Recent Advances in Mixed Integer Programming Modeling and Computation

Juan Pablo Vielma

Massachusetts Institute of Technology

MIT Center for Transportation & Logistics.

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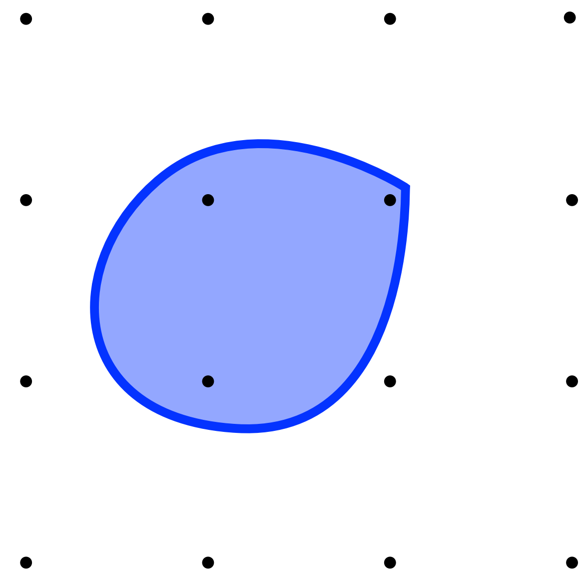
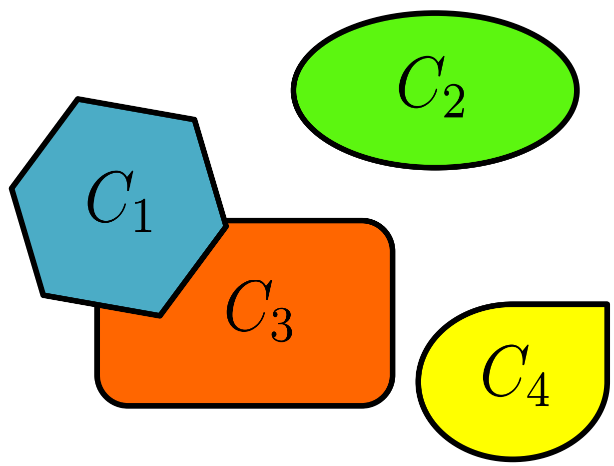
(Nonlinear) Mixed Integer Programming (MIP)

$$\min \quad f(x)$$
$$s.t.$$
$$x \in C$$
$$x_i \in \mathbb{Z} \quad i \in I$$



Mostly convex f and C.



NP-hard = Challenge Accepted!

50+ Years of MIP = Significant Solver Speedups

- Algorithmic Improvements (**Machine Independent**):
  - CPLEX → ILOG → IBM
    - v1.2 (1991) – v11 (2007): **29,000 x** speedup
  - GUROBI OPTIMIZATION
    - v1 (2009) – v6.5 (2015): **48.7 x** speedup

  ≈ **1.9 x / year**

- Also **convex nonlinear**:
  - GUROBI OPTIMIZATION
    - v6.0 (2014) – v6.5 (2015) quadratic: **4.43 x**
      (**V.**, Dunning, Huchette, Lubin, 2015)

Widespread Use of Linear/Quadratic MIP Solvers

http://www.gurobi.com/company/example-customers

State of MIP Solvers

- Mature: Linear and Quadratic (Conic Quadratic/SOCP)
  - Commercial:

GUROBI OPTIMIZATION   IBM CPLEX   optimized by Xpress MP FICO

  - “Open Source”

SCIP   COIN|OR CBC   GNU GLPK

- Emerging: Convex Nonlinear (e.g. SDP)
  - Open-Source + Commercial linear MIP Solver > Commercial

KNITRO < COIN|OR Bonmin <

Accessing MIP Solvers = Modelling Languages

- User-friendly algebraic modelling languages (AML):

GAMS
AMPL

PYOMO ↔ python™
CVX RESEARCH ↔ MATLAB®

Standalone and Fast | Based on General Language and Versatile

- Fast and Versatile, but complicated
  - Proprietary low-level C/C++ solver interphases.
  - C/C++ Coin-OR interphases and frameworks
- 21st Century AMLs: JuMP ↔ julia

Outline

- JuMP ↔ julia overview.
- Advanced MIP formulations.
- Convex nonlinear MIP solvers.
- Optimal Control with Julia, JuMP and Pajarito.
- Other applications if time permits.

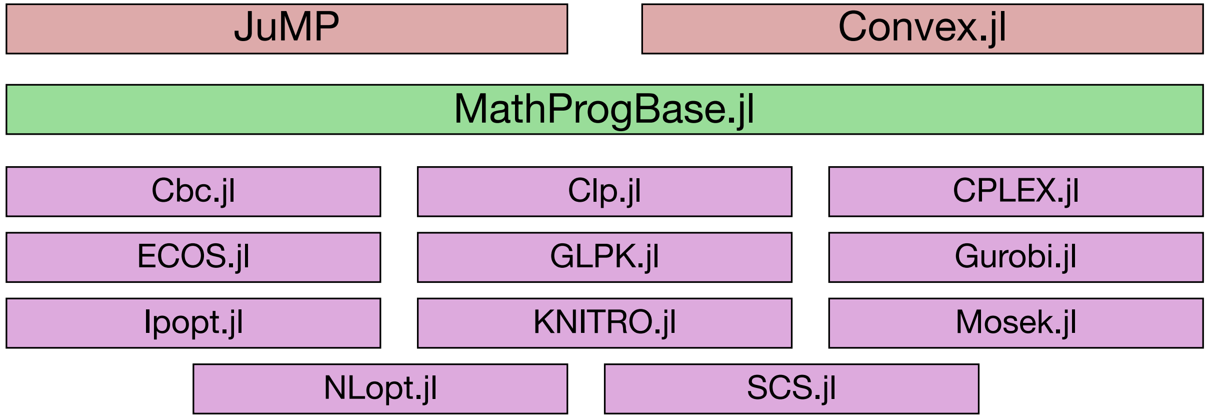
JuMP ⟷ julia

Why julia and JuMP ?

- julia http://julialang.org
  - 21^{st} century programming language
  - MIT licensed (and developed): free and open source
  - (Almost) as fast as C (LLVM JIT) and as easy as Matlab
- JuMP
  - Julia-based algebraic modelling language for optimization
  - Easy and natural syntax for linear, quadratic and conic (e.g. SDP) mixed-integer optimization.
  - Modular, extensible, easy to embed (e.g. simulation, visualization, etc.) and FAST.
  - Solver-independent access to advanced MIP features (e.g cutting plane callbacks)

Extensive Stack of Modelling and Solver Packages

JuliaOpt



Solvers

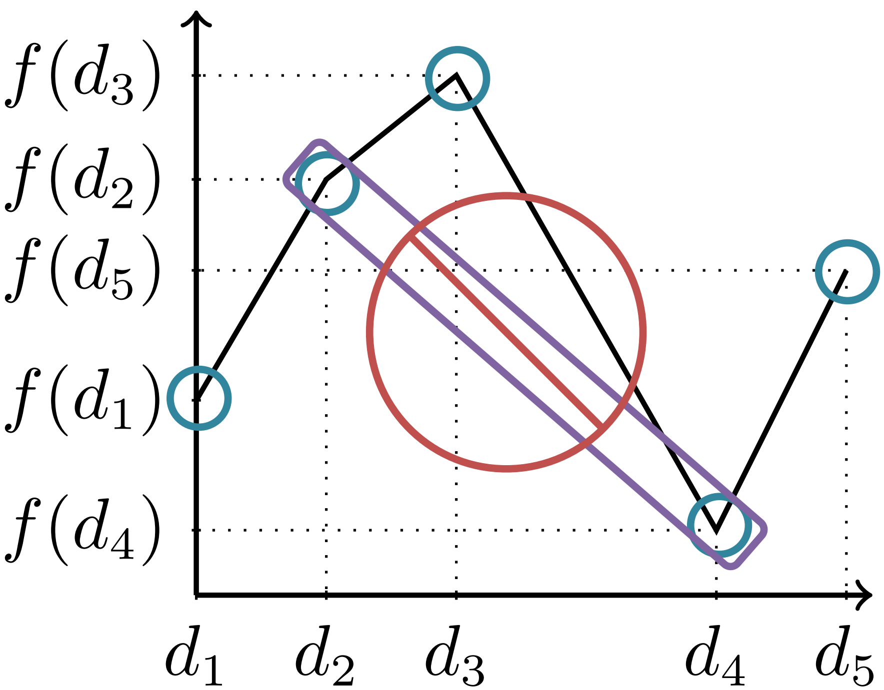
http://www.juliaopt.org

- JuMP extensions for: block stochastic optimization, robust optimization, chance constraints, piecewise linear optimization, polynomial optimization, multi-objective optimization, discrete time stochastic optimal control, sum of squares optimization, etc.
- Useful Julia Packages: Multivariate Polynomials, etc.

Advanced MIP Formulations

Simple Formulation for Univariate Functions

$$z = f(x)$$



$$\text{Size} = O(\#\text{ of segments})$$

Non-Ideal: Fractional Extreme Points

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^{5} \lambda_j, \quad \lambda_j \geq 0$$

$$y \in \{0,1\}^4, \quad \sum_{i=1}^{4} y_i = 1$$

$$0 \leq \lambda_1 \leq y_1$$

$$0 \leq \lambda_2 \leq y_1 + y_2$$

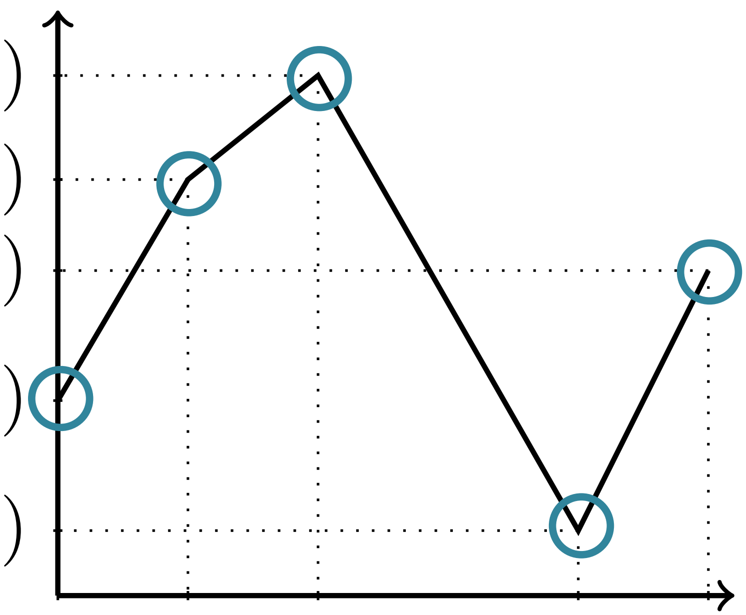
$$0 \leq \lambda_3 \leq y_2 + y_3$$

$$0 \leq \lambda_4 \leq y_3 + y_4$$

$$0 \leq \lambda_5 \leq y_4$$

Advanced Formulation for Univariate Functions

$$z = f(x)$$



$$\text{Size} = O\left(\log_2 \# \text{ of segments}\right)$$

Ideal: Integral Extreme Points

- V. and Nemhauser 2011.

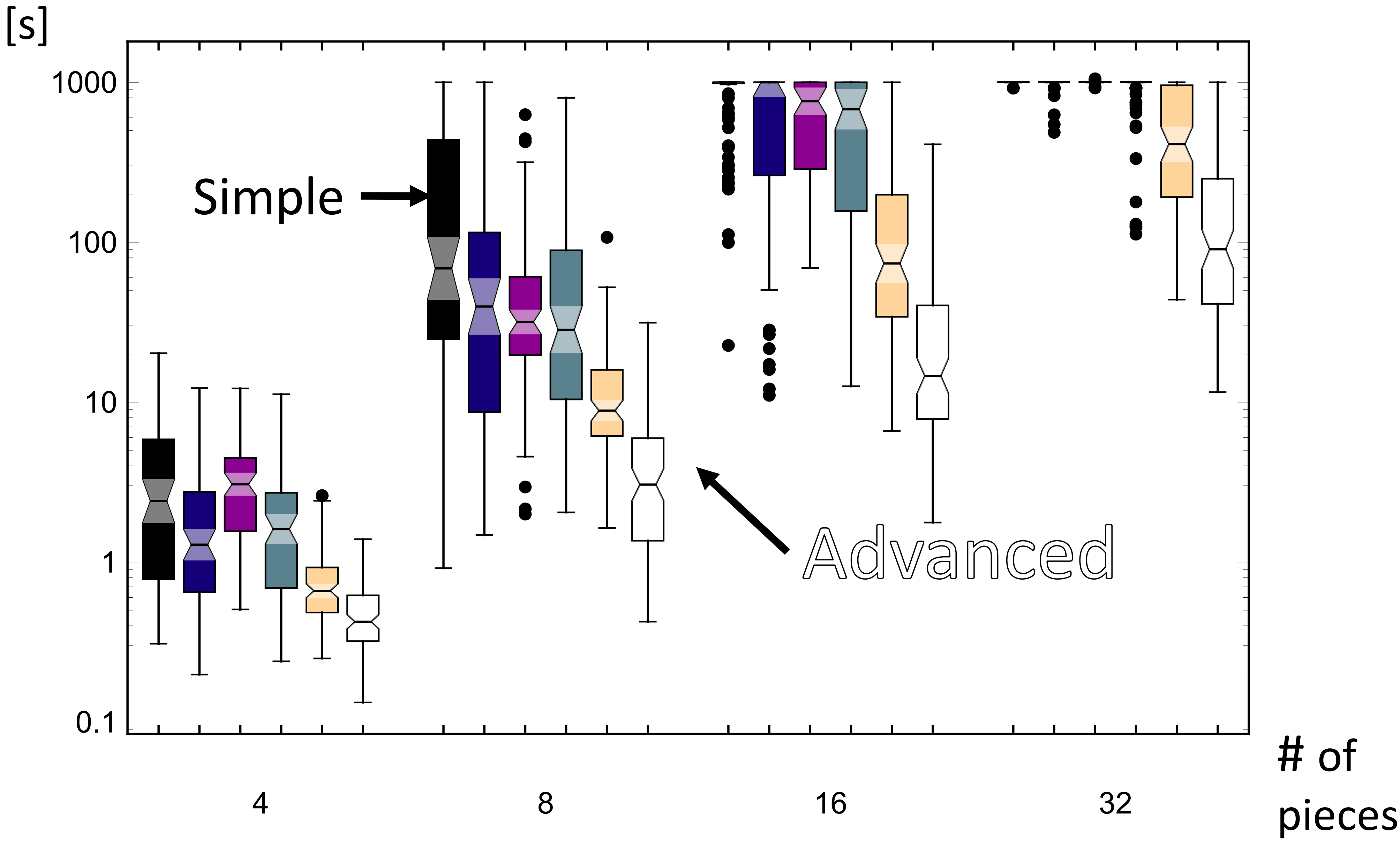
$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^{5} \lambda_j, \quad \lambda_j \geq 0$$

$$y \in \{0,1\}^2$$

$$\begin{aligned} 0 \leq \lambda_1 + \lambda_5 &\leq 1 - y_1 \\ 0 \leq \lambda_3 \quad\quad &\leq y_1 \\ 0 \leq \lambda_4 + \lambda_5 &\leq 1 - y_2 \\ 0 \leq \lambda_1 + \lambda_2 &\leq y_2 \end{aligned}$$

Formulation Improvements can be Significant

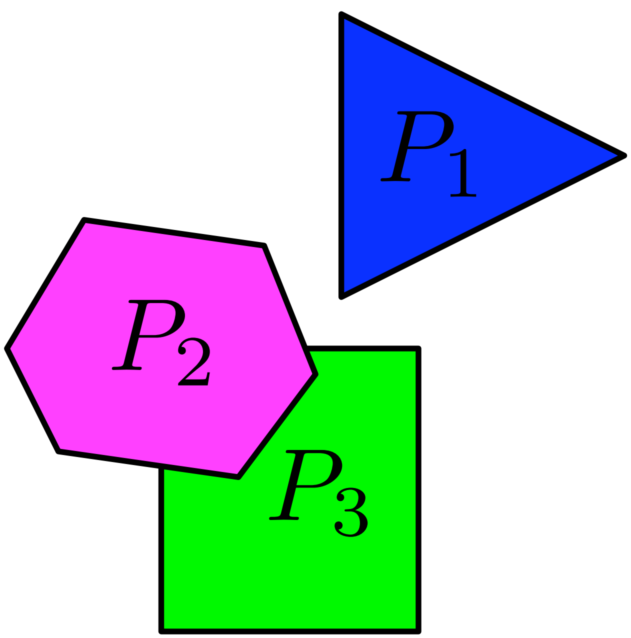


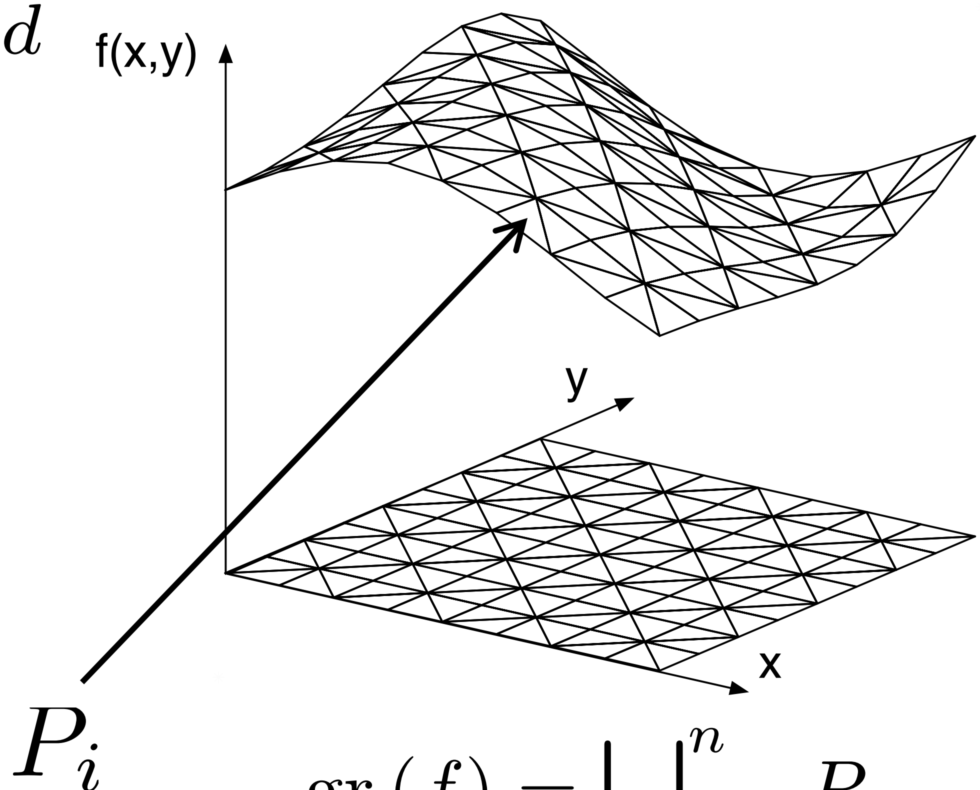
Transportation Problems with Concave Costs using GUROBI

More Advanced Small/Strong Formulation

- Modeling Finite Alternatives = Unions of Polyhedra

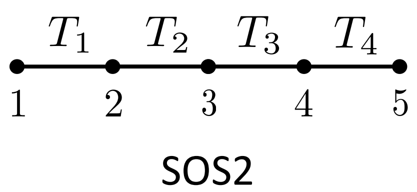
$$x \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^d$$

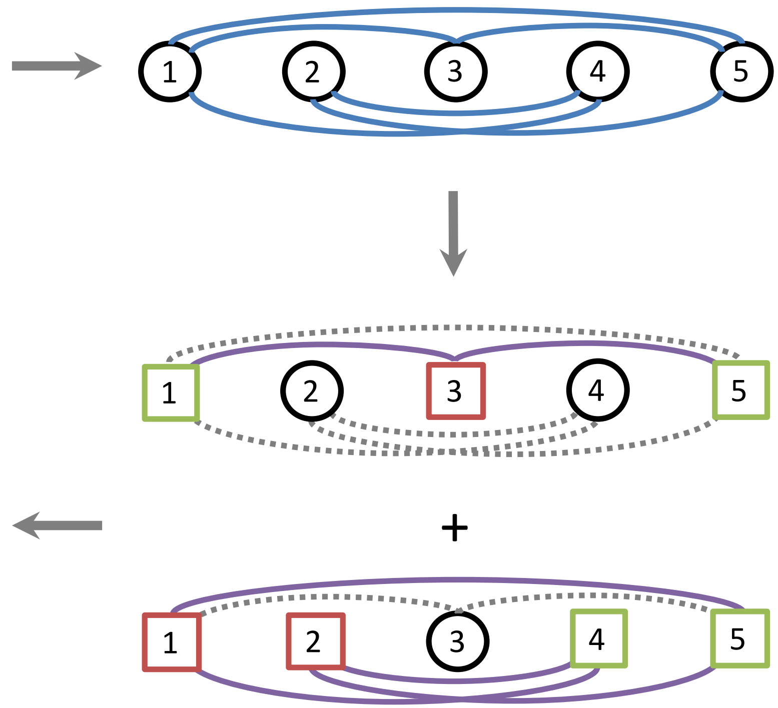


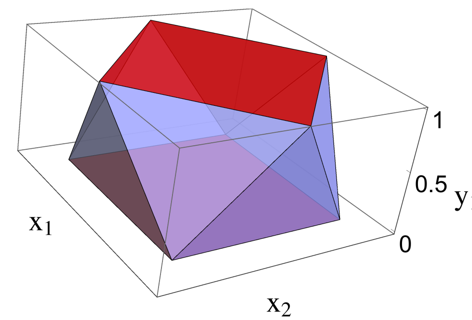


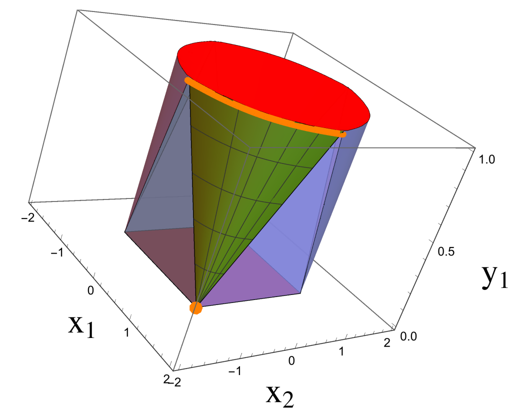
$$\operatorname{gr}(f) = \bigcup_{i=1}^{n} P_i$$

Many Techniques Based on Geometry/Graphs









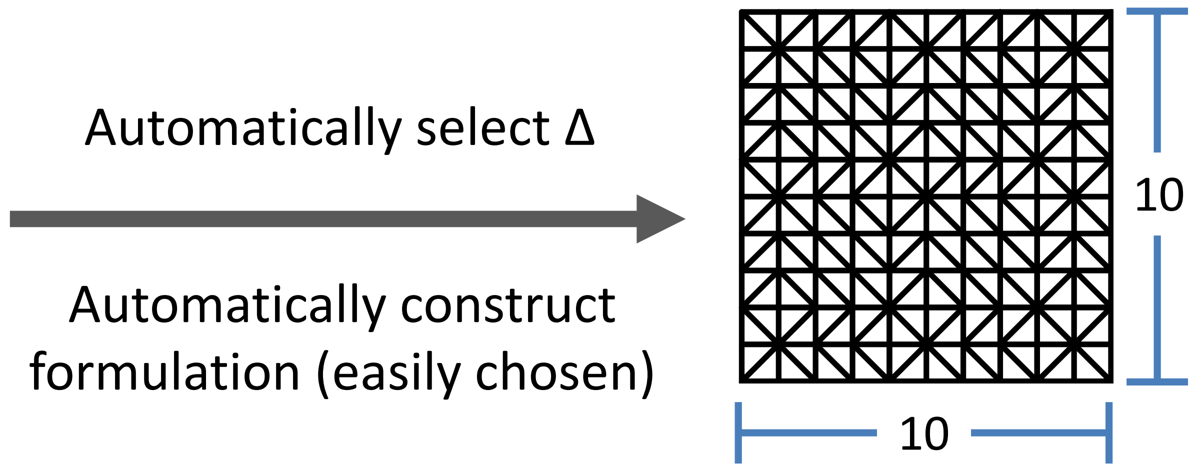
$$
\begin{aligned}
0 \le \lambda_1 + \lambda_5 &\le 1 - y_1 \\
0 \le \lambda_3 \quad\quad &\le y_1 \\
0 \le \lambda_4 + \lambda_5 &\le 1 - y_2 \\
0 \le \lambda_1 + \lambda_2 &\le y_2
\end{aligned}
$$

- Somewhat complicated, but worth it!
- Also nonlinear MIP formulations.
- V. ’15; Huchette, Dey and V. ’16, Huchette and V. ‘16; Huchette and V. ‘17; V. ‘17a and V. ’17b.

Some Easily Accessible Through JuMP Extensions

- PiecewiseLinearOpt.jl (Huchette and V. 2017)

$$\begin{aligned} \min \quad & \exp(x+y) \\ s.t. \quad & \\ & x, y \in [0,1] \end{aligned}$$




```julia
using JuMP, PiecewiseLinearOpt
m = Model()
@variable(m, x)
@variable(m, y)

z = piecewiselinear(m, x, y, 0:0.1:1, 0:0.1:1, (u,v) -> exp(u+v))
@objective(m, Min, z)
```

Convex Nonlinear MIP Solvers

Nonlinear MIP B&B Algorithms

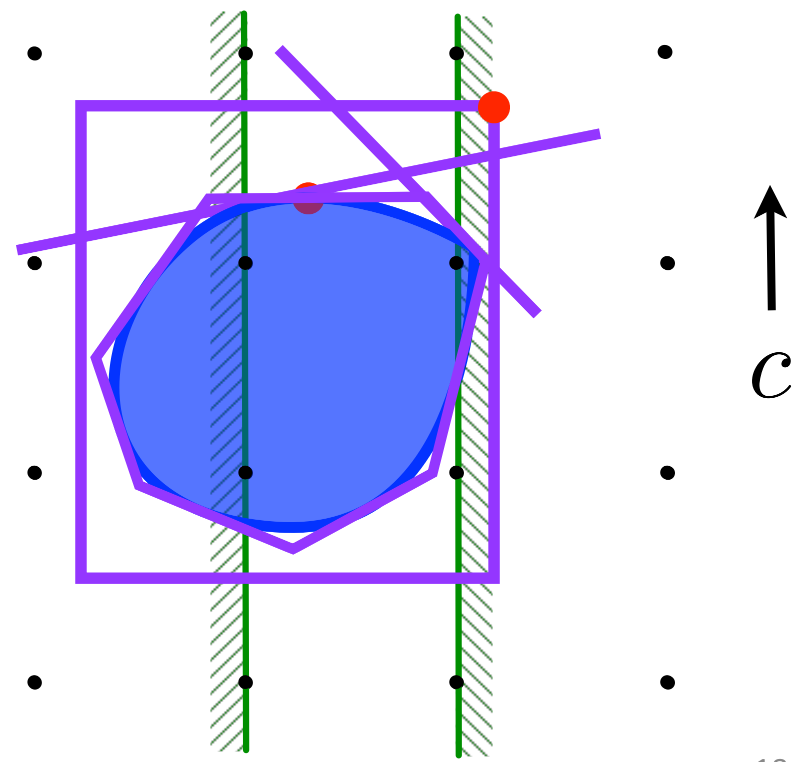
- NLP (QCP) Based B&B
- (Dynamic) LP Based B&B
  - Few cuts = high speed.
  - Possible slow convergence.
- Lifted LP B&B
  - Extended or Lifted relaxation.
  - Static relaxation
    - Mimic NLP B&B.
  - Dynamic relaxation
    - Standard LP B&B

$$\max \quad \sum_{i=1}^{n} c_i x_i$$

$$s.t. \quad Ax + Dz \leq b,$$

$$g_i(x) \leq 0,\ i \in I, \quad x \in \mathbb{Z}^n$$

$$x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$$



Second Order Conic or Conic Quadratic Problems

- Problems using Euclidean norm:
  - e.g. Portfolio Optimization Problems

$$\max \quad \bar{a}x$$

$$s.t.$$

$$\left\| Q^{1/2} x \right\|_2 \leq \sigma$$

$$\sum_{j=1}^{n} x_j = 1, \quad x \in \mathbb{R}_+^n$$

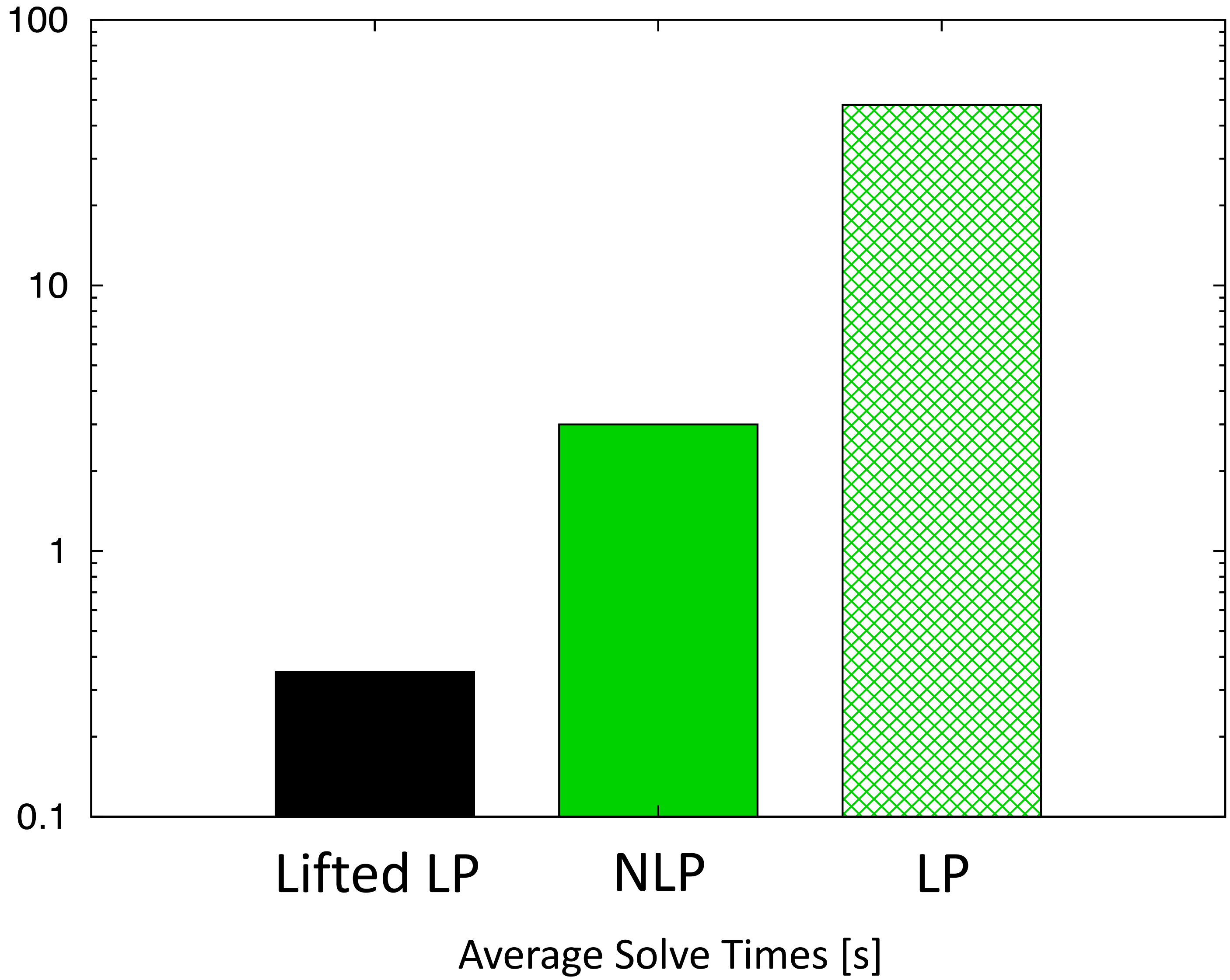
$$x_j \leq z_j \quad \forall j \in [n]$$

$$\sum_{j=1}^{n} z_j \leq K, \quad z \in \{0,1\}^n$$

- $\bar{a}$ expected returns.
- $Q^{1/2}$ square root of covariance matrix.
- K maximum number of assets.
- σ maximum risk.

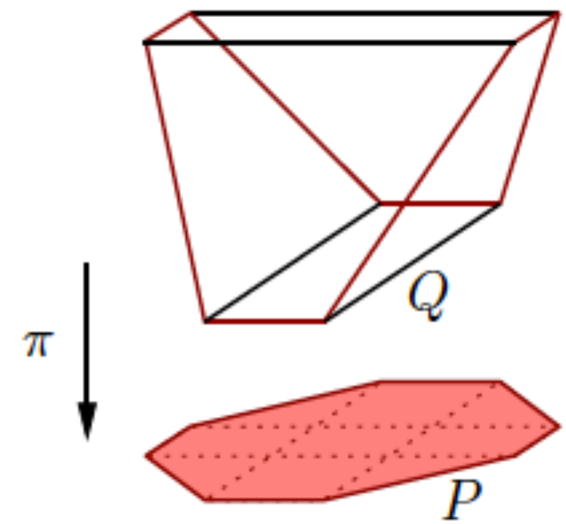
LP v/s NLP B&B for CPLEX v11 for n = 20 and 30

- Results from V., Ahmed and Nemhauser 2008.



Lifted or Extended Approximations

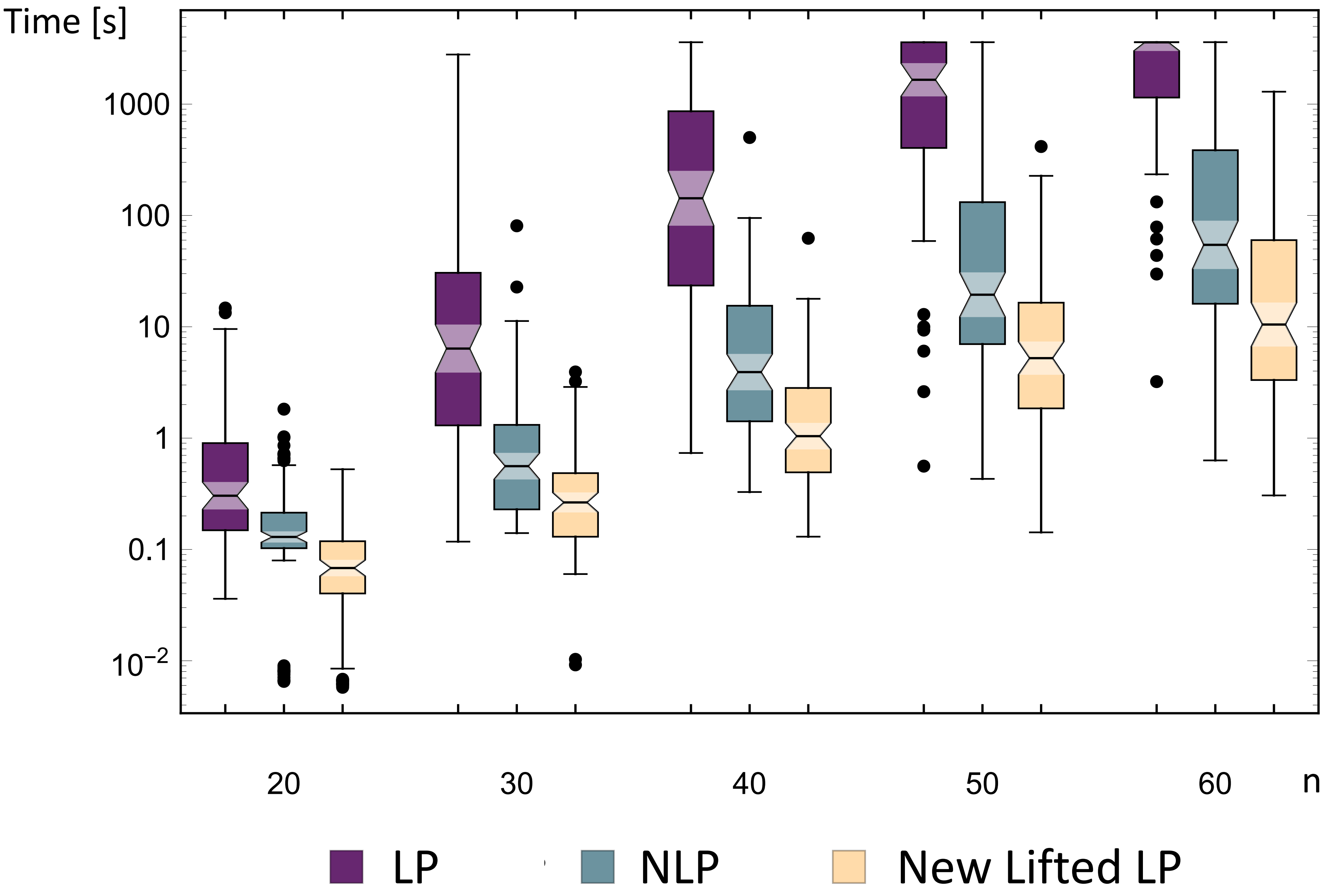
- Projection = multiply constraints.
- V., A. and N. 2008:
  - Extremely accurate, but static and complex approximation by Ben-Tal and Nemirovski
- V., Dunning, Huchette and Lubin 2016: Simple, dynamic and good approximation:



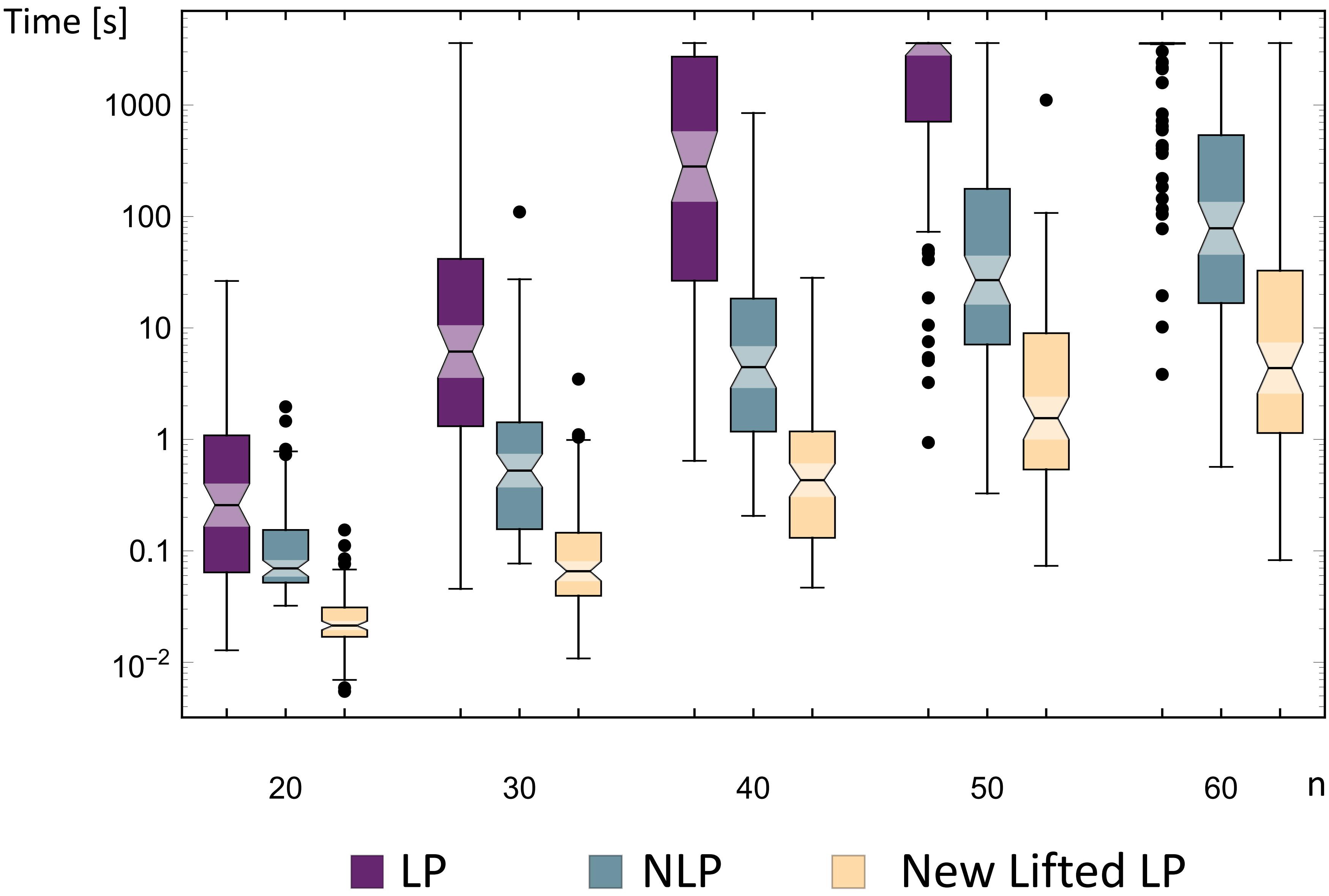
$$\|y\|_2 \leq y_0 \quad \longrightarrow \quad \begin{array}{l} y_i^2 \leq z_i \cdot y_0 \quad \forall i \in [n] \\ \sum_{i=1}^{n} z_i \leq y_0 \end{array}$$

Image from Lipton and Regan, https://rjlipton.wordpress.com

CPLEX v12.6 for n = 20, 30, 40, 50 and 60



Gurobi v5.6.3 for n = 20, 30, 40, 50 and 60



All Major Solvers Now Implement Lifted LP

- First Talks: julia / JuMP
  - SIAM Optimization (SIOPT), May 2014 ≈ two weeks coding.
  - IBM Thomas J. Watson Research Center, December 2014.
- Paper in arxive, May 28, 2015.

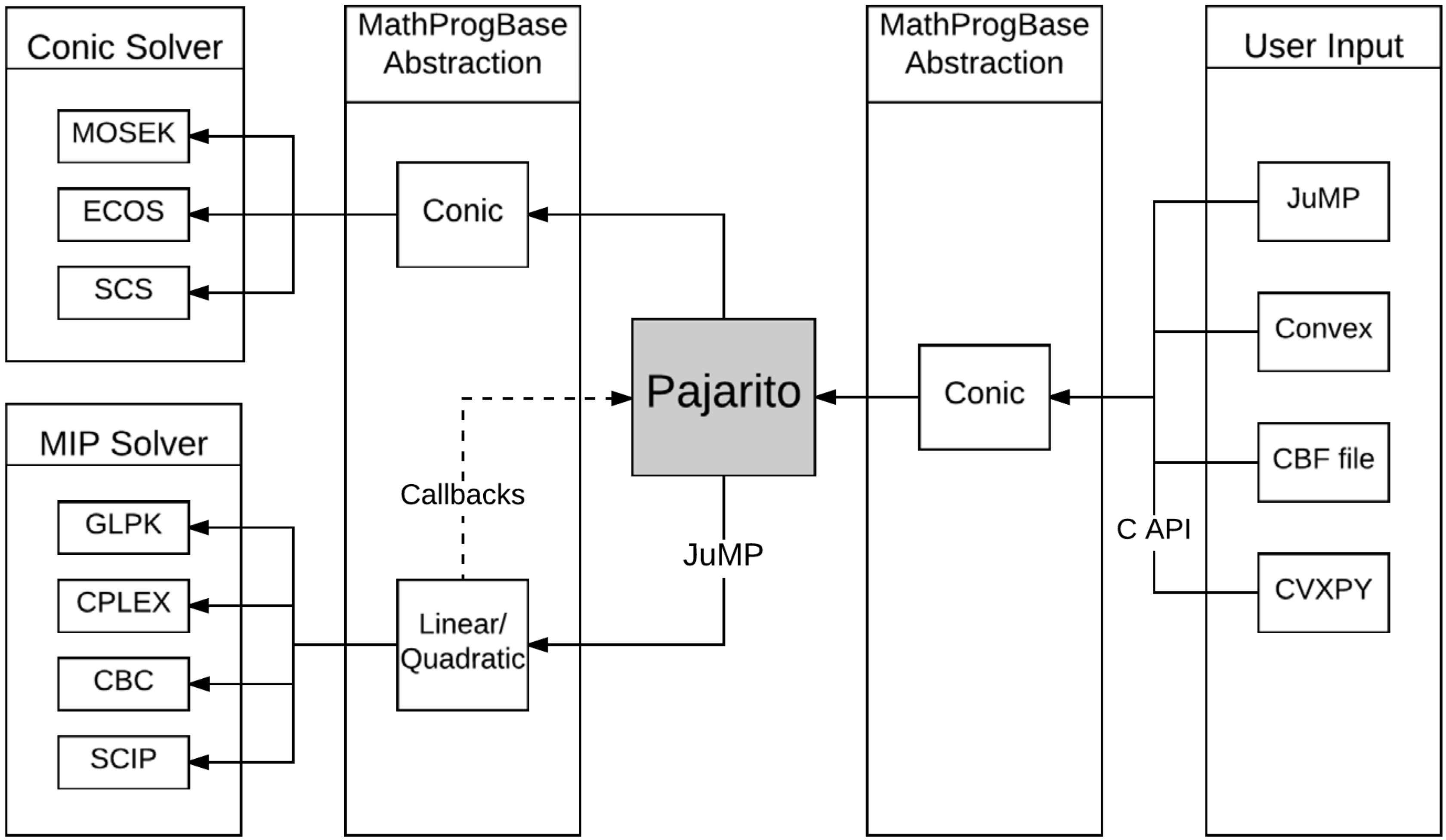
  Two weeks!
- IBM CPLEX v12.6.2, June 12, 2015.
- GUROBI OPTIMIZATION v6.5, October 2015.
- optimized by Xpress MP FICO v8.0, May 2016.
- SCIP v4.0, March 2017.

However... We Can Sill Beat CPLEX!

- julia / JuMP-based solver Pajarito
  - Lubin, Yamangil, Bent and V. '16 and Coey, Lubin and V. '17.

solver	termination status counts				time(s)
	conv	wrong	not conv	limit	
SCIP	78	1	0	41	43.36
CPLEX	96	3	5	16	14.30
Paj-iter	96	1	0	23	38.70
Paj-MSD	101	0	0	19	18.12

Flexible Architecture Thanks to Julia-Opt Stack

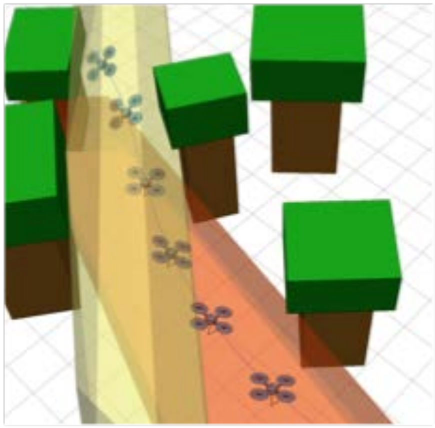


- Fastest Open Source MISOCP Solver!
- Pajarito can also solve MISDPs and MI-“EXP”

Optimal Control with Julia, JuMP and Pajarito

Joey Huchette ≈ two weeks for SIOPT '17

Trajectory Planning with Collision Avoidance



- Motivating: Steering a quadcopter through obstacles [Deits/Tedrake:2015]
- Position described by polynomials:
  - $(p^x(t), p^y(t))_{t \in [0,1]}$
  - avoid obstacles
  - initial/terminal conditions
  - minimize "jerk" of path
- Solution approach:
  - split domain into "safe" polyhedrons + discretize time into intervals
  - "smooth" piecewise polynomial trajectories in each interval, which chose polyhedron

variables = polynomials

↓

Mixed-Integer Polynomial Programming

Disjunctive *Polynomial* Optimization Formulation

Variables = Polynomials : $\{p_i : [T_i, T_{i+1}] \to \mathbb{R}^2\}_{i=1}^N$

$$\min_{p} \quad \sum_{i=1}^{N} ||p_i'''(t)||^2$$

$$\text{s.t.} \quad p_1(0) = X_0,\ p'(0) = X_0',\ p''(0) = X_0''$$

$$p_N(1) = X_f,\ p_N'(1) = X_f',\ p_N''(1) = X_f''$$

Initial/Terminal Conditions

$$p_i(T_{i+1}) = p_{i+1}(T_{i+1}) \quad \forall i \in \{1, \ldots, N\}$$

$$p_i'(T_{i+1}) = p_{i+1}'(T_{i+1}) \quad \forall i \in \{1, \ldots, N\}$$

$$p_i''(T_{i+1}) = p_{i+1}''(T_{i+1}) \quad \forall i \in \{1, \ldots, N\}$$

Interstitial Smoothing Conditions

$$\bigvee_{r=1}^{R} [A^r p_i(t) \leq b^r] \ \text{ for } t \in [T_i, T_{i+1}] \quad \forall i \in \{1, \ldots, N\}$$

Avoid Collision = Remain in Safe Regions

~~Disjunctive~~ *Polynomial* Optimization Formulation
Mixed-Integer

Variables = Polynomials : $\{p_i : [T_i, T_{i+1}] \to \mathbb{R}^2\}_{i=1}^N$

$$\min_{p} \quad \sum_{i=1}^{N} ||p_i'''(t)||^2$$

s.t.

$$p_1(0) = X_0,\ p'(0) = X_0',\ p''(0) = X_0''$$
$$p_N(1) = X_f,\ p_N'(1) = X_f',\ p_N''(1) = X_f''$$

Initial/Terminal Conditions

$$p_i(T_{i+1}) = p_{i+1}(T_{i+1}) \quad \forall i \in \{1, \ldots, N\}$$
$$p_i'(T_{i+1}) = p_{i+1}'(T_{i+1}) \quad \forall i \in \{1, \ldots, N\}$$
$$p_i''(T_{i+1}) = p_{i+1}''(T_{i+1}) \quad \forall i \in \{1, \ldots, N\}$$

Interstitial Smoothing Conditions

$$b_j^r + M_j^r(1 - z_{i,r}) - A_j^r p_i(t) \geq 0 \qquad \text{for } t \in [T_i, T_{i+1}] \quad \forall i, j, r$$

$$\sum_{r=1}^{R} z_{i,r} = 1 \quad \forall i, z \in \{0,1\}^{N \times R}$$

Avoid Collision = Remain in Safe Regions

~~Disjunctive~~ ~~*Polynomial*~~ Optimization Formulation
Mixed-Integer Sum-of-Squares

Variables = Polynomials : $\{p_i : [T_i, T_{i+1}] \to \mathbb{R}^2\}_{i=1}^N$

$$\min_{p} \quad \sum_{i=1}^{N} ||p_i'''(t)||^2$$

s.t.

$$p_1(0) = X_0,\ p'(0) = X_0',\ p''(0) = X_0''$$
$$p_N(1) = X_f,\ p_N'(1) = X_f',\ p_N''(1) = X_f''$$

Initial/Terminal Conditions

$$p_i(T_{i+1}) = p_{i+1}(T_{i+1}) \quad \forall i \in \{1, \dots, N\}$$
$$p_i'(T_{i+1}) = p_{i+1}'(T_{i+1}) \quad \forall i \in \{1, \dots, N\}$$
$$p_i''(T_{i+1}) = p_{i+1}''(T_{i+1}) \quad \forall i \in \{1, \dots, N\}$$

Interstitial Smoothing Conditions

$$b_j^r + M_j^r(1 - z_{i,r}) - A_j^r p_i(t) \text{ is SOS for } t \in [T_i, T_{i+1}] \quad \forall i, j, r$$

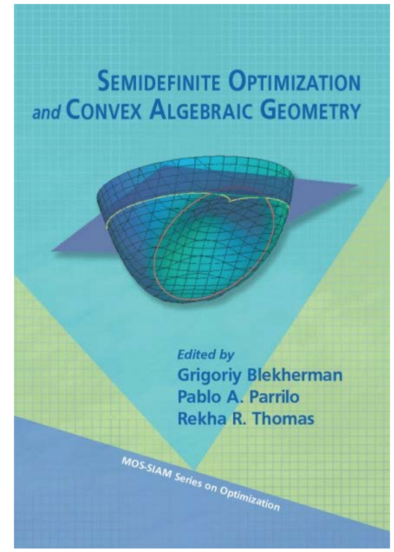
$$\sum_{r=1}^{R} z_{i,r} = 1 \quad \forall i,\ z \in \{0,1\}^{N \times R}$$

Avoid Collision = Remain in Safe Regions

From Sum of Squares to Semidefinite Programming

- Sufficient condition for non-negative polynomial:
  - Sum of Squares : $f(x) = \sum_i g_i^2(x)$
  - SDP representable for fixed degree:

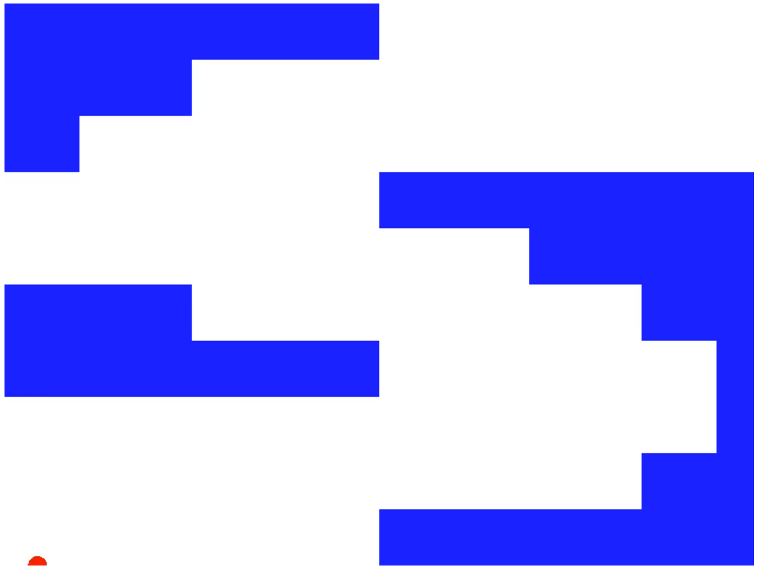
$$\text{degree} \leq k \quad \rightarrow \quad (k-1) \times (k-1) \text{ matrices}$$



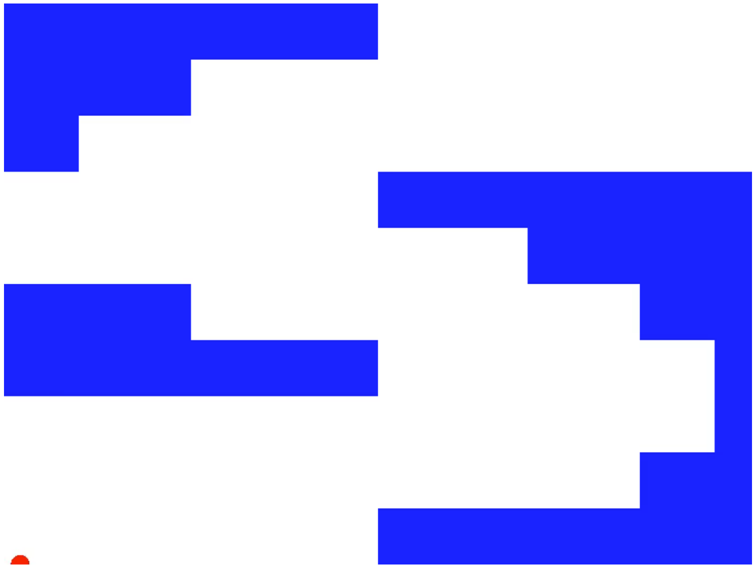
- MI-SOS:
  - Low degree polynomials (≤3):
    - MI-SOCP: solvable by Gurobi/CPLEX
    - Deits/Tedrake:2015
  - Higher degree polynomials:
    - MI-SDP: solvable by Pajarito

Results for 9 Regions and 8 time steps

- Infeasible for degree ≤ 3 (MI-SOCP)
- Pajarito results for degree 5:



First Feasible Solution:
58 seconds



Optimal Solution:
651 seconds

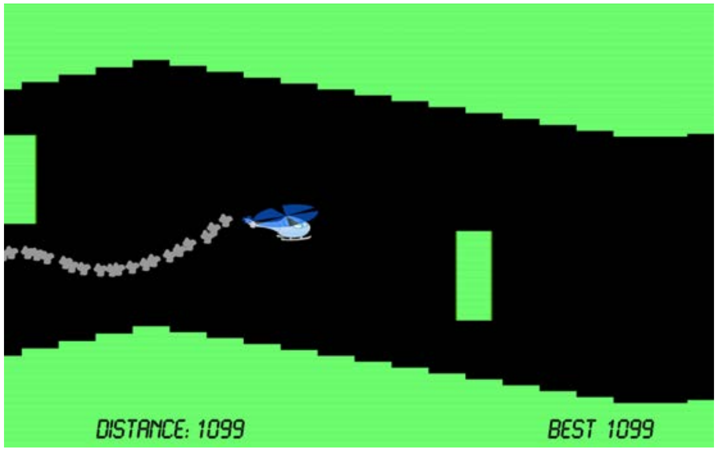
```
function eval_poly(r)
    for i in 1:N
        if T[i] <= r <= T[i+1]
            return PP[(:x,i)]([r], [t]), PP[(:y,i)]([r], [t])
            break
        end
    end
end
```

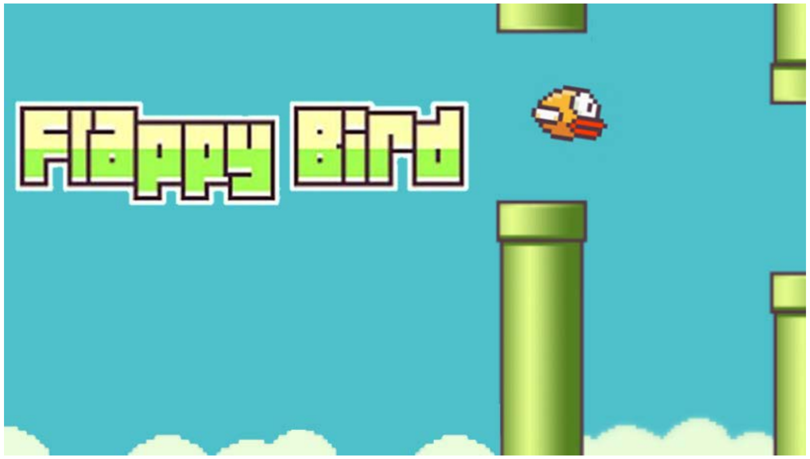
```
model = SOSModel(solver=PajaritoSolver())

@polyvar(t)
Z = monomials([t], 0:r)

@variable(model, H[1:N,boxes], Bin)

p = Dict()
for j in 1:N
   @constraint(model, sum(H[j,box] for box in boxes) == 1)
   p[(:x,j)] = @polyvariable(model, _, Z)
   p[(:y,j)] = @polyvariable(model, _, Z)
   for box in boxes
       xl, xu, yl, yu = box.xl, box.xu, box.yl, box.yu
       @polyconstraint(model, p[(:x,j)] >= Mxl + (xl-Mxl)*H[j,box], domain = (t >= T[j] && t <= T[j+1]))
       @polyconstraint(model, p[(:x,j)] <= Mxu + (xu-Mxu)*H[j,box], domain = (t >= T[j] && t <= T[j+1]))
       @polyconstraint(model, p[(:y,j)] >= Myl + (yl-Myl)*H[j,box], domain = (t >= T[j] && t <= T[j+1]))
       @polyconstraint(model, p[(:y,j)] <= Myu + (yu-Myu)*H[j,box], domain = (t >= T[j] && t <= T[j+1]))
   end
end

for ax in (:x,:y)
   @constraint(model,                p[(ax,1)      ]([0], [t]) == X₀[ax])
   @constraint(model, differentiate(p[(ax,1)], t   )([0], [t]) == X₀'[ax])
   @constraint(model, differentiate(p[(ax,1)], t, 2)([0], [t]) == X₀''[ax])
   for j in 1:N-1
       @constraint(model,               p[(ax,j)    ]([T[j+1]],[t]) ==               p[(ax,j+1)    ]([T[j+1]],[t]))
       @constraint(model, differentiate(p[(ax,j)],t  )([T[j+1]],[t]) == differentiate(p[(ax,j+1)],t  )([T[j+1]],[t]))
       @constraint(model, differentiate(p[(ax,j)],t,2)([T[j+1]],[t]) == differentiate(p[(ax,j+1)],t,2)([T[j+1]],[t]))
   end
   @constraint(model,                p[(ax,N)      ]([1], [t]) == X₁[ax])
   @constraint(model, differentiate(p[(ax,N)], t   )([1], [t]) == X₁'[ax])
   @constraint(model, differentiate(p[(ax,N)], t, 2)([1], [t]) == X₁''[ax])
end

@variable(model, γ[keys(p)] ≥ 0)
for (key,val) in p
   @constraint(model, γ[key] ≥ norm(differentiate(val, t, 3)))
end
@objective(model, Min, sum(γ))
```

```
using SFML

const window_width = 800
const window_height = 600

window = RenderWindow("Helicopter",
                       window_width, window_height)
event = Event()

rects = RectangleShape[]
for box in boxes
    rect = RectangleShape()
    xl = (window_width/M)*box.xl
    xu = (window_width/M)*box.xu
    yl = window_height*(domain.yu-box.yl)
    yu = window_height*(domain.yu-box.yu)
    set_size(rect, Vector2f(xu-xl, yu-yl))
    set_position(rect, Vector2f(xl, yl))
    set_fillcolor(rect, SFML.white)
    push!(rects, rect)
end

type Helicopter
    shape::CircleShape
    past_path::Vector{Vector2f}
    path_func::Function
end

const radius = 10

heli = Helicopter(CircleShape(),
                  Vector2f[Vector2f(X0[:x]*window_width,
                       X0[:y]*window_height)], eval_poly)
set_position(heli.shape, Vector2f(window_width/2,
             window_height/2))
set_radius(heli.shape, radius)
set_fillcolor(heli.shape, SFML.red)
set_origin(heli.shape, Vector2f(radius, radius))
```

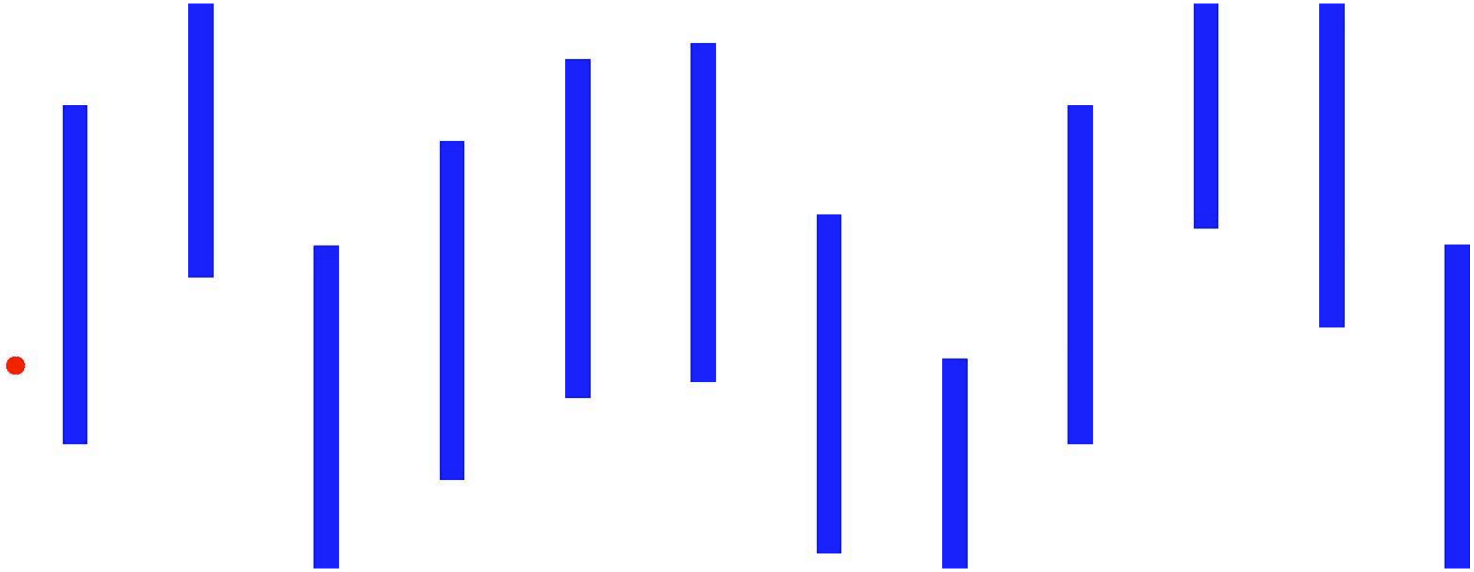
```
function update_heli!(heli::Helicopter, tm)
    (_x,_y) = heli.path_func(tm)
    x = window_width / M * _x
    y = window_height * (1-_y)
    pt = Vector2f(x,y)
    set_position(heli.shape, pt)
    # move(heli.shape, pt-heli.past_path[end])
    push!(heli.past_path, pt)
    get_position(heli.shape)
    nothing
end

const maxtime = 10.0

make_gif(window, window_width, window_height,
         1.05*maxtime, "foobarbat.gif", 0.05)

clock = Clock()
restart(clock)

while isopen(window)
    frametime = as_seconds(get_elapsed_time(clock))
    @show normalizedtime = Tmin +
                (frametime / maxtime)*(Tmax-Tmin)
    (normalizedtime >= Tmax) && break
    while pollevent(window, event)
        if get_type(event) == EventType.CLOSED
            close(window)
        end
    end
    clear(window, SFML.blue)
    for rect in rects
        draw(window, rect)
    end
    update_heli!(heli, normalizedtime)
    draw(window, heli.shape)
    display(window)
end
```

Helicopter Game / Flappy Bird





- 60 horizontal segments, obstacle every 5 = 80 sec. to opt.



Summary

- Advances in MIP = Advanced Formulations + Advanced Solvers + Easy Access Through JuMP
- More information:
  - Advanced Formulations: 15.083 +
    - Mixed integer linear programming formulation techniques. V. '15.
  - Algorithms/Solvers: 15.083 +
    - M. Lubin's thesis defense: Monday, June 5, 1:00 PM, E62-550
  - Julia and JuMP: 15.083 + webpages +
    - JuMP Developers Meetup: June 12-16, 2017, E62: Advanced, but some talks first 2 days. http://www.juliaopt.org/developersmeetup
- 15.083: Integer Programming and Combinatorial Optimization
  - Spring 2018: Formulations + Algorithms + julia

References: Available at www.mit.edu/~jvielma/

- MIP Formulations Survey:
  - “Mixed integer linear programming formulation techniques”. J. P. Vielma. SIAM Review 57, 2015. pp. 3–57.
- Other advanced MIP formulation techniques:
  - “Embedding Formulations and Complexity for Unions of Polyhedra”. J. P. Vielma. To appear in Management Science, 2017.
  - “Strong mixed-integer formulations for the floor layout problem”. J. Huchette, S. S. Dey and J. P. Vielma. Submitted for publication, 2016.
  - “Small and Strong Formulations for Unions of Convex Sets from the Cayley Embedding”. J. P. Vielma. Submitted for publication, 2017.
  - “Small independent branching formulations for unions of V-polyhedra”. J. Huchette and J. P. Vielma. Submitted for publication, 2016.
  - “Mixed-integer models for nonseparable piecewise linear optimization: unifying framework and extensions”. J. P. Vielma, S. Ahmed and G. Nemhauser. Operations Research 58, 2010. pp. 303–315.

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- Convex Nonlinear MIP Solvers:
  - "Extended Formulations in Mixed Integer Conic Quadratic Programming". J. P. Vielma, I. Dunning, J. Huchette and M. Lubin. To appear in Mathematical Programming Computation, 2016.
  - "Extended Formulations in Mixed-Integer Convex Programming". M. Lubin, E. Yamangil, R. Bent and J. P. Vielma. In Q. Louveaux and M. Skutella, editors, Proceedings of the 18th Conference on Integer Programming and Combinatorial Optimization (IPCO 2016), Lecture Notes in Computer Science 9682, 2016. pp. 102–113.
  - "Polyhedral approximation in mixed-integer convex optimization". M. Lubin, E. Yamangil, R. Bent and J. P. Vielma. Submitted for publication, 2016.

References

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  - "Julia: A fresh approach to numerical computing". J. Bezanson, A. Edelman, S. Karpinski and V. B. Shah. SIAM Review 59, 2017. pp. 65–98. https://julialang.org/publications/julia-fresh-approach-BEKS.pdf
- JuMP:
  - https://github.com/JuliaOpt/JuMP.jl
  - "JuMP: A modeling language for mathematical optimization". I. Dunning, J. Huchette, and Miles Lubin. SIAM Review 59, 2017. pp. 295-320. https://arxiv.org/abs/1508.01982