Mixed Integer Programming Models for Non-Separable Piecewise Linear Cost Functions

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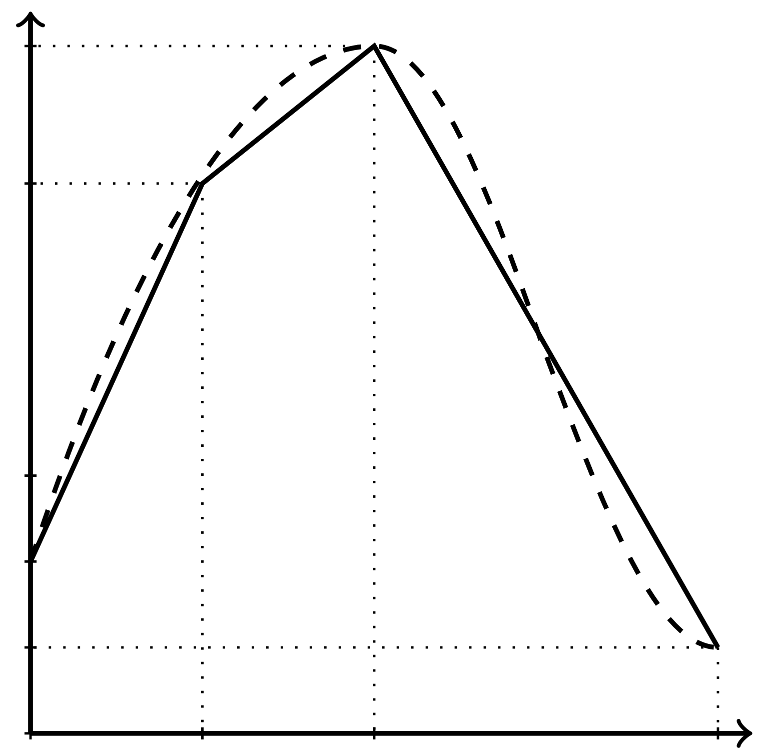
Piecewise Linear Optimization

$$\begin{aligned} \min \quad & f_0(x) \\ s.t. \quad & \\ & f_i(x) \leq 0 \quad \forall i \in I \\ & x \in X \subset \mathbb{R}^n \end{aligned}$$



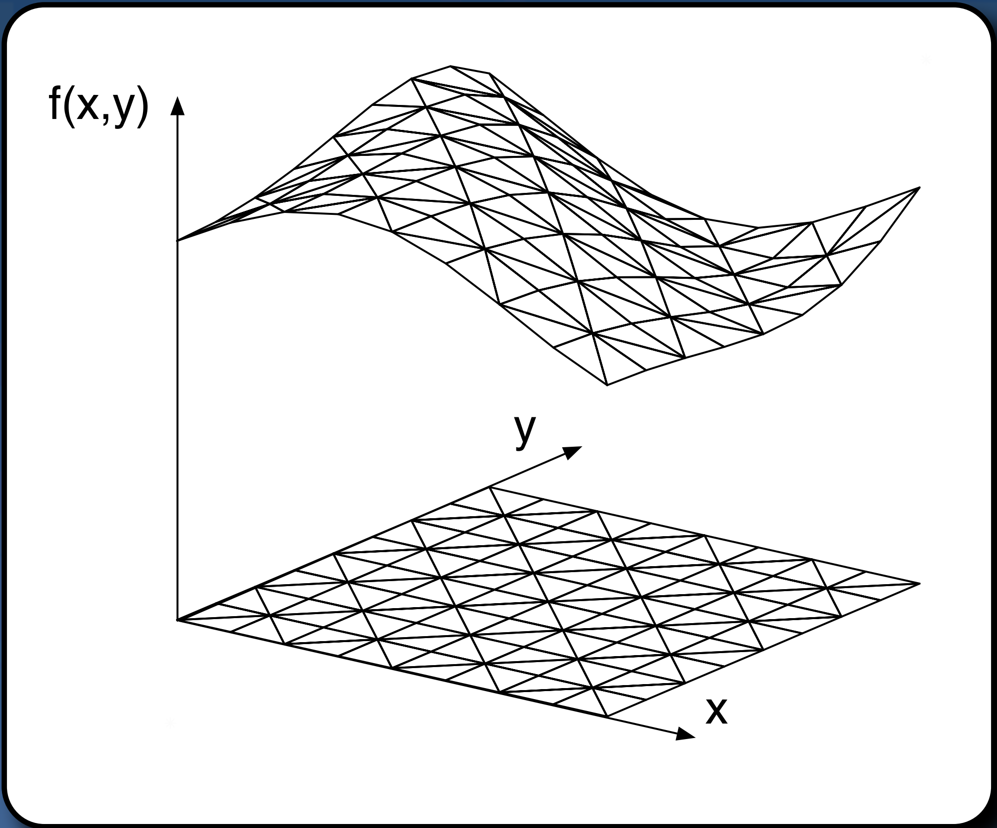
- $\forall i \in \{0\} \cup I \quad f_i(x) : D \longrightarrow \mathbb{R}$ is a piecewise linear function (PLF) and X is any compact set.
- Convex = Linear Programming. Non-Convex = NP Hard.
- Specialized algorithms (Tomlin 1981, ..., de Farias et al. 2008) or Mixed Integer Programming Models (12+ papers).

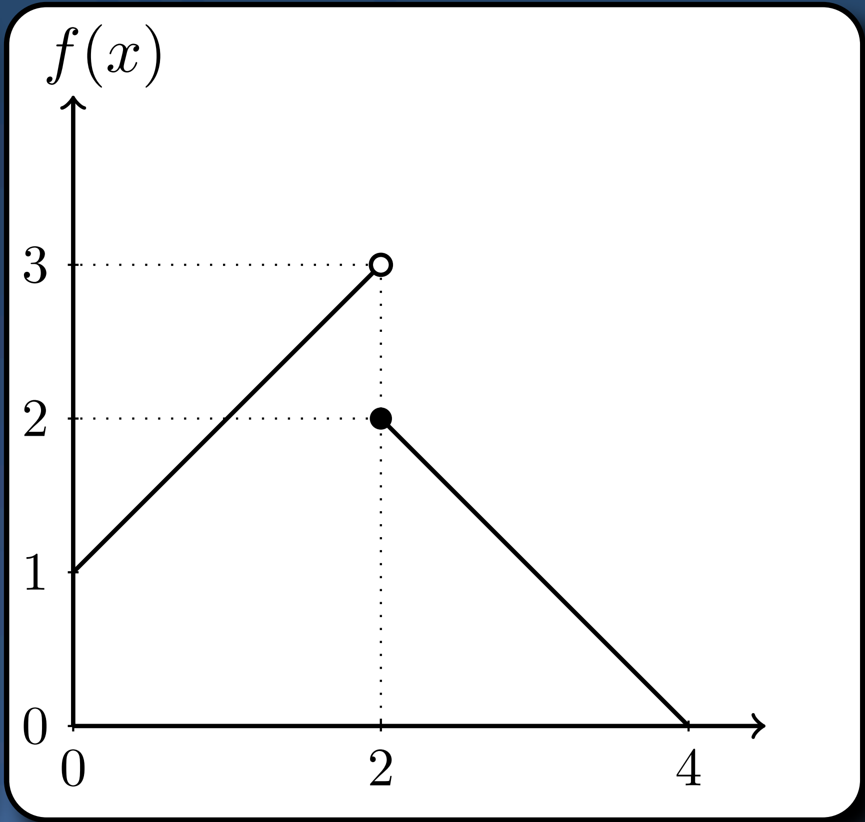
Mixed Integer Models for PLFs

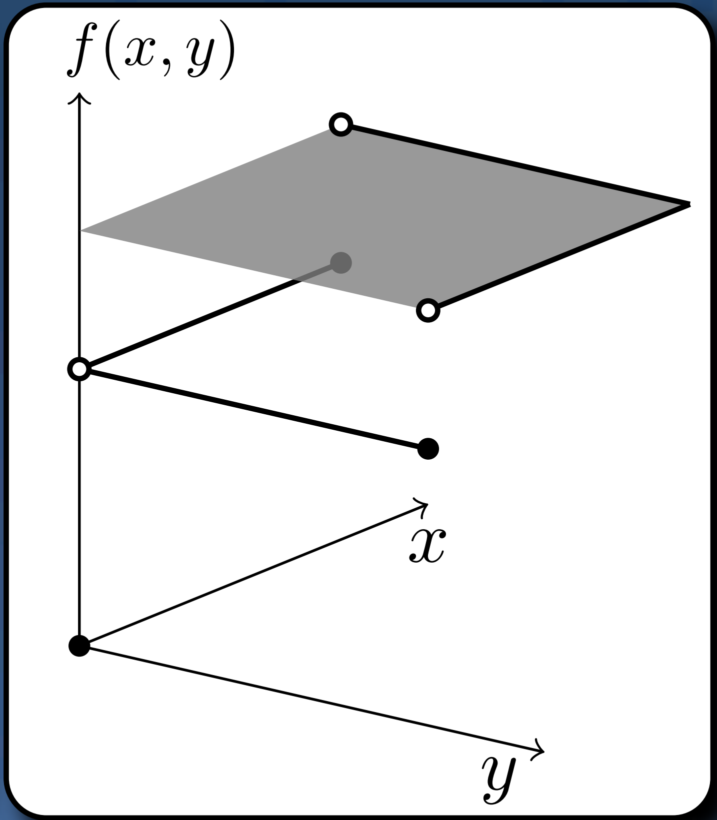
- Existing studies are for separable functions:

$$f(x) = \sum_{j=1}^{n} f_j(x_j) \text{ for } f_j(x_j) : \mathbb{R} \to \mathbb{R}$$

- We emphasize Non-Separable and/or discontinuous:



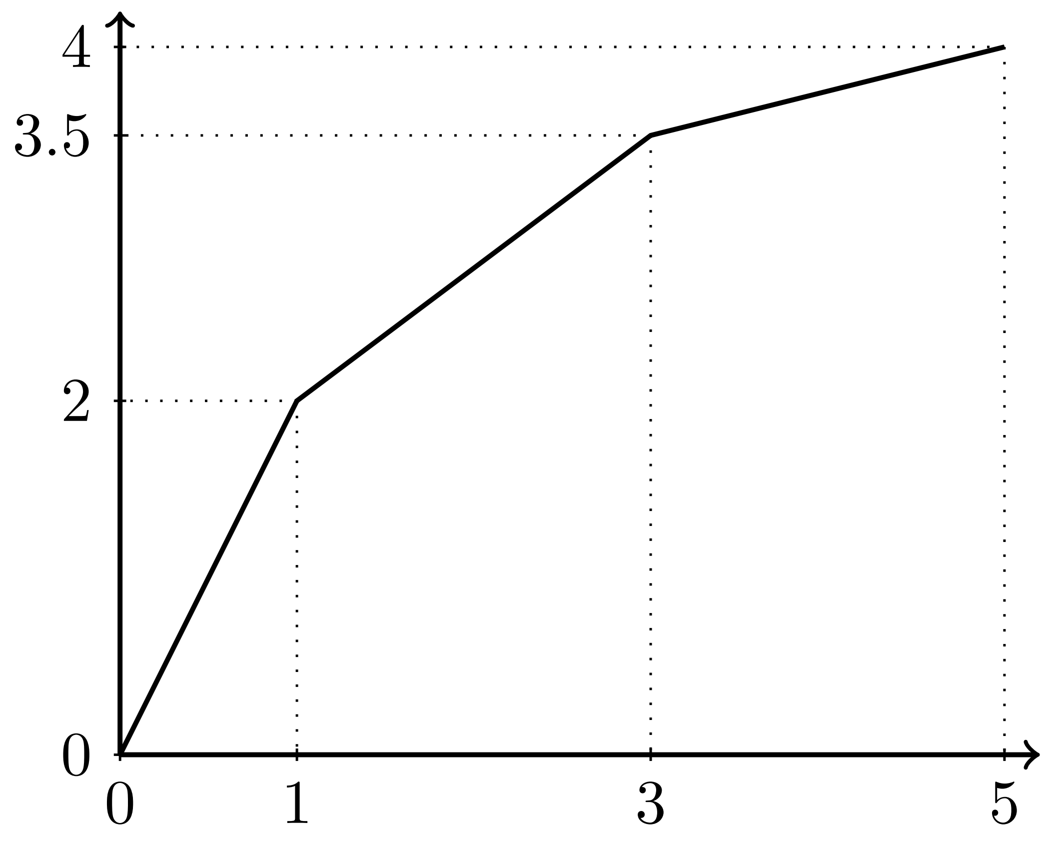


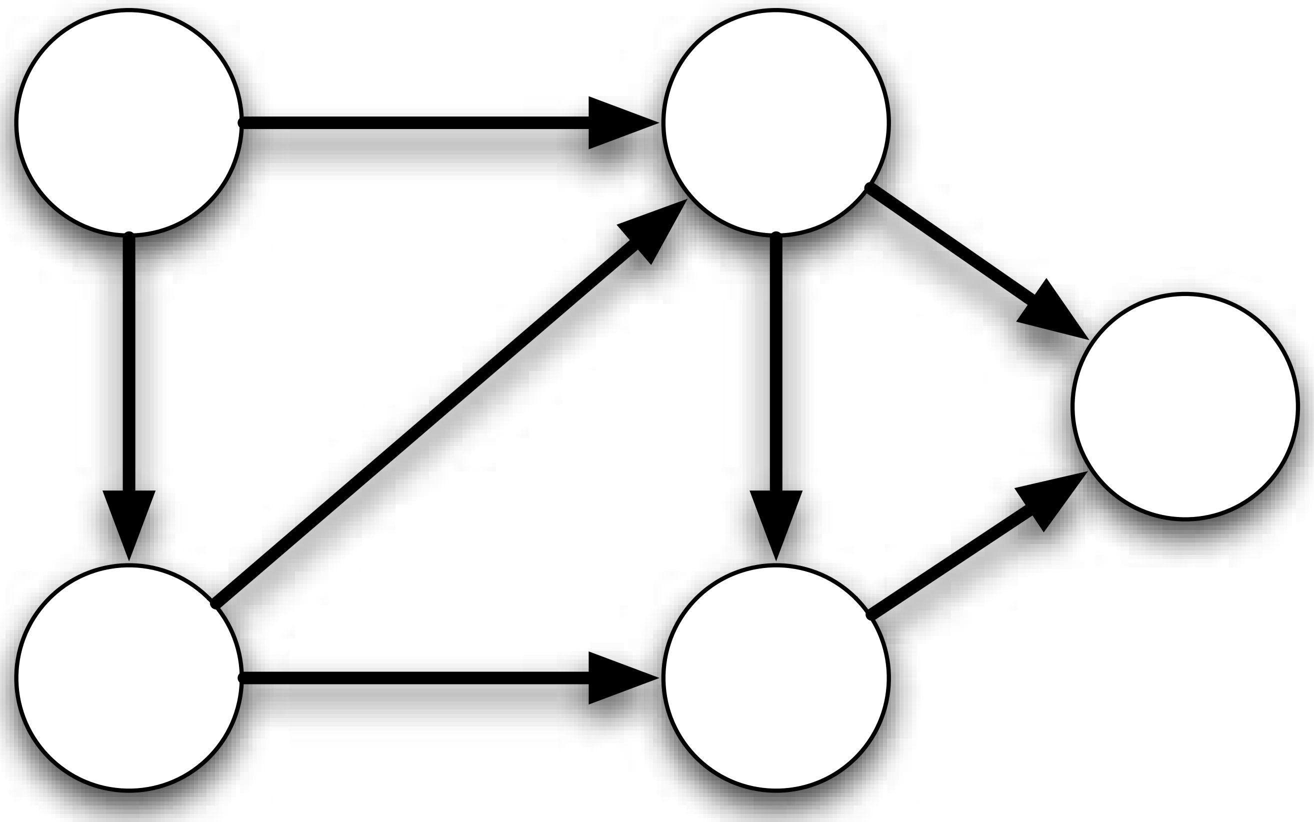


Outline

- Applications of Piecewise Linear Functions.
- Modeling Piecewise Linear Functions.
- Computational Results.
- Extension to Discontinuous Functions.
- Final Remarks.

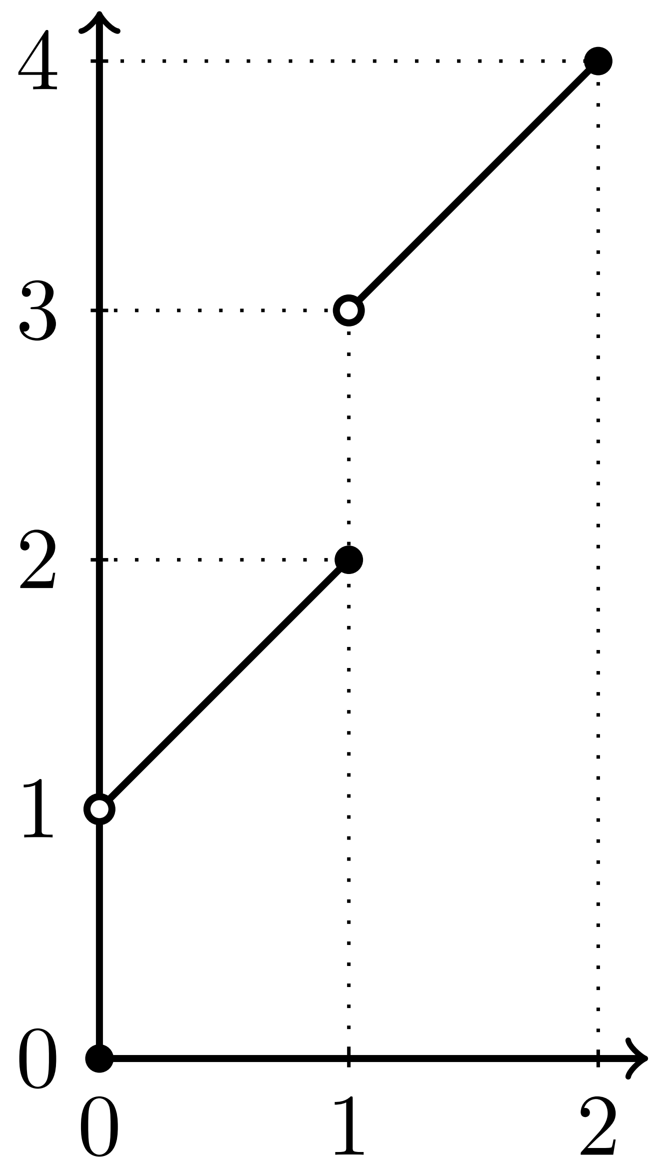
Economies of Scale: Concave





- Single and multi-commodity network flow.
- Applications in telecommunications, transportation, and logistics.
- (Balakrishnan and Graves 1989, ..., Croxton, et al. 2007).

Fixed Charges and Discounts

1.


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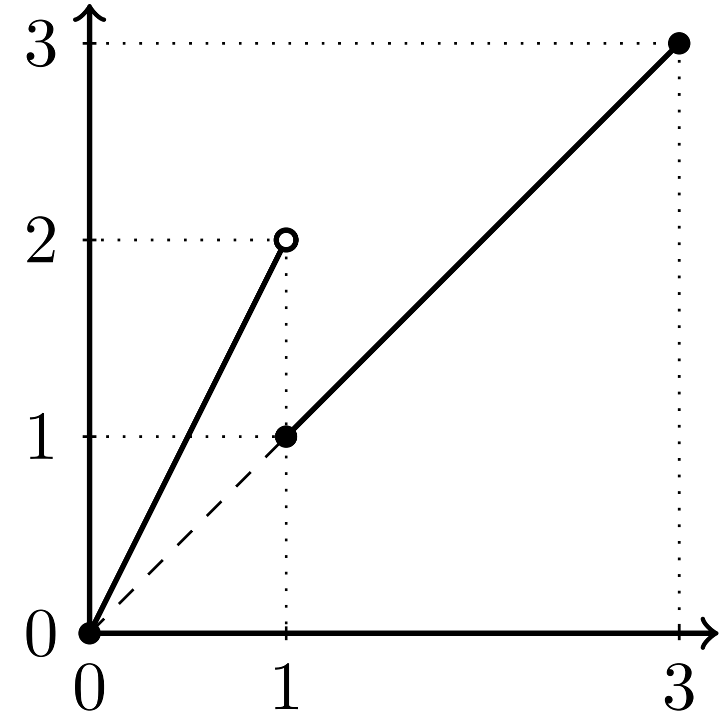

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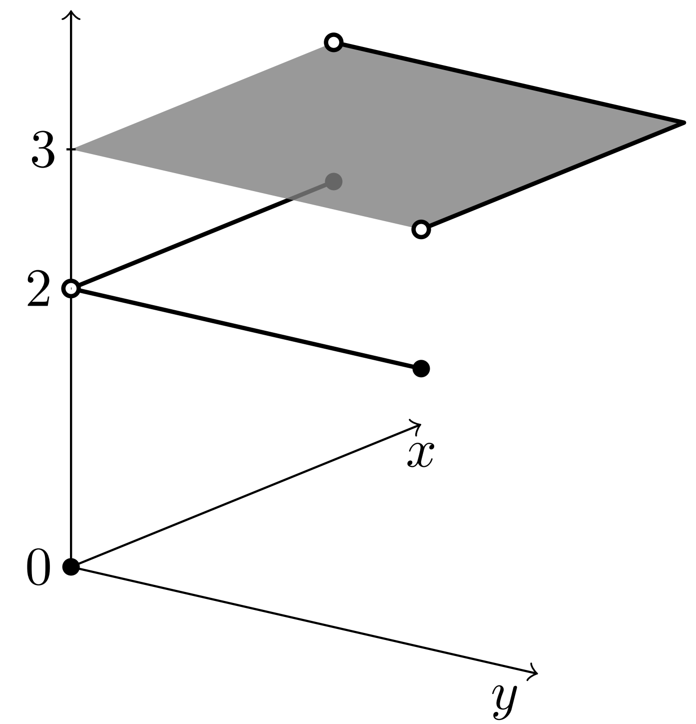
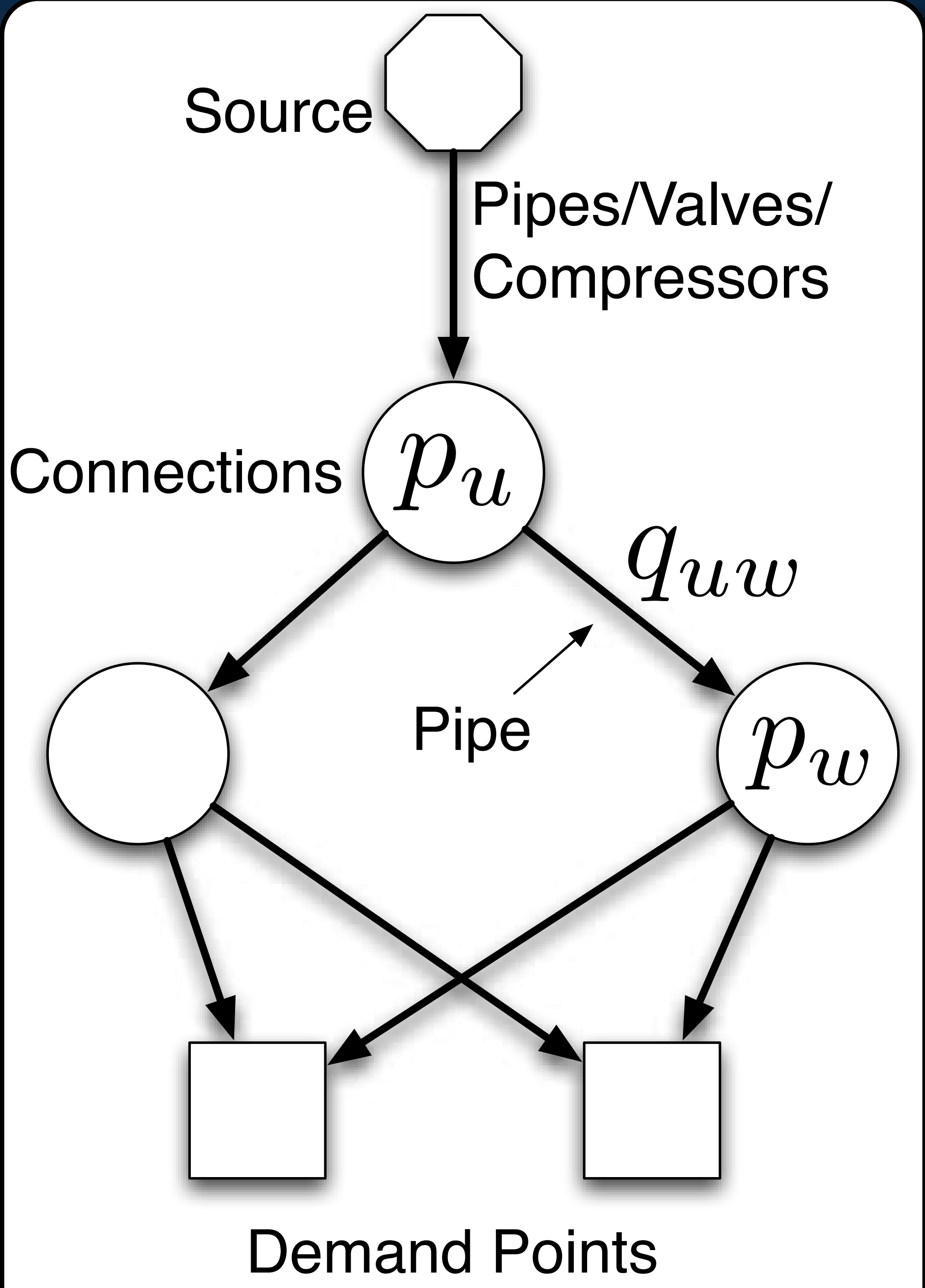


1. Fixed Costs in Logistics.
2. Discounts (e.g. Auctions: Sandholm, et al. 2006, CombineNet).
3. Discounts in fixed charges (Lowe 1984).

Non-Linear and PDE Constraints



$$p(x,t) = \text{gas pressure}$$

$$q(x,t) = \text{gas volume flow}$$

$$A\,\frac{\partial \rho}{\partial t} + \rho_0\,\frac{\partial q}{\partial x} = 0,$$

$$\frac{\partial p}{\partial x} = -\lambda\frac{|v|v}{2D}\,\rho.$$

- Gas Network Optimization (Martin et al. 2006).

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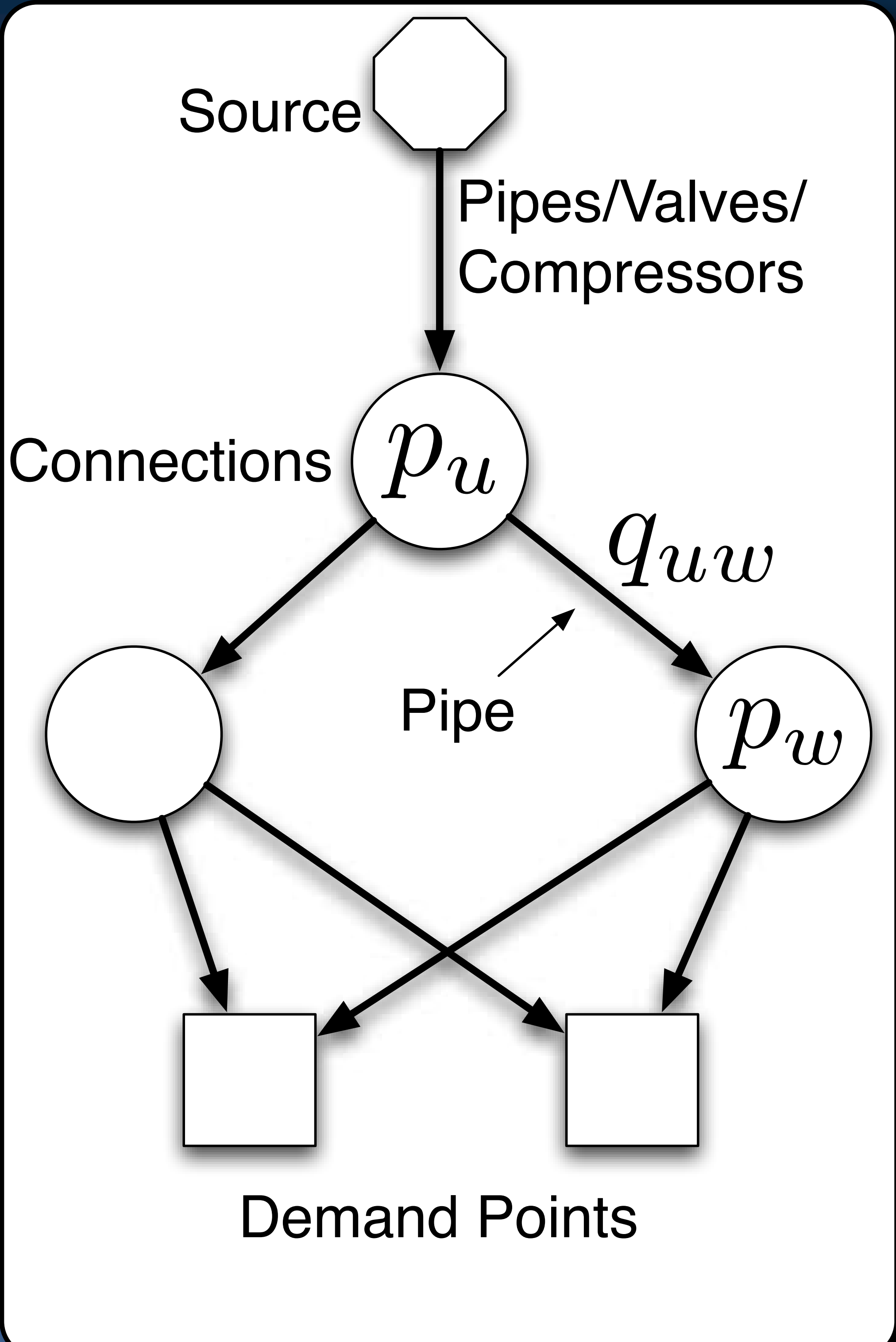

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Discretize non-linear stationary solution $p_v = g(p_u, q_{uv})$

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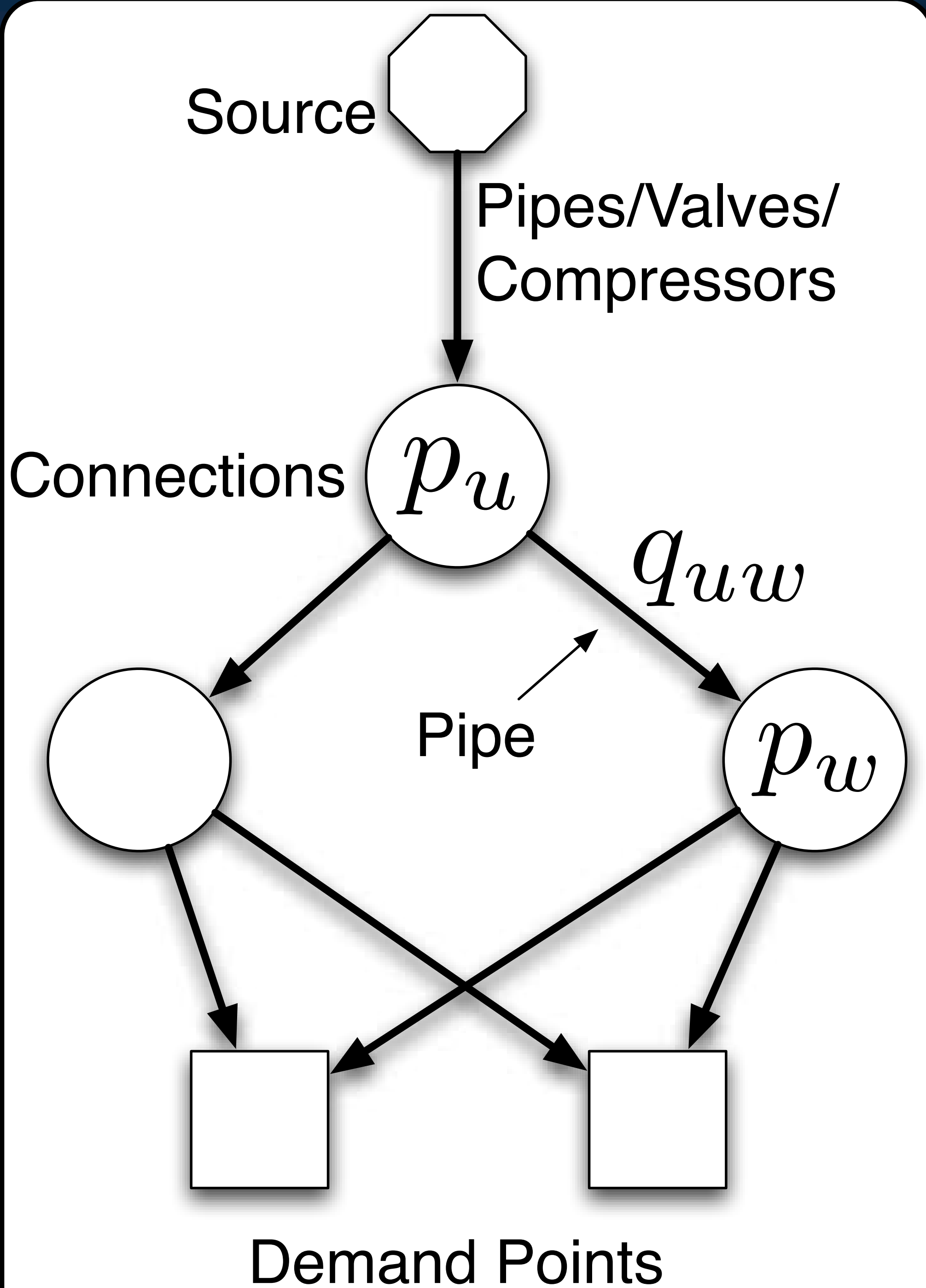

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Discretize PDE

(Fügenschuh, et al. 2008)

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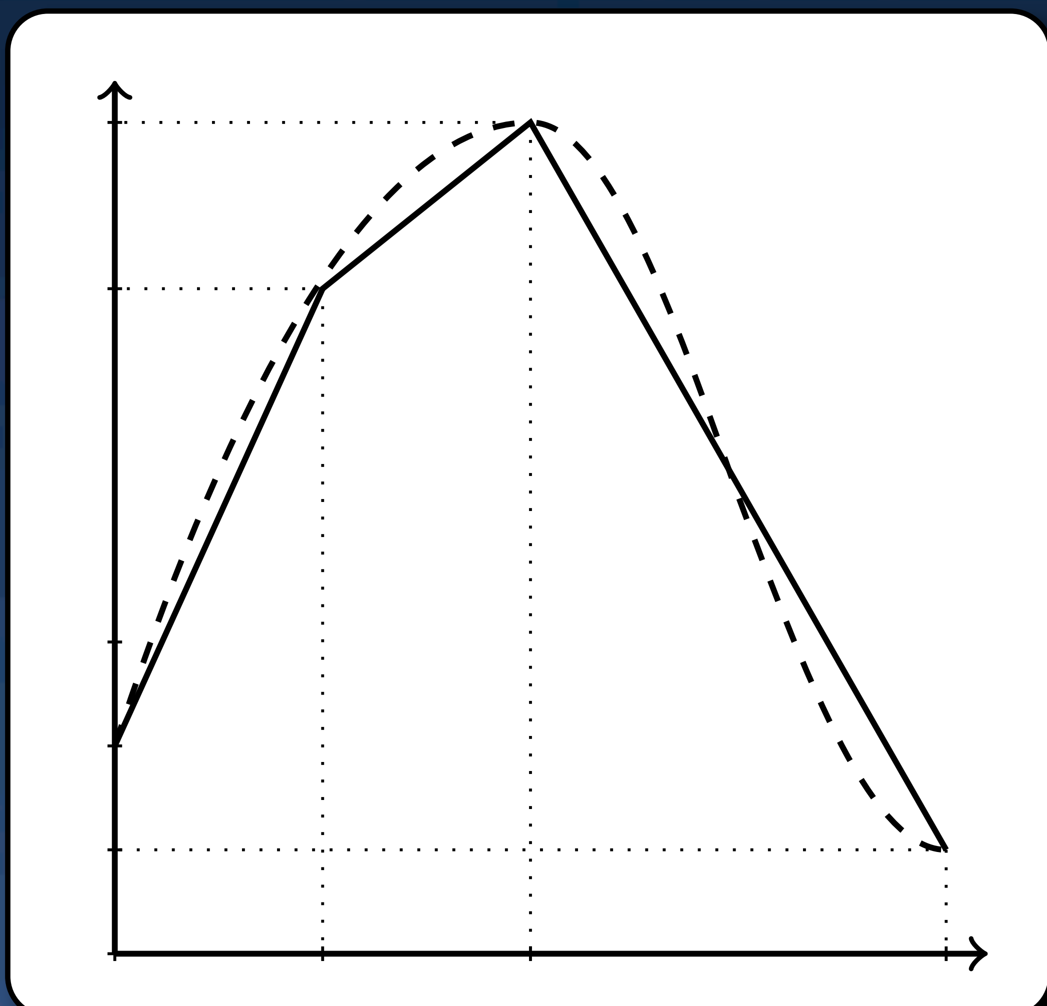
Numerically Exact Global Optimization



- Process engineering (Bergamini et al. 2005, 2008, Computers and Chemical Eng.)

- Wetland restoration (Stralberg et al. 2009).

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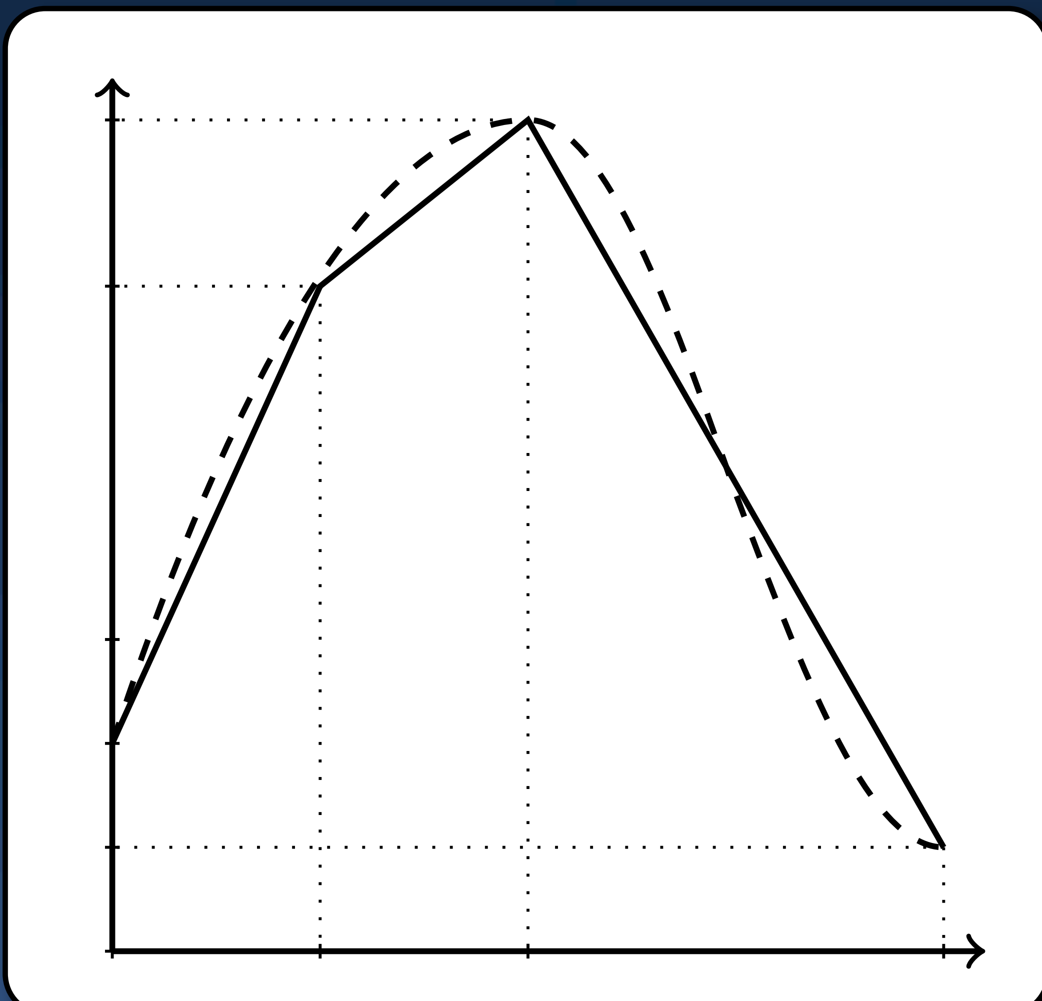

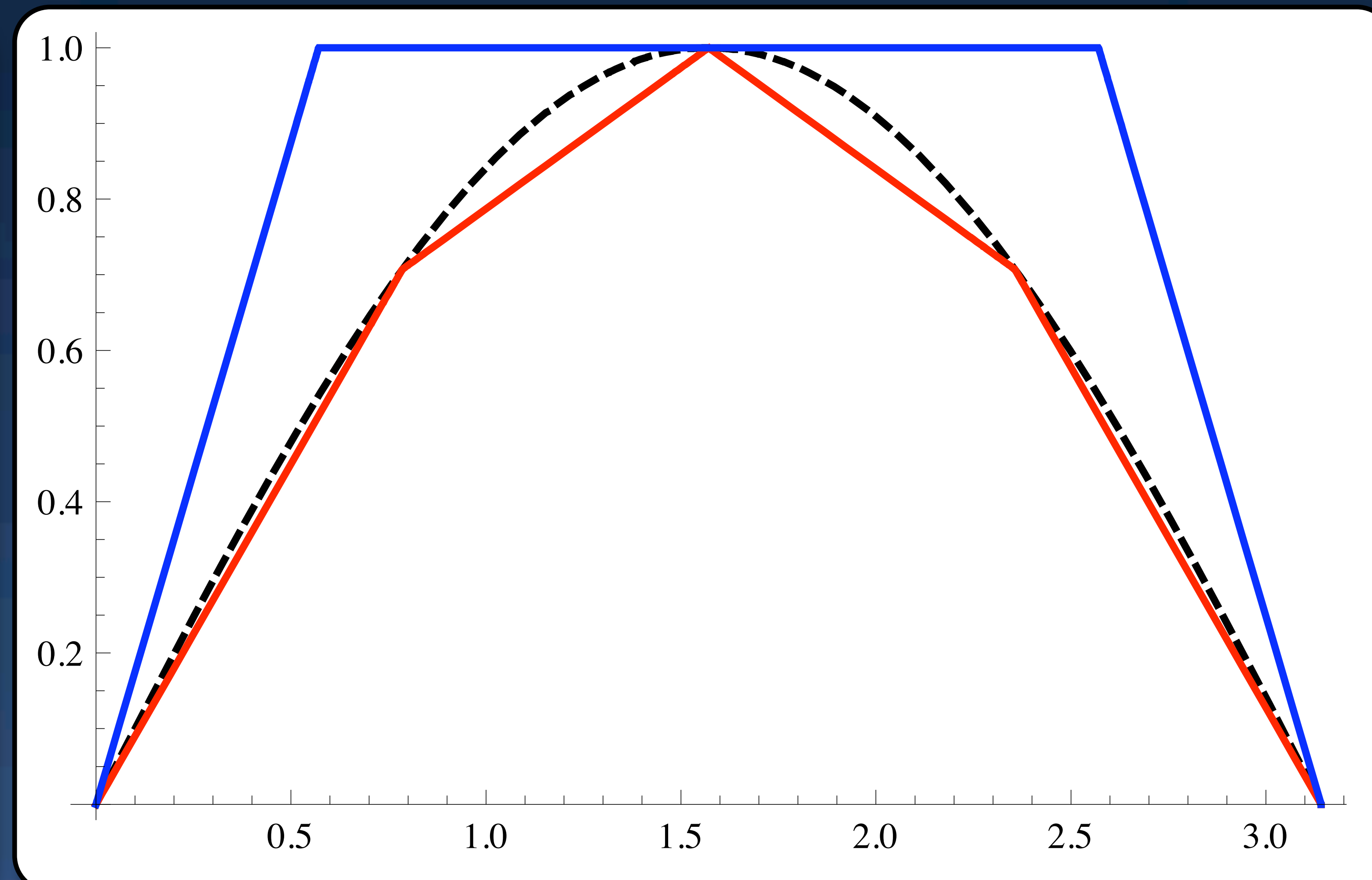

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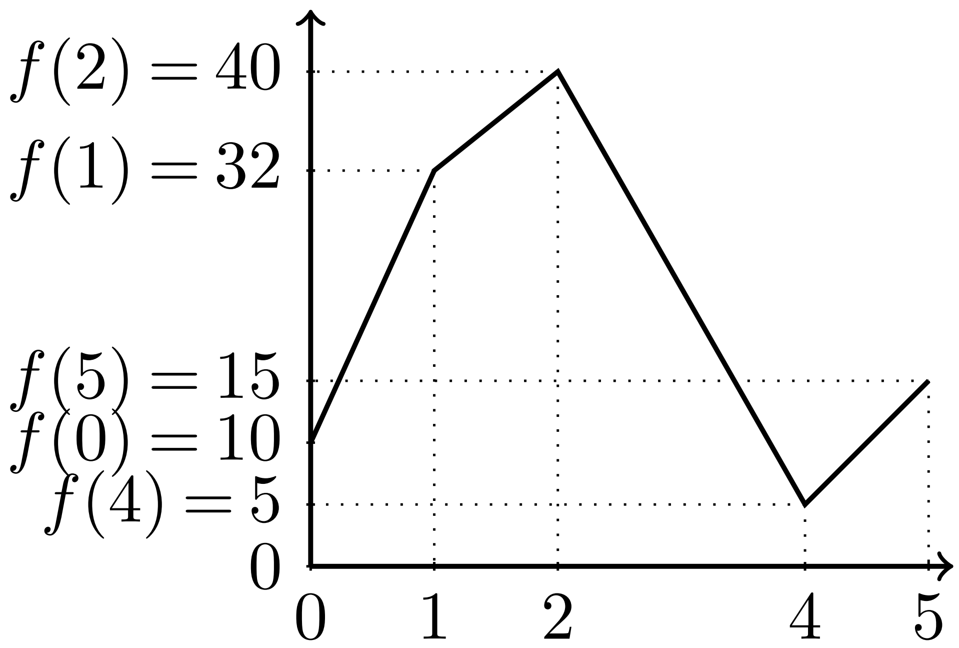
Piecewise Linear Functions: Definition



$$f(x) := \begin{cases} 22x + 10 & x \in [0,1] \\ 8x + 24 & x \in [1,2] \\ -17.5x + 75 & x \in [2,4] \\ 10x - 35 & x \in [4,5] \end{cases}$$

Definition 1. Piecewise Linear $f : D \subset \mathbb{R}^n \to \mathbb{R}$:

$$f(x) := \left\{ m_P x + c_P \quad x \in P \quad \forall P \in \mathcal{P}. \right.$$

for finite family of polytopes $\mathcal{P}$ such that $D = \bigcup_{P \in \mathcal{P}} P$

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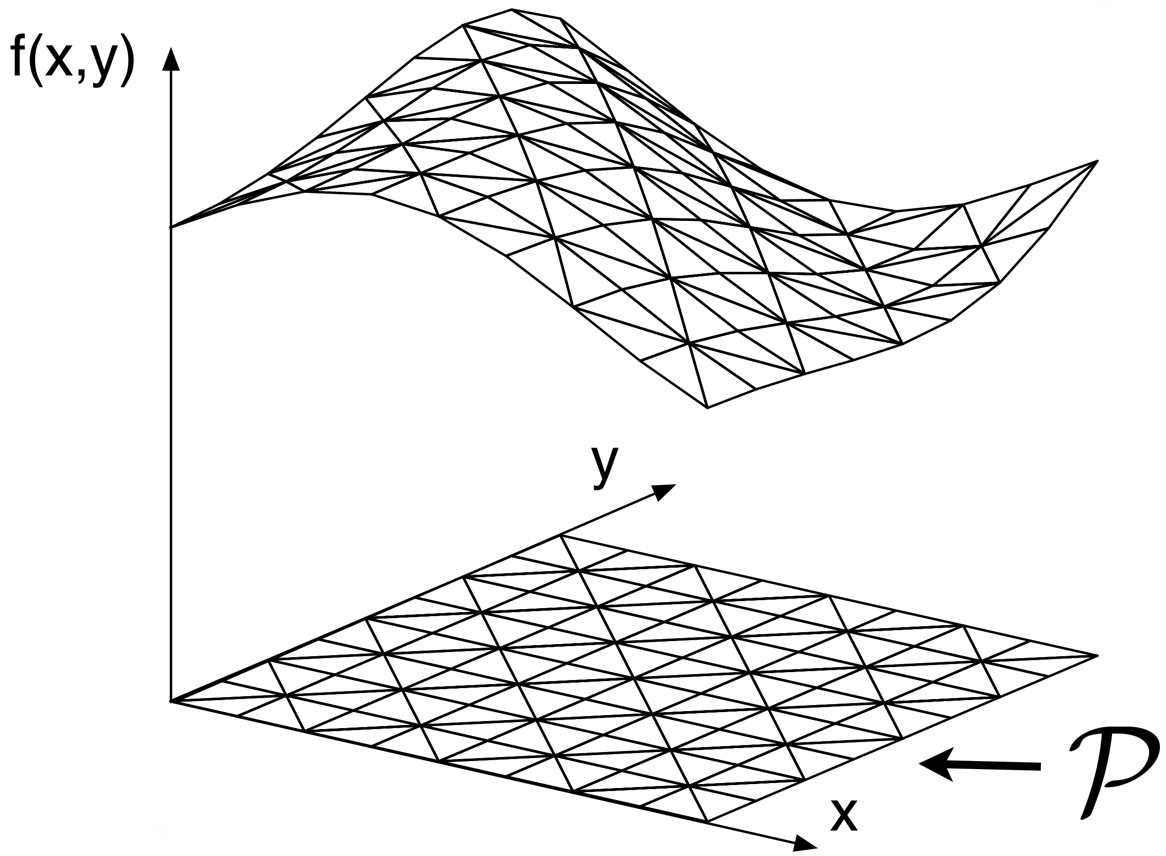

$$f(x,y) := \begin{cases} 0.48x + 0.03y + 6 & (x,y) \in P_1 \\ & \vdots \\ -0.4x - 0.04y + 8.45 & (x,y) \in P_{128} \end{cases}$$

$$P_1 := \{(x,y) \in \mathbb{R} \,:\, y \geq 0,\ x \leq 1, y - x \leq 0\}$$

$$\vdots$$

$$P_{128} := \{(x,y) \in \mathbb{R} \,:\, y \geq 0,\ x \geq 7, x + y \leq 8\}$$

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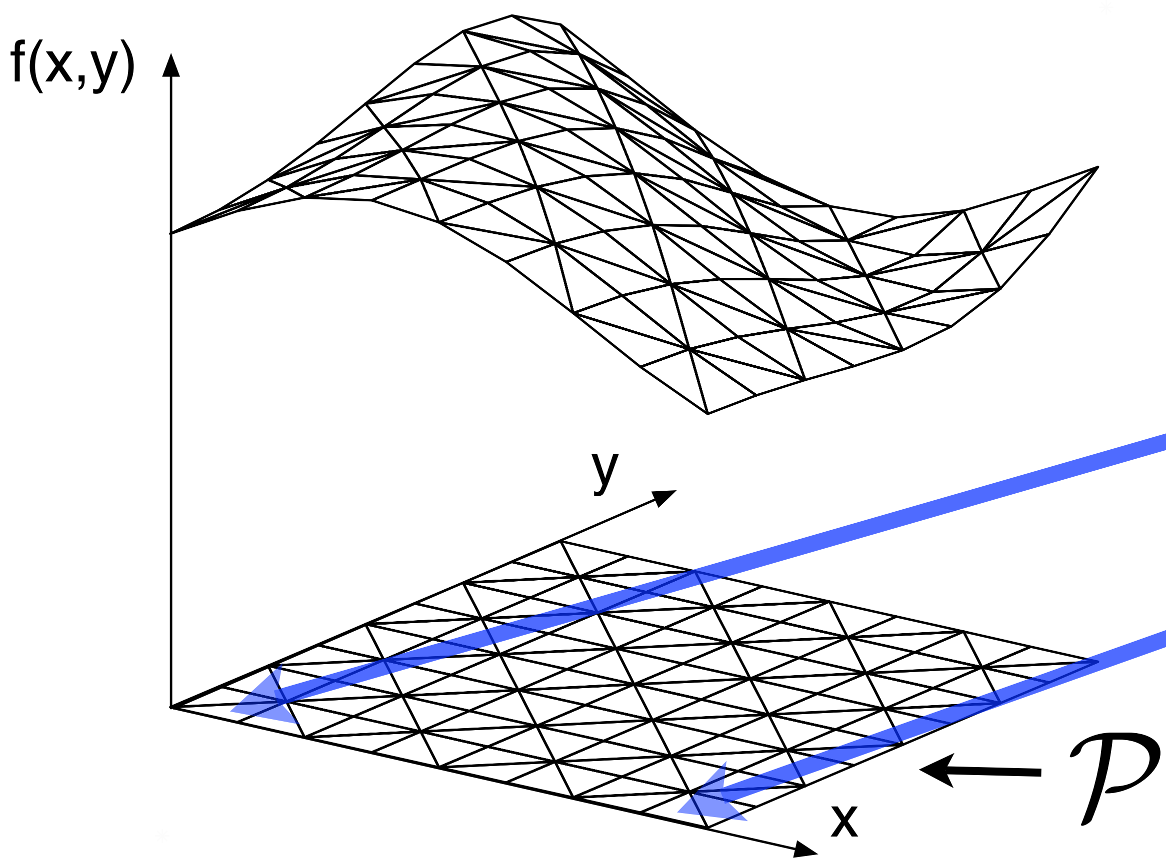

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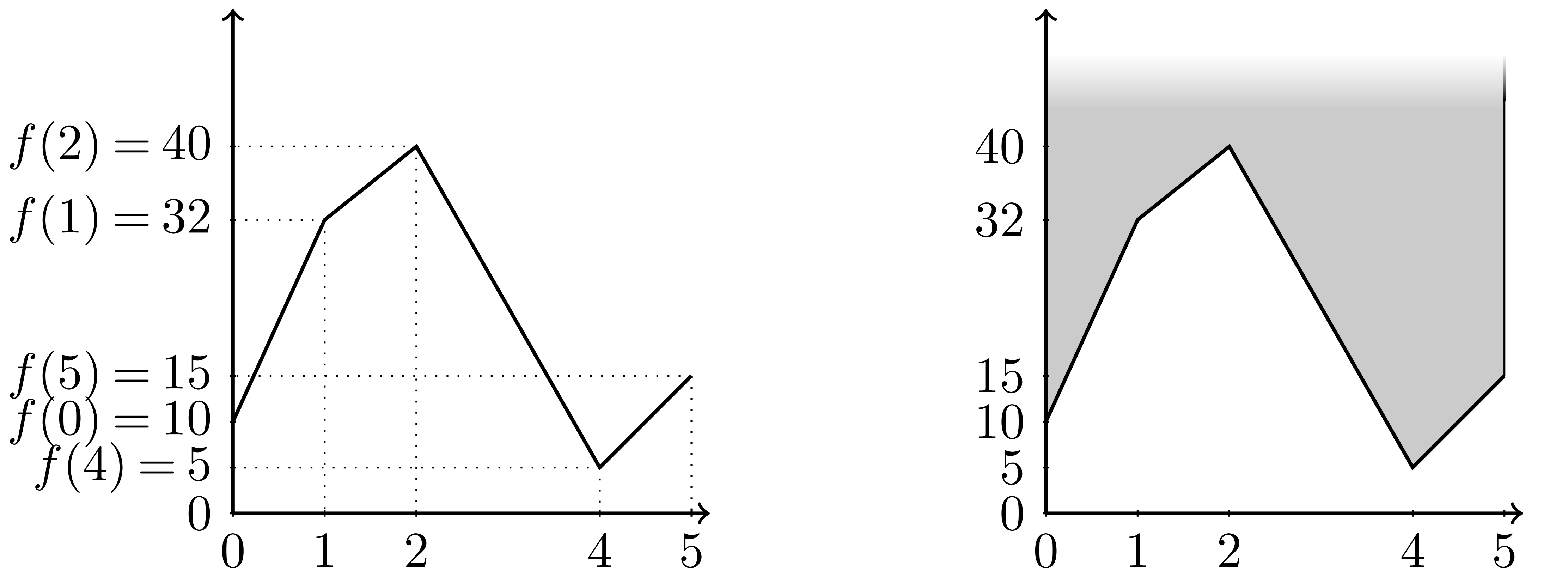
Modeling Function = Epigraph

- $\operatorname{epi}(f) := \{(x,z) \in D \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R} : f(x) \le z\}.$



(a) f.

(b) $\operatorname{epi}(f)$.

- Example: $f(x) \le 0 \Leftrightarrow (x,z) \in \operatorname{epi}(f),\ z \le 0$

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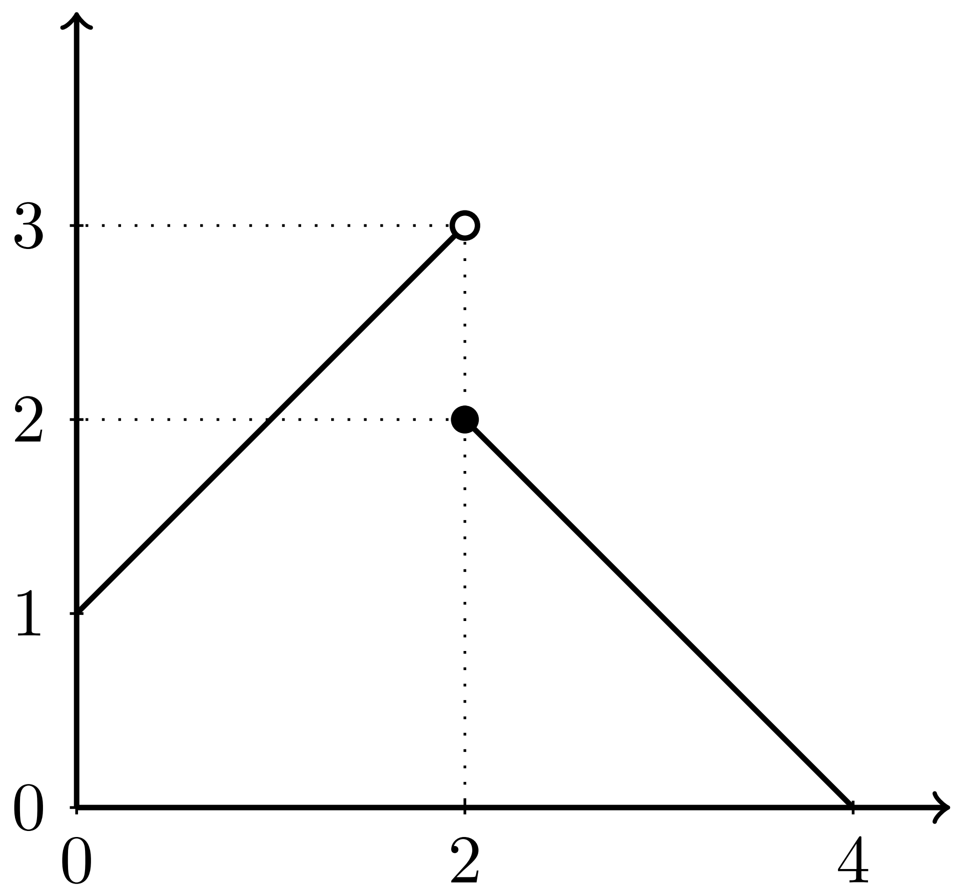

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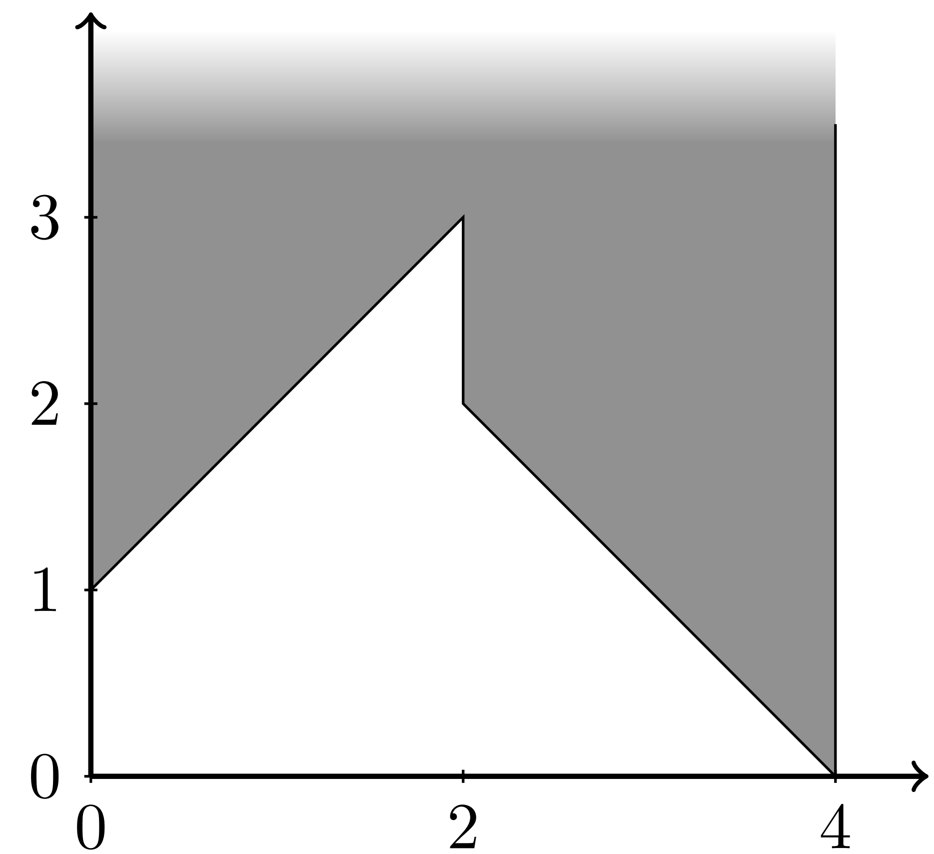

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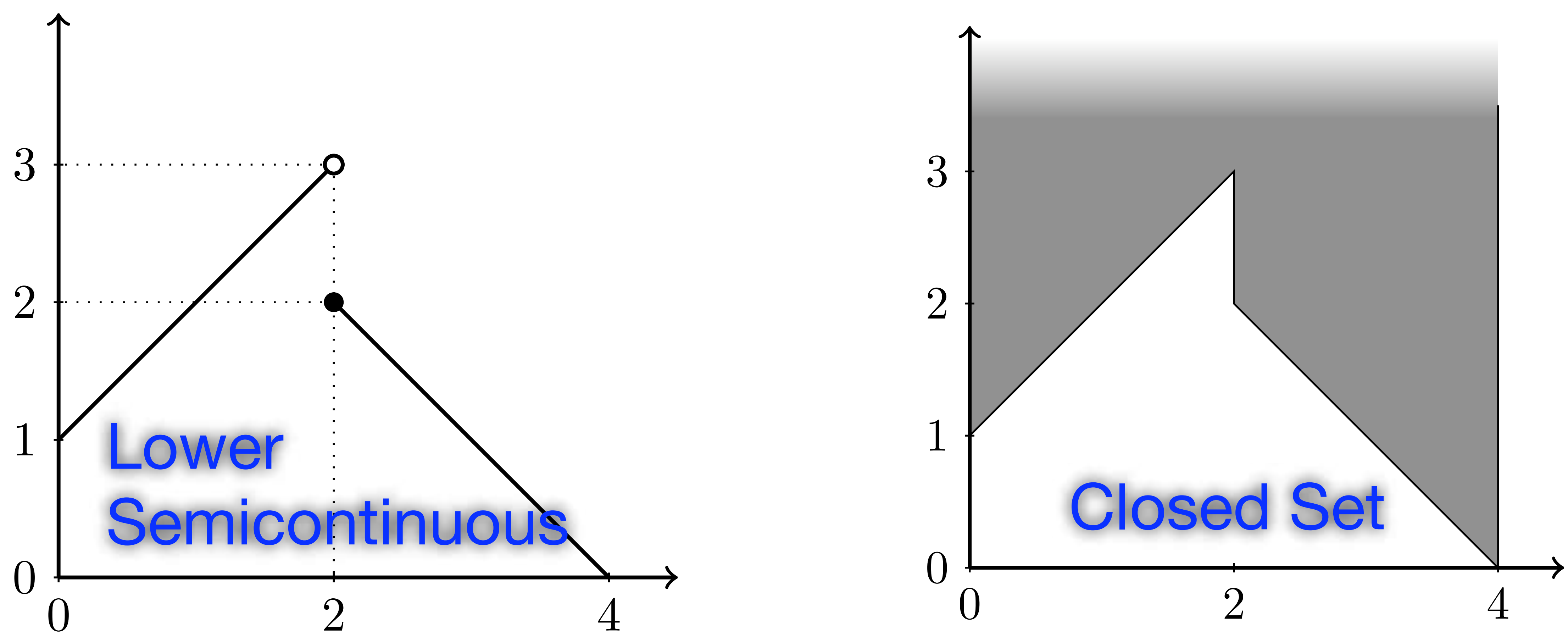

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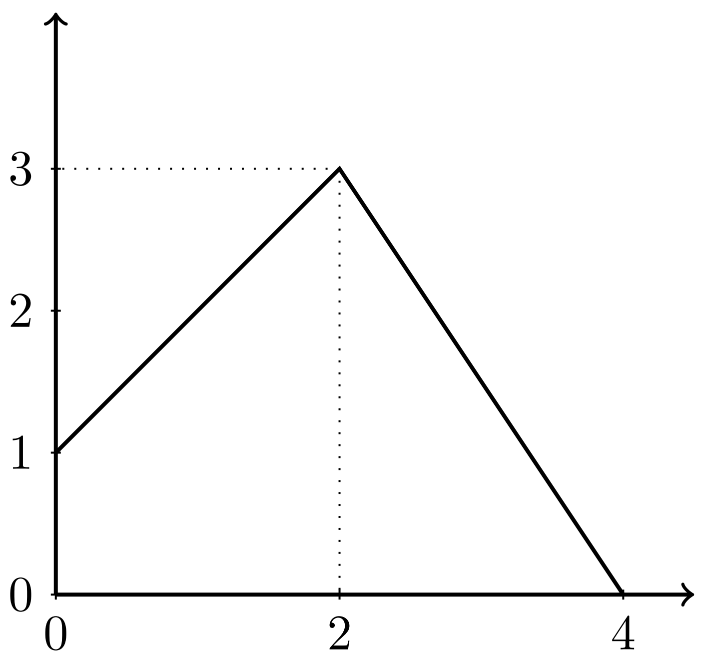
Convex Comb. Univariate (Dantzig, 1960)



$$f(x) := \begin{cases} x+1 & x \in [0,2] \leftarrow P_1 \\ 6 - 3/2x & x \in [2,4] \leftarrow P_2 \end{cases}$$

$V(P) =$ vertices of P.

$\mathcal{V}(\mathcal{P}) := V(P_1) \cup V(P_2) = \{0, 2, 4\}.$

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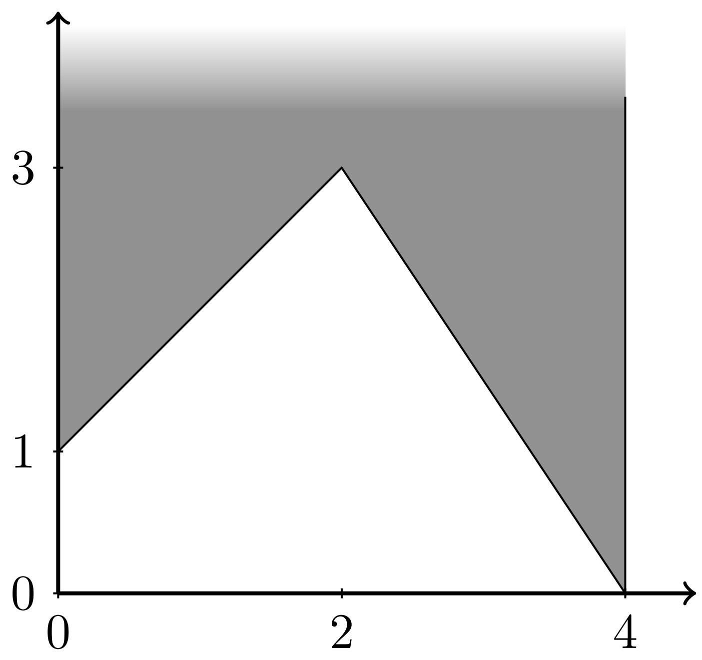

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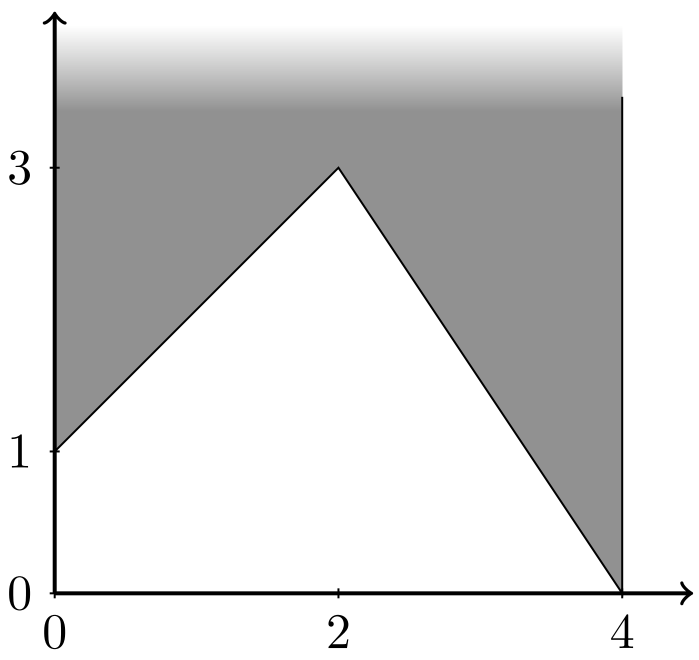

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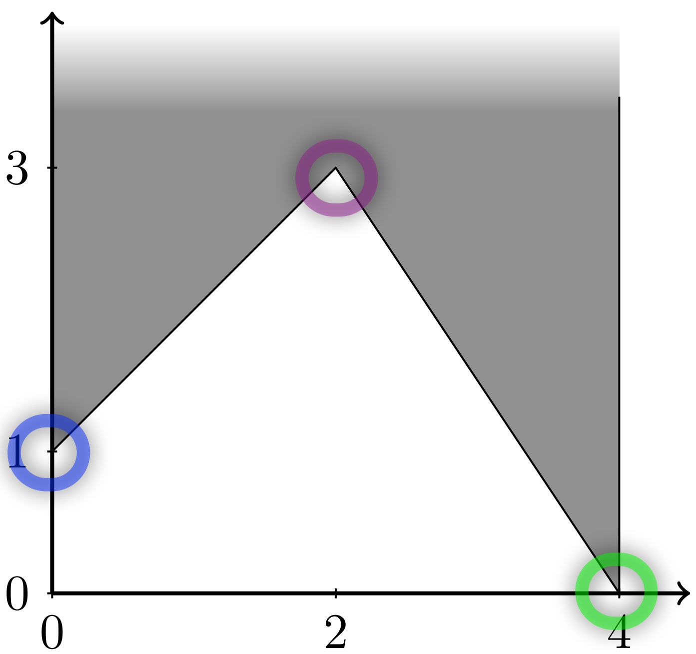

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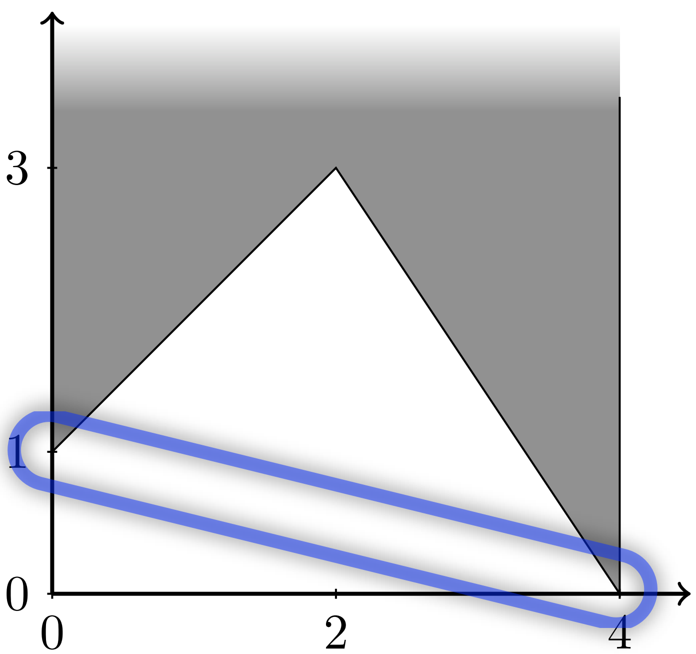

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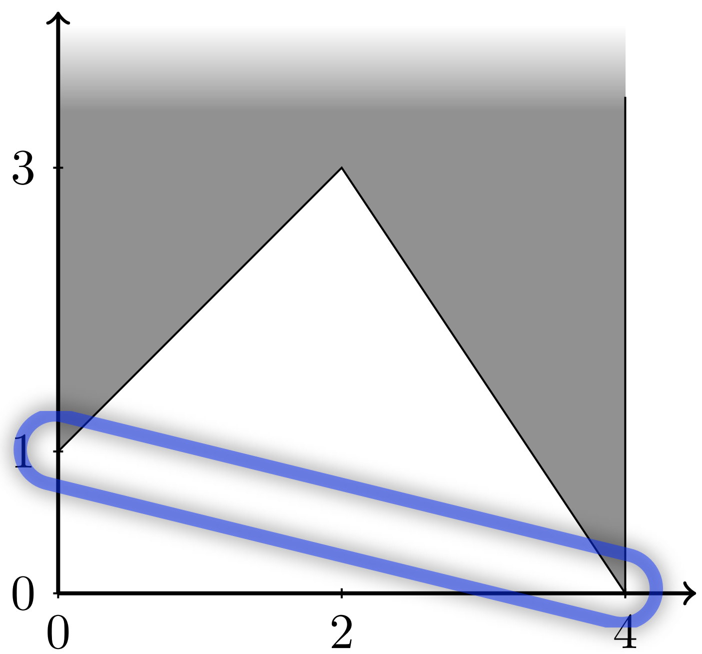

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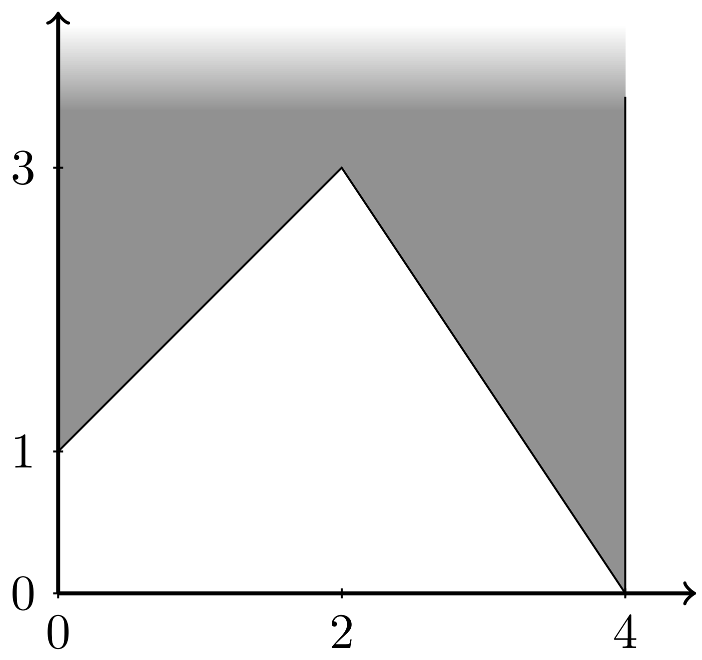

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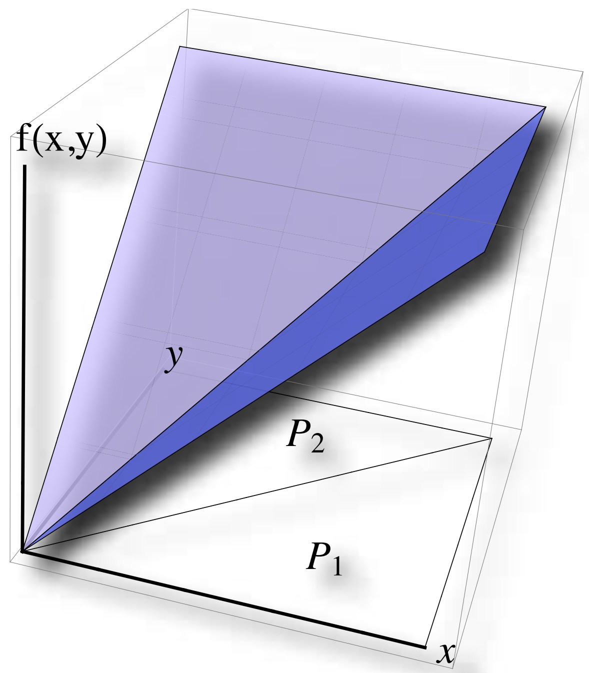
CC Multivariate (Lee and Wilson, 2001)



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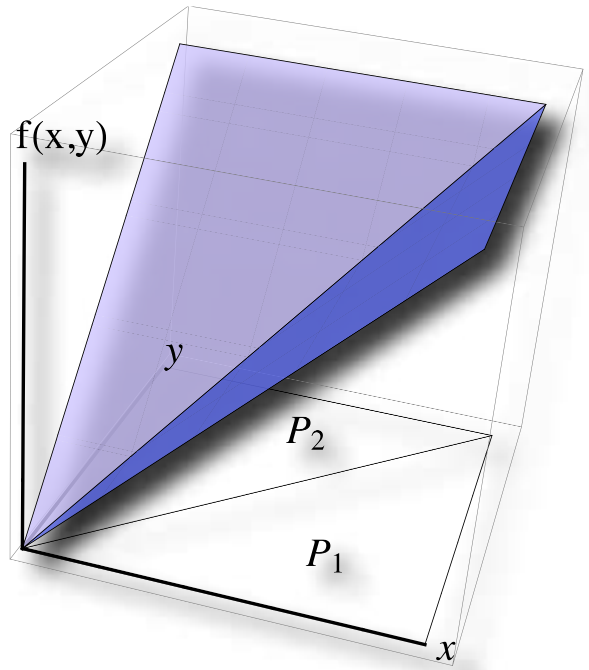

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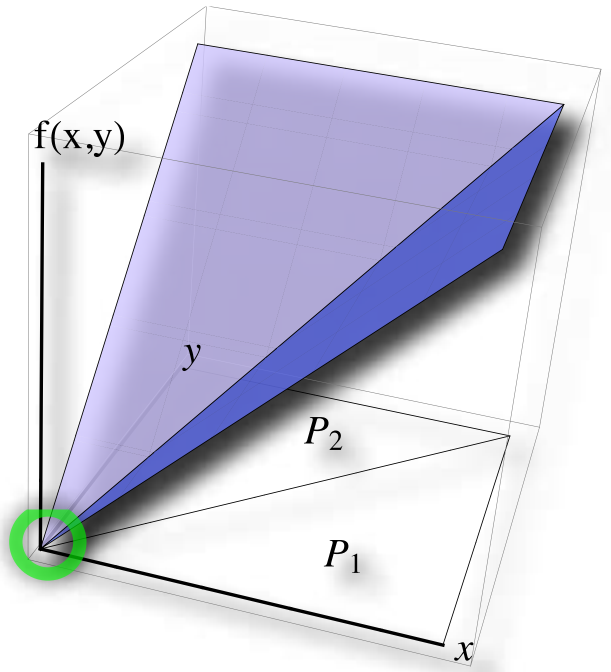

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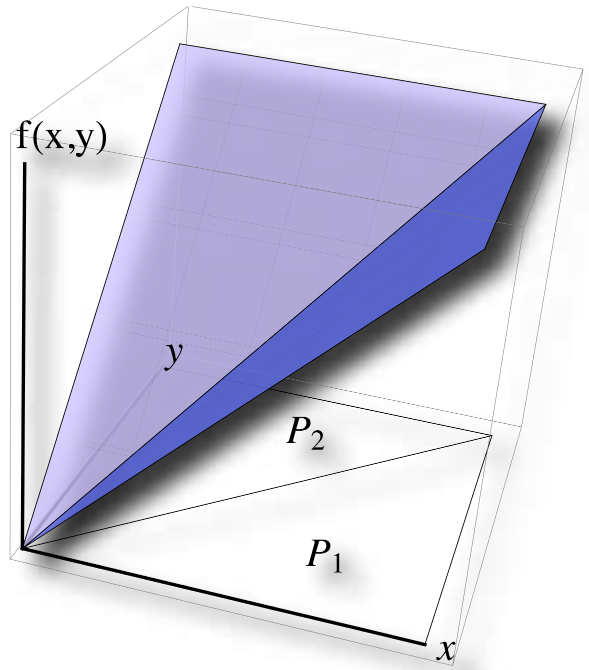

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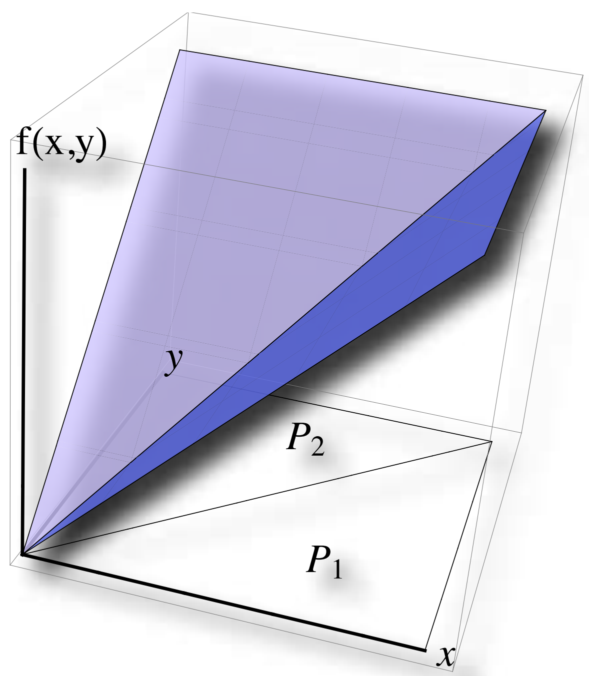
$$\lambda_{(0,0)} \le y_{P_1} + y_{P_1} \dots \quad y_{P_1} + y_{P_1} = 1, \quad y_{P_1}, y_{P_1} \in \{0,1\}$$

CC Multivariate (Lee and Wilson, 2001)



$$f(x,y) := \begin{cases} x & (x,y) \in P_1 \\ y & (x,y) \in P_1 \end{cases}$$

$$P_1 := \{(x,y) \,:\, x \leq 1,\, 0 \leq y \leq x\}, \quad P_2 := \ldots$$

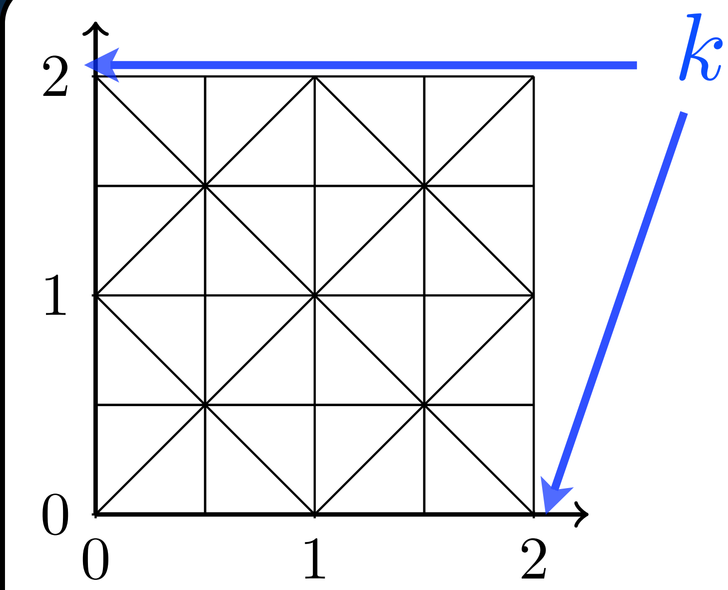
$$\mathcal{V}(\mathcal{P}) := \{(0,0),(1,0),(0,1),(1,1)\}.$$

$$x = 0\lambda_{(0,0)} + 1\lambda_{(1,0)} + 0\lambda_{(0,1)} + 1\lambda_{(1,1)}$$
$$y = 0\lambda_{(0,0)} + 0\lambda_{(1,0)} + 1\lambda_{(0,1)} + 1\lambda_{(1,1)}$$
$$z \geq 0\lambda_{(0,0)} + 1\lambda_{(1,0)} + 1\lambda_{(0,1)} + 1\lambda_{(1,1)}$$
$$1 = \lambda_{(0,0)} + \lambda_{(1,0)} + \lambda_{(0,1)} + \lambda_{(1,1)}, \quad \lambda_{(0,0)}, \ldots, \lambda_{(1,1)} \geq 0$$
$$\lambda_{(0,0)} \leq y_{P_1} + y_{P_1} \ldots \quad y_{P_1} + y_{P_1} = 1, \quad y_{P_1}, y_{P_1} \in \{0,1\}$$

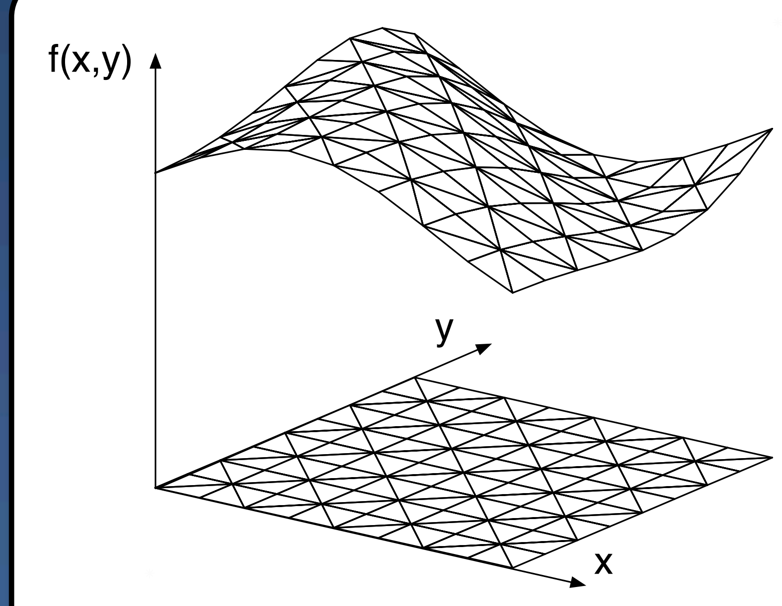
Polytopes that have (0,0) as a vertex.

Other issues: Log models & Strength

- Strength:
  - Popular models are strong.
  - Standard MILP techniques can yield weak models.
- Size of existing models is linear in $|\mathcal{P}|$:
  - We can get models with "size" logarithmic in $|\mathcal{P}|$ (Vielma and Nemhauser 2008, Vielma et al. 2009).

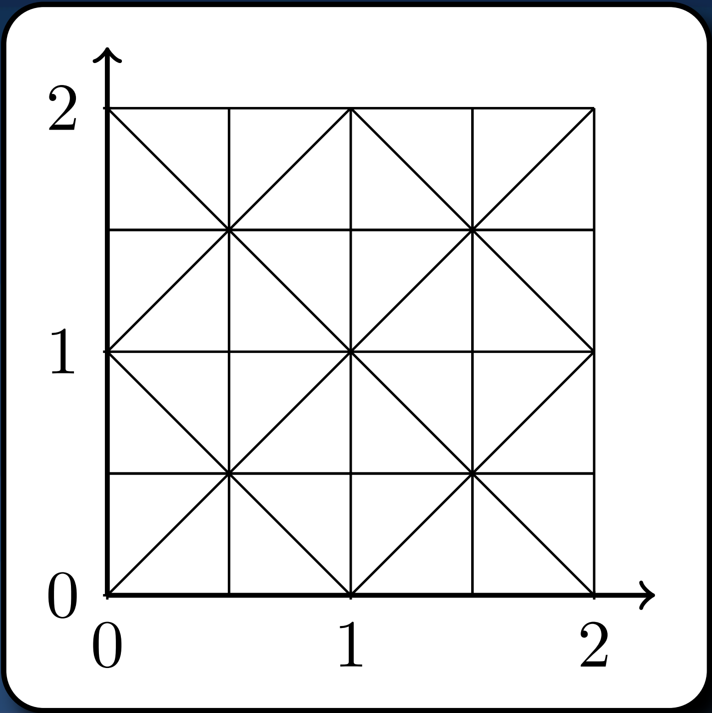


$$|\mathcal{P}| = O(k^2).$$



Computation: Transportation Problems

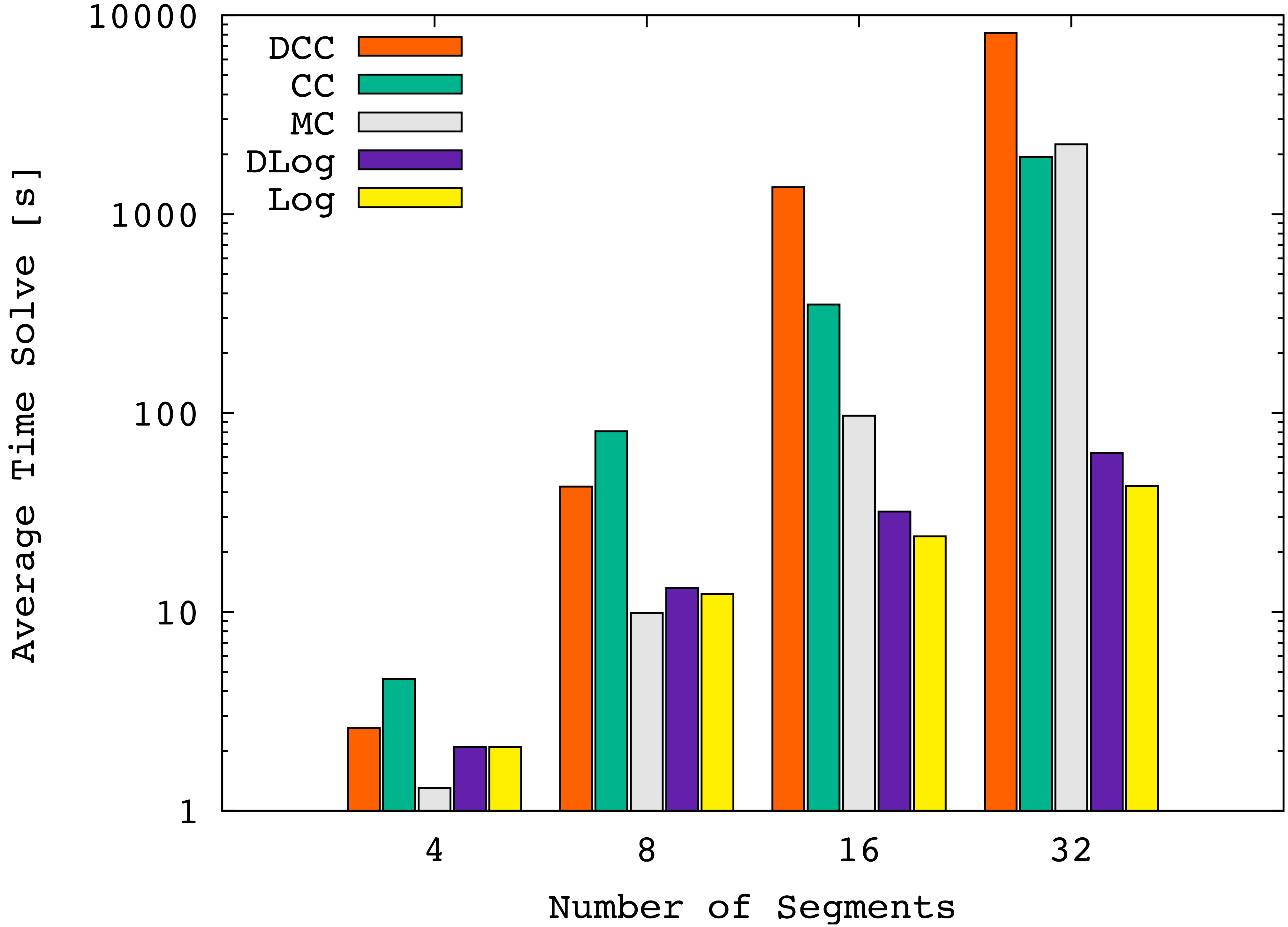
- Minimization Problems:
  - Univariate Objective: Sum of 100 univariate functions, each affine in k segments.
  - Multivariate Objective: Sum of 10 bivariate functions, each affine in a k x k grid.
- Solver: CPLEX 11 on 2.4Ghz machine.



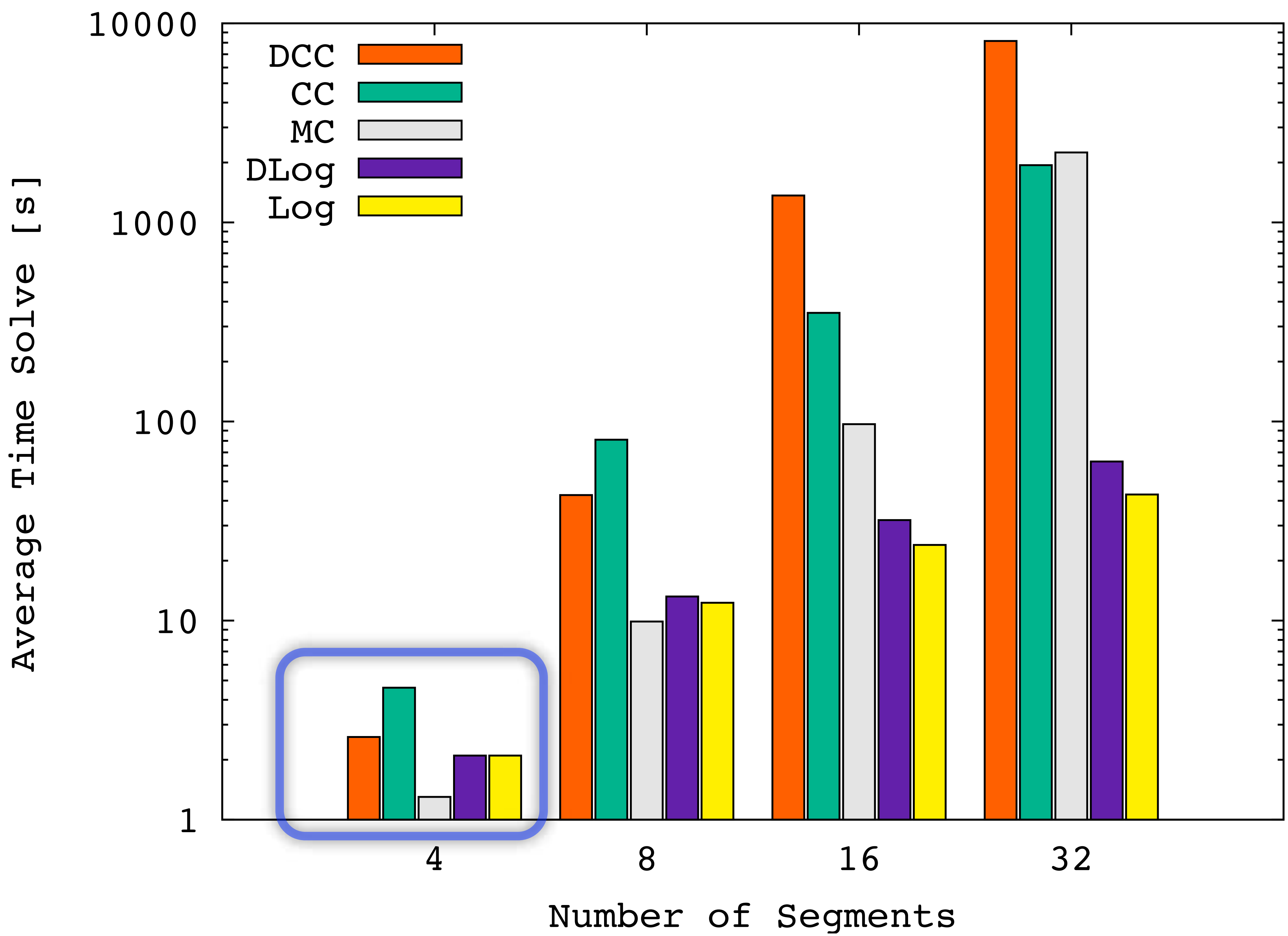
$(x, y) \to g(\|(x, y)\|)$

Concave PLF $g(\cdot)$

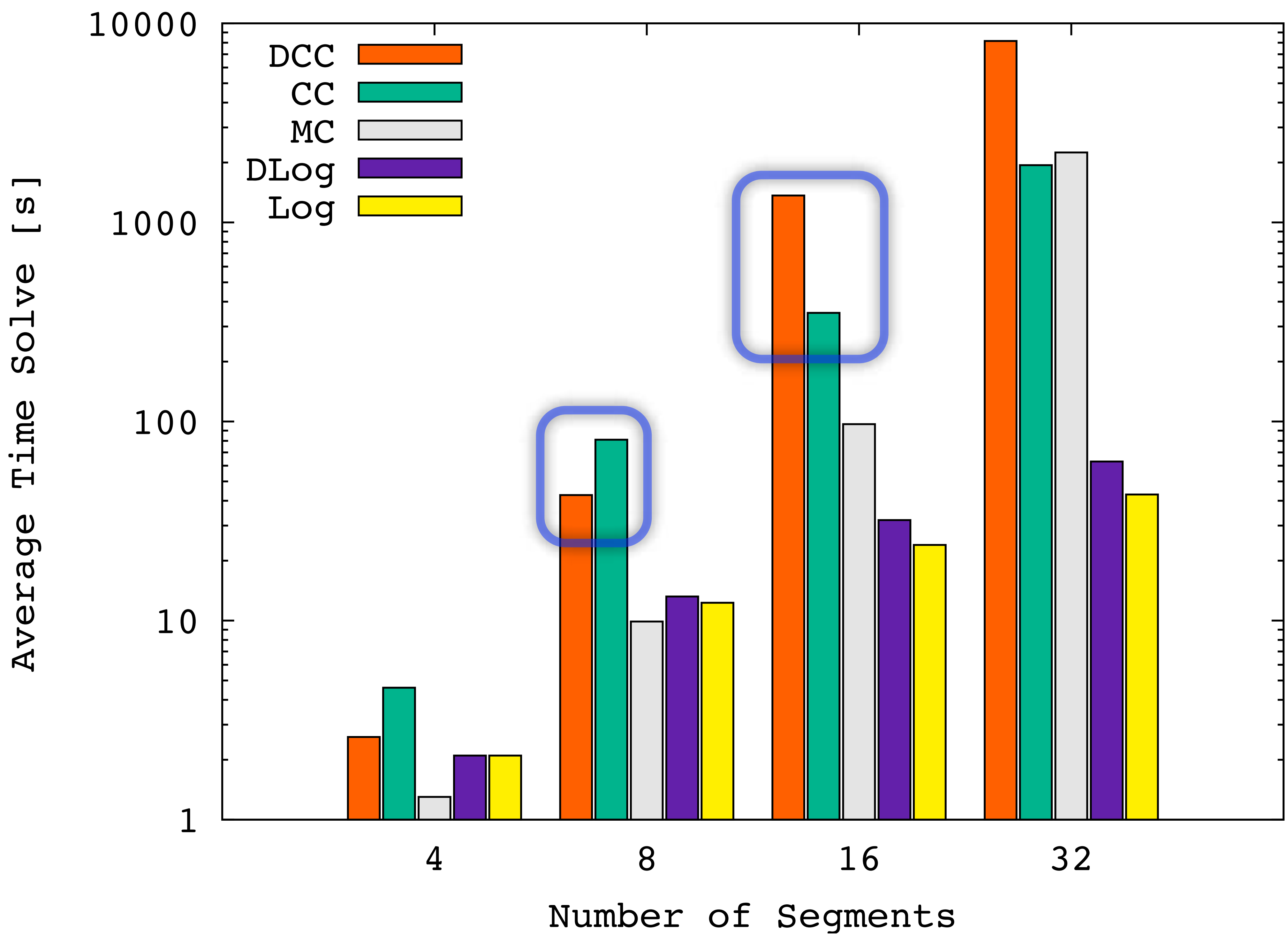
Univariate Case (Separable)



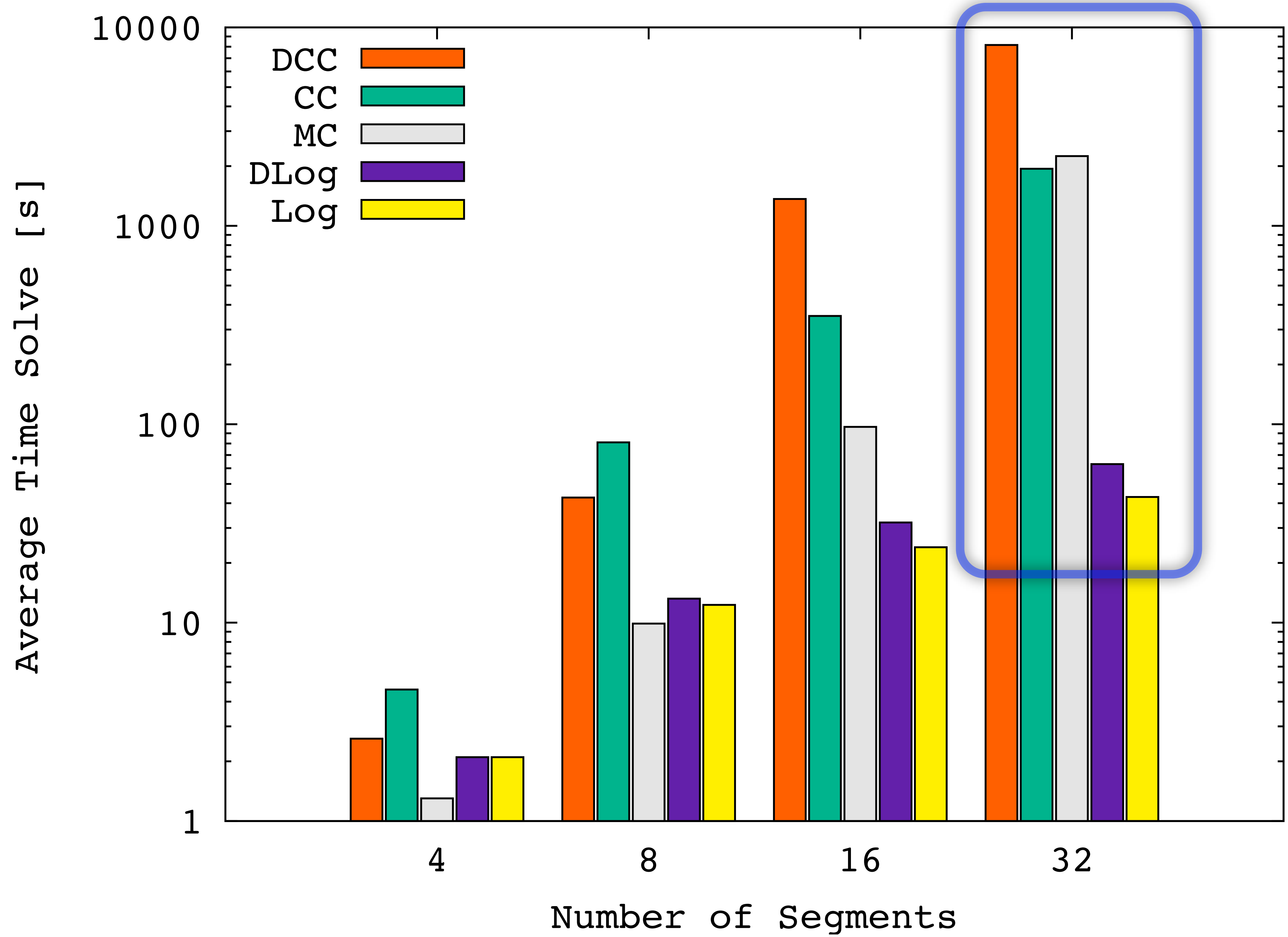
Univariate Case (Separable)



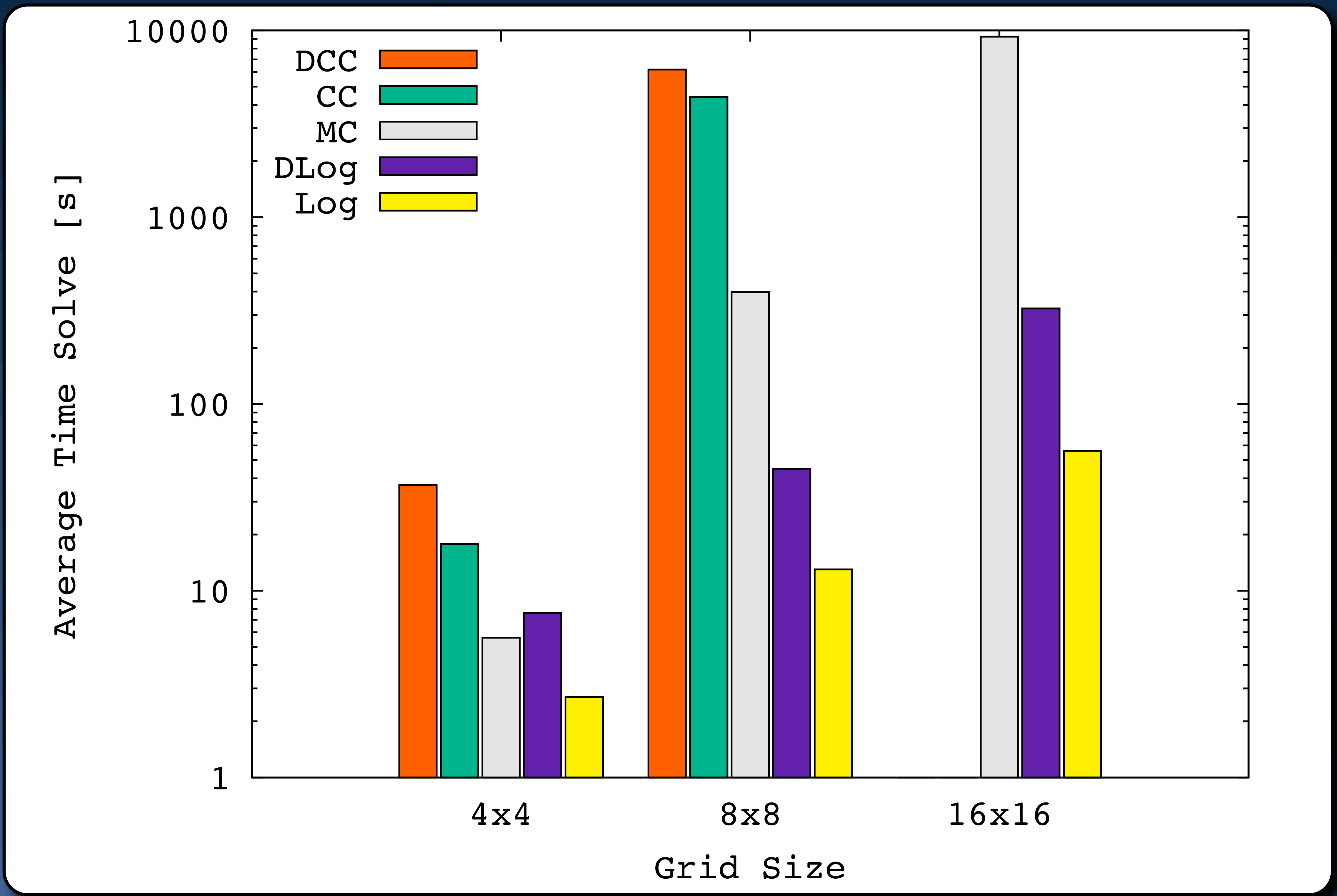
Univariate Case (Separable)



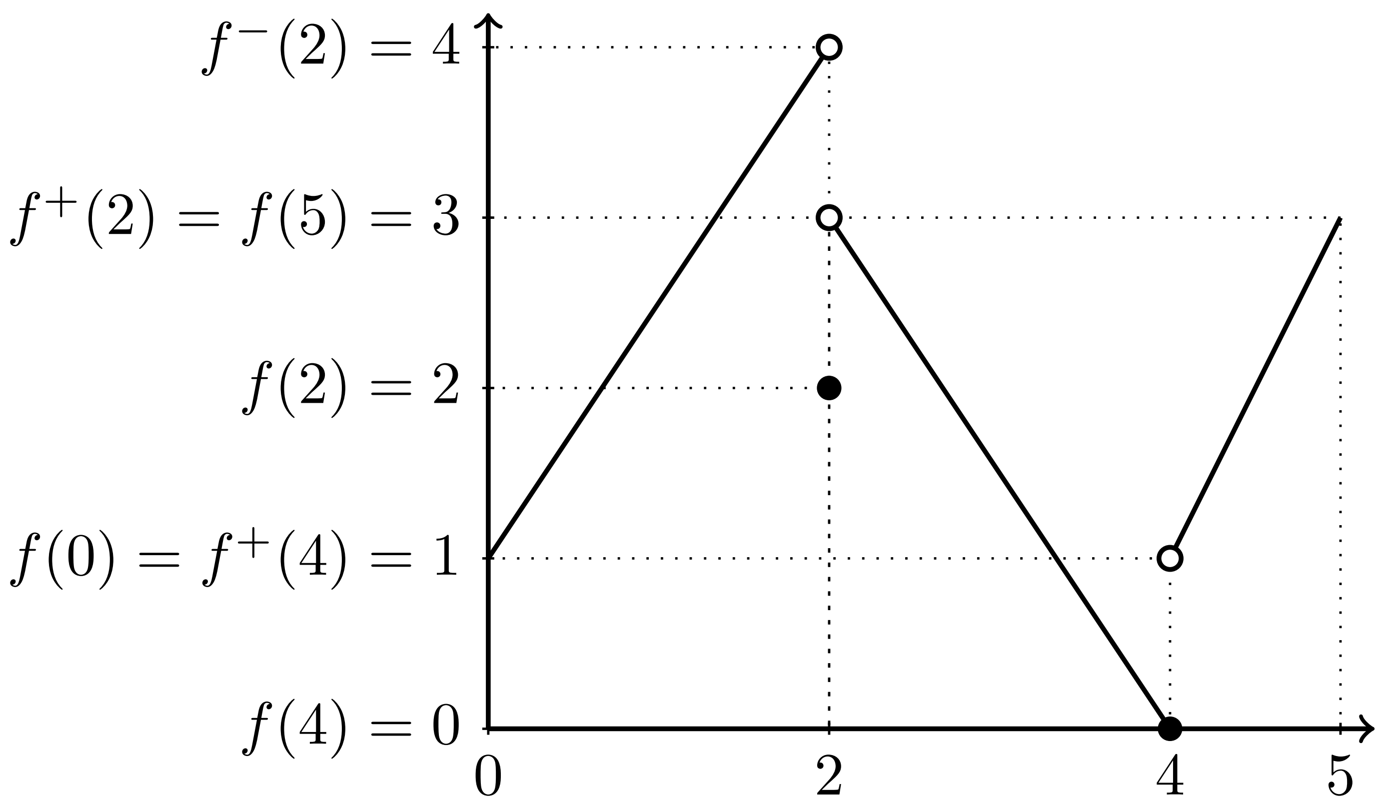
Univariate Case (Separable)



Multivariate Case (Non-Separable)



Lower Semicontinuity (LSC)



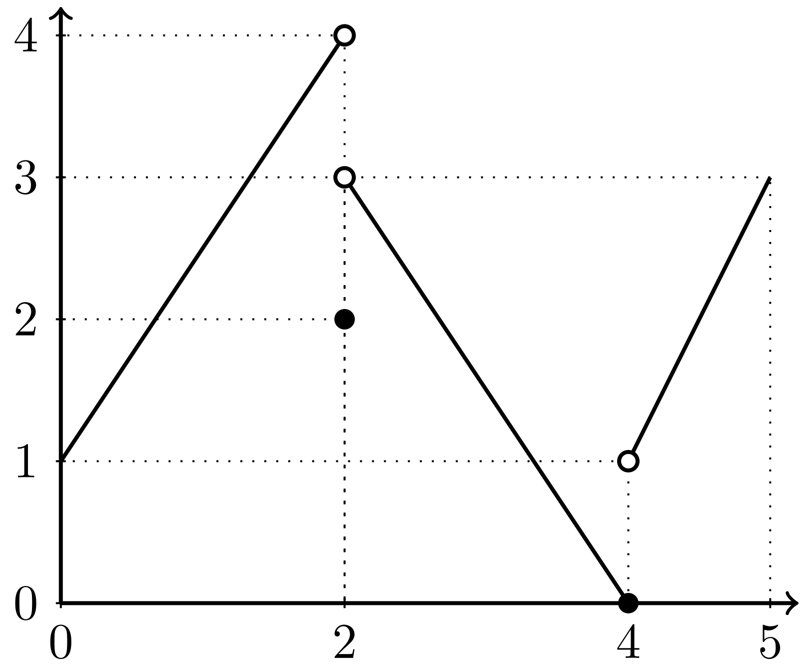
$$f^-(d) = \lim_{\substack{x \to d \\ x \le d}} f(x)$$

$$f^+(d) = \lim_{\substack{x \to d \\ x \ge d}} f(x).$$

- Lower Semicontinuity:

$$f(x) \le f^-(x), f^+(x) \Leftrightarrow \text{epi}(f) \text{ is closed.}$$

Lower Semicontinuous PLFs



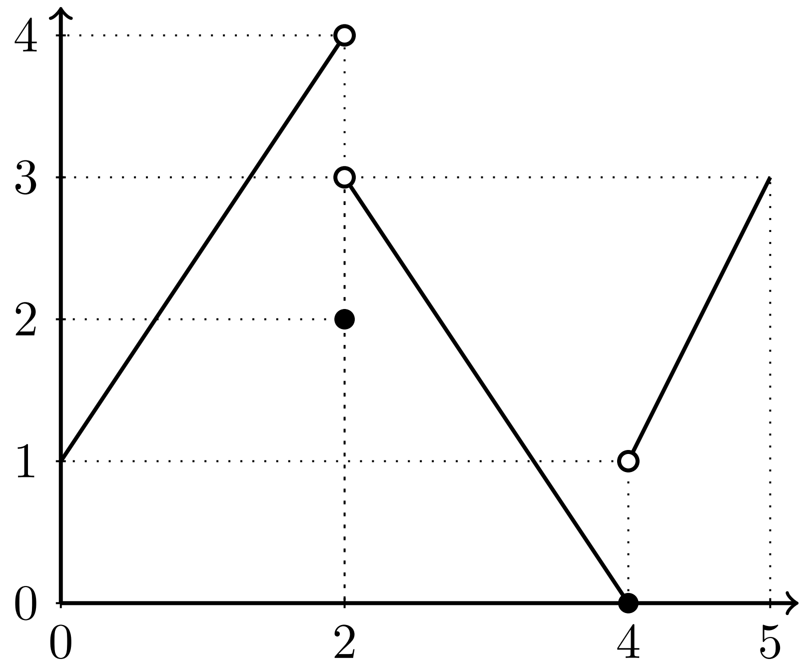
$$f(x) := \begin{cases} 1.5x + 1 & x \in [0, 2) \\ 2 & x \in [2, 2] \\ -1.5x + 6 & x \in (2, 4] \\ 2x - 7 & x \in (4, 5] \end{cases}$$

$$f(x) := \{ m_P x + c_P \quad x \in P \quad \forall P \in \mathcal{P}$$

Finite family of copolytopes

$$P = \{x \in \mathbb{R}^n \ : \ a_i x \leq b_i \, \forall i \in \{1, \dots, p\}, \\ a_i x < b_i \, \forall i \in \{p, \dots, m\}\}$$

Lower Semicontinuous PLFs



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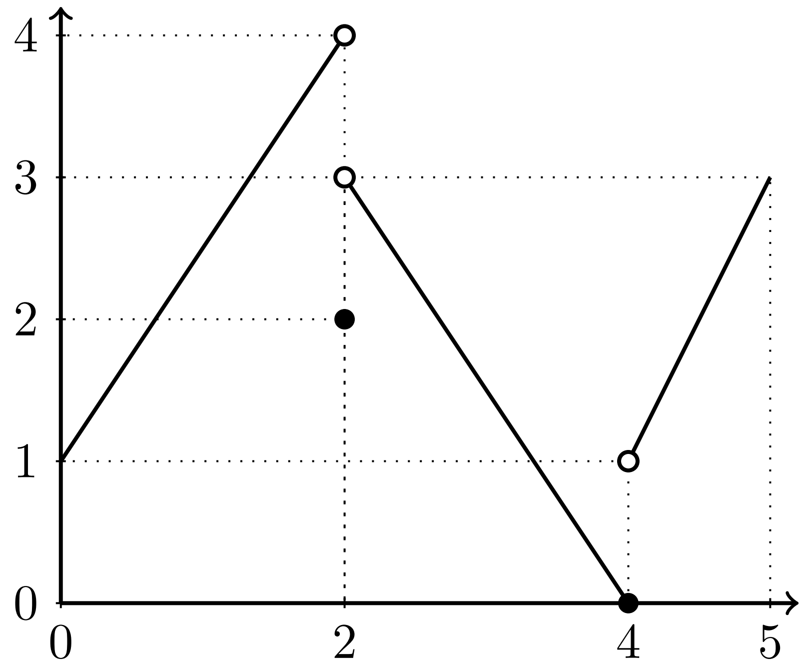
Lower Semicontinuous PLFs



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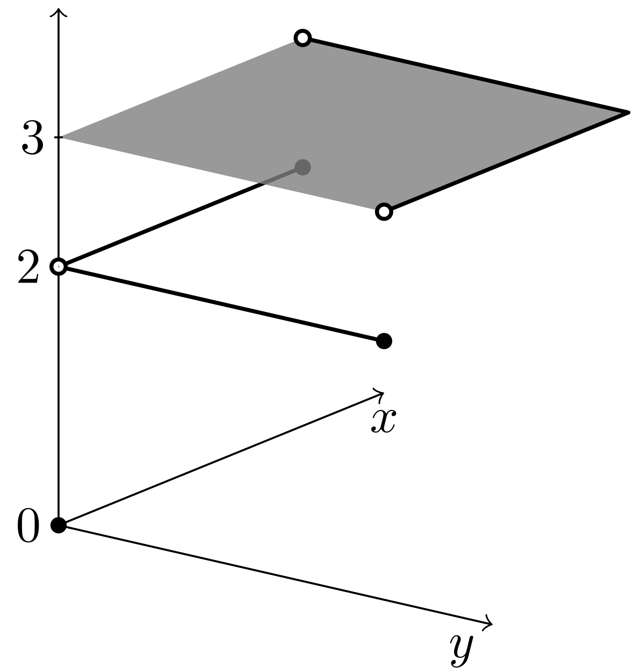
Lower Semicontinuous PLFs



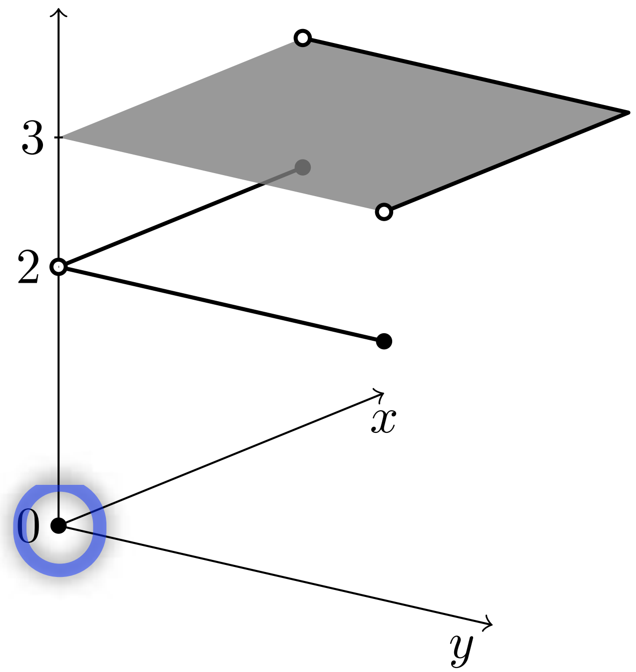
$$f(x,y) := \begin{cases} 3 & (x,y) \in (0,1]^2 \\ 2 & (x,y) \in \{(x,y) \in \mathbb{R}^2 \,:\, x = 0,\ y > 0\} \\ 2 & (x,y) \in \{(x,y) \in \mathbb{R}^2 \,:\, y = 0,\ x > 0\} \\ 0 & (x,y) \in \{(0,0)\}. \end{cases}$$

$$f(x) := \begin{cases} m_P x + c_P & x \in P \quad \forall P \in \mathcal{P} \end{cases}$$

Finite family of copolytopes

$$P = \{x \in \mathbb{R}^n \,:\, a_i x \leq b_i \, \forall i \in \{1, \ldots, p\}, \quad a_i x < b_i \, \forall i \in \{p, \ldots, m\}\}$$

Lower Semicontinuous PLFs



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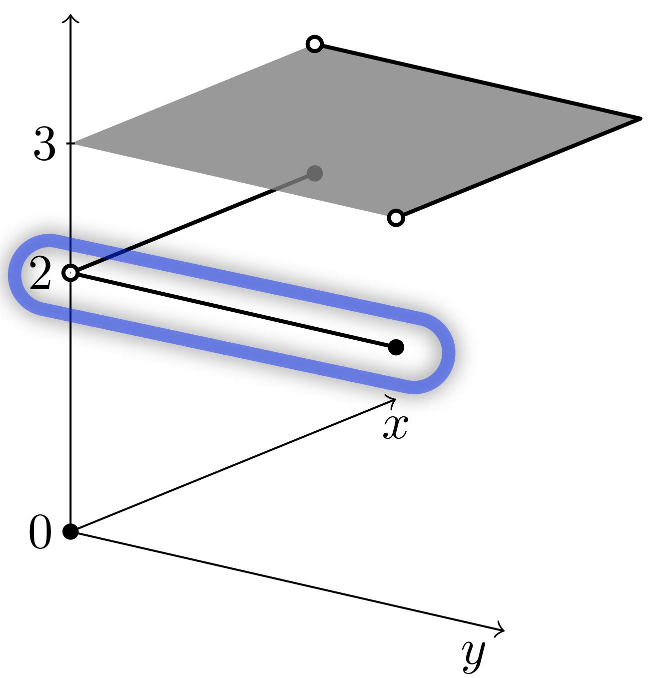
Lower Semicontinuous PLFs



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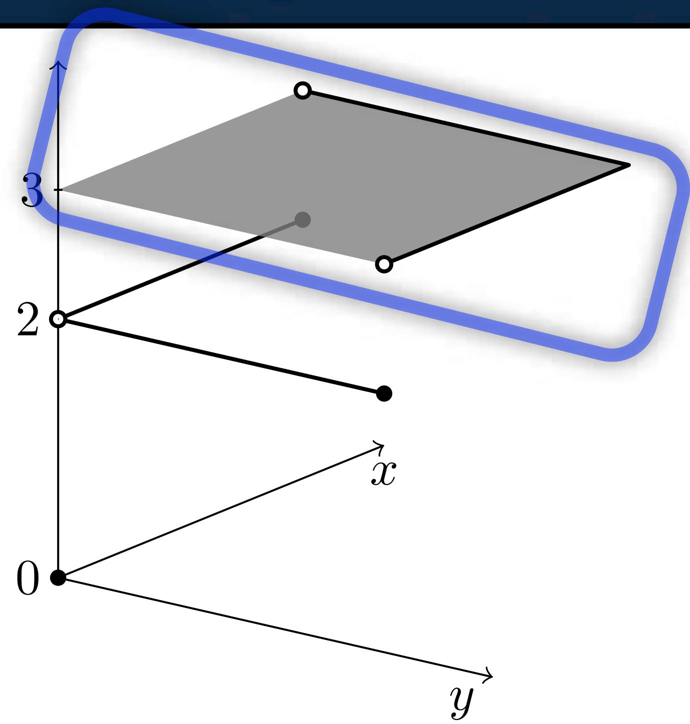
Lower Semicontinuous PLFs



$$f(x,y) := \begin{cases} 3 & (x,y) \in (0,1]^2 \\ 2 & (x,y) \in \{(x,y) \in \mathbb{R}^2 \ : \ x = 0, \ y > 0\} \\ 2 & (x,y) \in \{(x,y) \in \mathbb{R}^2 \ : \ y = 0, \ x > 0\} \\ 0 & (x,y) \in \{(0,0)\}. \end{cases}$$

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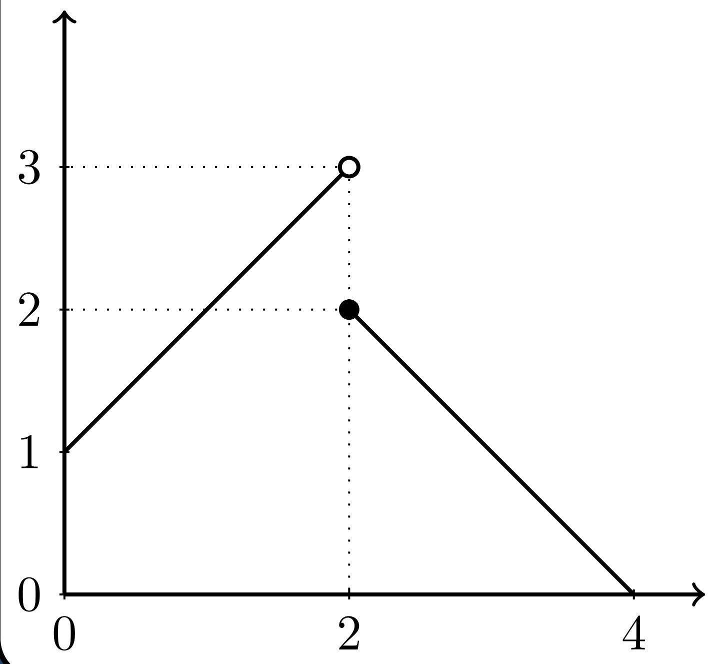
Finite family of copolytopes

$$P = \{x \in \mathbb{R}^n \ : \ a_i x \leq b_i \, \forall i \in \{1, \ldots, p\}, \\ a_i x < b_i \, \forall i \in \{p, \ldots, m\}\}$$

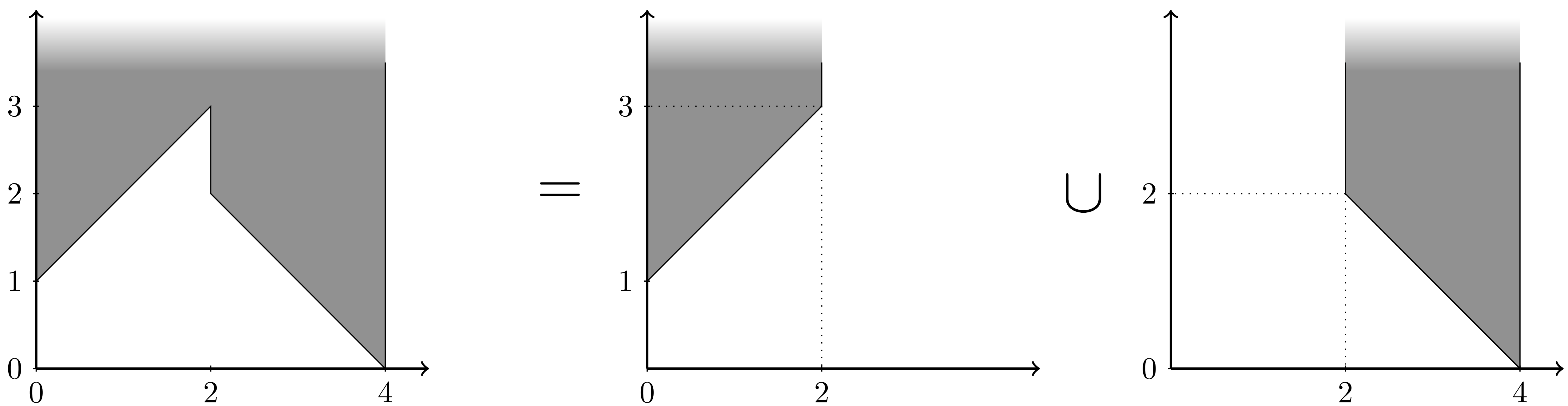
Disaggregated CC (Jeroslow and Lowe, 1984)



(DCC)

$$f(x) := \begin{cases} x+1 & x \in [0,2) \\ 4-x & x \in [2,4] \end{cases}$$

Disaggregated CC (Jeroslow and Lowe, 1984)



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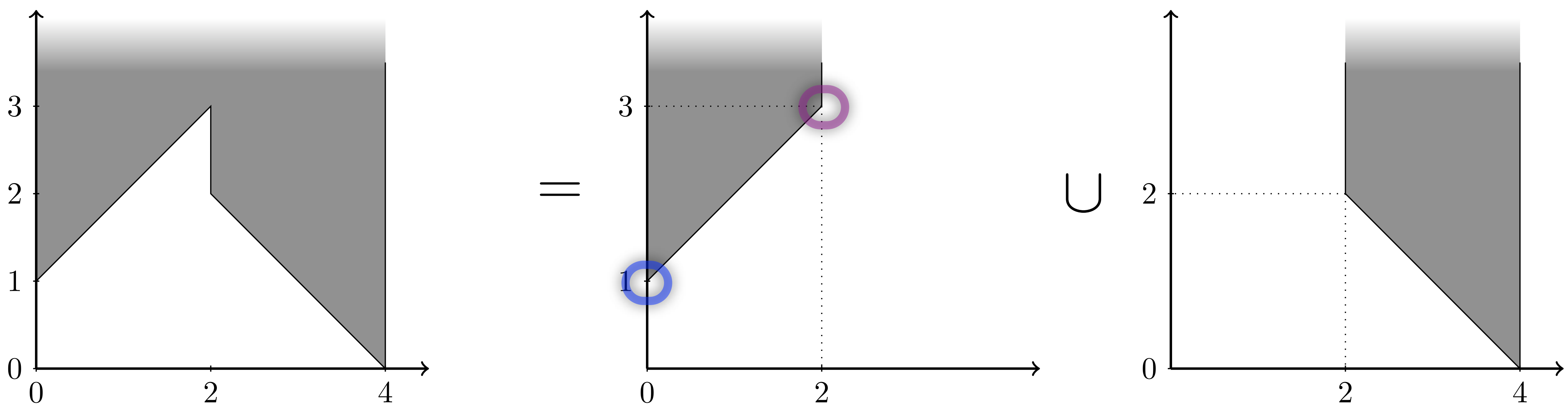


(DCC)

P_1

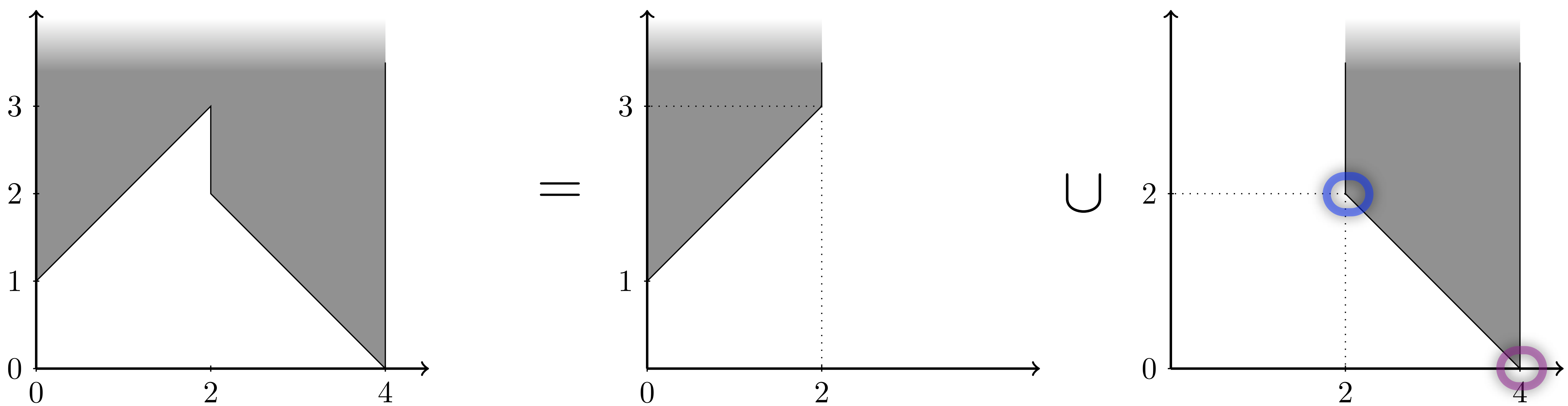
$$f(x) := \begin{cases} x+1 & x \in [0,2) \\ 4-x & x \in [2,4] \end{cases}$$

$$x = 0\lambda_{P_1,0} + 2\lambda_{P_1,2}$$

$$z \geq 1\lambda_{P_1,0} + 3\lambda_{P_1,2}$$

$$1 = \lambda_{P_1,0} + \lambda_{P_1,2}, \quad \lambda_{P_1,0}, \lambda_{P_1,2} \geq 0$$

Disaggregated CC (Jeroslow and Lowe, 1984)



(DCC)

$$f(x) := \begin{cases} x+1 & x \in [0,2) \\ 4-x & x \in [2,4] \end{cases}$$

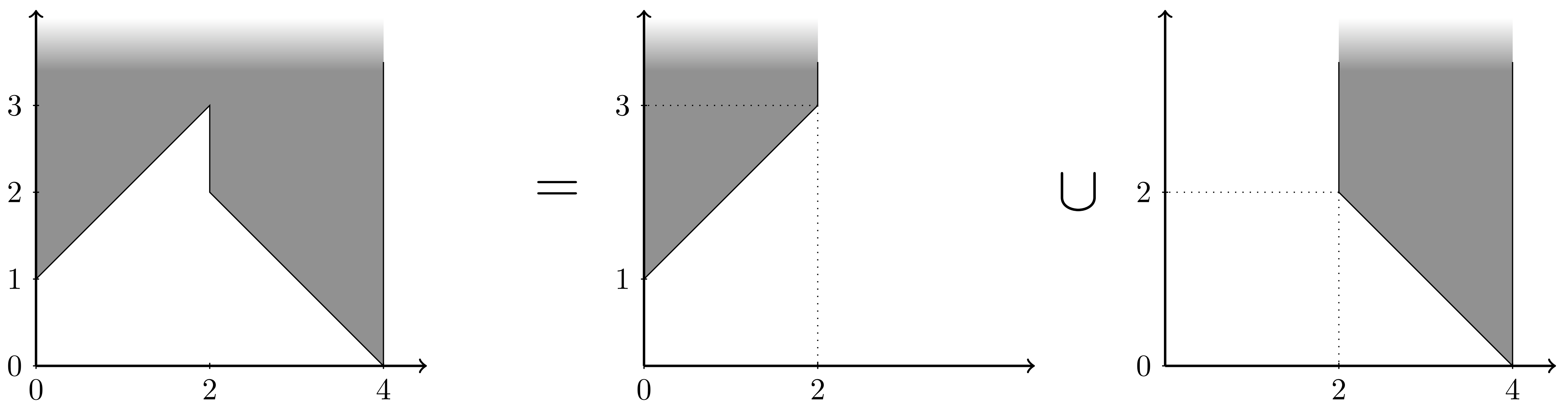
P_2

$$x = 2\lambda_{P_2,2} + 4\lambda_{P_2,4}$$

$$z \geq 2\lambda_{P_2,2} + 0\lambda_{P_2,4}$$

$$1 = \lambda_{P_2,2} + \lambda_{P_2,4}, \quad \lambda_{P_2,2}, \lambda_{P_2,4} \geq 0$$

Disaggregated CC (Jeroslow and Lowe, 1984)



(DCC)

$$f(x) := \begin{cases} x+1 & x \in [0,2) \\ 4-x & x \in [2,4] \end{cases}$$

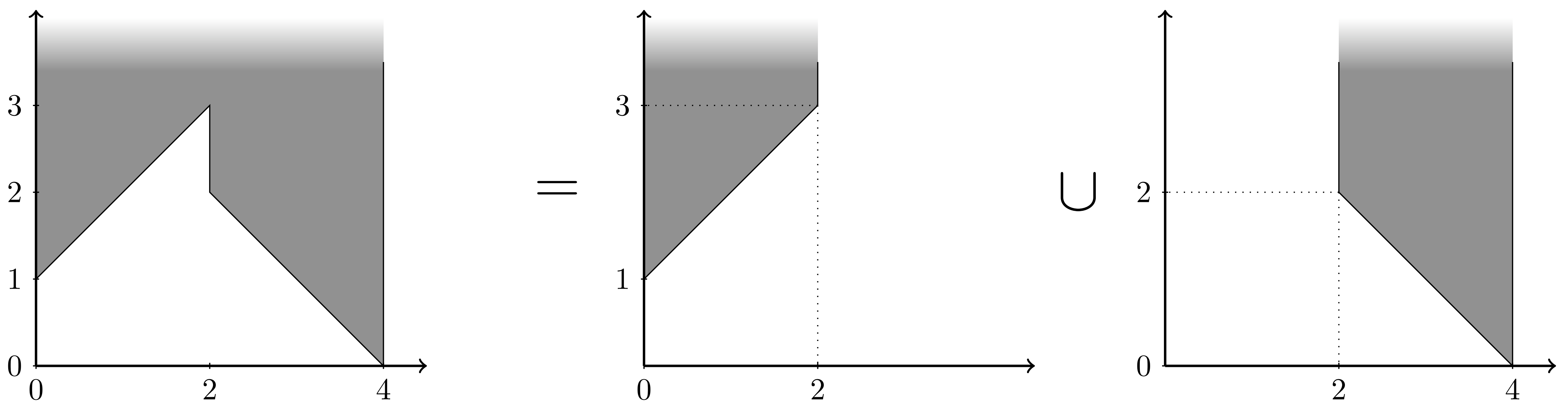
$$x = 0\lambda_{P_1,0} + 2\lambda_{P_1,2} + 2\lambda_{P_2,2} + 4\lambda_{P_2,4}$$

$$z \geq 1\lambda_{P_1,0} + 3\lambda_{P_1,2} + 2\lambda_{P_2,2} + 0\lambda_{P_2,4}$$

$$1 = \lambda_{P_1,0} + \lambda_{P_1,2}, \quad \lambda_{P_1,0}, \lambda_{P_1,2} \geq 0$$

$$1 = \lambda_{P_2,2} + \lambda_{P_2,4}, \quad \lambda_{P_2,2}, \lambda_{P_2,4} \geq 0$$

Disaggregated CC (Jeroslow and Lowe, 1984)



(DCC)

$$f(x) := \begin{cases} x+1 & x \in [0,2) \\ 4-x & x \in [2,4] \end{cases}$$

$$x = 0\lambda_{P_1,0} + 2\lambda_{P_1,2} + 2\lambda_{P_2,2} + 4\lambda_{P_2,4}$$

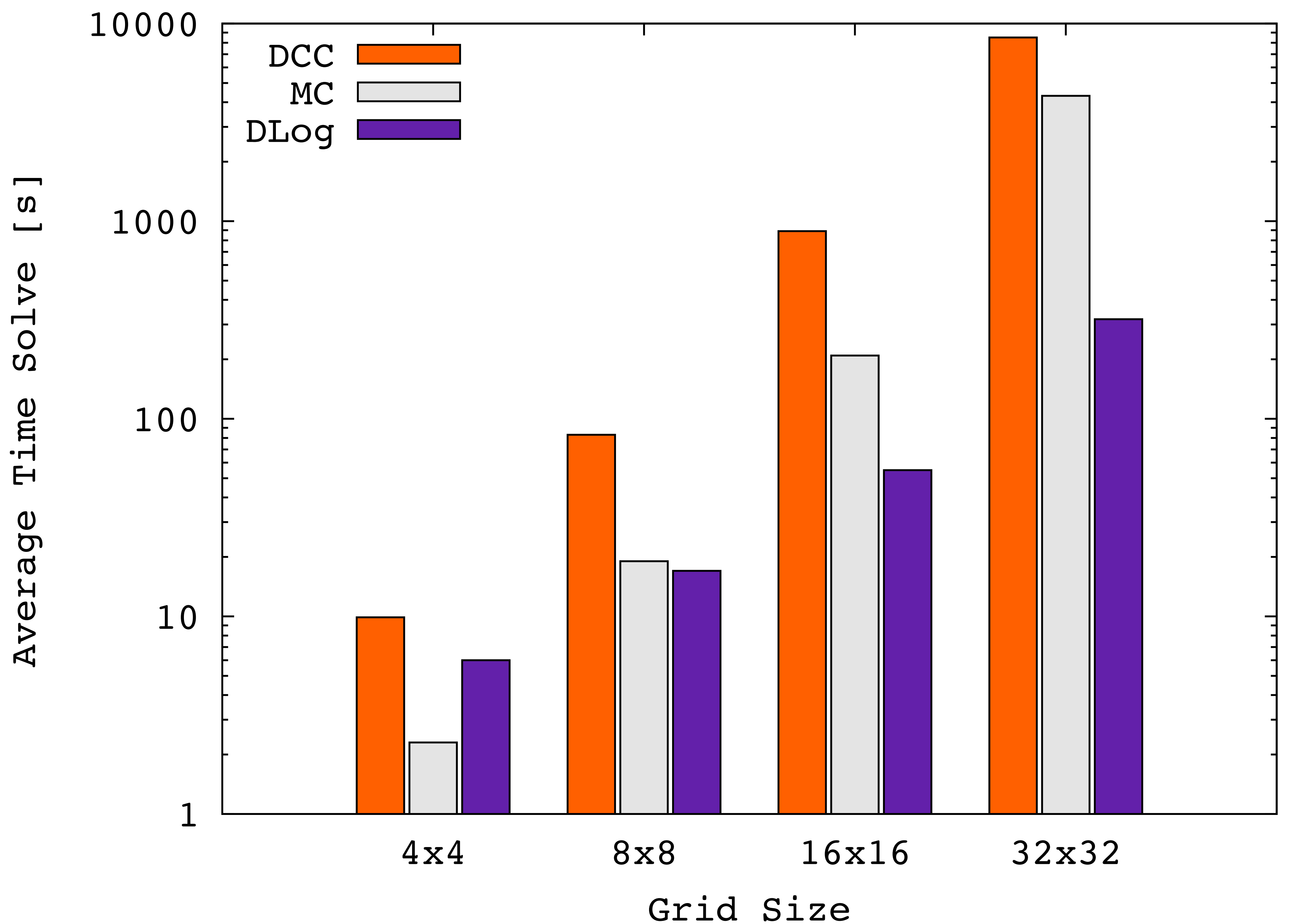
$$z \geq 1\lambda_{P_1,0} + 3\lambda_{P_1,2} + 2\lambda_{P_2,2} + 0\lambda_{P_2,4}$$

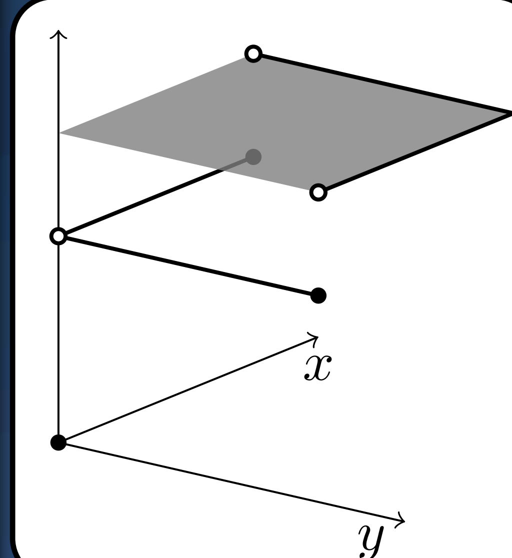
$$y_{P_1} = \lambda_{P_1,0} + \lambda_{P_1,2}, \quad \lambda_{P_1,0}, \lambda_{P_1,2} \geq 0$$

$$y_{P_2} = \lambda_{P_2,2} + \lambda_{P_2,4}, \quad \lambda_{P_2,2}, \lambda_{P_2,4} \geq 0$$

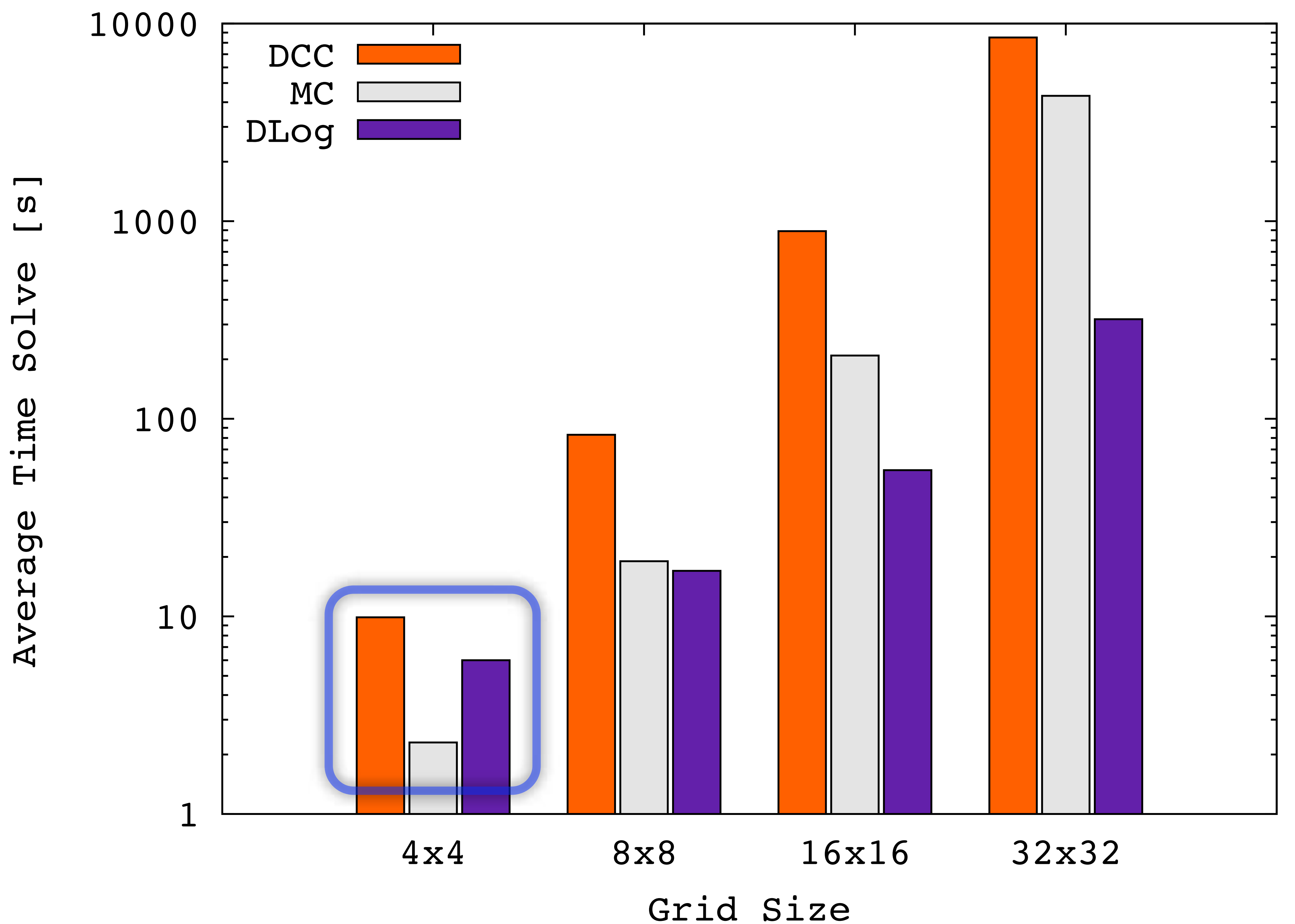
$$1 = y_{P_1} + y_{P_2}, \quad y_{P_1}, y_{P_2} \in \{0,1\}$$

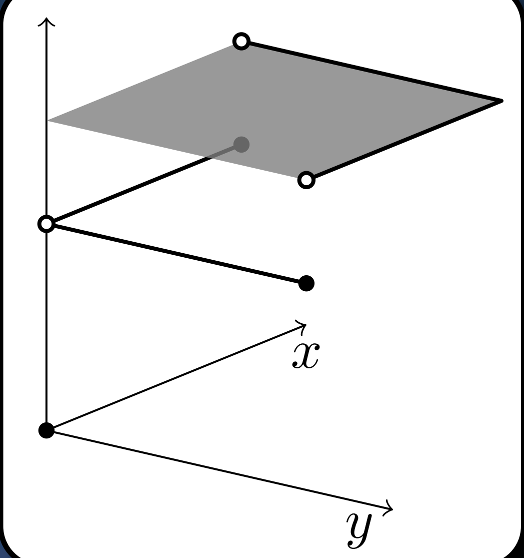
Multivariate Lower Semicontinuous



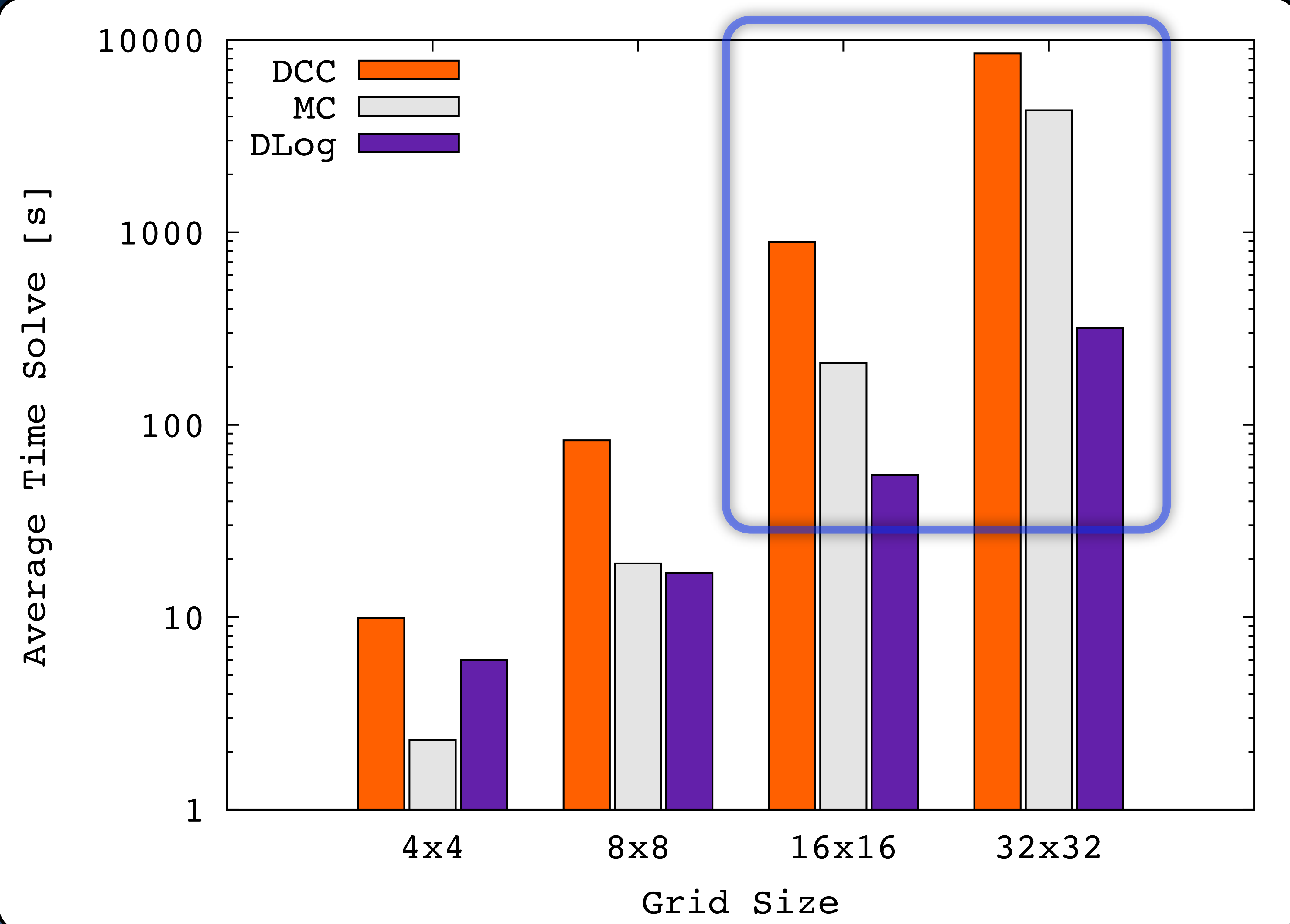


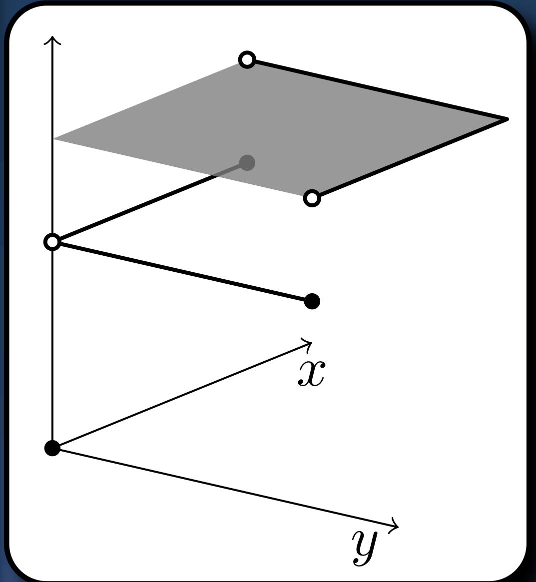
Multivariate Lower Semicontinuous





Multivariate Lower Semicontinuous



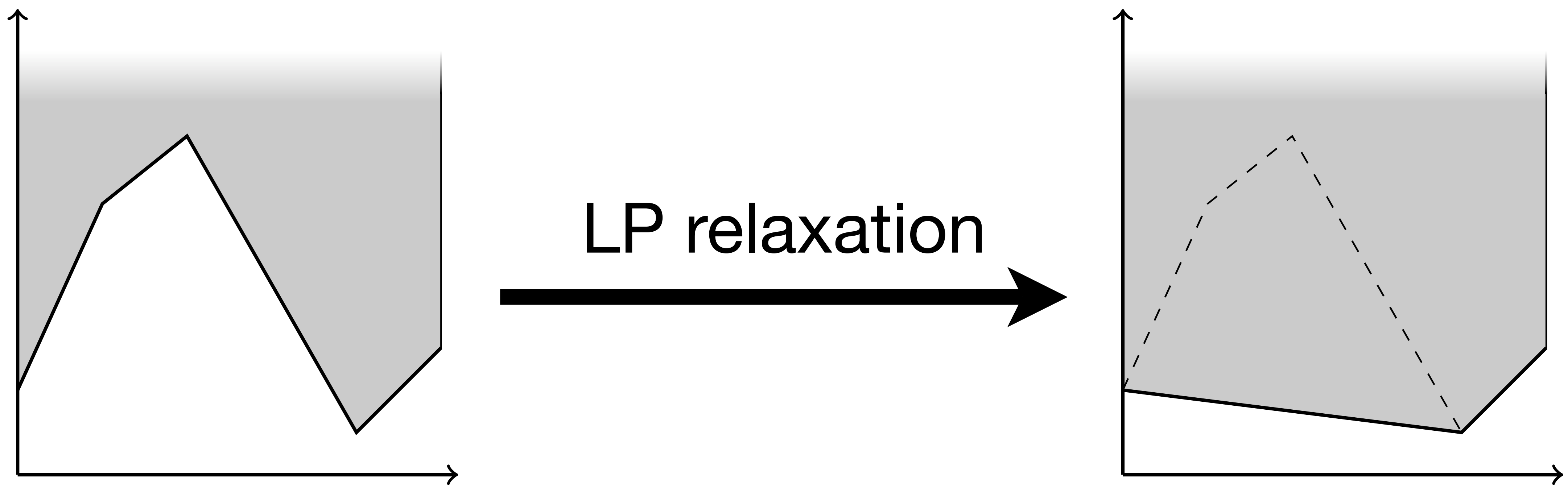


Final Remarks

- Many MILP models for PLF: Most are simple.
- Popular models are strong.
- Caution: Standard MILP techniques can weak models.
- Best model varies: e.g. Log's best for fine grids.
- Advertisement: Papers at http://www2.isye.gatech.edu/~jvielma

Strength of LP Relaxations

- *Sharp* Models: LP = lower convex envelope.

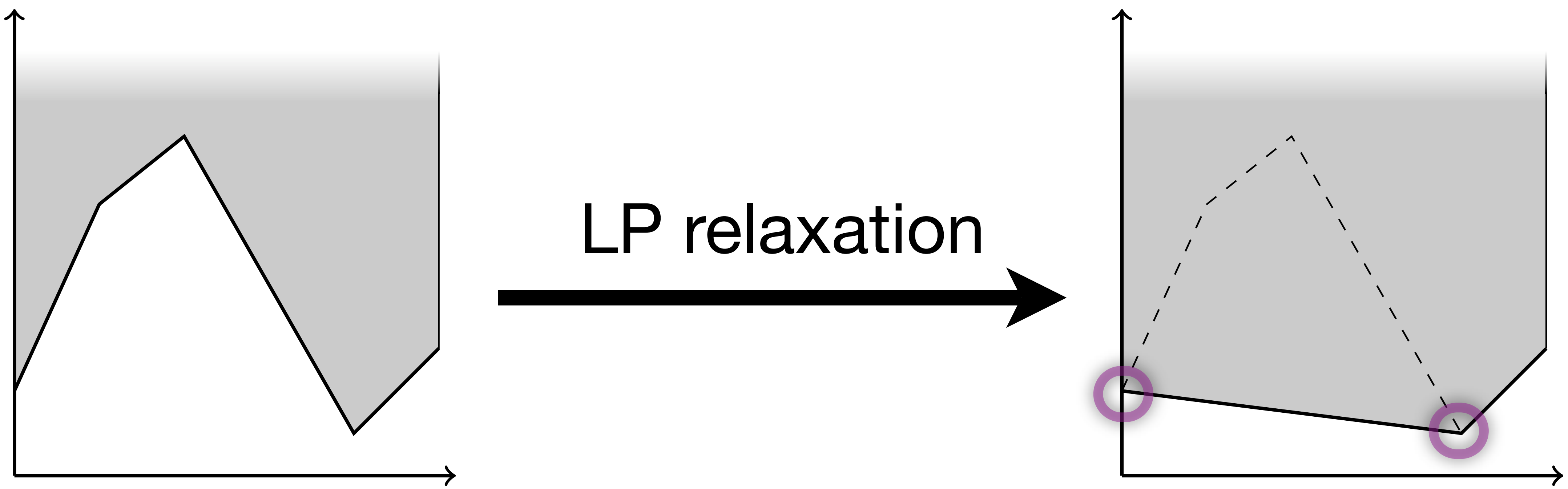


(a) $\mathrm{epi}(f)$. (b) $\mathrm{conv}(\mathrm{epi}(f))$.

  - All popular models are sharp.
- *Locally Ideal*: LP = Integral (All but CC, even Log).
- *Locally ideal* implies *Sharp*.

Strength of LP Relaxations

- *Sharp* Models: LP = lower convex envelope.



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  - All popular models are sharp.
- *Locally Ideal*: LP = Integral (All but CC, even Log).
- *Locally ideal* implies *Sharp*.

For Multivariate Functions

(DCC)

$$\sum_{P\in\mathcal{P}}\sum_{v\in V(\overline{P})}\lambda_{P,v}v=x,\quad \sum_{P\in\mathcal{P}}\sum_{v\in V(\overline{P})}\lambda_{P,v}\left(m_P v+c_P\right)\leq z$$

$$\lambda_{P,v}\geq 0\quad \forall P\in\mathcal{P},\, v\in V(\overline{P}),\quad \sum_{P\in\mathcal{P}}\sum_{v(\overline{P})}\lambda_{P,v}=1$$

$$\sum_{v\in V(\overline{P})}\lambda_{P,v}=y_P\quad \forall P\in\mathcal{P},\quad \sum_{P\in\mathcal{P}}y_P=1,\quad y_P\in\{0,1\}\quad \forall P\in\mathcal{P}$$

(CC)

$$\sum_{v\in\mathcal{V}(\mathcal{P})}\lambda_v v=x,\quad \sum_{v\in\mathcal{V}(\mathcal{P})}\lambda_v\left(m_P v+c_P\right)\leq z$$

$$\lambda_v\geq 0\quad \forall v\in\mathcal{V}(\mathcal{P}):=\bigcup_{P\in\mathcal{P}}V(P),\quad \sum_{v\in\mathcal{V}(\mathcal{P})}\lambda_v=1$$

$$\lambda_v\leq \sum_{\{P\in\mathcal{P}:v\in P\}}y_P\quad \forall v\in\mathcal{V}(\mathcal{P}),\quad \sum_{P\in\mathcal{P}}y_P=1,\quad y_P\in\{0,1\}\quad \forall P\in\mathcal{P}$$

For Multivariate Functions

$$\sum_{P\in\mathcal{P}}\sum_{v\in V(\overline{P})}\lambda_{P,v}v = x, \quad \sum_{P\in\mathcal{P}}\sum_{v\in V(\overline{P})}\lambda_{P,v}\left(m_P v + c_P\right) \le z$$

(DCC)

$$\lambda_{P,v} \ge 0 \quad \forall P\in\mathcal{P},\, v\in V(\overline{P}), \quad \sum_{P\in\mathcal{P}}\sum_{v(\overline{P})}\lambda_{P,v} = 1$$

$$\sum_{v\in V(\overline{P})}\lambda_{P,v} = y_P \quad \forall P\in\mathcal{P}, \quad \sum_{P\in\mathcal{P}} y_P = 1, \quad y_P\in\{0,1\} \quad \forall P\in\mathcal{P}$$

$$\sum_{v\in\mathcal{V}(\mathcal{P})}\lambda_v v = x, \quad \sum_{v\in\mathcal{V}(\mathcal{P})}\lambda_v\left(m_P v + c_P\right) \le z$$

(CC)

$$\lambda_v \ge 0 \quad \forall v\in\mathcal{V}(\mathcal{P}) := \bigcup_{P\in\mathcal{P}} V(P), \quad \sum_{v\in\mathcal{V}(\mathcal{P})}\lambda_v = 1$$

$$\lambda_v \le \sum_{\{P\in\mathcal{P}\,:\,v\in P\}} y_P \quad \forall v\in\mathcal{V}(\mathcal{P}), \quad \sum_{P\in\mathcal{P}} y_P = 1, \quad y_P\in\{0,1\} \quad \forall P\in\mathcal{P}$$

For Multivariate Functions

(DCC)

$$\sum_{P\in\mathcal{P}}\sum_{v\in V(\overline{P})}\lambda_{P,v}v = x, \quad \sum_{P\in\mathcal{P}}\sum_{v\in V(\overline{P})}\lambda_{P,v}\left(m_P v + c_P\right) \leq z$$

$$\lambda_{P,v} \geq 0 \quad \forall P\in\mathcal{P},\, v\in V(\overline{P}), \quad \sum_{P\in\mathcal{P}}\sum_{v(\overline{P})}\lambda_{P,v} = 1$$

$$\sum_{v\in V(\overline{P})}\lambda_{P,v} = y_P \quad \forall P\in\mathcal{P}, \quad \sum_{P\in\mathcal{P}} y_P = 1, \quad y_P\in\{0,1\} \quad \forall P\in\mathcal{P}$$

(CC)

$$\sum_{v\in\mathcal{V}(\mathcal{P})}\lambda_v v = x, \quad \sum_{v\in\mathcal{V}(\mathcal{P})}\lambda_v\left(m_P v + c_P\right) \leq z$$

$$\lambda_v \geq 0 \quad \forall v\in\mathcal{V}(\mathcal{P}) := \bigcup_{P\in\mathcal{P}} V(P), \quad \sum_{v\in\mathcal{V}(\mathcal{P})}\lambda_v = 1$$

$$\lambda_v \leq \sum_{\{P\in\mathcal{P}\,:\,v\in P\}} y_P \quad \forall v\in\mathcal{V}(\mathcal{P}), \quad \sum_{P\in\mathcal{P}} y_P = 1, \quad y_P\in\{0,1\} \quad \forall P\in\mathcal{P}$$

Logarithmic DCC (DLog)

$$\sum_{P\in\mathcal{P}}\sum_{v\in V(P)} \lambda_{P,v} v = x, \qquad \sum_{P\in\mathcal{P}}\sum_{v\in V(P)} \lambda_{P,v}\left(m_P v + c_P\right) \le z$$

$$\lambda_{P,v} \ge 0 \quad \forall P\in\mathcal{P},\ v\in V(P), \quad \sum_{P\in\mathcal{P}}\sum_{v\in V(P)} \lambda_{P,v} = 1$$

$$\sum_{P\in\mathcal{P}^+(B,l)}\sum_{v\in V(P)} \lambda_{P,v} \le y_l, \sum_{P\in\mathcal{P}^0(B,l)}\sum_{v\in V(P)} \lambda_{P,v} \le (1-y_l), y_l \in \{0,1\} \quad \forall l \in L(\mathcal{P})$$

where $B : \mathcal{P} \to \{0,1\}^{\lceil \log_2 |\mathcal{P}| \rceil}$ is any injective function, $L(\mathcal{P}) := \{1,\ldots,\lceil \log_2 |\mathcal{P}| \rceil\}$,

$\mathcal{P}^+(B,l) := \{P \in \mathcal{P} : B(P)_l = 1\}$ and $\mathcal{P}^0(B,l) := \{P \in \mathcal{P} : B(P)_l = 0\}$.

- New? Direct from ideas in Ibaraki (1976), Vielma and Nemhauser (2008)

Logarithmic DCC (DLog)

$$\sum_{P\in\mathcal{P}}\sum_{v\in V(P)}\lambda_{P,v}v = x, \qquad \sum_{P\in\mathcal{P}}\sum_{v\in V(P)}\lambda_{P,v}\left(m_P v + c_P\right) \leq z$$

$$\lambda_{P,v} \geq 0 \quad \forall P\in\mathcal{P},\, v\in V(P), \quad \sum_{P\in\mathcal{P}}\sum_{v\in V(P)}\lambda_{P,v} = 1$$

$$\sum_{P\in\mathcal{P}^+(B,l)}\sum_{v\in V(P)}\lambda_{P,v} \leq y_l, \sum_{P\in\mathcal{P}^0(B,l)}\sum_{v\in V(P)}\lambda_{P,v} \leq (1-y_l), y_l\in\{0,1\} \quad \forall l\in L(\mathcal{P})$$

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- New? Direct from ideas in Ibaraki (1976), Vielma and Nemhauser (2008)

Logarithmic Conv. Comb. (Log)

$$\sum_{v\in\mathcal{V}(\mathcal{P})} \lambda_v v = x, \qquad \sum_{v\in\mathcal{V}(\mathcal{P})} \lambda_v \left(m_P v + c_P\right) \le z$$

$$\lambda_v \ge 0 \quad \forall v \in \mathcal{V}(\mathcal{P}), \qquad \sum_{v\in\mathcal{V}(\mathcal{P})} \lambda_v = 1$$

$$\sum_{v\in L_s} \lambda_v \le y_s, \qquad \sum_{v\in R_s} \lambda_v \le (1 - y_s), \qquad y_s \in \{0,1\} \quad \forall s \in \mathcal{S}.$$

- Requires Independent Branching Scheme.
- Vielma and Nemhauser (2008).

Logarithmic Conv. Comb. (Log)

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- Requires Independent Branching Scheme.
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Multiple Choice (MC)

$$\sum_{P \in \mathcal{P}} x^P = x, \qquad \sum_{P \in \mathcal{P}} \left(m_P x^P + c_P y_P \right) \leq z$$

$$A_P x^P \leq y_P b_P \quad \forall P \in \mathcal{P}$$

$$\sum_{P \in \mathcal{P}} y_P = 1, \qquad y_P \in \{0, 1\} \quad \forall P \in \mathcal{P},$$

where $A_P x \leq b_P$ is the set of linear inequalities describing P.

- Balakrishnan and Graves (1989), Croxton et al. (2003a), Jeroslow and Lowe (1984) and Lowe (1984)

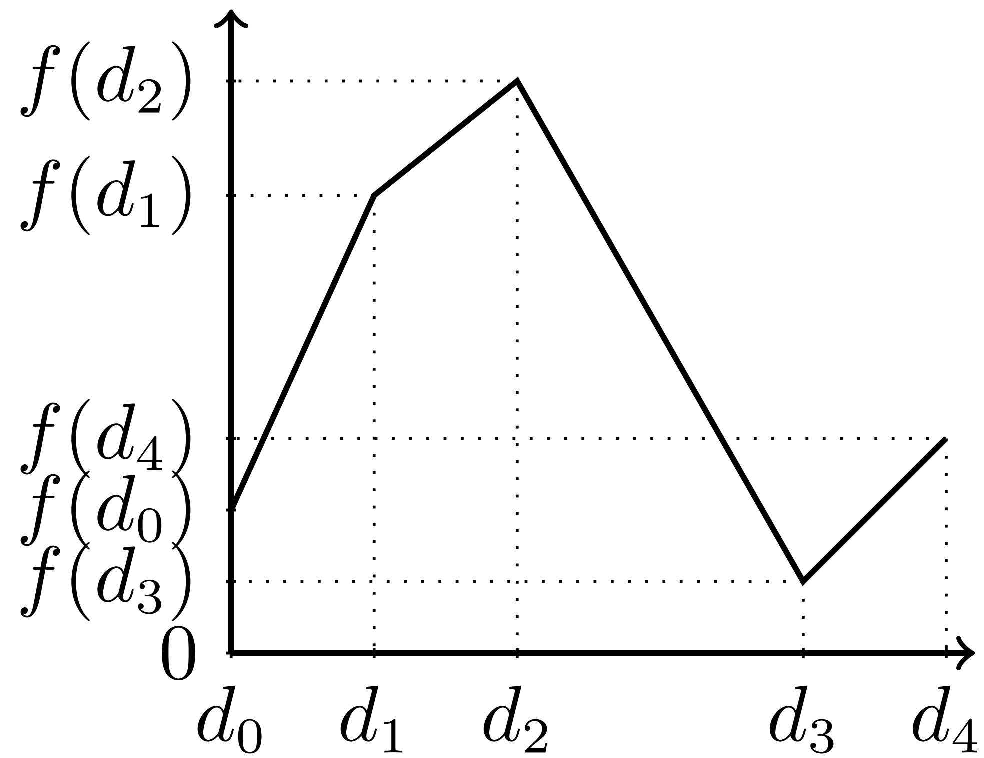
Incremental or Delta (Inc)

$$d_0 + \sum_{k=1}^{K} \delta_k \left(d_k - d_{k-1}\right) = x$$

$$f(d_0) + \sum_{k=1}^{K} \delta_k \left(f(d_k) - f(d_{k-1})\right) \leq z$$

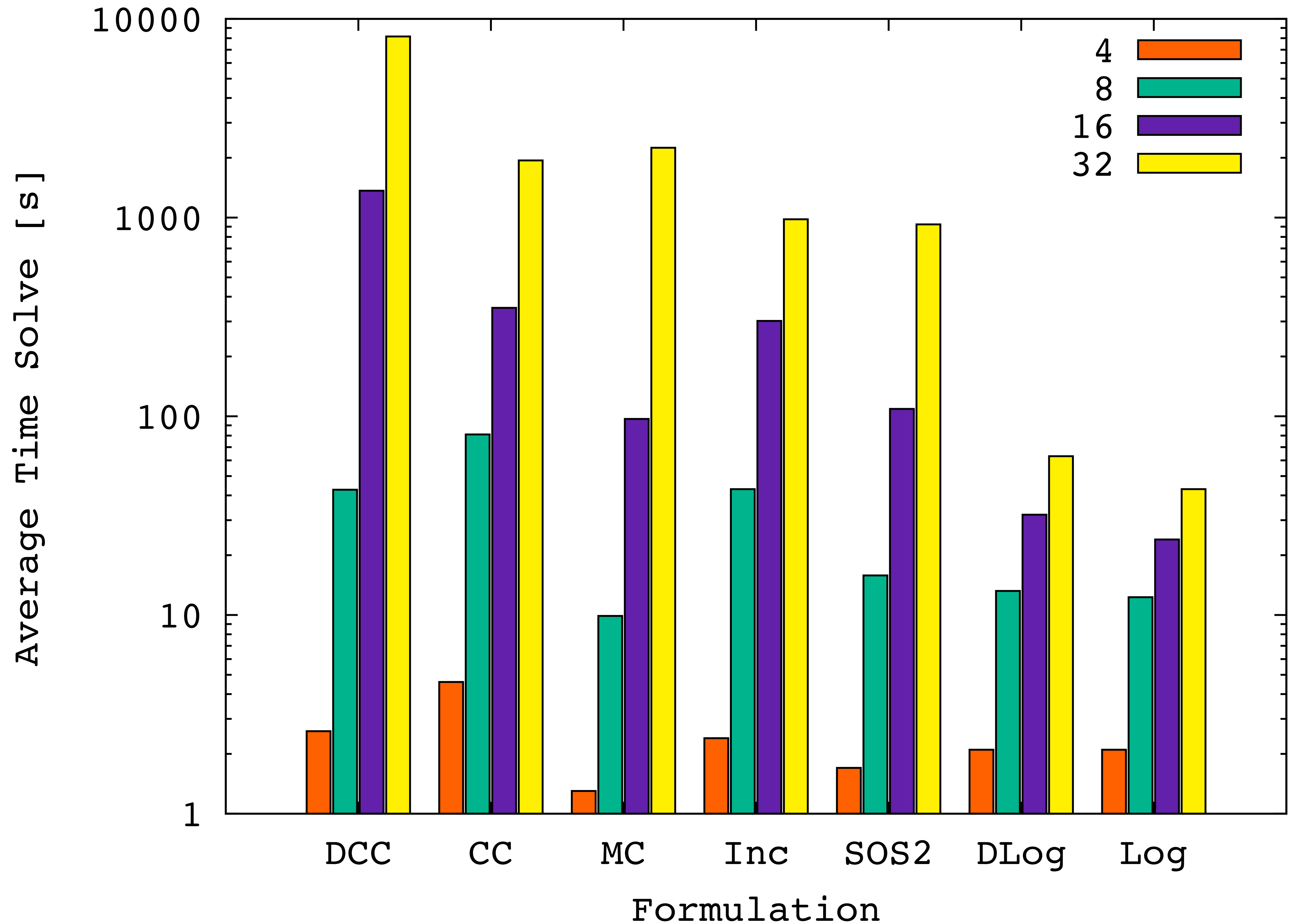
$$\delta_1 \leq 1, \quad \delta_K \geq 0, \quad \delta_{k+1} \leq y_k \leq \delta_k,$$

$$y_k \in \{0,1\} \quad \forall k \in \{1,\ldots,K-1\}.$$

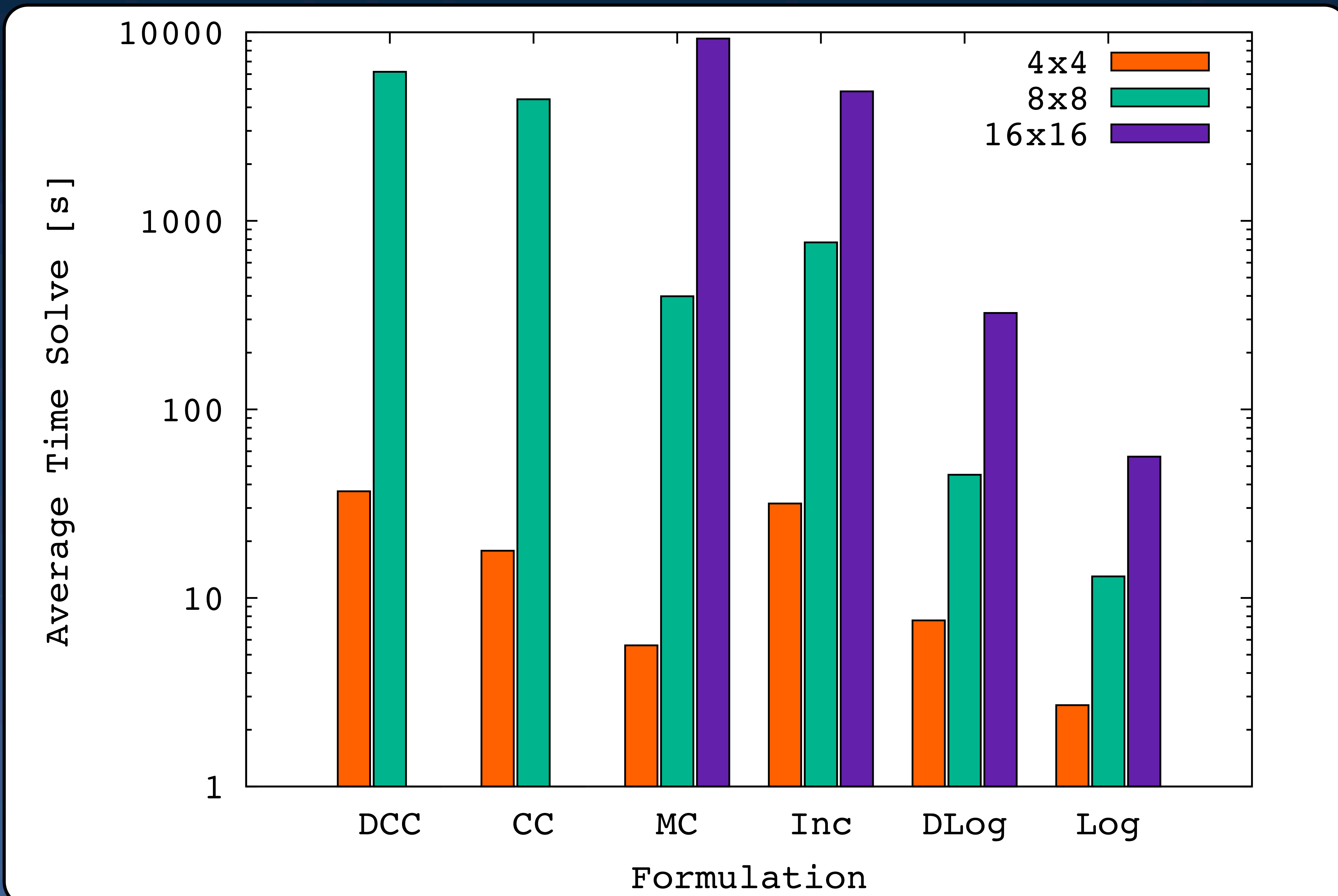


- Similar for multivariate functions.
- Croxton et al. (2003a), Dantzig (1963, 1960), Keha et al. (2004), Markowitz and Manne (1957), Padberg (2000), Sherali (2001), Vajda (1964) and Wilson (1998).

Univariate Case (Separable)



Multivariate Case (Non-Separable)



Multivariate Lower Semicontinuous