Mixed-Integer Convex Representability

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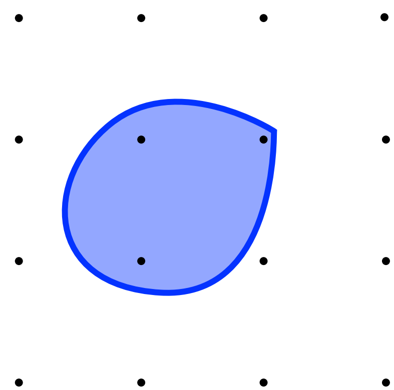
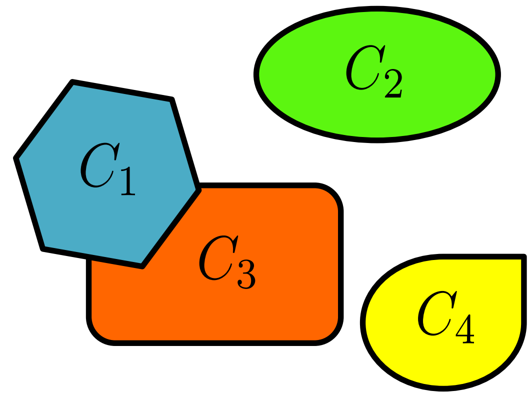
Mixed Integer Convex Optimization (MICP)

$$\begin{aligned} \min \quad & f(x,z) \\ s.t. \quad & (x,z) \in C \\ & z \in \mathbb{Z}^d \end{aligned}$$



Pure-integer



Mixed-integer

f and C = closed and convex

- Subclasses: MIQCQP, MISOCP, MISDP, ...
- Solvers: CPLEX, Gurobi, Xpress, Bonmin, Pajarito, FilMINT, Knitro, Mosek, ...

MICP Formulations and Representability

- A set $S \subseteq \mathbb{R}^n$ is MICP representable (MICPR) if it has an MICP formulation:
  - A closed convex set $M \subseteq \mathbb{R}^{n+p+d}$
  - auxiliary continuous variables $y \in \mathbb{R}^p$
  - auxiliary integer variables $z \in \mathbb{Z}^d$

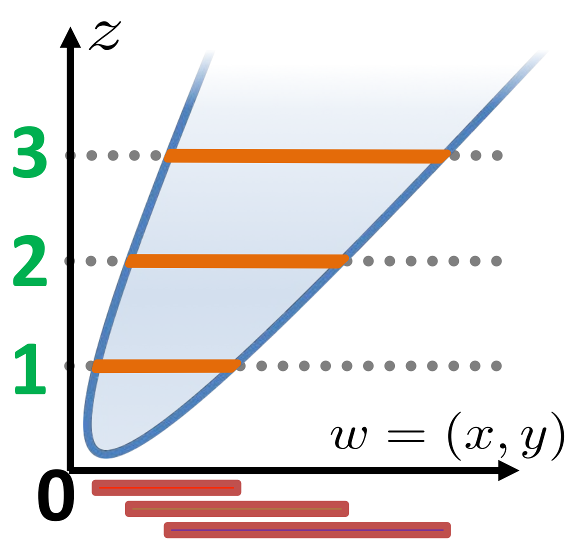
$$x \in S \quad \Leftrightarrow \quad \begin{array}{c} \exists (y, z) \in \mathbb{R}^p \times \mathbb{Z}^d \text{ s.t.} \\ (x, y, z) \in M \end{array}$$

or equivalently

$$S = \operatorname{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

MICPR = Convex Sets Indexed by Integers in Convex

$$S = \mathrm{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$



$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \mathrm{proj}_x \left(B_z \right)$$

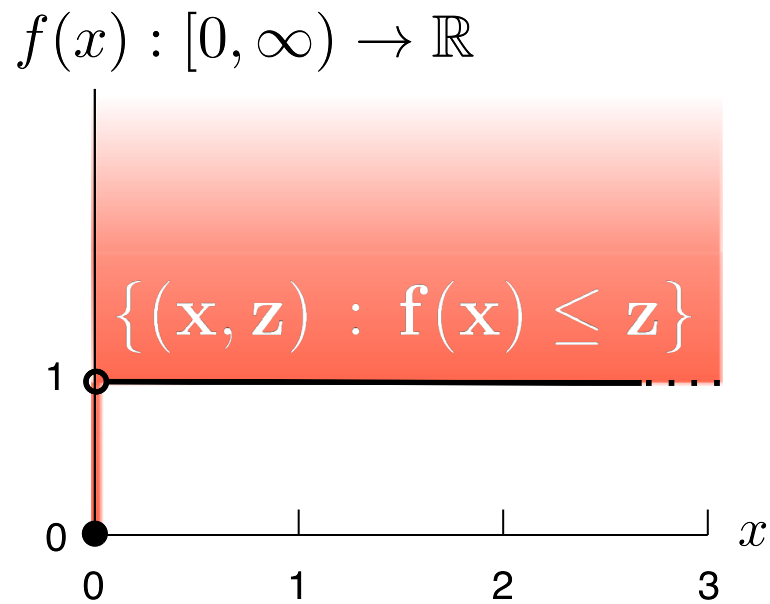
$$I = \mathrm{proj}_z \left(M \right) \text{ convex}$$

Structured ***Countably Infinite*** Union of Convex Sets

Known Results for 0-1 Integer Variables

$$S = \bigcup_{i=1}^{k} P_i$$

- M = Rational Polyhedron (⇔) :
  - P_i = rational polyhedra with same recession cone (Jeroslow and Lowe '84)



✗ $f(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}$

- M = Closed Convex (⇐) :
  - P_i = closed convex sets with same recession cone (e.g. Ceria & Soares '99)

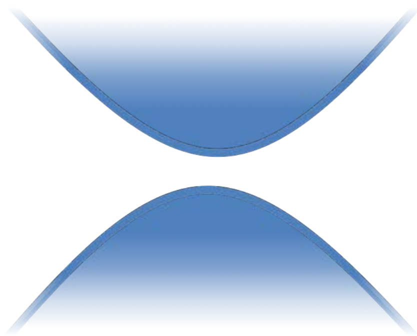
Known Results for General Integer Variables (⇔)

$$S = \bigcup_{i=1}^{k} P_i + \left\{ \sum_{i=1}^{t} \lambda_i r^i \ : \ \lambda \in \mathbb{Z}_+^t \right\}$$

- M = Rational Polyhedron :
  - P_i = rational polytopes (Jeroslow & Lowe '84)
- $M = \{x \in \mathbb{Z}^2 \ : \ x_1 \cdot x_2 \geq \alpha\}$:
  - P_i = points (Dey & Moran '13)
- M = Rational Polyhedron ∩ "Rational" Ellipsoidal Cylinder :
  - P_i = Rational Ellipsoid ∩ Polytope (Del Pia & Poskin '16)
- M = Compact Convex + Rational Polyhedron Cone :
  - P_i = Compact Convex (Lubin, Zadik & V. '17)

What Sets are MICP Representable (MICPR) ?

Two sheet hyperbola?



$$\{x \in \mathbb{R}^2 \,:\, 1 + x_1^2 \leq x_2^2\}$$

Spherical shell?

$$\{x \in \mathbb{R}^2 \,:\, 1 \leq \|x\| \leq 2\}$$

- Integer points in parabola $\{(x, x^2) : x \in \mathbb{Z}\}$?
- The set of $n \times n$ matrices with rank $\leq k$?
- Set of prime numbers?

0-1 (Binary) MICPR Characterization

- $S \subseteq \mathbb{R}^n$ is 0-1 MICPR $\Leftrightarrow \exists$ closed convex sets $T_1, \ldots, T_k \subseteq \mathbb{R}^{n+p}$ such that
  - $S = \bigcup_{i=1}^{k} \operatorname{proj}_x(T_i)$

- An (ideal) formulation of $\boldsymbol{x} \in S \subseteq \mathbb{R}^n$:
  - $(\boldsymbol{x}^i, \boldsymbol{y}^i, z_i) \in \overline{\operatorname{cone}}(T_i \times \{1\}) \qquad \forall i \in \{1, \ldots, k\}$

    $\left\|\boldsymbol{x}^i\right\|_2^2 \leq z_i t_i. \qquad \forall i \in \{1, \ldots, k\}$

    $\sum_{i=1}^{k} \boldsymbol{x}^i = \boldsymbol{x},$

    $\sum_{i=1}^{k} z_i = 1, \qquad \boldsymbol{z} \in \{0,1\}^k$

    $t \in \mathbb{R}_+^k, \ \boldsymbol{x}^i \in \mathbb{R}^n, \ \boldsymbol{y}^i \in \mathbb{R}^p \qquad \forall i \in \{1, \ldots, k\}$

A Simple Lemma for non-MICP Representability

- Obstruction for MICP representability of S :

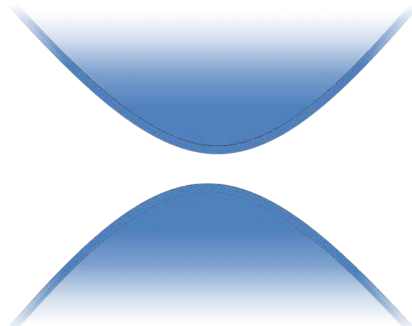
$$\text{infinite } R \subseteq S \quad \text{s.t.}$$

$$\frac{u+v}{2} \notin S \quad \forall u, v \in R,\ u \neq v$$

✗ **Spherical shell** $\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$

What Sets are MICP Representable (MICPR) ?

Two sheet hyperbola?

✓ 

$$\{x \in \mathbb{R}^2 \ : \ 1 + x_1^2 \leq x_2^2\}$$

Spherical shell?

✗ 

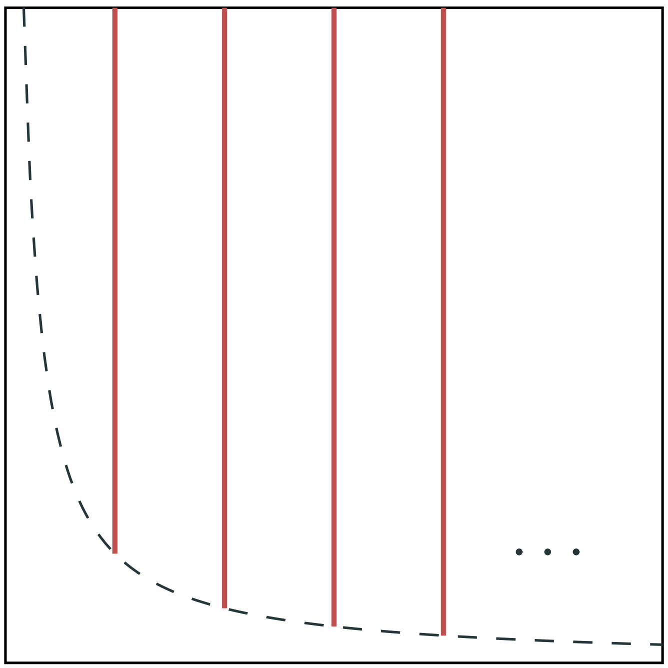
$$\{x \in \mathbb{R}^2 \ : \ 1 \leq \|x\| \leq 2\}$$

- ✗ Integer points in parabola $\{(x, x^2) : x \in \mathbb{Z}\}$?
- ✗ The set of $n \times n$ matrices with rank $\leq k$?
- ✗ Set of prime numbers?

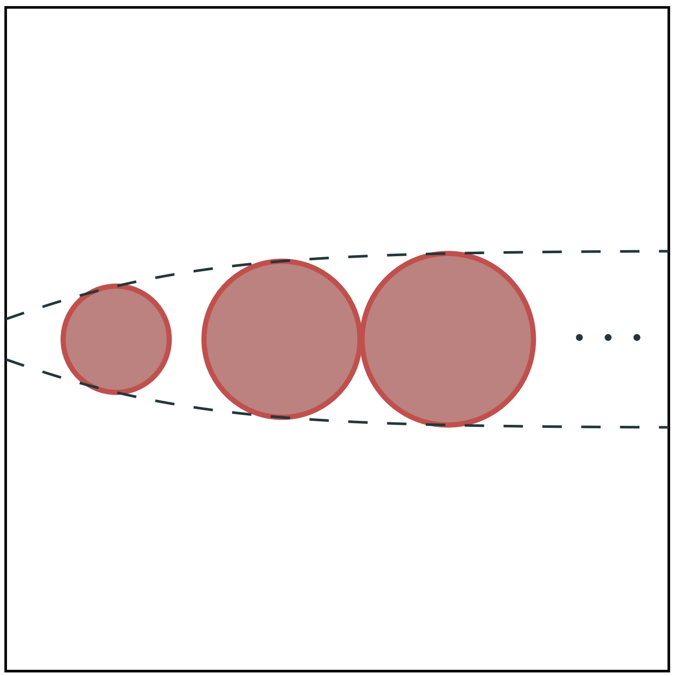
Does have non-convex polynomial MIP formulation

Structured ***Countably Infinite*** Unions of Convex

- How strange can they be?

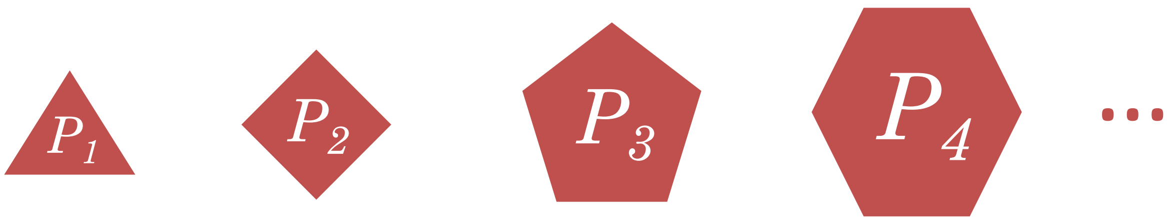


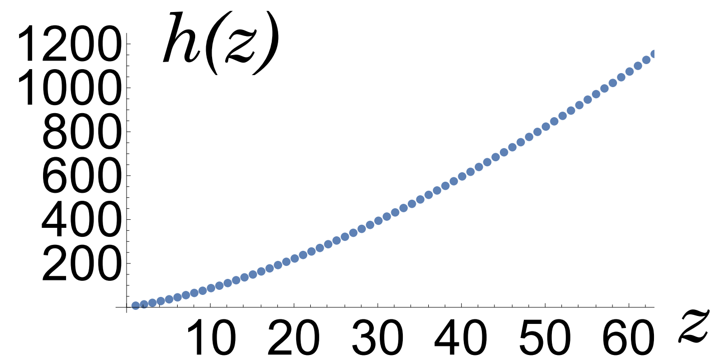
$$x_2 \geq 1/x_1 \geq 0, \quad x_1 \in \mathbb{Z}$$



$$\sqrt{(x_1 - 2z)^2 + x_2^2} \leq 1 - 1/z, \quad z \geq 1, \quad z \in \mathbb{Z}$$

Strange MICPR Set: Infinite Shapes





- There exist an increasing function h such that:
  - $P_z \subseteq \mathbb{R}^2$ regular $h(z)$-gon centered at $(z, 0)$
  - $P_z \cap P_{z'} = \emptyset, \quad z \neq z'$
  - $S = \bigcup_{z=1}^{\infty} P_z$ is MICPR
- Equal volume ⇒ Finite # of Shapes

Even Stranger MICPR Set s: Non-Periodic

- An **infinite set** S is **periodic** if and only if:

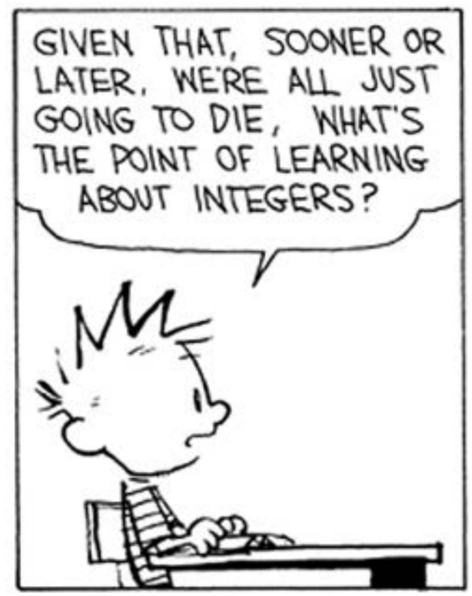
$$\exists r \in \mathbb{R}^n \quad \forall \lambda \in \mathbb{Z}_+,\ x \in S \quad x + \lambda r \in S$$

- Non-periodic MICPR sets
  - Dense discrete set

$$\left\{ \sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \,:\, x \in \mathbb{N} \right\} \subseteq [0,1]$$

  - Set of naturals

$$\left\{ x \in \mathbb{N} \,:\, \sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \notin \left(\varepsilon, 1 - \sqrt{2}\varepsilon \right) \right\}$$



"God made the integers, all else is the work of man”

- Leopold Kronecker

A Definition Rational MICPR (R-MICPR)

- Formulation for $\left\{\sqrt{2}x - \left\lfloor\sqrt{2}x\right\rfloor \,:\, x \in \mathbb{N}\right\} \subseteq [0,1]$:

$$\|(z_1, z_1)\|_2 \le z_2 + 1, \quad \|(z_2, z_2)\|_2 \le 2z_1, \quad x_1 = y_1 - z_2,$$
$$\|(z_1, z_1)\|_2 \le y_1, \quad \|(y_1, y_1)\|_2 \le 2z_1, \quad z \in \mathbb{Z}^2$$

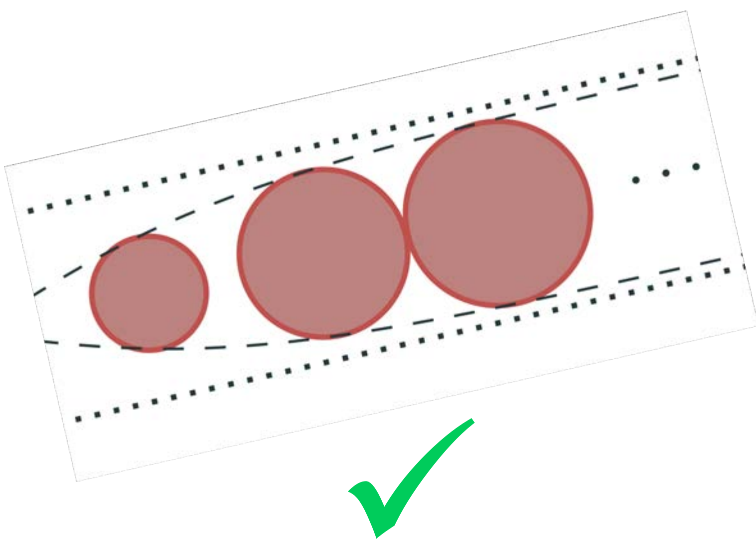
- Rational MICP Formulation :

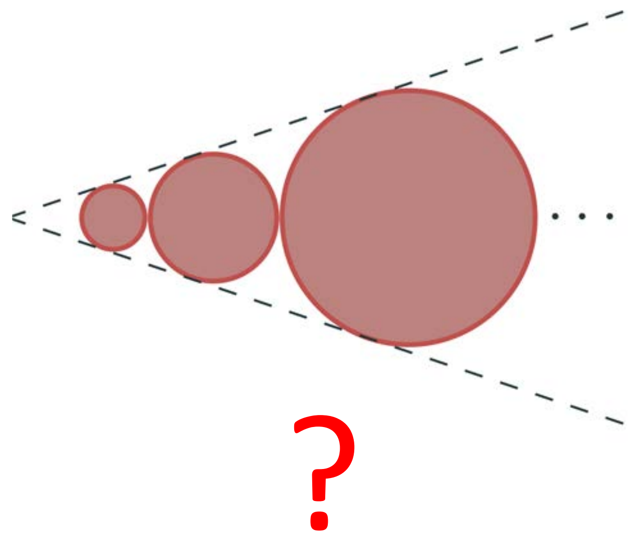
$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \operatorname{proj}_x(B_z) \qquad \begin{aligned} S &= \operatorname{proj}_x\left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d\right)\right) \\ I &= \operatorname{proj}_z(M) \end{aligned}$$

- Any rational affine mapping of index set I :
  – Is bounded, or
  – Has an integer (rational) recession direction

(Some) Rational MICPR Sets are Periodic

- Jeroslow & Lowe ’84 : **rational MILP** = **periodic**
- **Theorem:** A **rational MICPR set** S is a **finite union** of **periodic** sets if:
  - S **is closed** and the **maximal convex subsets of** S **are <u>uniformly bounded</u>**





  - S **is union of points** (S **not** necessarily **closed**)

Corollaries and Other Properties

- Both strange discrete sets are not R-MICPR.

$$\left\{\sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \,:\, x \in \mathbb{N}\right\} \subseteq [0,1]$$

$$\left\{x \in \mathbb{N} \,:\, \sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \notin \left(\varepsilon, 1 - \sqrt{2}\varepsilon\right)\right\}$$

- If S is **R-MICPR** and **compact**, then S is a **finite union** of **compact sets** (and hence 0-1 MICPR).
- If $S \subseteq \mathbb{N}$ is **R-MICPR**, then S is a **finite union of points** and a **MILP representable set**.
- MICPR is **closed under finite union, cartesian product and sum**, but **NOT closed under intersection**

Summary on General MICPR

- General MICPR:
  - Infinite union of convex w. $\neq$ recession cones
  - Can't be **too** non-convex (e.g. Primes not MICPR)
  - Non-polyhedrality crucial for $\neq$ recessions and closure under union (e.g. MI-SOCP formulation for unbounded SOS2 constraints on Chris' talk yesterday)
- MICPR sets can be very strange:
  - Infinite # of Shapes: controlled by equal volume
  - Non-periodic sets: controlled by rational unboundedness (R-MICPR)
- OBS: R-MICPR can fail by hidden rays (cf. affine map)

A Definition Rational MICPR (R-MICPR)

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \operatorname{proj}_x (B_z) \qquad S = \operatorname{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

$$I = \operatorname{proj}_z (M)$$

- Any rational affine mapping of index set I :
  - Is bounded, or
  - Has an integer (rational) recession direction
- Irrational directions can hide!
  - R-MICPR $\not\Leftrightarrow$ $\operatorname{span}(\operatorname{rec}(I))$ and/or $\operatorname{aff}(I)$ = rational space

$$\left(z_1 + \sqrt{2} z_2\right)^2 \leq z_3 \qquad \operatorname{span}(\operatorname{rec}(I)) = \operatorname{span}(\{\mathbf{e}_3\})$$

$$\left(z_2 - \sqrt{2} z_1\right)^2 \leq 1 \qquad \operatorname{rec}(\operatorname{proj}_{z_1, z_2}(I)) = \operatorname{span}(\{(1, \sqrt{2})\})$$

A Simple Lemma for non-MICP Representability

- Obstruction for MICP representability of S :

$$\text{infinite } R \subseteq S \quad \text{s.t.} \quad \frac{u+v}{2} \notin S \quad \forall u, v \in R,\ u \neq v$$

Proof: Assume for contradiction there exists M such that:

$$S = \operatorname{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

$$\begin{array}{l} (u, y_u, z_u) \in M \\ (v, y_v, z_v) \in M \end{array} \quad \Rightarrow \quad \frac{z_u + z_v}{2} \notin \mathbb{Z}^d$$

$$z_u \equiv z_v \pmod 2 \text{ component-wise} \quad \Rightarrow \quad \frac{z_u + z_v}{2} \in \mathbb{Z}^d$$

$$\text{component-wise parity classes} = 2^d < |R| = \infty \qquad \Rightarrow\!\Leftarrow$$