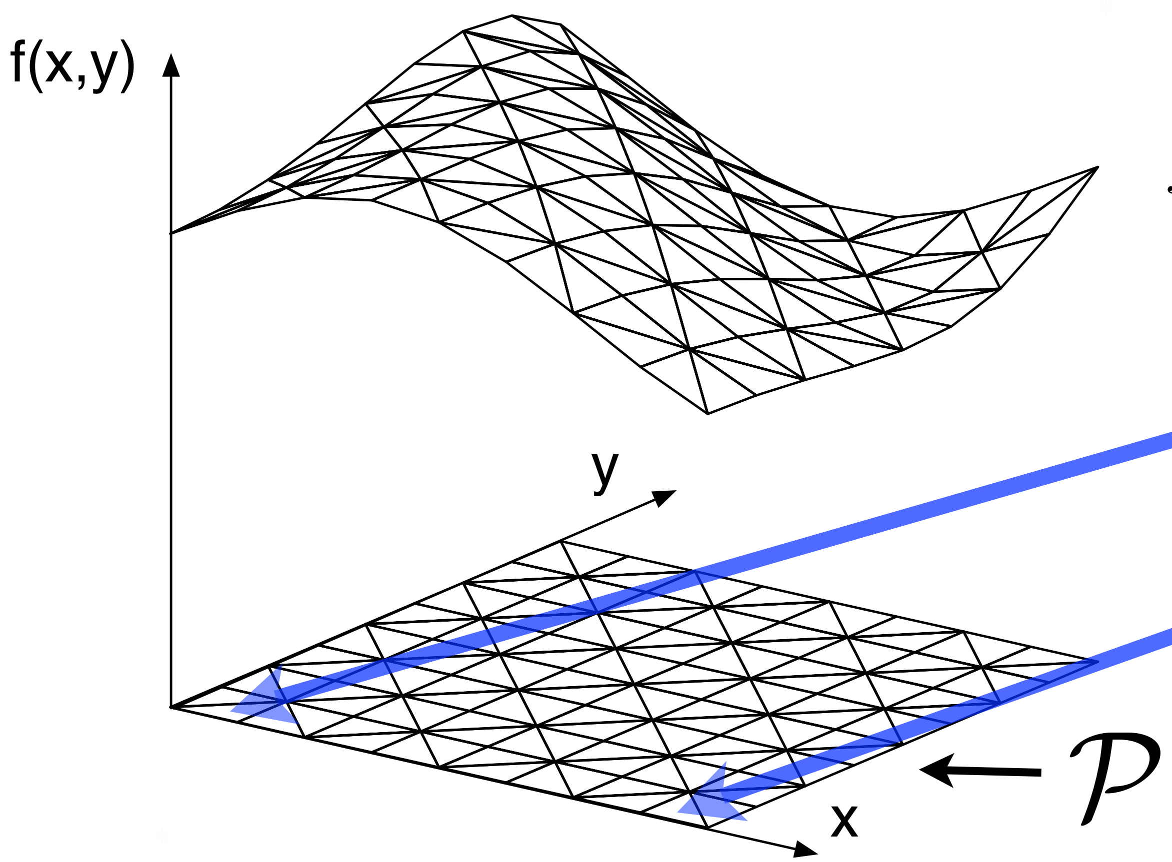

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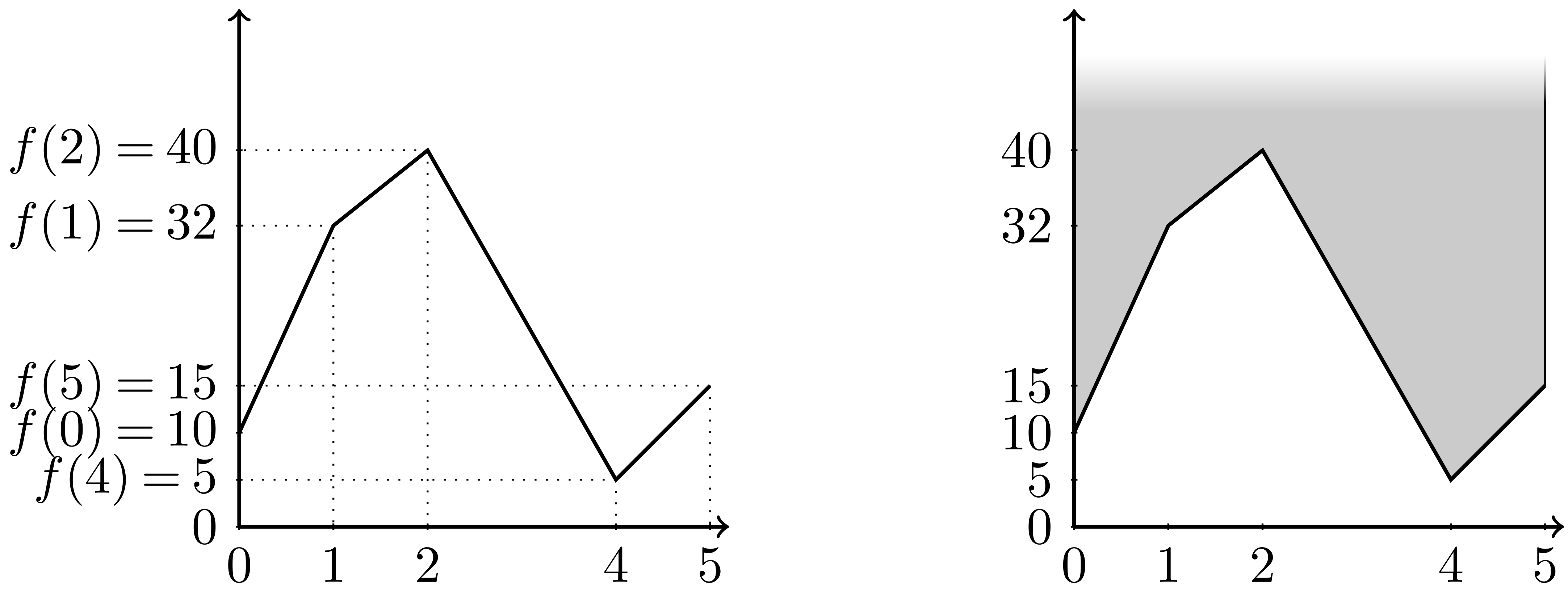
Modeling Function = Epigraph

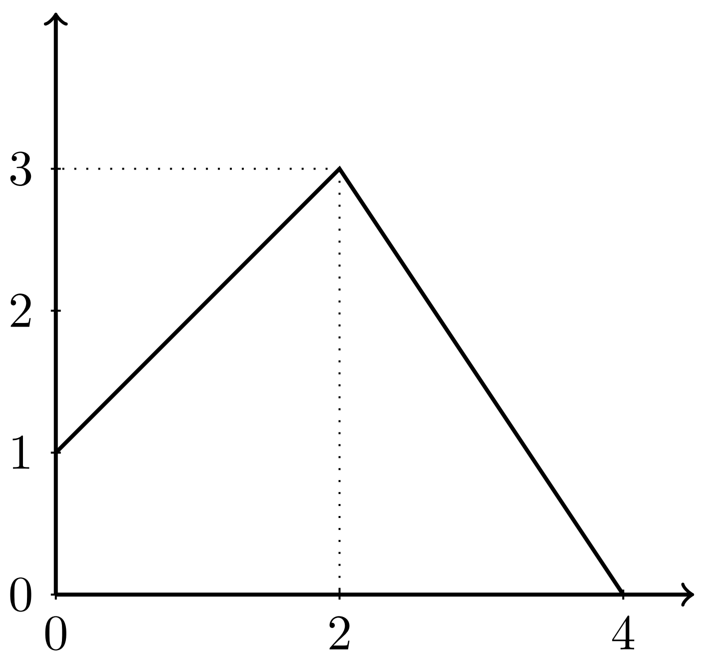
- $\mathrm{epi}(f) := \{(x, z) \in D \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R} \,:\, f(x) \leq z\}$.



(a) f. (b) $\mathrm{epi}(f)$.

- Example: $f(x) \leq 0 \Leftrightarrow (x, z) \in \mathrm{epi}(f),\ z \leq 0$

Convex Combination (CC): Univariate



$$f(x) := \begin{cases} x+1 & x \in [0,2] \leftarrow P_1 \\ 6 - 3/2x & x \in [2,4] \leftarrow P_2 \end{cases}$$

$V(P) =$ vertices of P.

$\mathcal{V}(\mathcal{P}) := V(P_1) \cup V(P_2) = \{0, 2, 4\}.$

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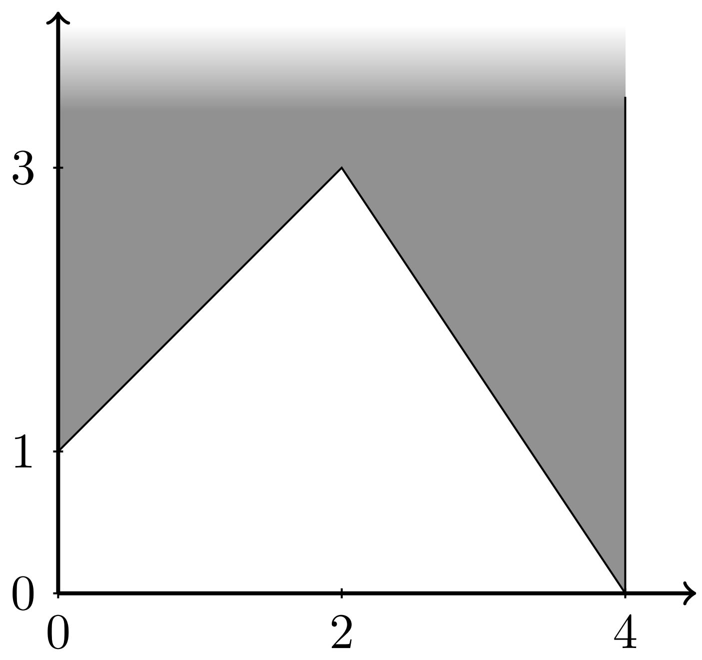

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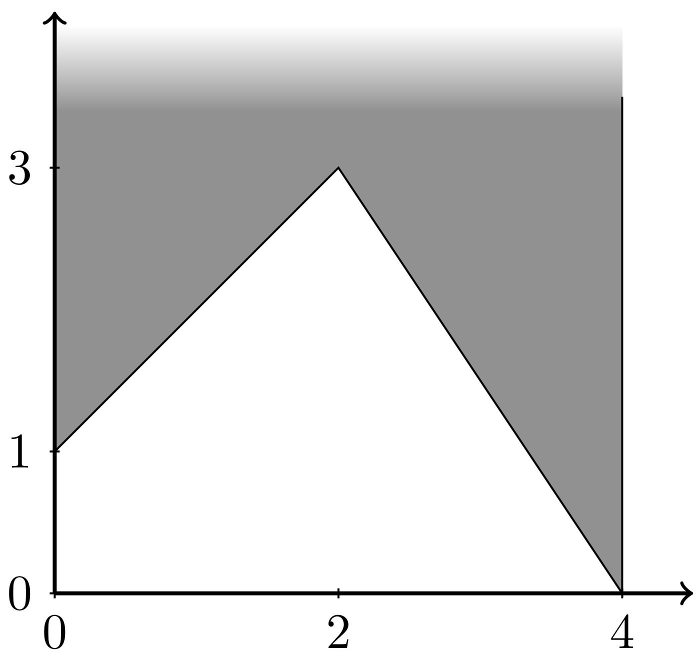

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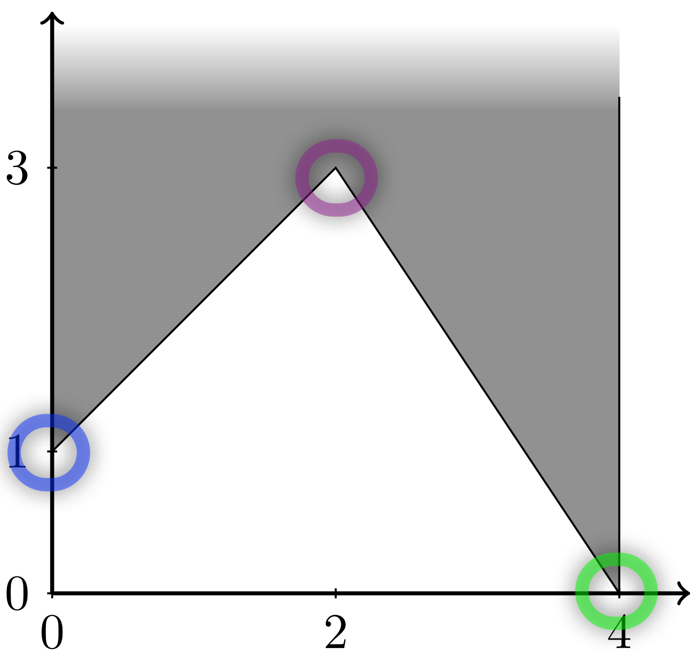

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$$z \geq 1\lambda_0 + 3\lambda_2 + 0\lambda_4$$

$$1 = \lambda_0 + \lambda_2 + \lambda_4, \quad \lambda_0, \lambda_2, \lambda_4 \geq 0$$

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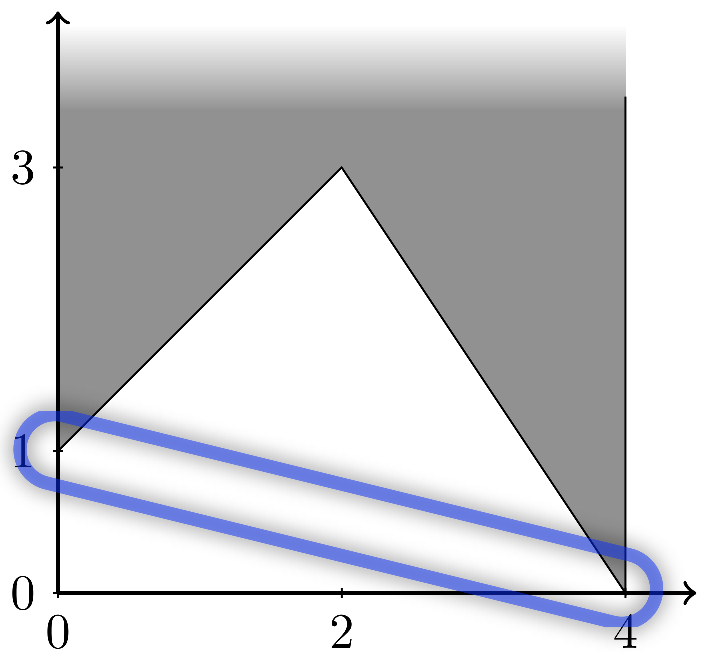

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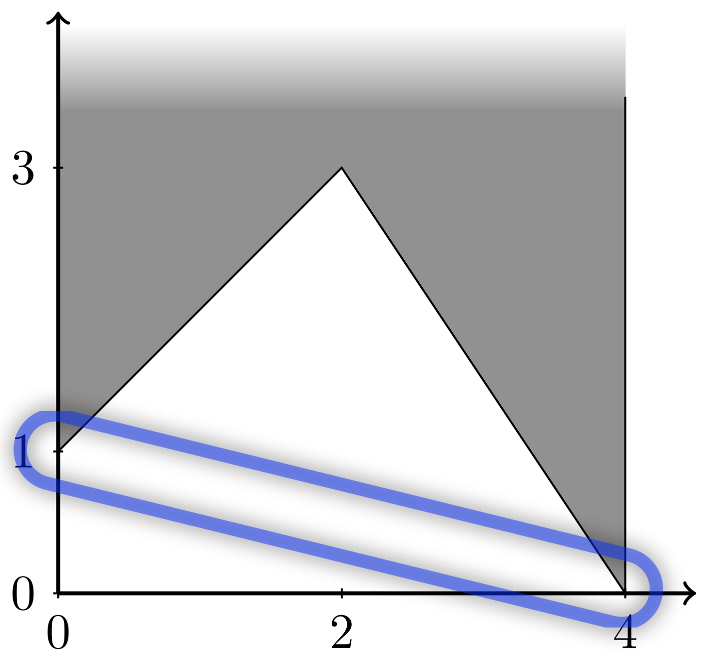

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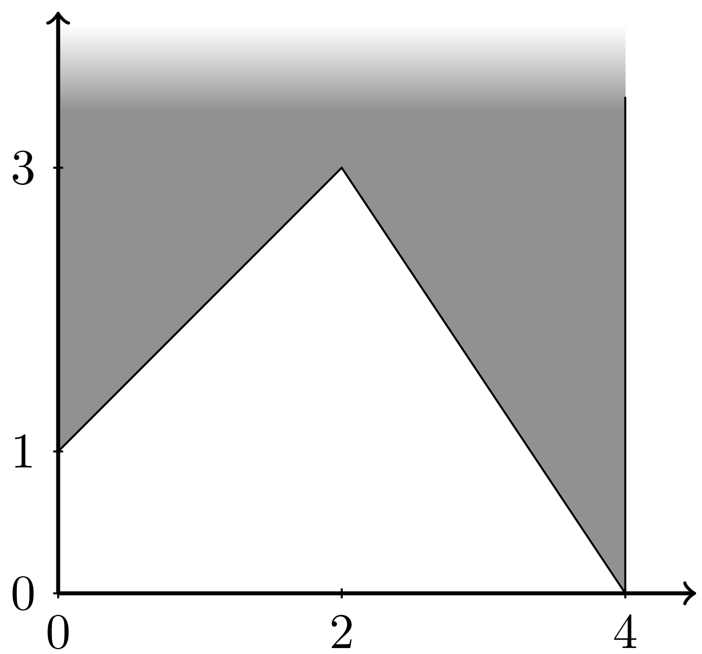

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λ's are SOS2 ⟶

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$$\lambda_v \ge 0 \quad \forall v \in \mathcal{V}(\mathcal{P}) := \bigcup_{P\in\mathcal{P}} V(P), \quad \sum_{v\in\mathcal{V}(\mathcal{P})} \lambda_v = 1$$

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SOS2 only for univariate

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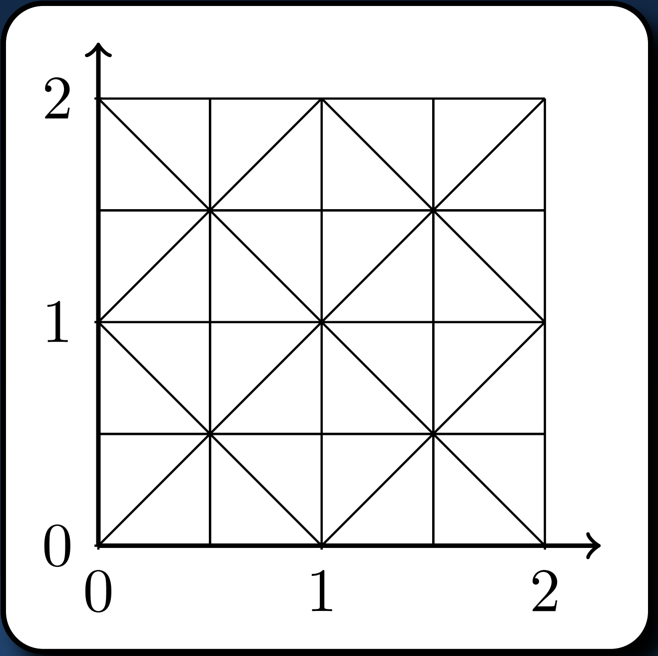
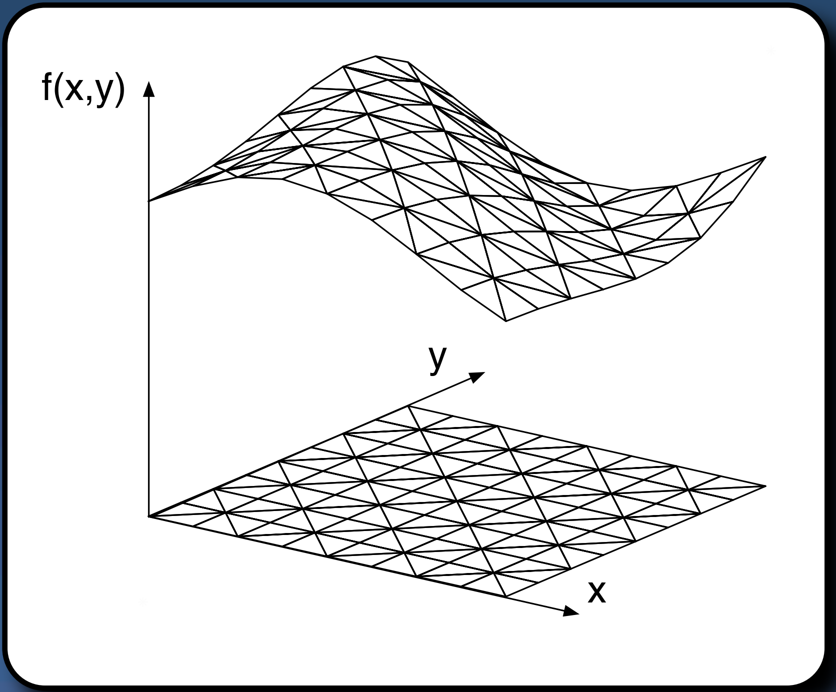
$$\exists P \in \mathcal{P} \text{ s.t. } \{v \in \mathcal{V}(\mathcal{P}) \, : \, \lambda_v > 0\} \subset V(P)$$

- Nonzero variables are associated to vertices of a single polytope.

Existing Models are Linear on $|\mathcal{P}|$

- Other models: Multiple Choice (MC), Incremental (Inc), Disaggregated Convex Combination (DCC).
- Number of binary variables and *combinatorial "extra" constraints* are linear in $|\mathcal{P}|$.
- For multivariate on a $k \times k$ grid $|\mathcal{P}| = O(k^2)$.
- Logarithmic sized formulations?





SOS1, SOS2 and CC constraints.

- SOS1-2 (Beale and Tomlin 1970):
  - SOS1: At most one variable is nonzero.
  - SOS2: Only 2 adjacent variables are nonzero.

    ✓ (0,1,1/2,0,0) ✕ (0,1,0,1/2,0)
- $(\lambda_i)_{i\in J} \in \mathbb{R}^J_+$, allowed sets $(S_i)_{i\in I}$, $S_i \subset J$.
  - SOS1: $I = J, \quad S_i = \{i\}$.
  - SOS2: $J = \{0,\ldots,m\}$, $I = J \setminus \{m\}$, $S_i = \{i, i+1\}$.
  - CC: $J = \mathcal{V}(\mathcal{P})$, $I = \mathcal{P}$, $S_P = V(P)$.

Logarithmic Formulation for SOS1

$\sum_{j=0}^{3} \lambda_j = 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3 \geq 0,$ at most 1 λ_j is nonzero.

Allowed sets: $S_0 = \{0\}$, $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$.

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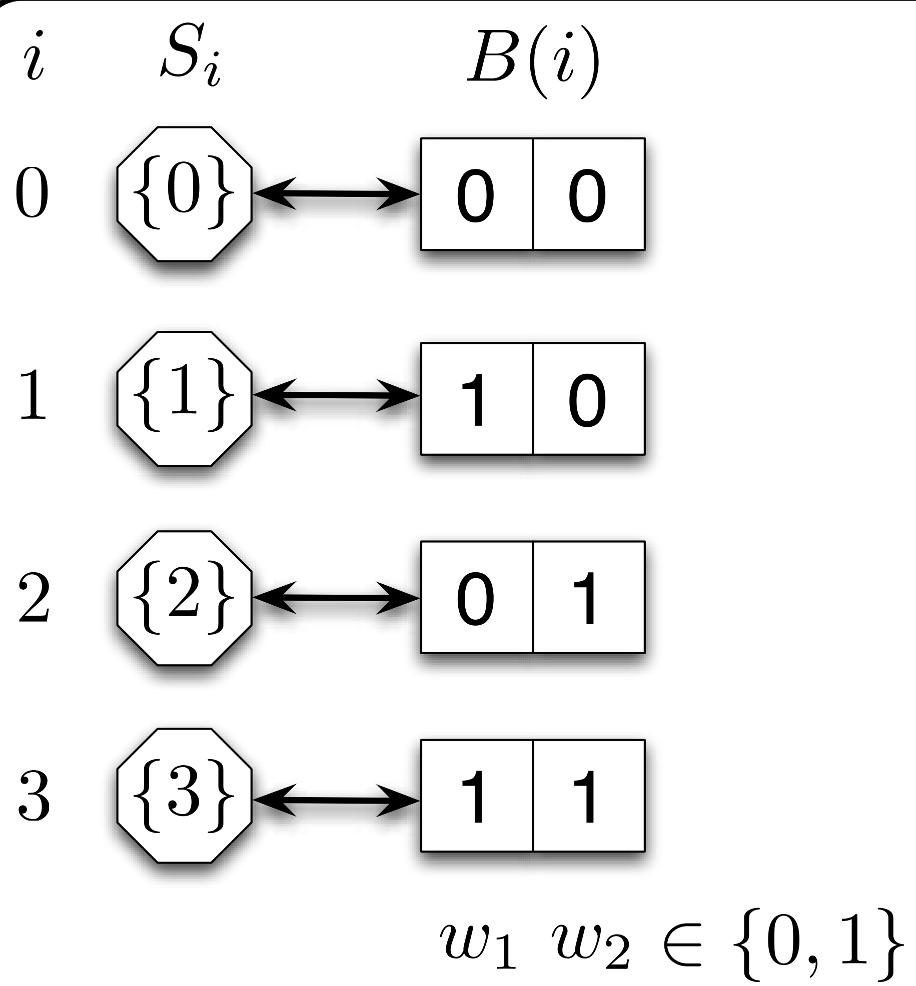

- Injective function:

$$B : \{0, \ldots, m-1\} \rightarrow \{0,1\}^{\lceil \log_2 m \rceil}$$

- Variables:

$$w \in \{0,1\}^{\lceil \log_2 m \rceil}$$

- Idea:

$$\lambda_j > 0 \Leftrightarrow w = B(j)$$

Logarithmic Formulation for SOS1

$$\sum_{j=0}^{3} \lambda_j = 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3 \geq 0, \text{ at most 1 } \lambda_j \text{ is nonzero.}$$

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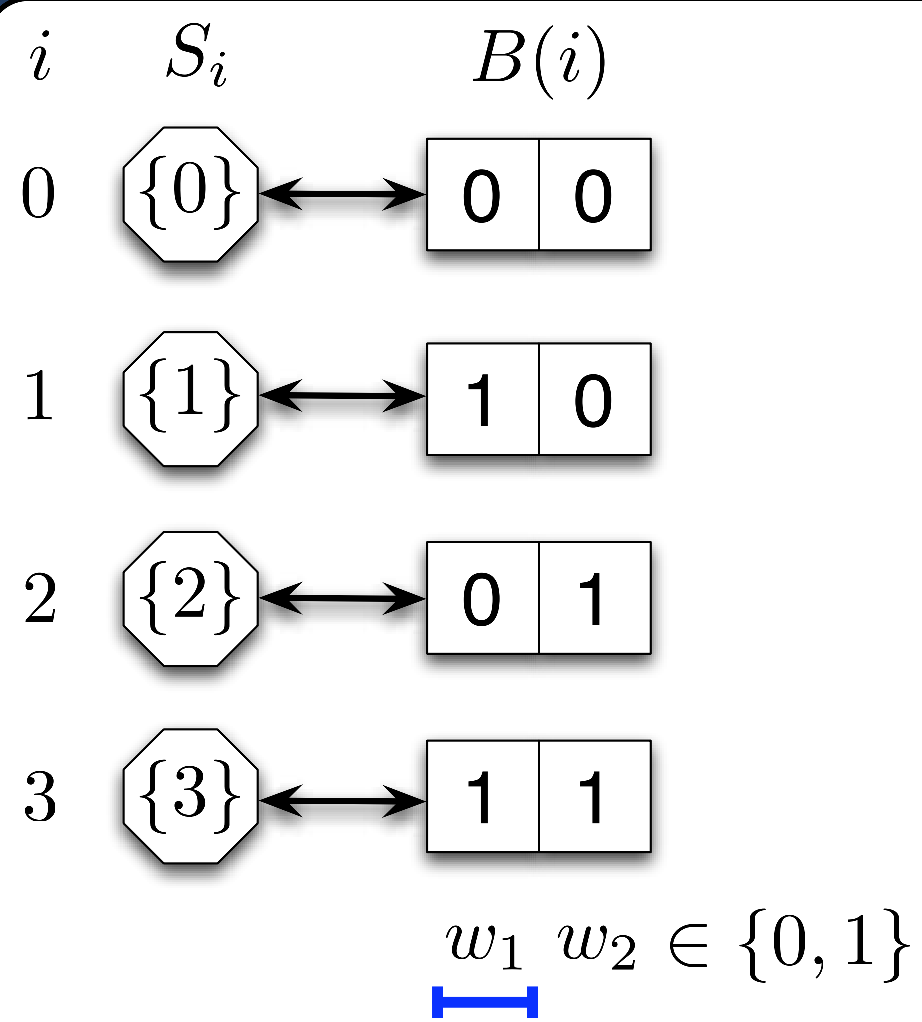

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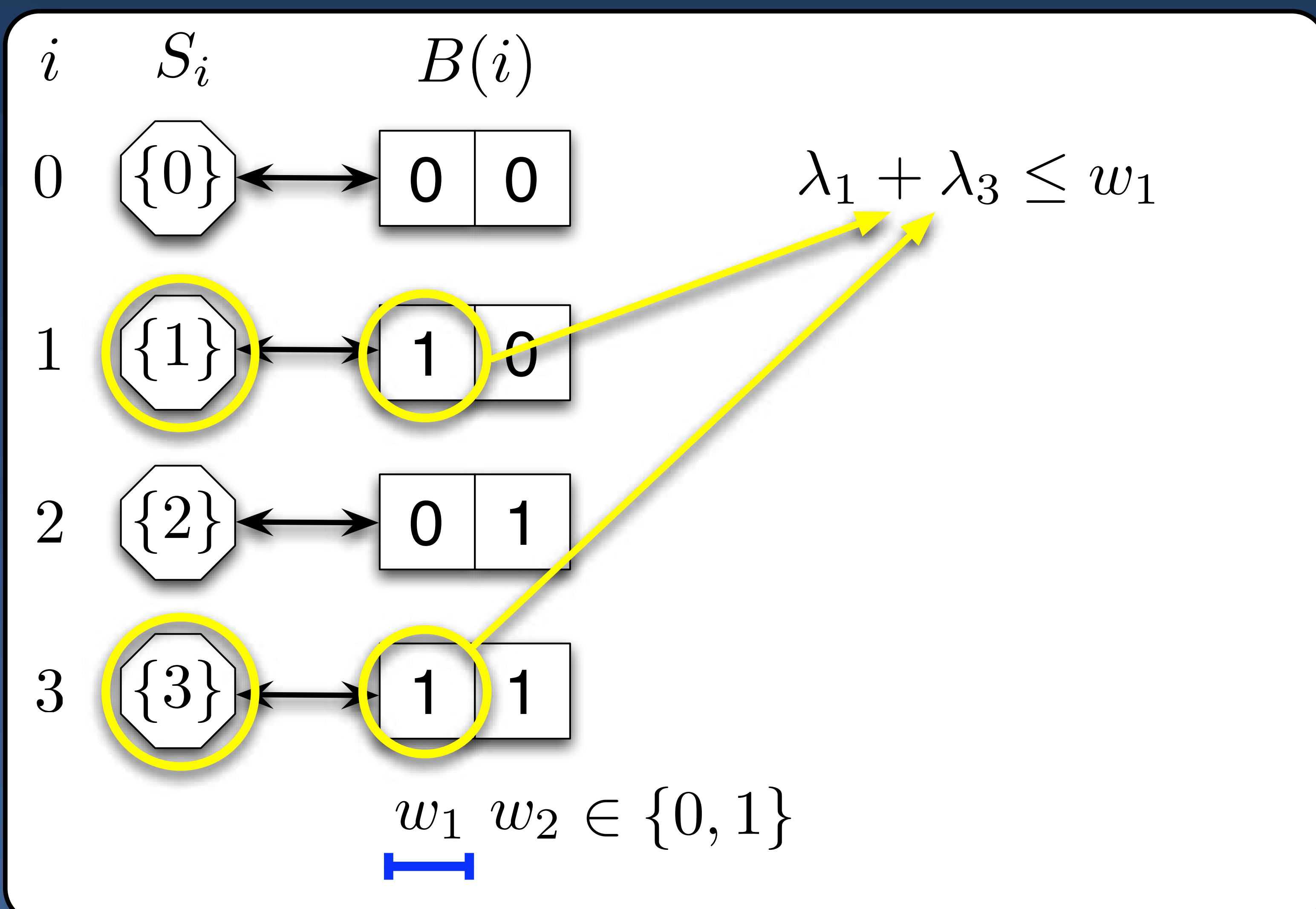

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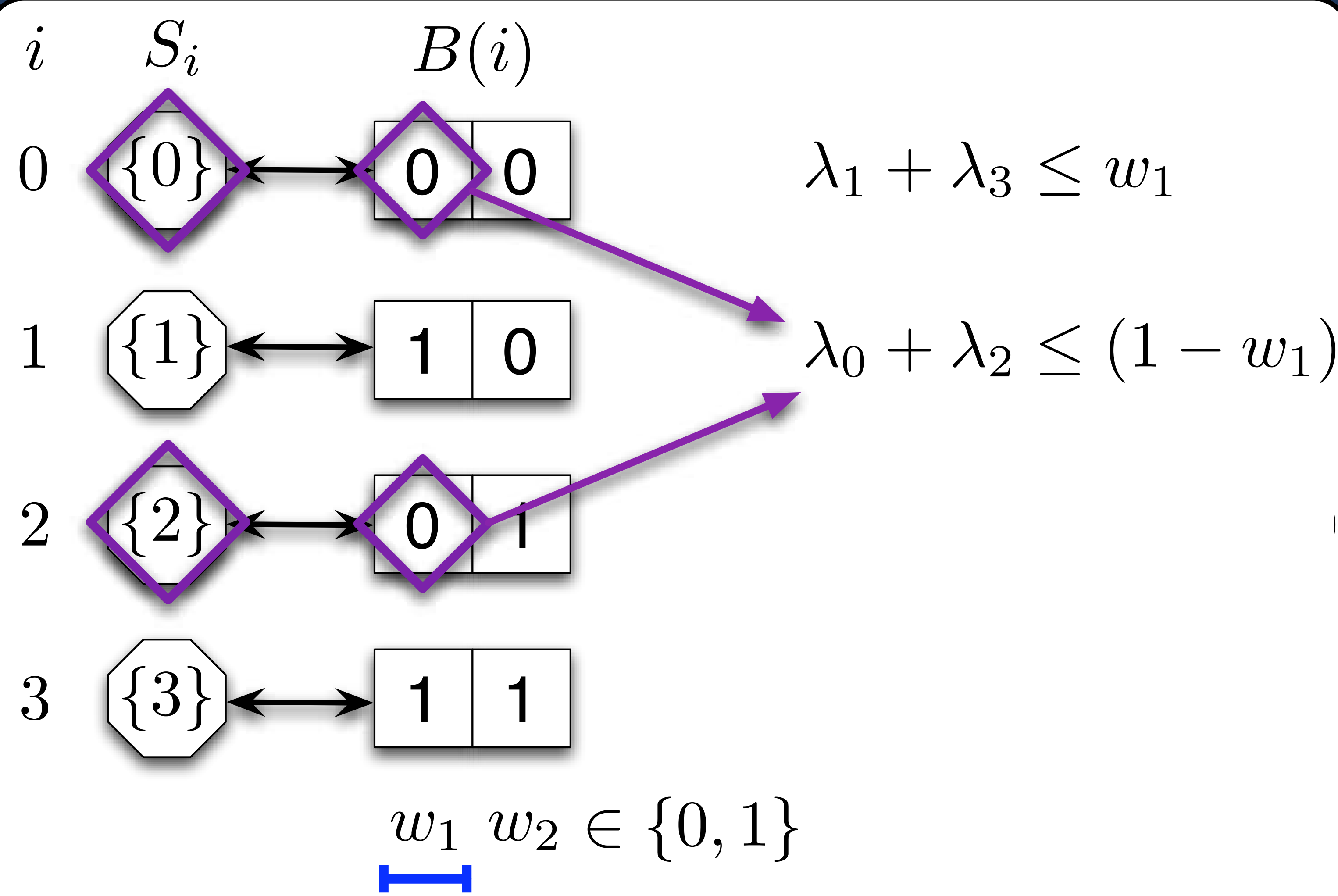

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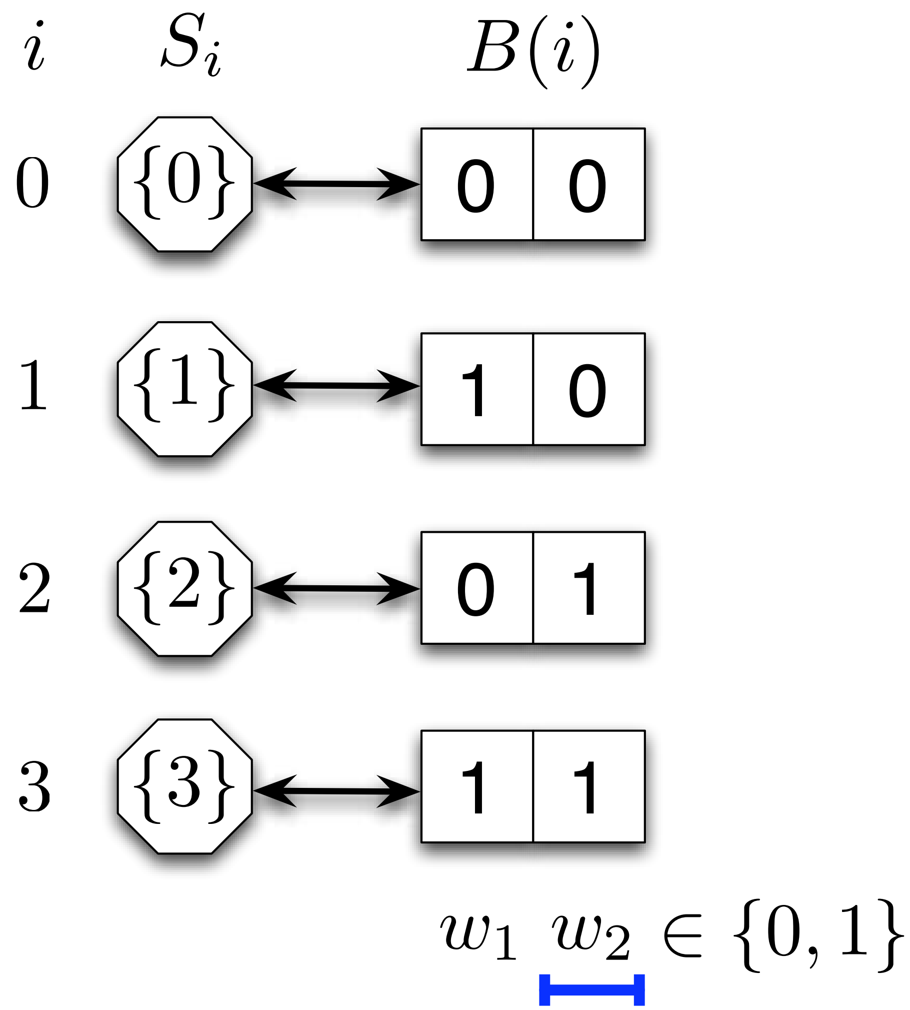

$\lambda_1 + \lambda_3 \leq w_1$

$\lambda_0 + \lambda_2 \leq (1 - w_1)$

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- In general:

$\lceil \log_2 m \rceil$ variables and $2\lceil \log_2 m \rceil$ constraints.

Logarithmic Formulation for SOS2

$\sum_{j=0}^{4} \lambda_j = 1, \quad \lambda_0, \ldots, \lambda_4 \geq 0,$ only 2 adjacent λ_j's ar nonzero.

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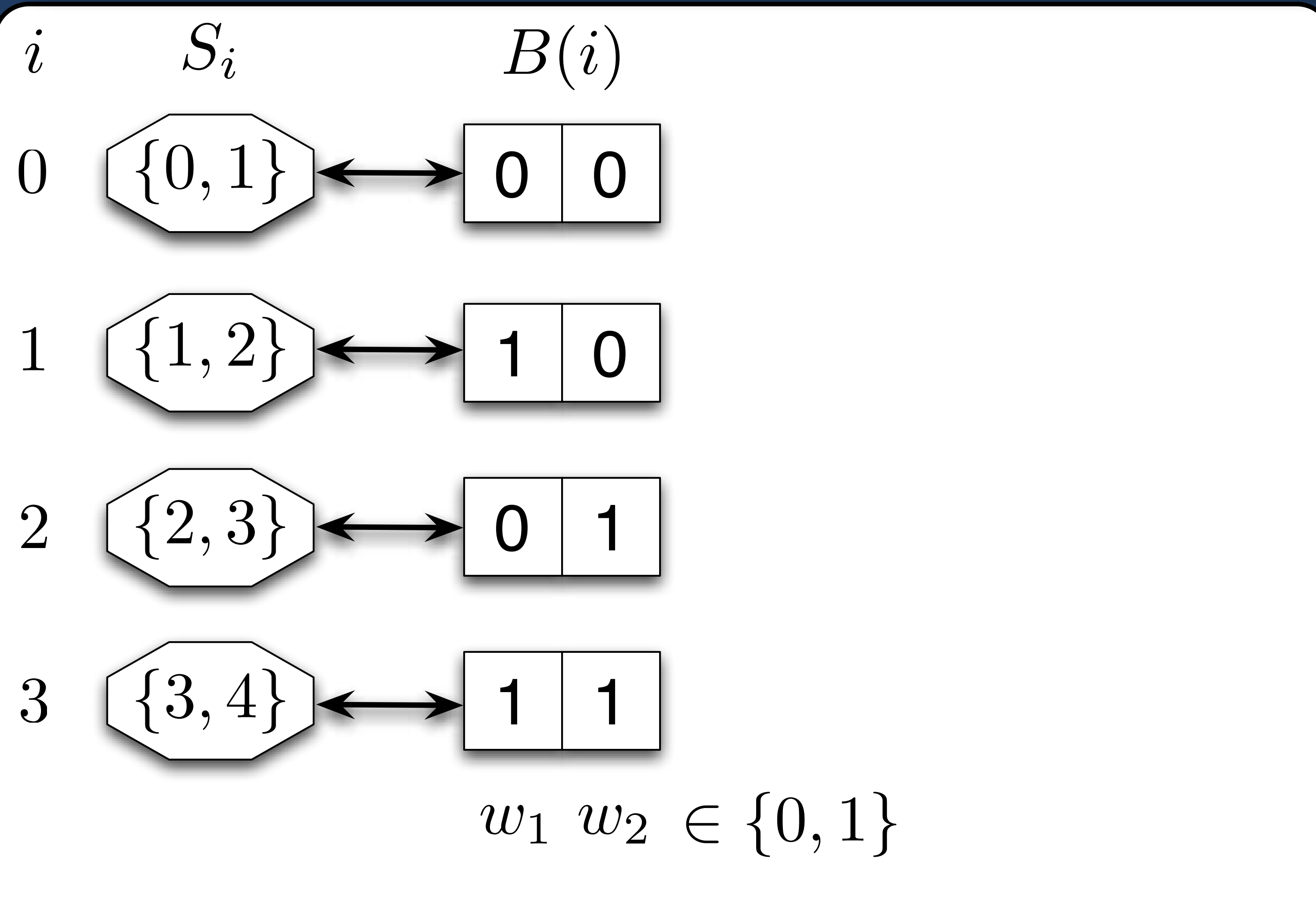

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$$\lambda_j, \lambda_{j+1} > 0 \Leftrightarrow w = B(j)$$

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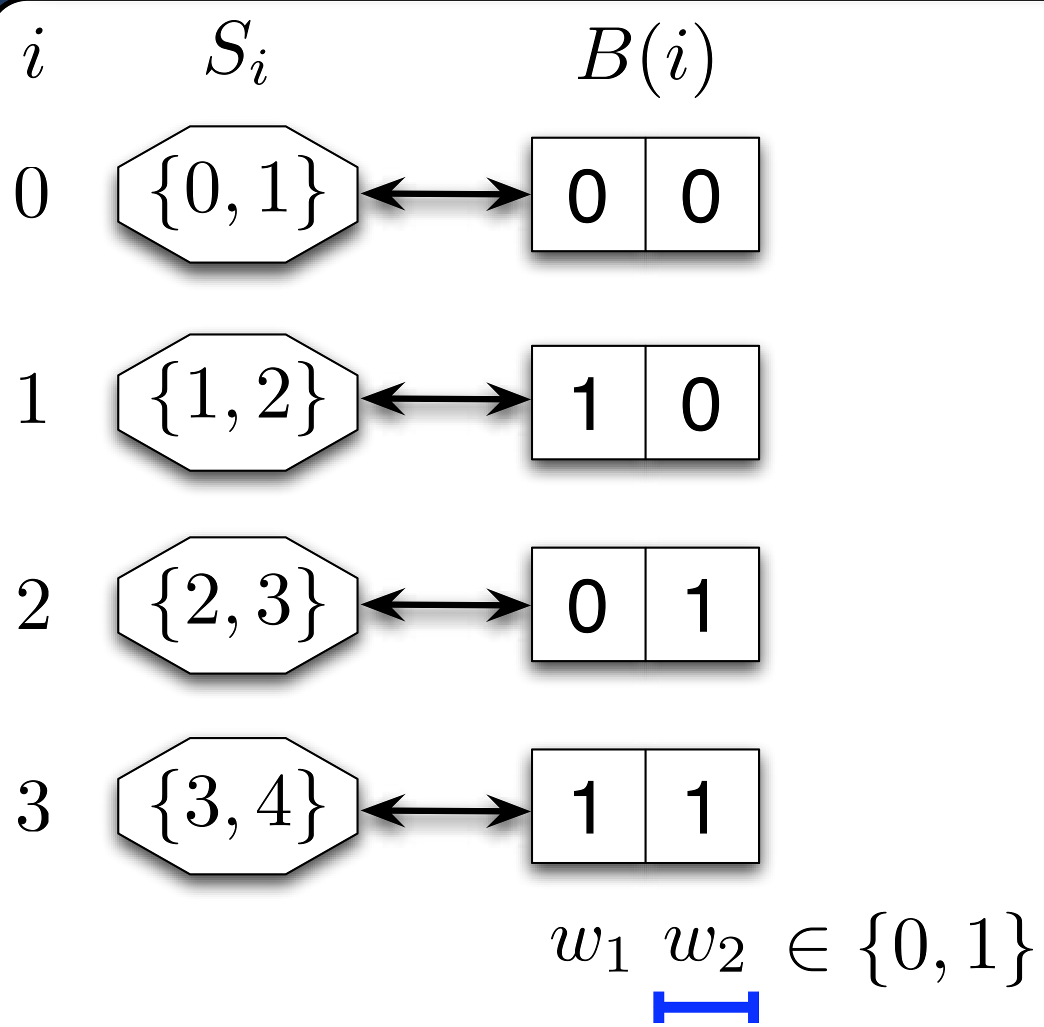

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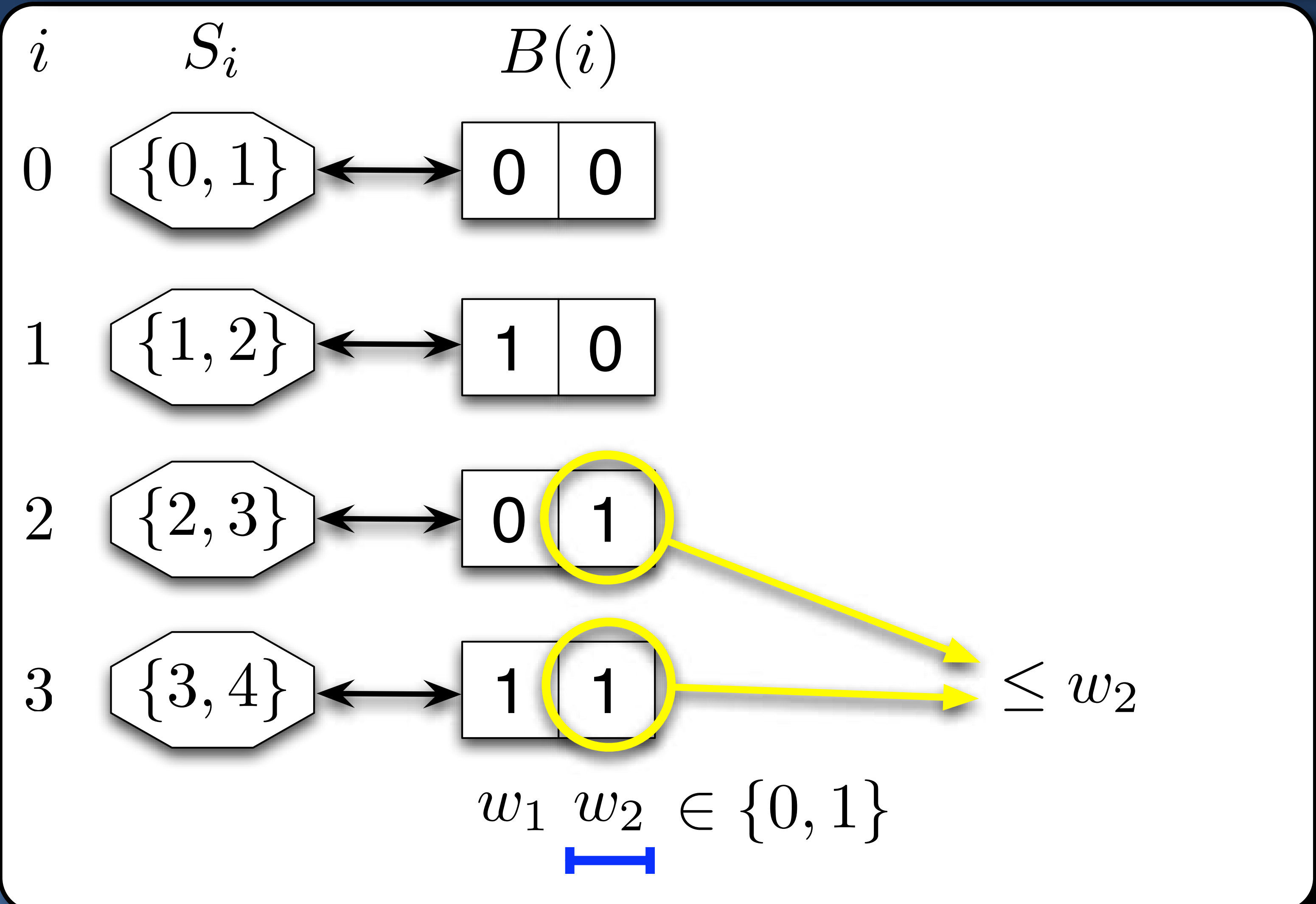

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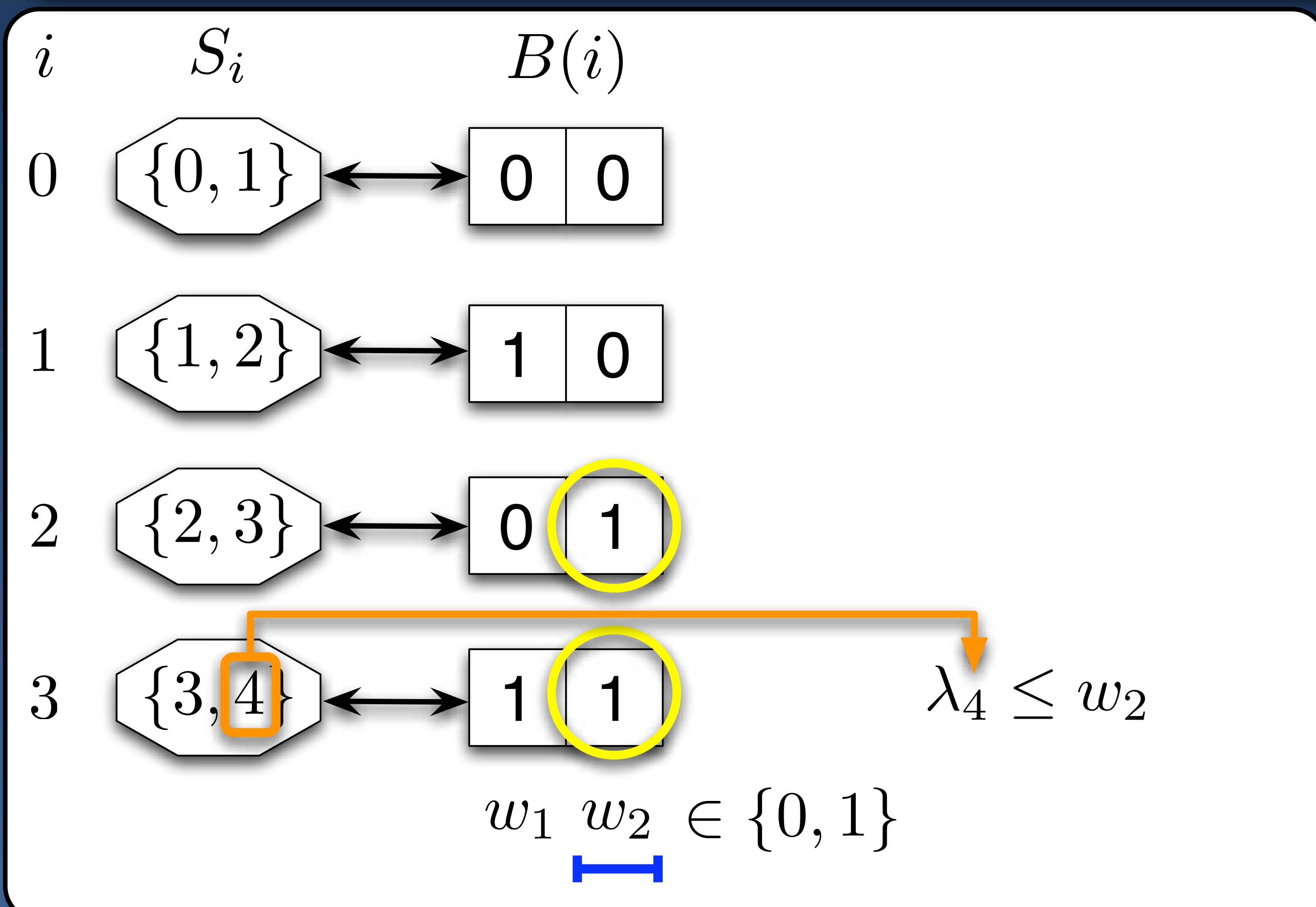

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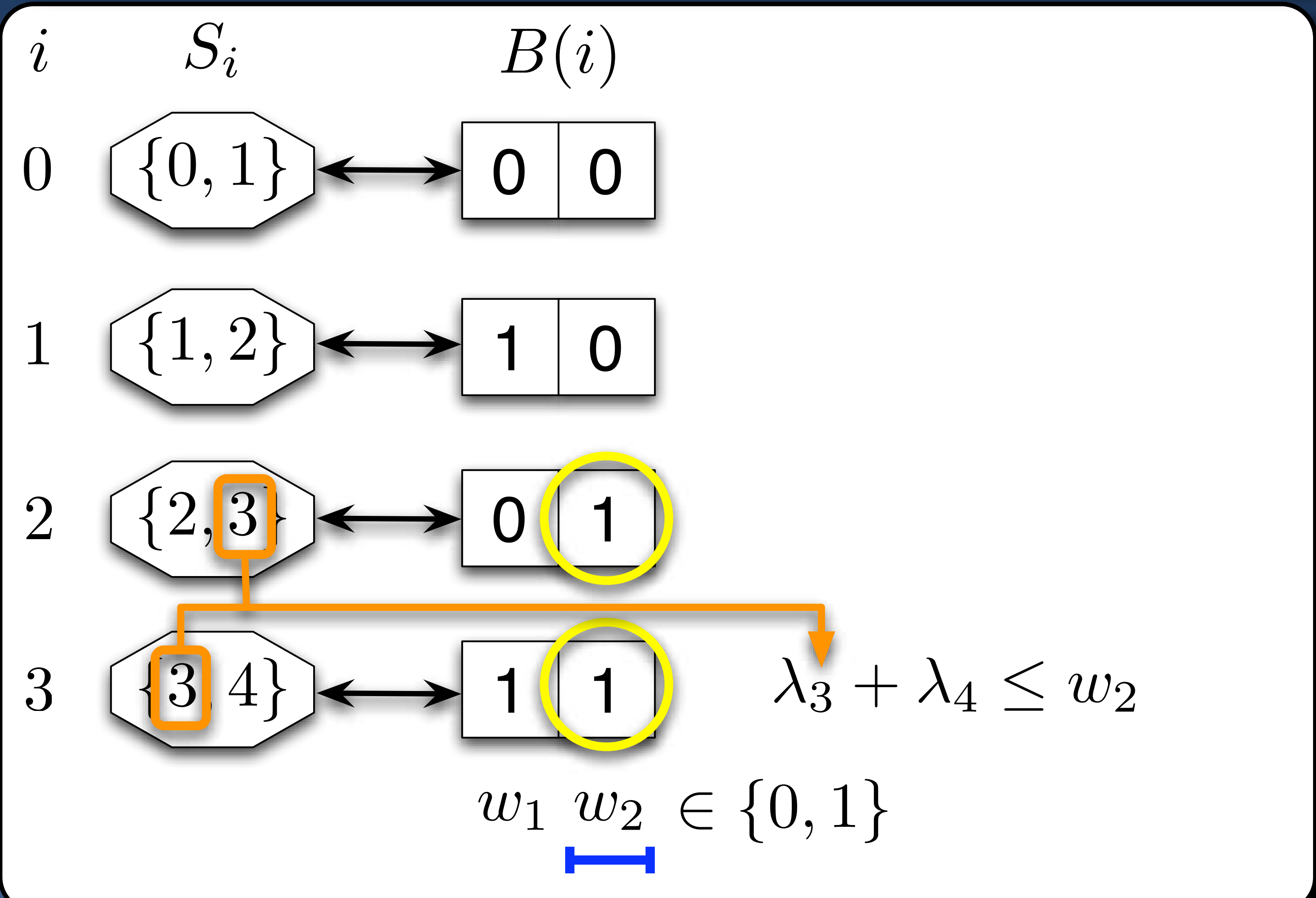

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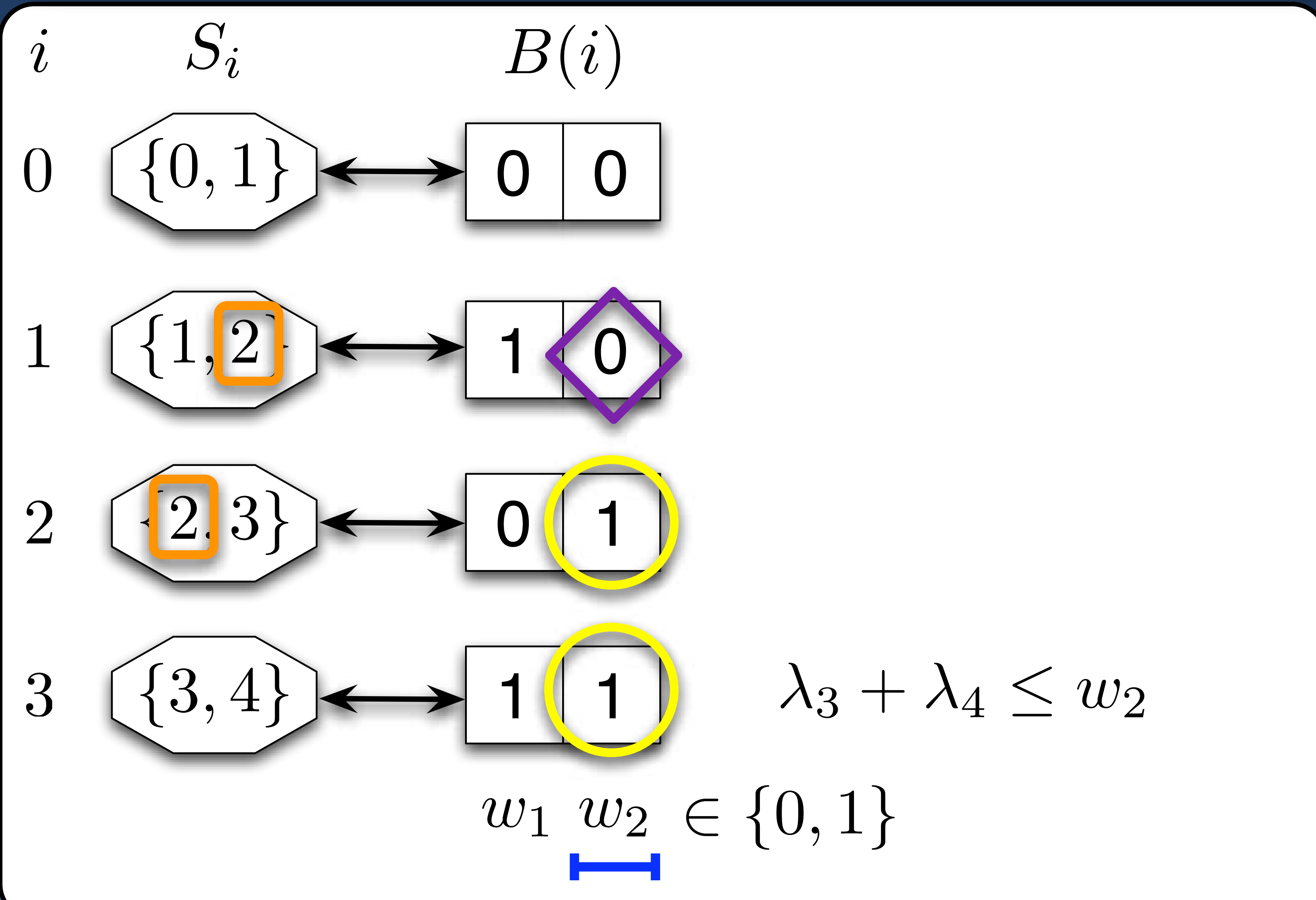

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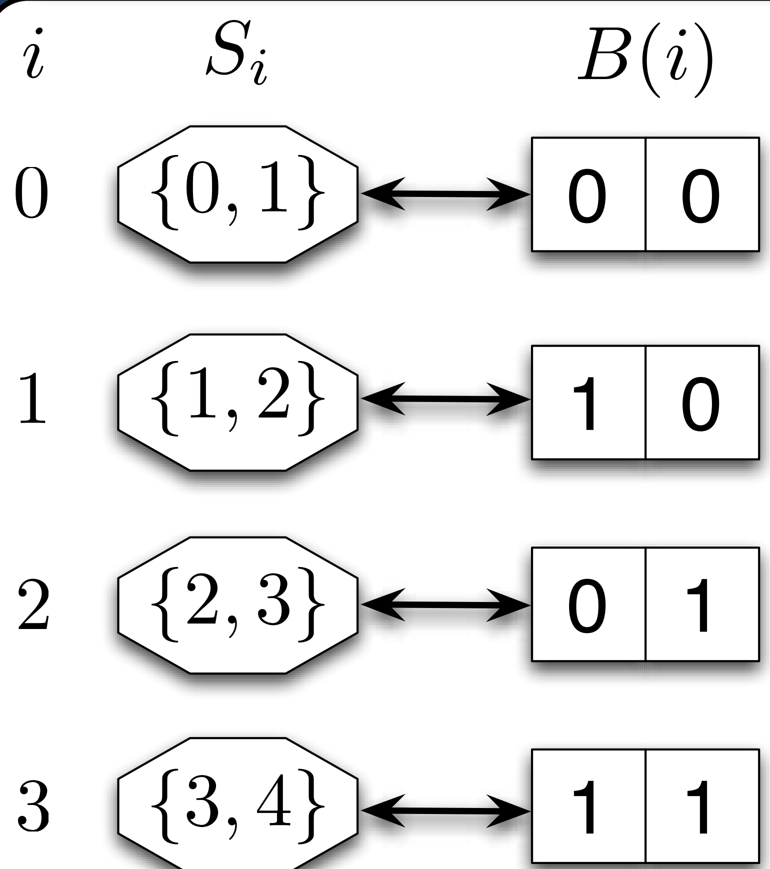

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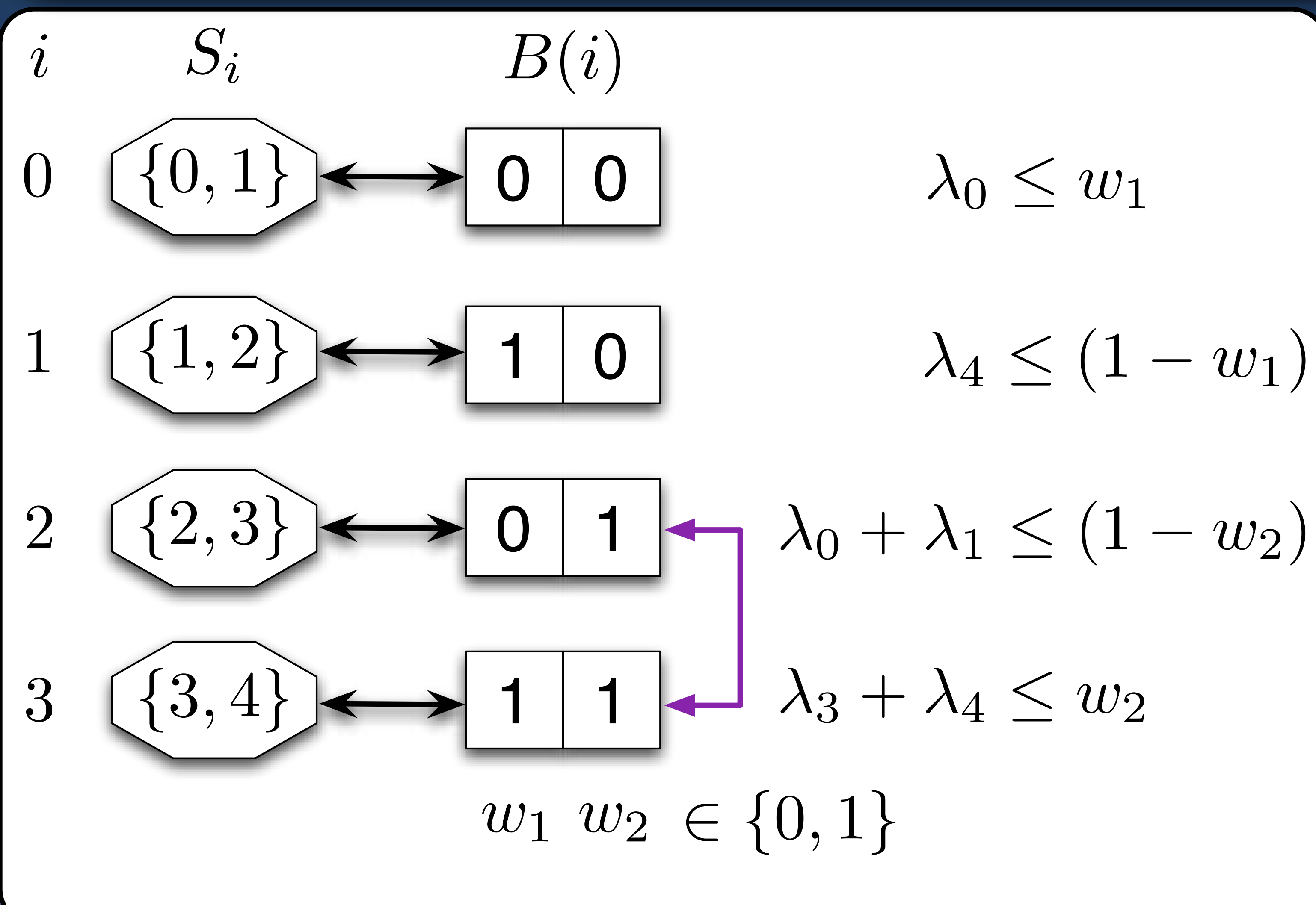

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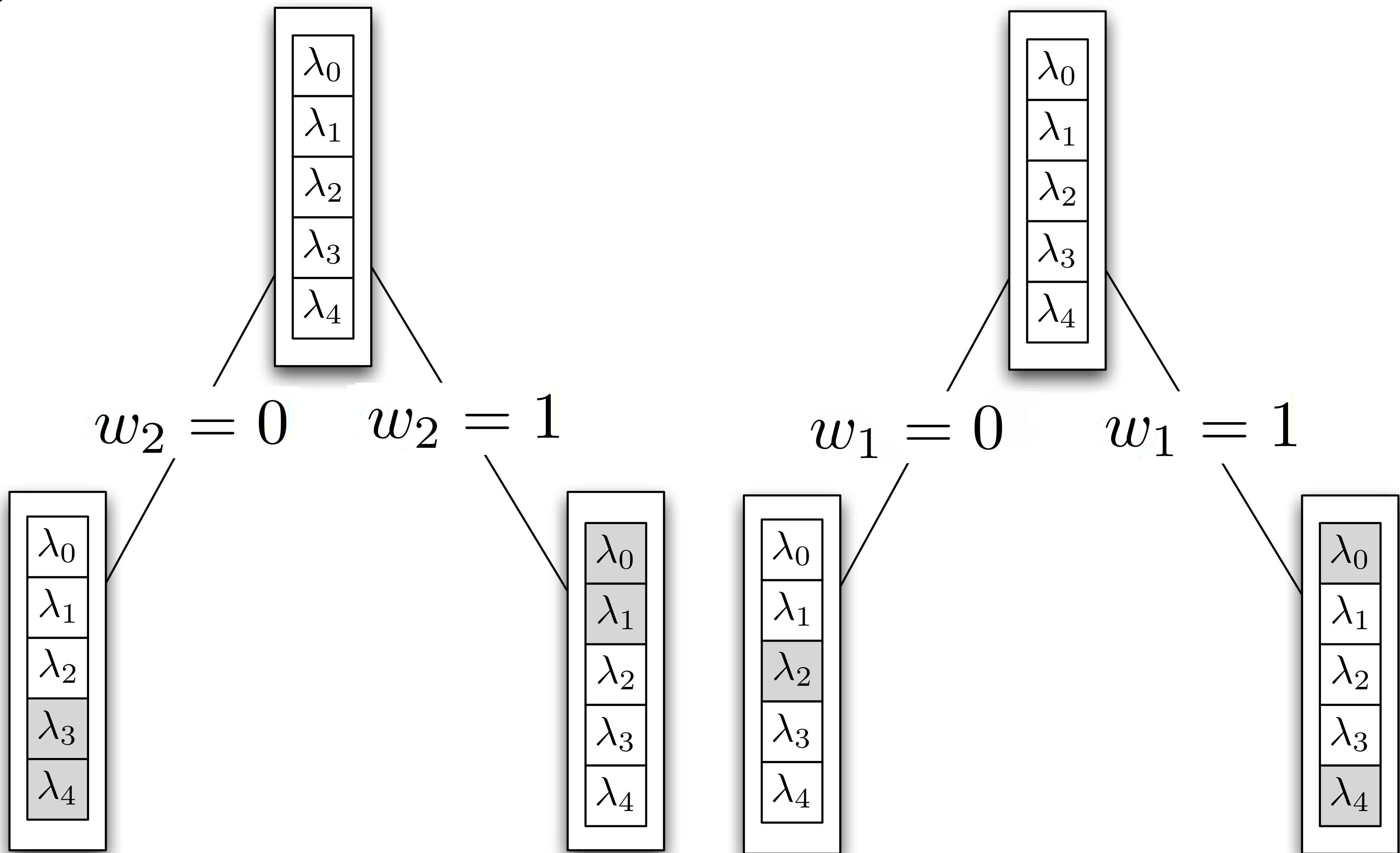
- In general:

  $B(i)$ and $B(i+1)$ differ in one component

- Gray Code.

Independent Branching: Dichotomies



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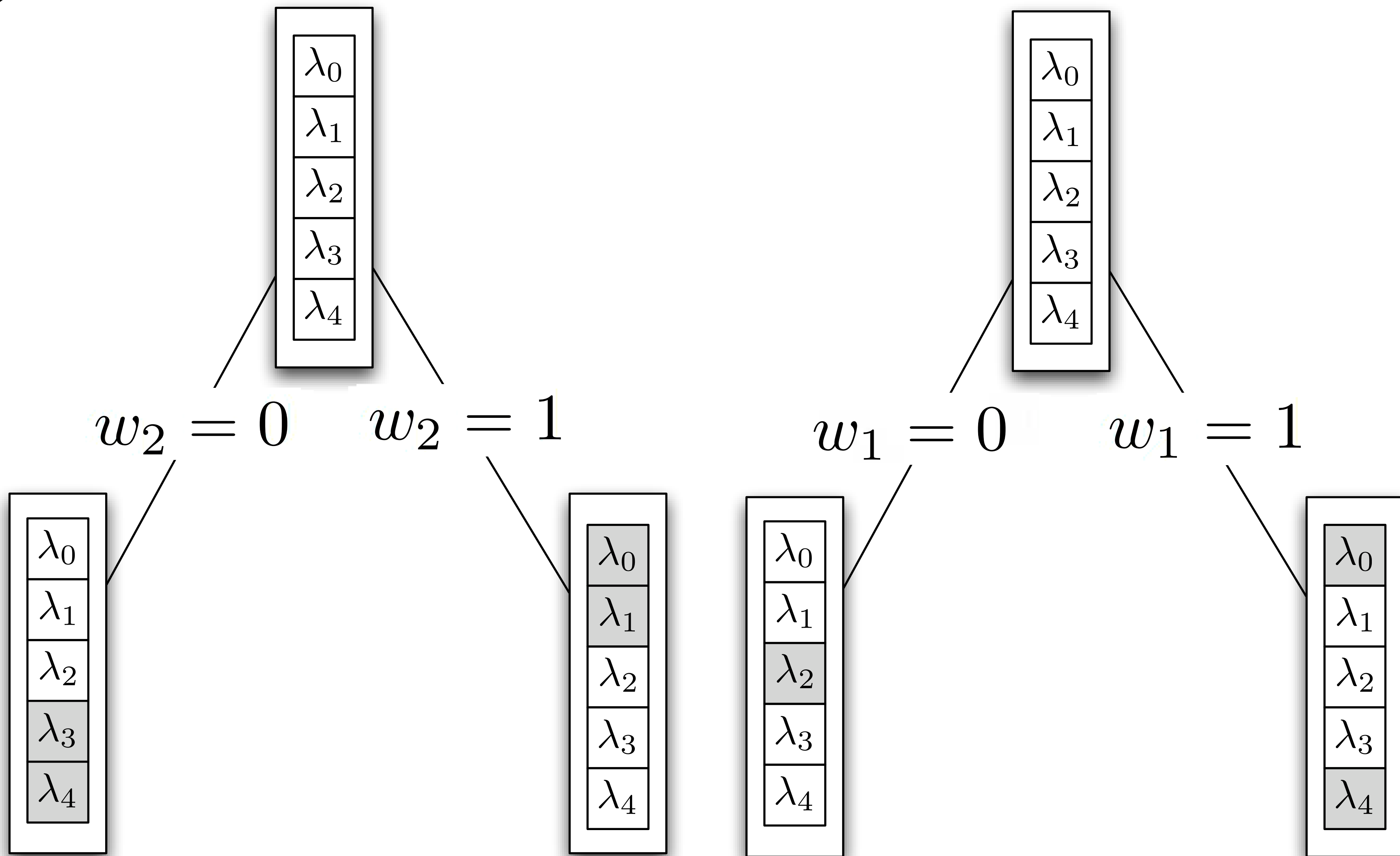

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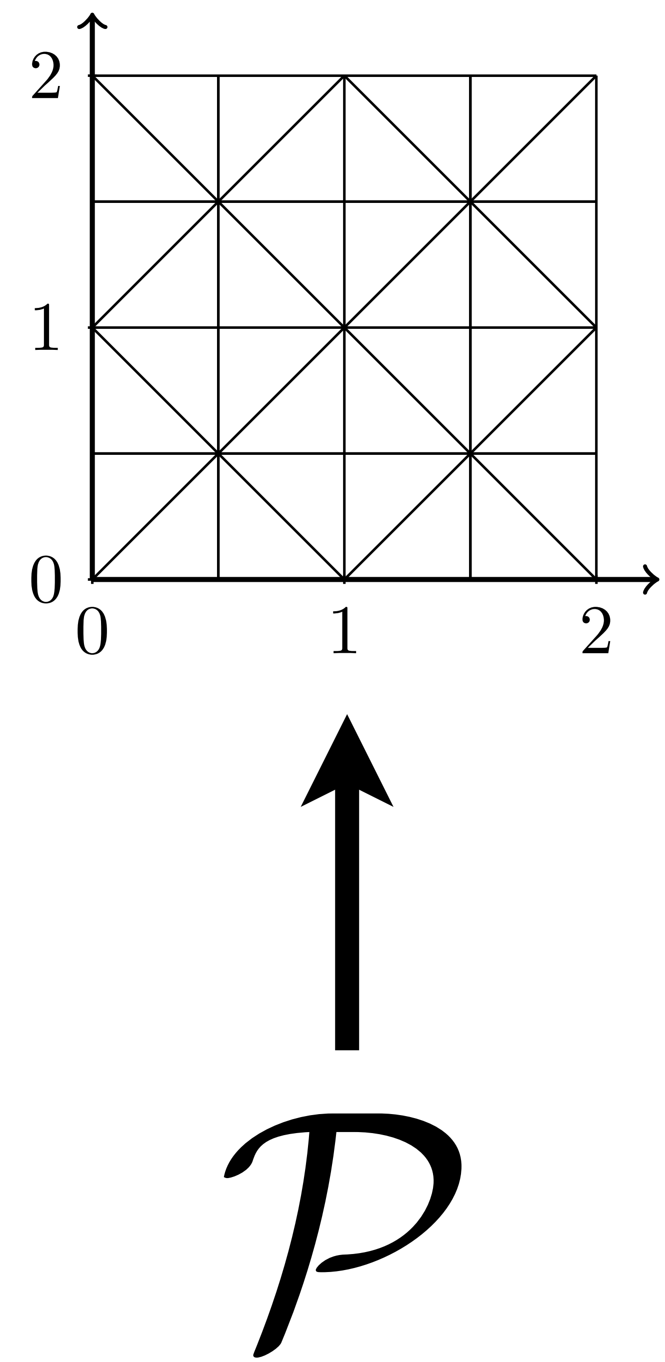
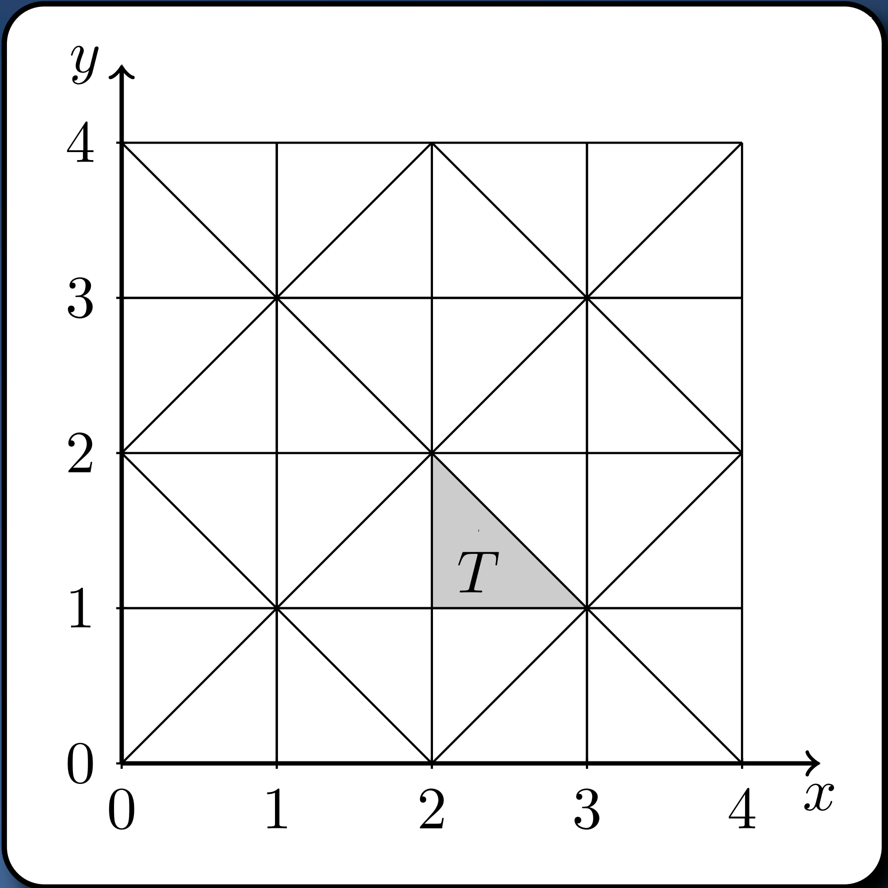


Independent Branching for 2 var CC

- Select Triangle by forbidding vertices.
- 2 stages:
  - Select Square by SOS2 on each variable.
  - Select 1 triangle from each square.



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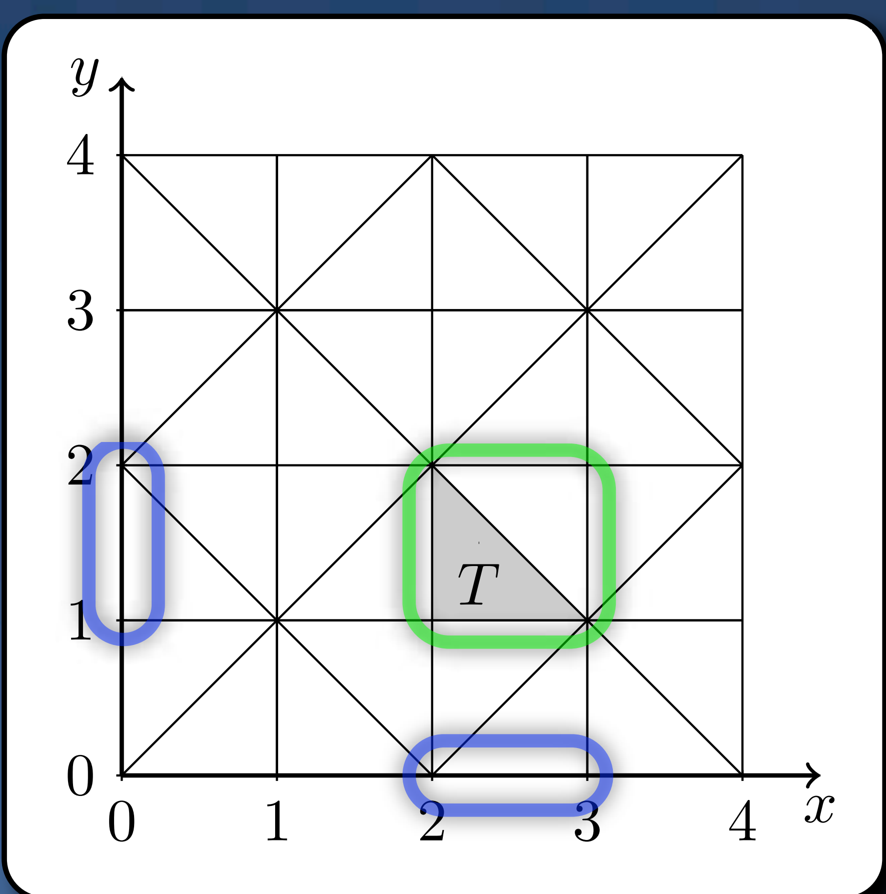

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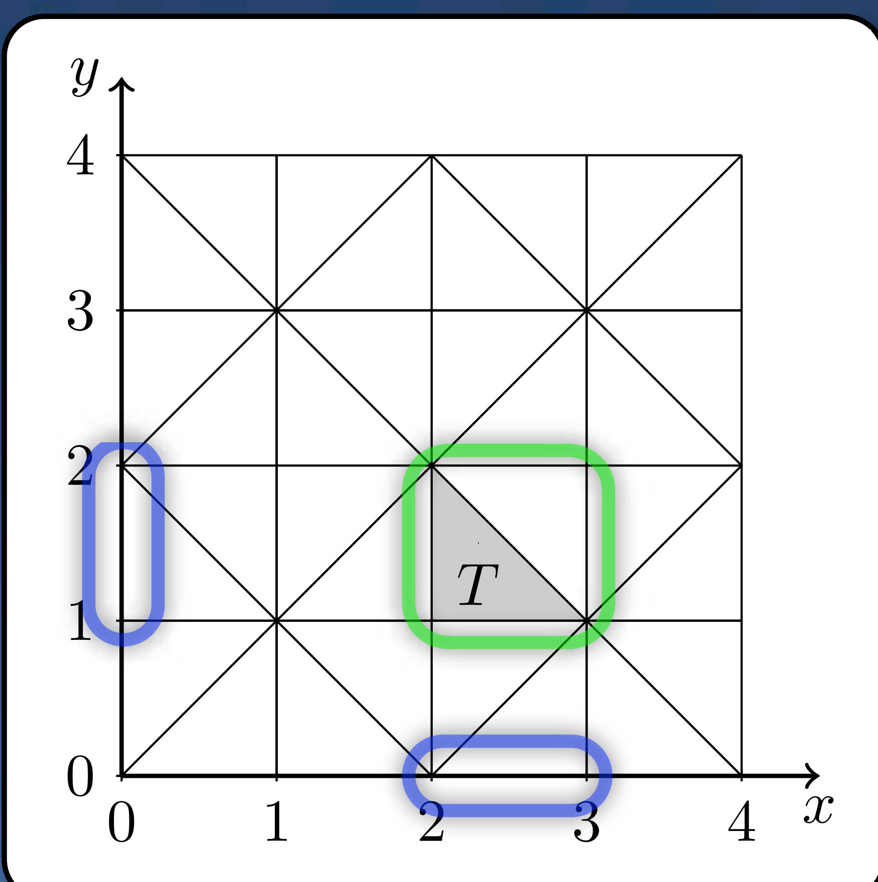
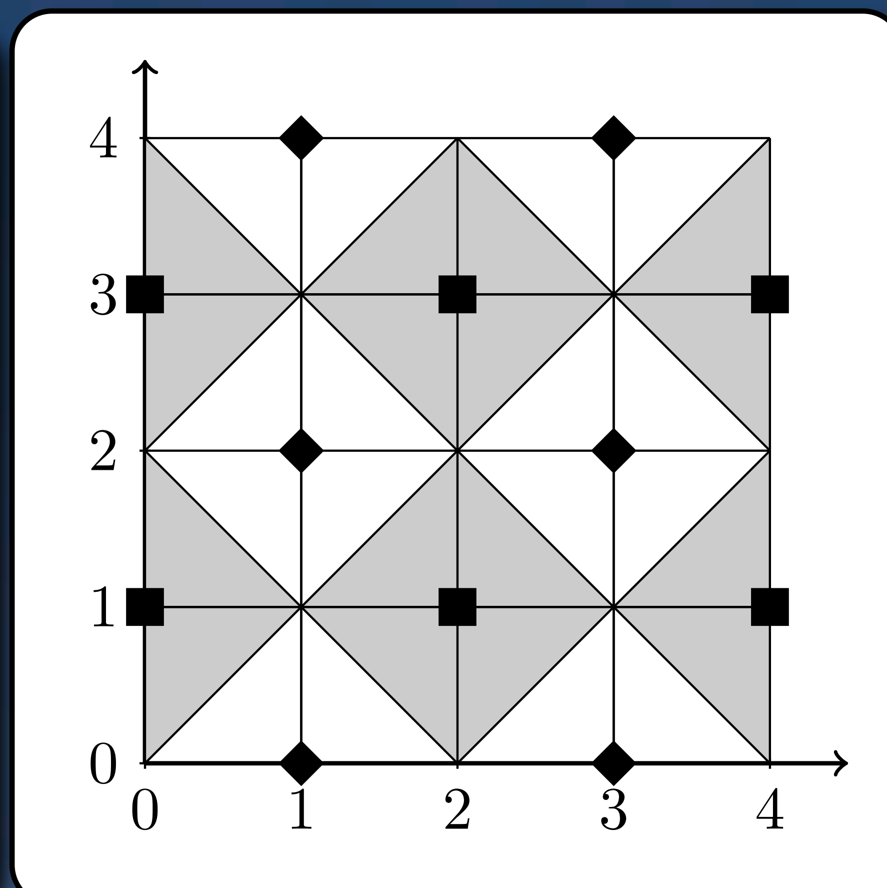


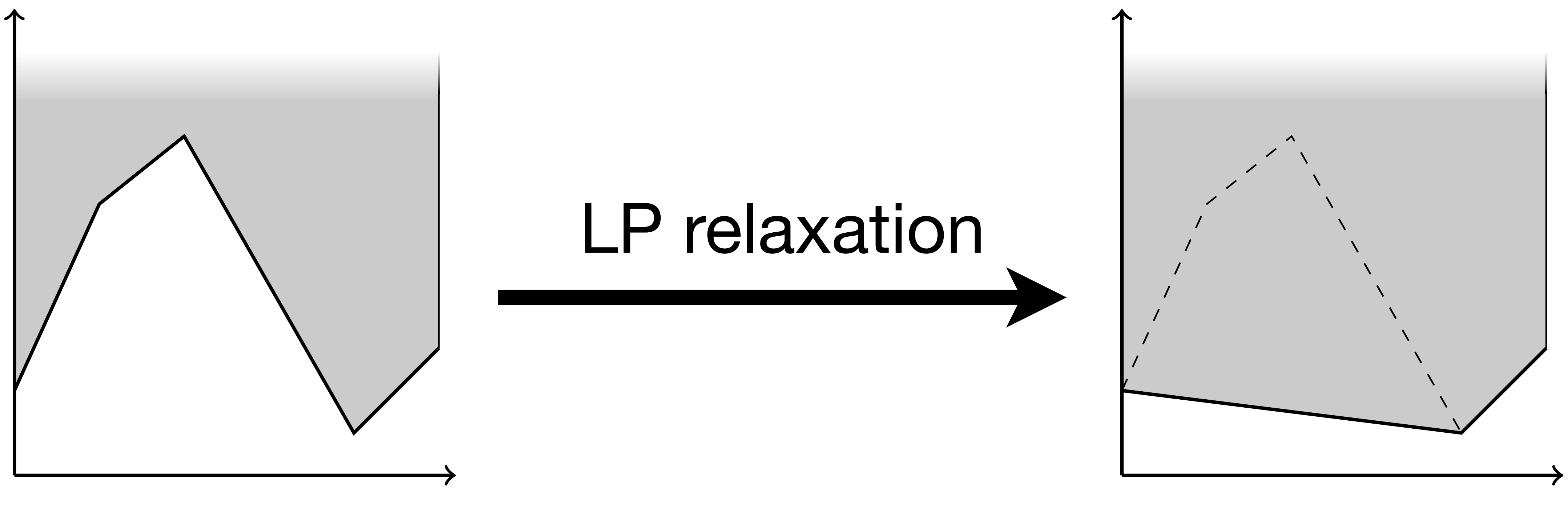


$$\bar{L} = \{(r,s) \in J \ : \ r \text{ even and } s \text{ odd}\} = \{\text{square vertices}\}$$

$$\bar{R} = \{(r,s) \in J \ : \ r \text{ odd and } s \text{ even}\} = \{\text{diamond vertices}\}$$

Strength of LP Relaxations

- *Sharp* Models: LP = lower convex envelope.



(a) $\mathrm{epi}(f)$. (b) $\mathrm{conv}(\mathrm{epi}(f))$.

  - All popular models are sharp.
- *Locally Ideal*: LP = Integral (All but CC, even Log).
- *Locally ideal* implies *Sharp*.

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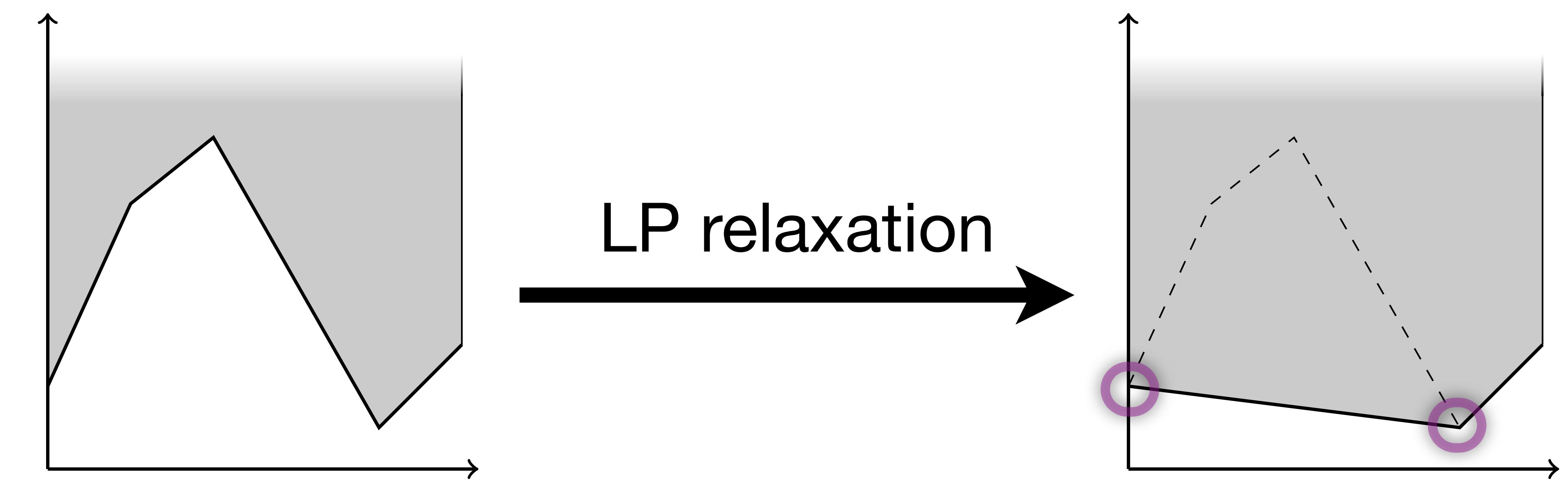

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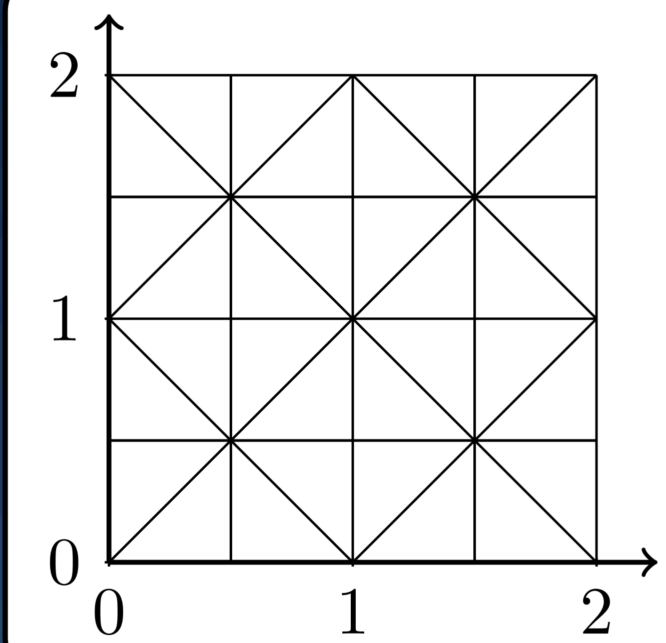
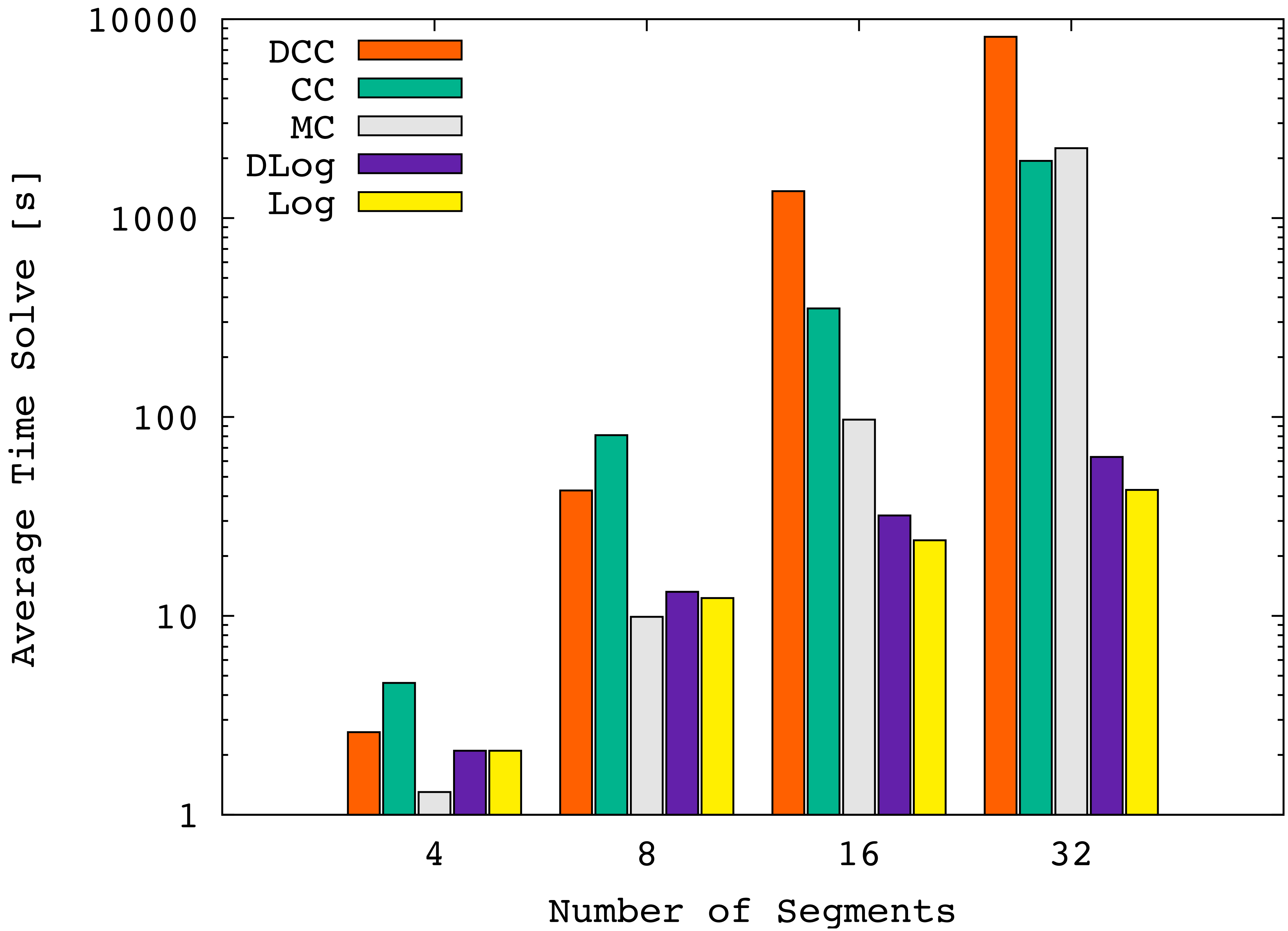
Computational Results

- Instances
  - Transportation problems (10x10 & 5x2).
  - Univariate: Concave Separable Objective.
  - Multivariate: 2-commodity. →
  - Functions: affine in k segments or k x k grid triangulation (100 instances per k).
- Solver: CPLEX 11 on 2.4Ghz machine.
- Logarithmic versions of CC = Log, DCC=DLog.



$(x, y) \rightarrow g(\|(x, y)\|)$
Concave PLF $g(\cdot)$

Univariate Case (Separable)



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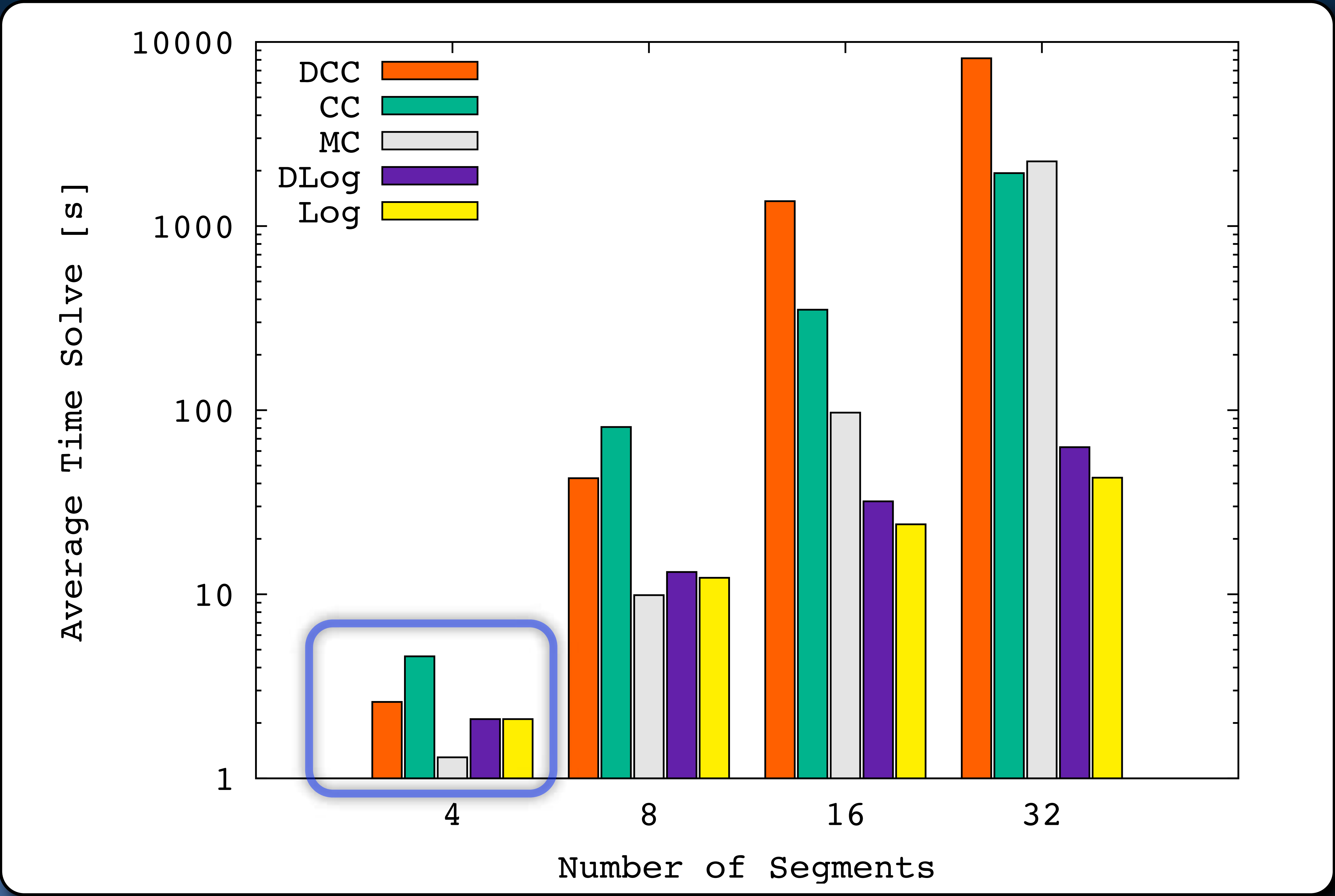

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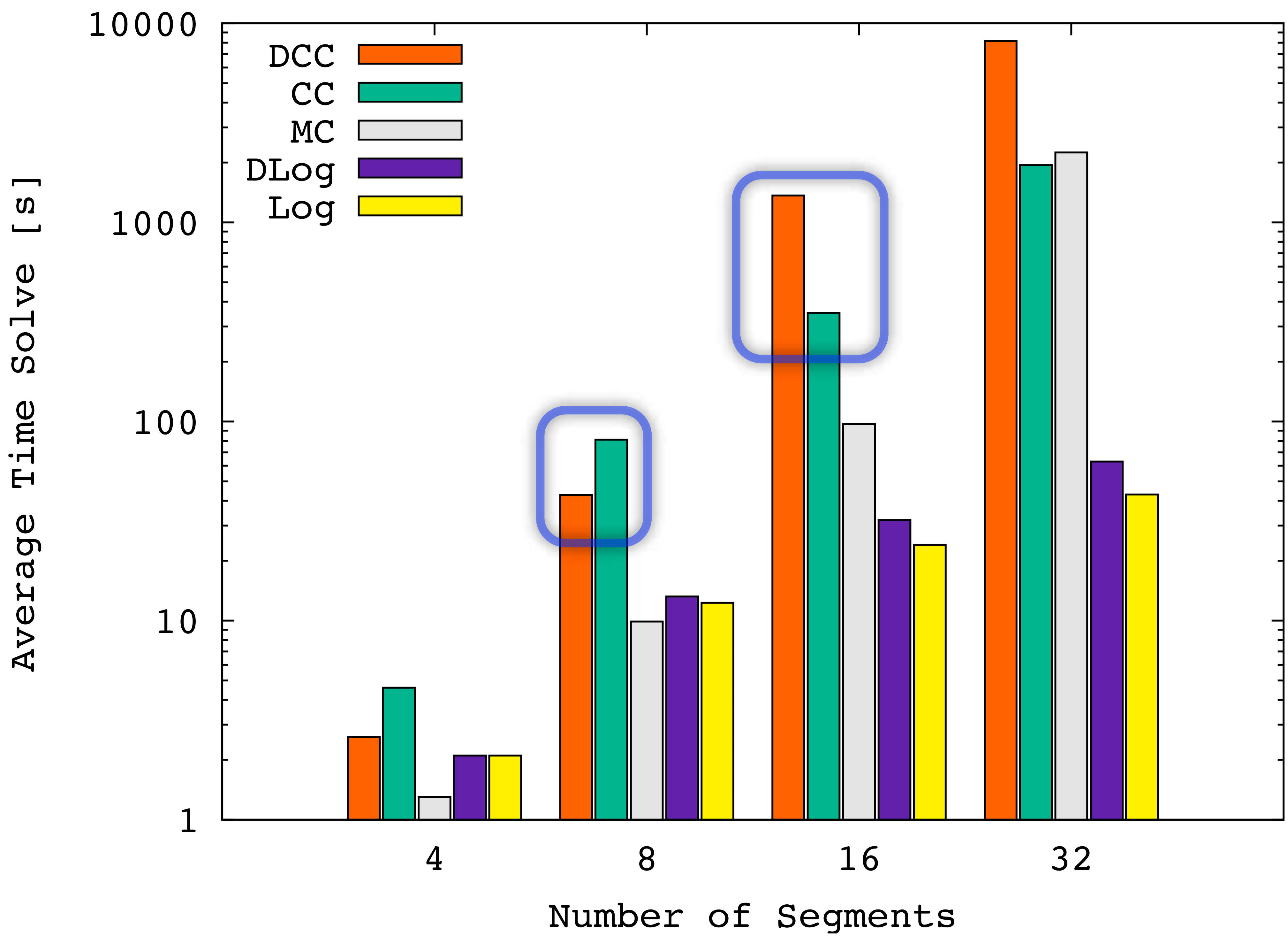

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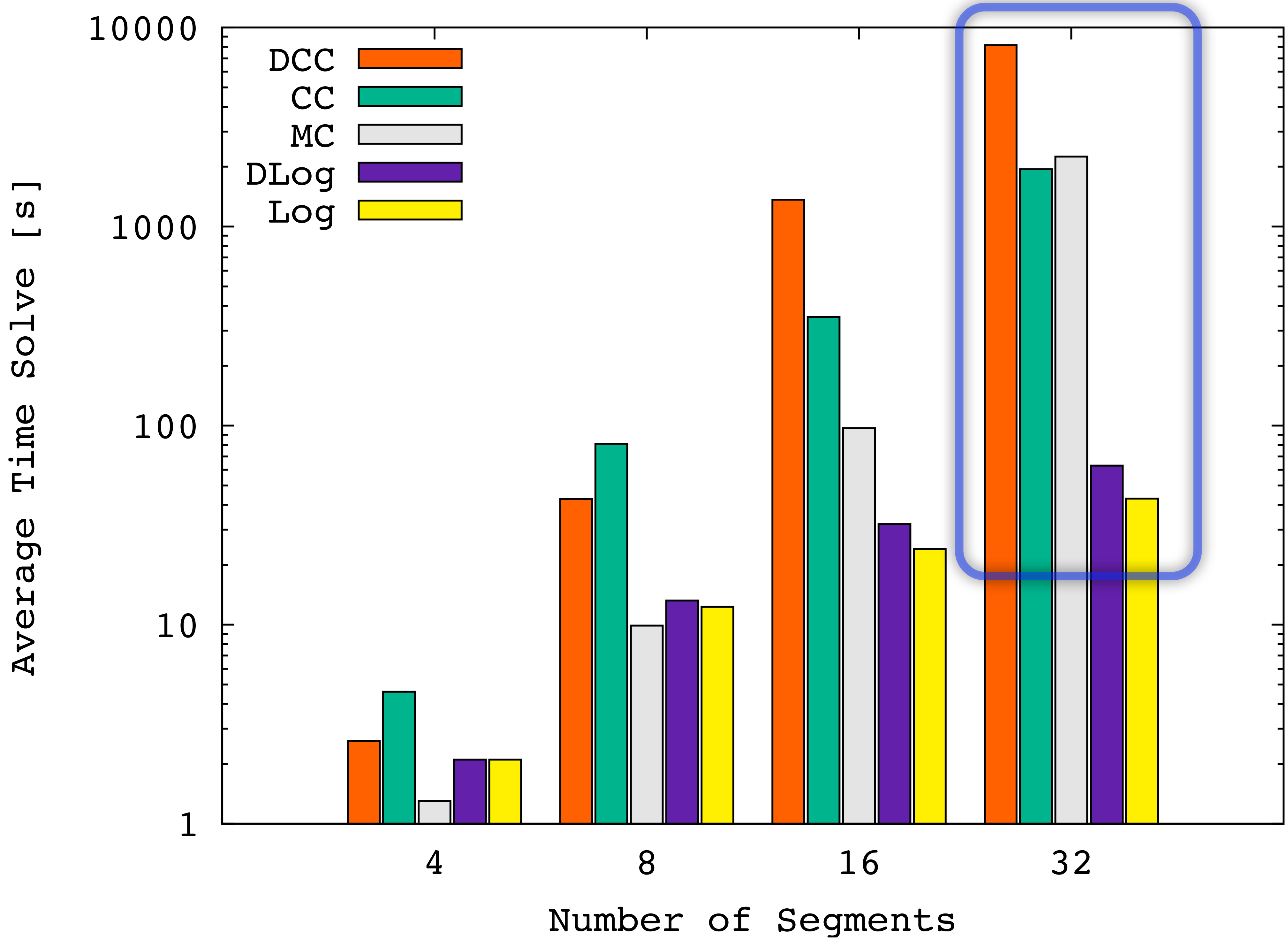
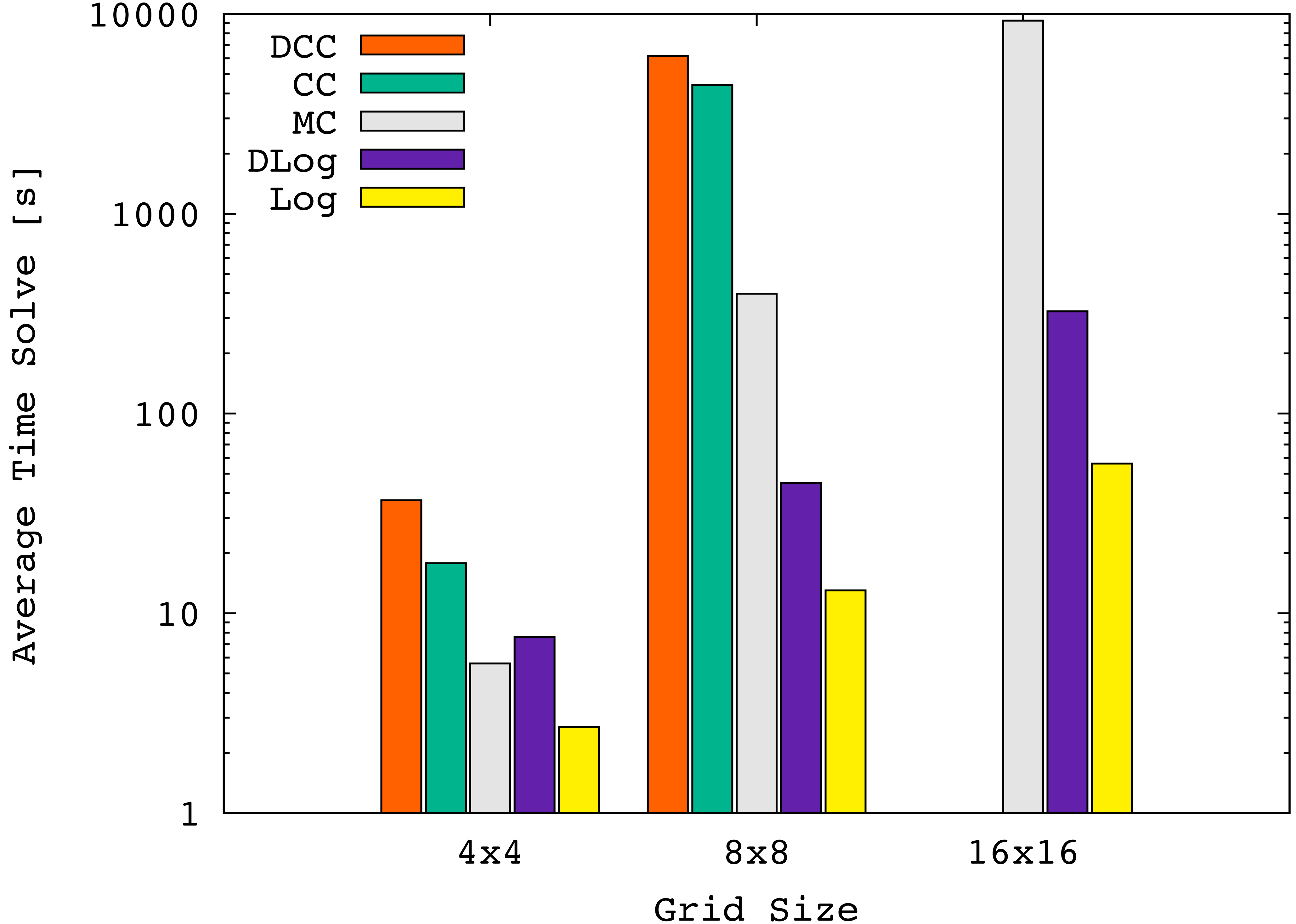
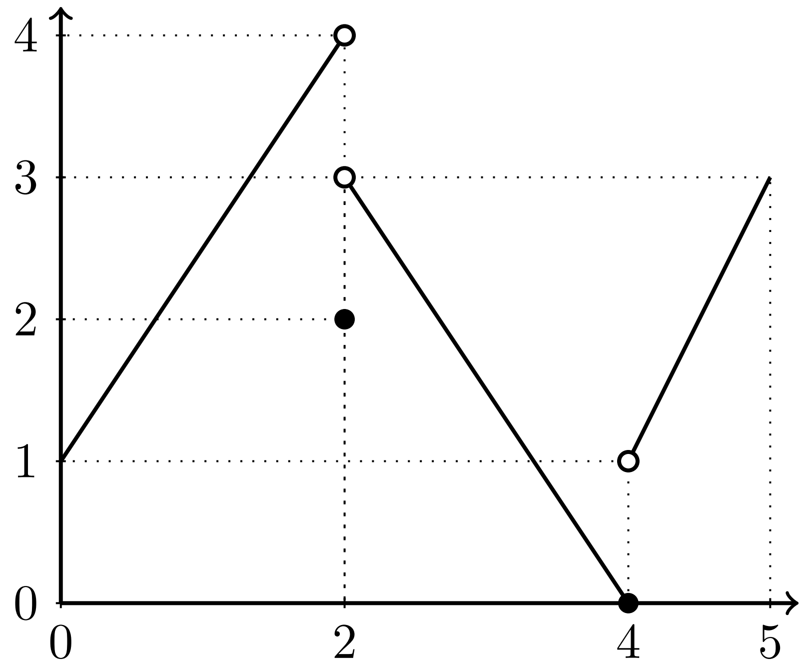


Multivariate Case (Non-Separable)



Lower Semicontinuous PLFs



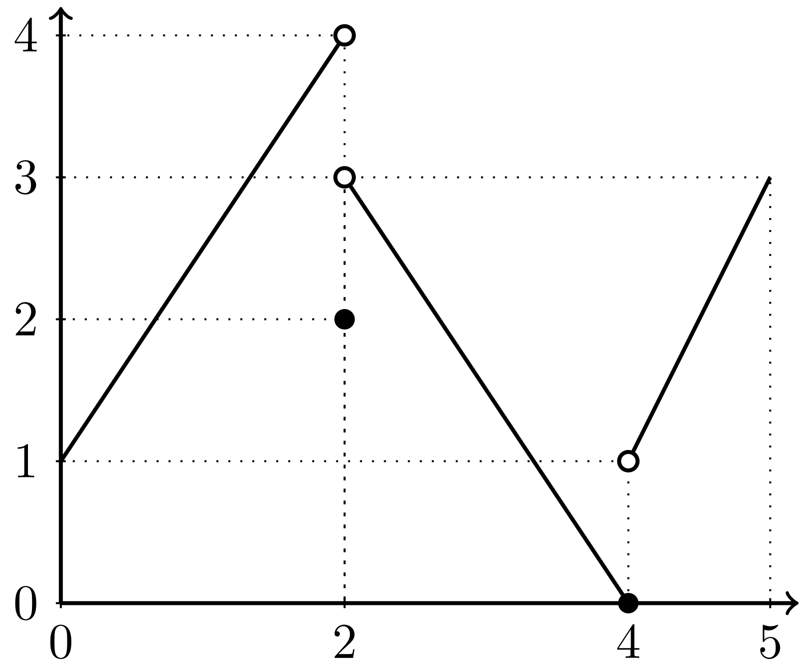
$$f(x) := \begin{cases} 1.5x + 1 & x \in [0,2) \\ 2 & x \in [2,2] \\ -1.5x + 6 & x \in (2,4] \\ 2x - 7 & x \in (4,5] \end{cases}$$

$$f(x) := \begin{cases} m_P x + c_P & x \in P \quad \forall P \in \mathcal{P} \end{cases}$$

Finite family of copolytopes

$$P = \{x \in \mathbb{R}^n \; : \; a_i x \leq b_i \, \forall i \in \{1, \ldots, p\}, \\ a_i x < b_i \, \forall i \in \{p, \ldots, m\}\}$$

Lower Semicontinuous PLFs



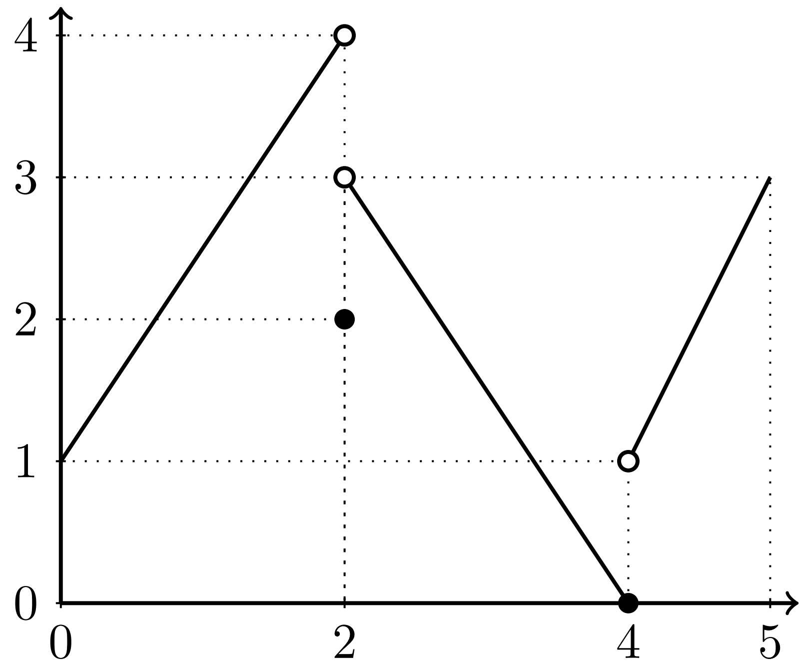
$$f(x) := \begin{cases} 1.5x + 1 & x \in [0, 2) \\ 2 & x \in [2, 2] \\ -1.5x + 6 & x \in (2, 4] \\ 2x - 7 & x \in (4, 5] \end{cases}$$

$$f(x) := \left\{ m_P x + c_P \quad x \in P \quad \forall P \in \mathcal{P} \right.$$

Finite family of copolytopes

$$P = \{x \in \mathbb{R}^n \ : \ a_i x \leq b_i \, \forall i \in \{1, \ldots, p\}, \\ a_i x < b_i \, \forall i \in \{p, \ldots, m\}\}$$

Lower Semicontinuous PLFs



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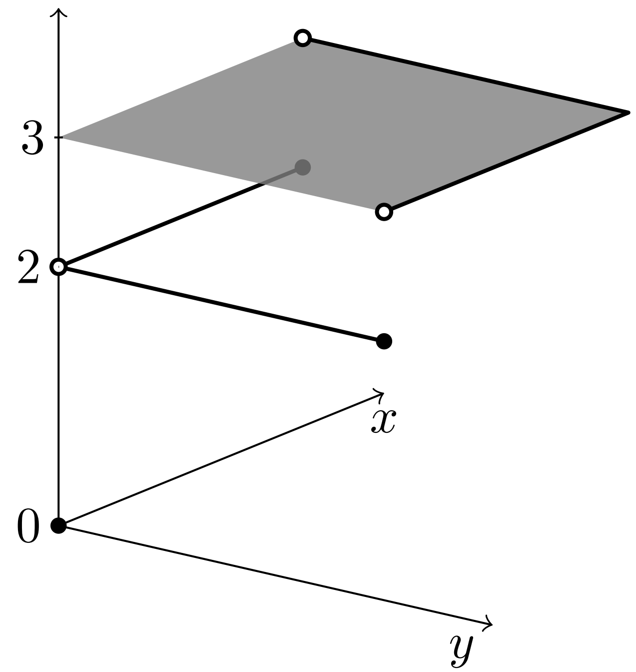

$$f(x,y) := \begin{cases} 3 & (x,y) \in (0,1]^2 \\ 2 & (x,y) \in \{(x,y) \in \mathbb{R}^2 \,:\, x = 0,\, y > 0\} \\ 2 & (x,y) \in \{(x,y) \in \mathbb{R}^2 \,:\, y = 0,\, x > 0\} \\ 0 & (x,y) \in \{(0,0)\}. \end{cases}$$

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Finite family of copolytopes

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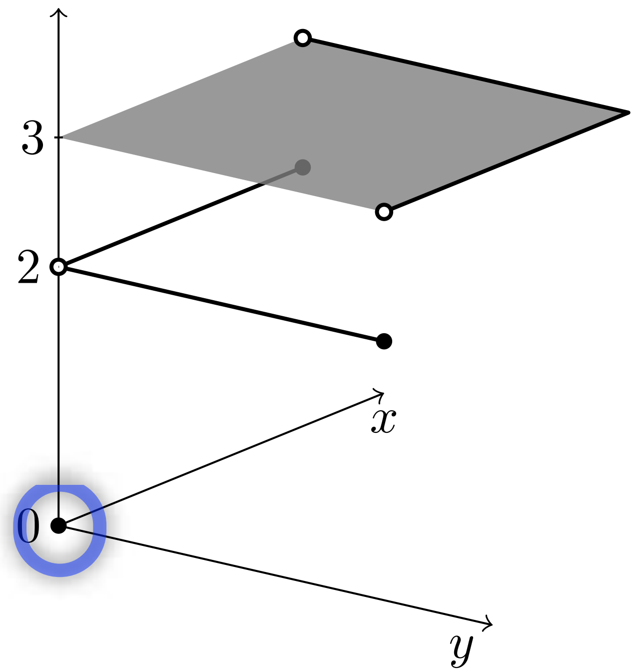
Lower Semicontinuous PLFs



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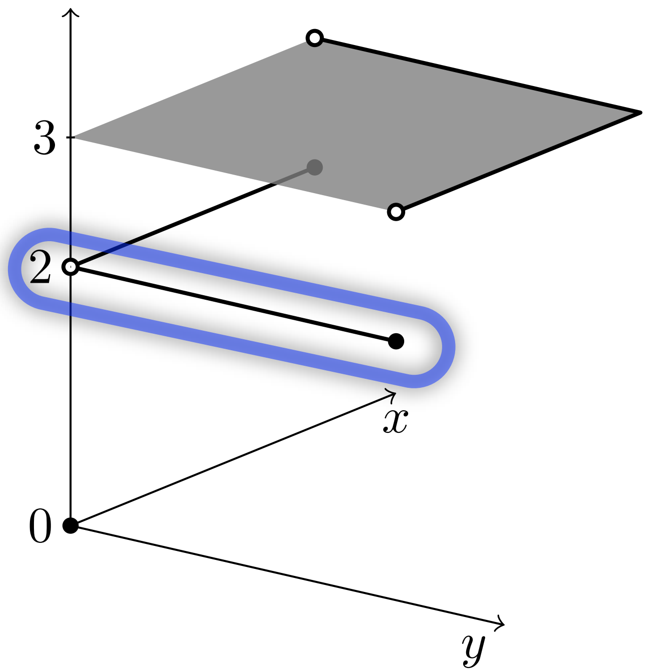
Finite family of copolytopes

Lower Semicontinuous PLFs



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Finite family of copolytopes

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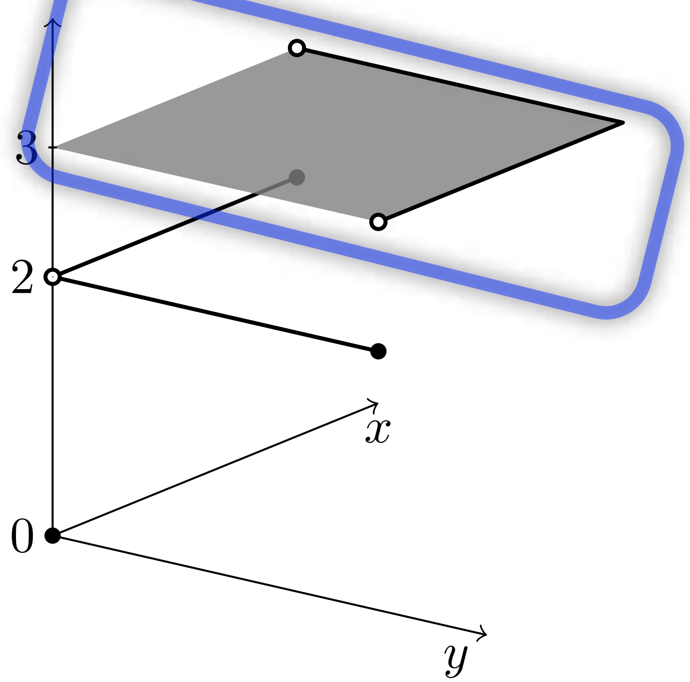
Lower Semicontinuous PLFs



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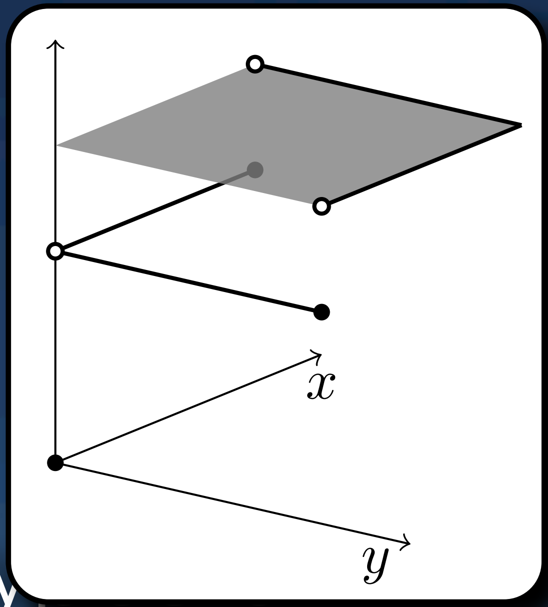
Finite family of copolytopes

$$P = \{x \in \mathbb{R}^n \,:\, a_i x \le b_i \,\forall i \in \{1,\dots,p\},\ a_i x < b_i \,\forall i \in \{p,\dots,m\}\}$$

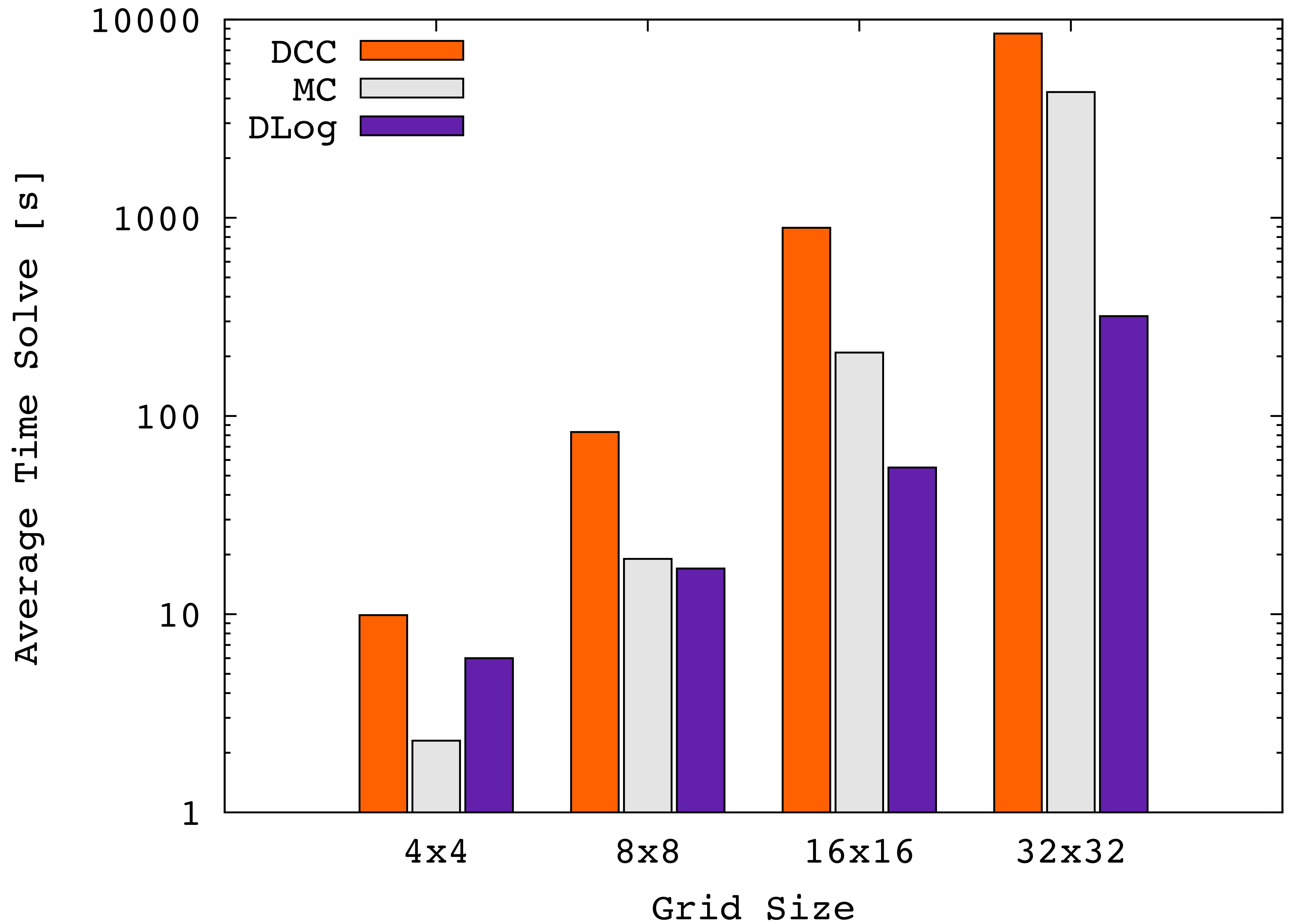
Lower Semicontinuous Models

- Direct from Disjunctive Programming (Jeroslow and Lowe)
  - “Extreme point” = DCC.
  - Traditional = Multiple Choice (MC).

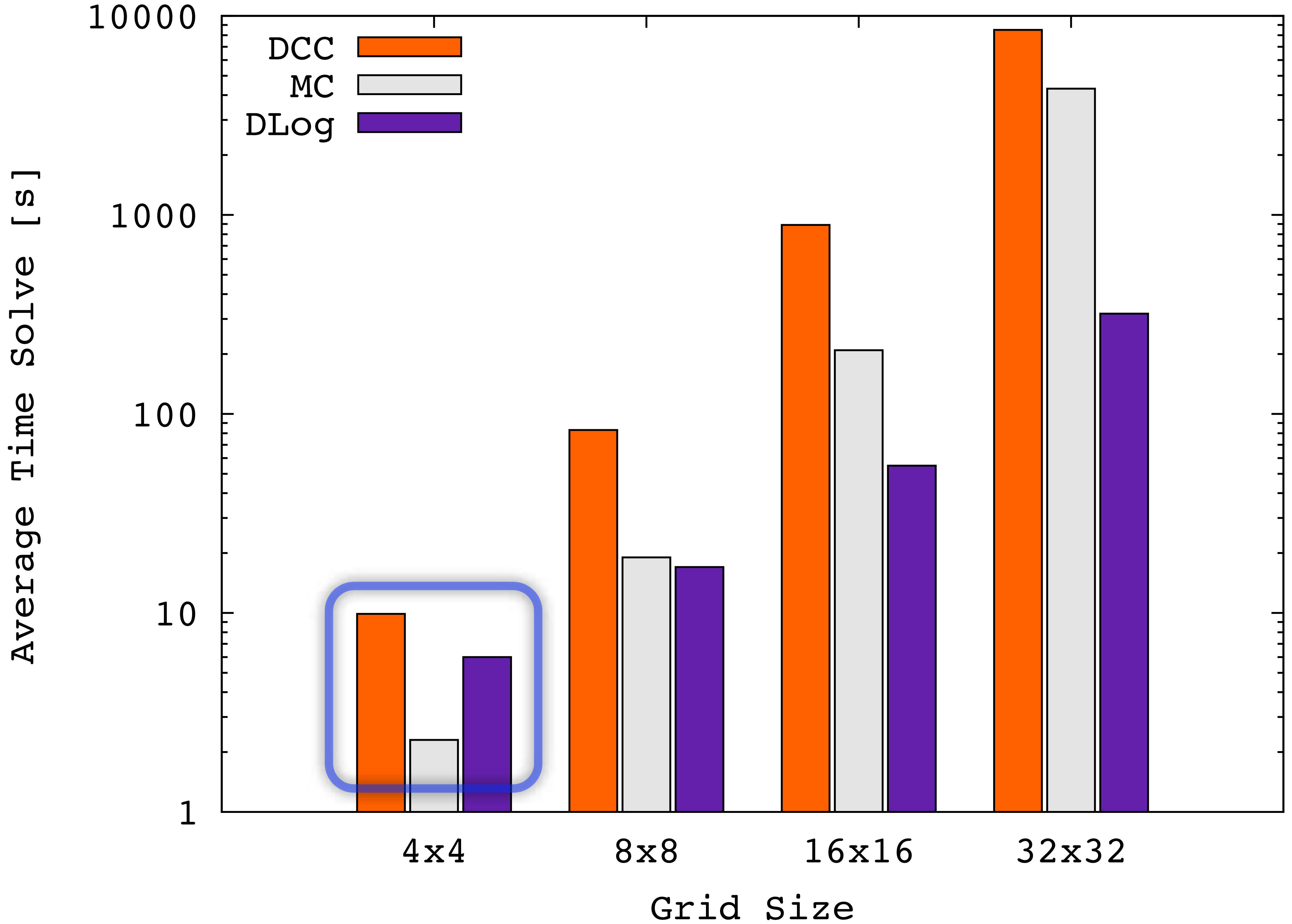


- Other models can be adapted to special ty discontinuities (e.g. simple fixed charges).
- MC, DCC, DLog are locally ideal and sharp.
- Computations: 2-commodity FC discount function.

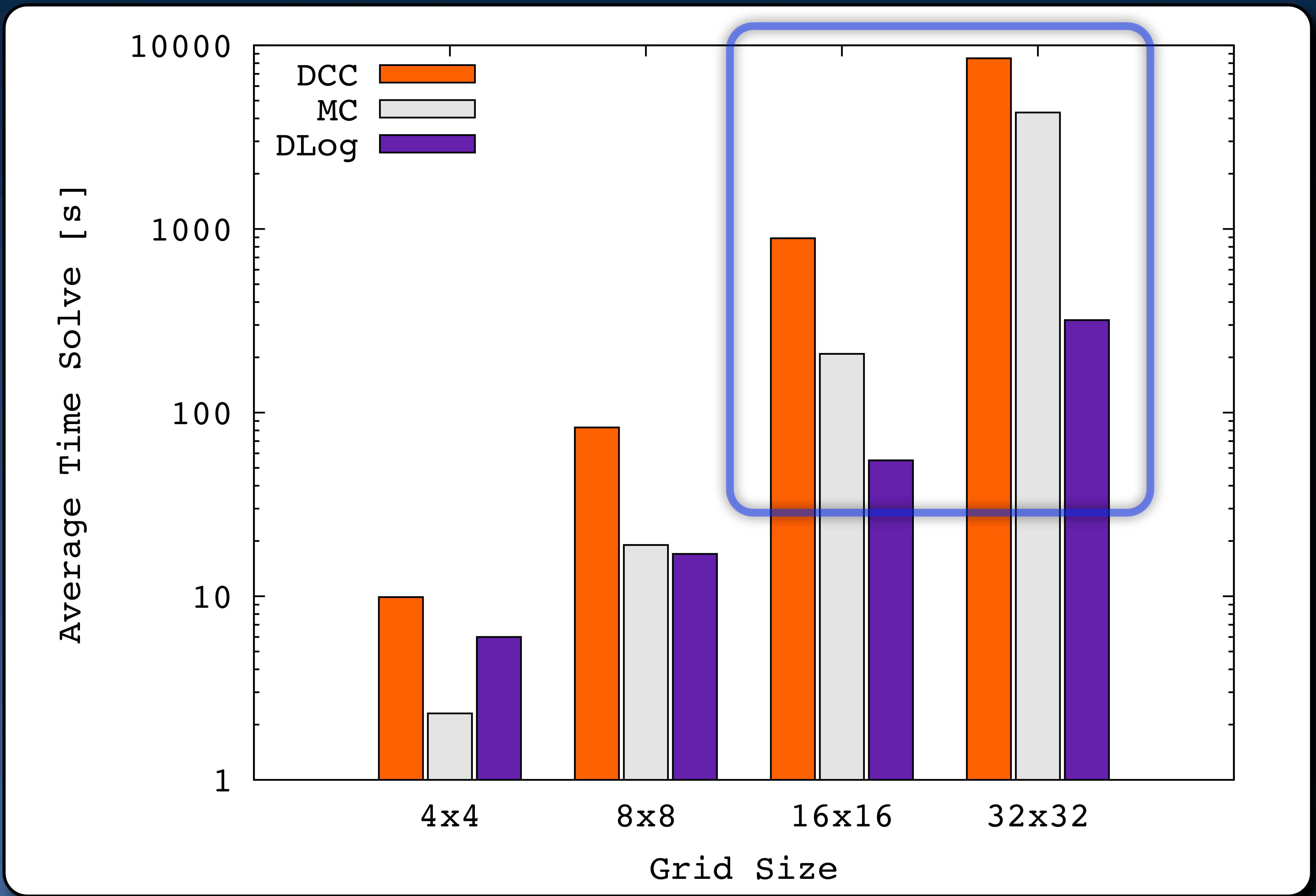
Multivariate Lower Semicontinuous



Multivariate Lower Semicontinuous



Multivariate Lower Semicontinuous



Final Remarks

- Unifying theoretical framework: allows for multivariate non-separable and lower semicontinuous functions.
- First logarithmic formulations: Theoretically strong and provides significant computational advantage for large $|\mathcal{P}|$.
- Revive forgotten formulations and functions: MC and fixed charge discount function.