Extended and Embedding Formulations for MINLP

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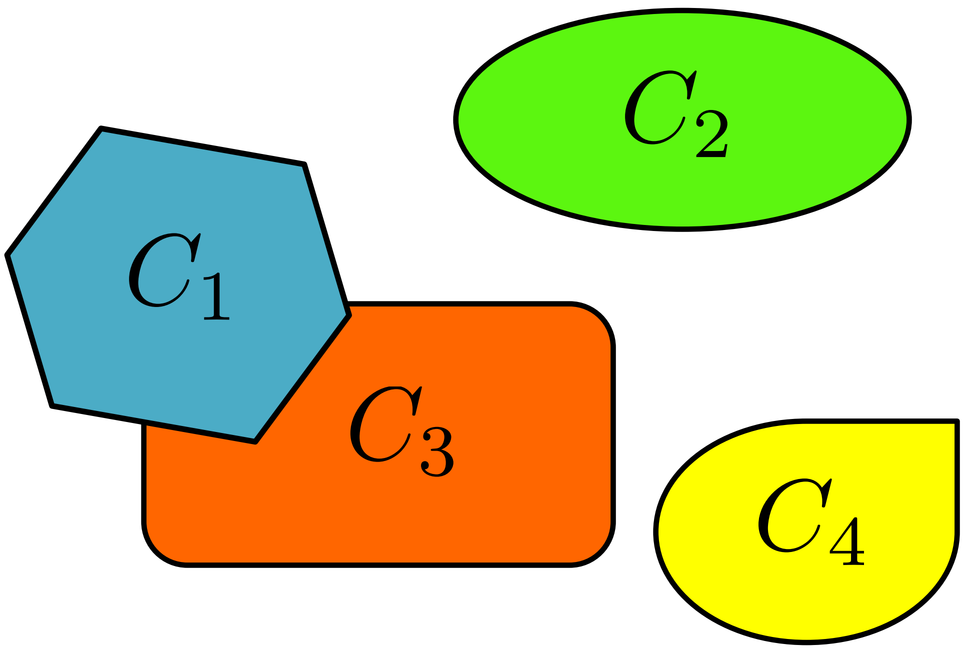
Nonlinear Mixed <u>0-1</u> Integer Formulations

- Modeling Finite Alternatives = Unions of Convex Sets

$$x \in \bigcup_{i=1}^{n} C_i \subseteq \mathbb{R}^d$$



Extended and Non-Extended Formulations for $\bigcup_{i=1}^{n} C_i$

$$C_i = \left\{x \in \mathbb{R}^d \;:\; f_i(x) \leq 0\right\}$$

Extended	Non-Extended
$\tilde{f}_i\left(x^i, y_i\right) \leq 0 \quad \forall i \in [n]$ $\sum_{i=1}^{n} x^i = x$ $\sum_{i=1}^{n} y_i = 1$ $y \in \{0,1\}^n$ $x,\ x^i \in \mathbb{R}^d \quad \forall i \in [n]$	$f_i(x) \leq M_i\left(1 - y_i\right) \quad \forall i \in [n]$ $\sum_{i=1}^{n} y_i = 1$ $y \in \{0,1\}^n$ $x \in \mathbb{R}^d \quad \forall i \in [n]$
Strong, but large	Small, but weak?

Outline

- Extended Formulations
  - Conic formulations, stability and outer-approximation
- Non-Extended Formulations
  - Embedding formulations = Strong non-extended

Extended Formulations:

Birdmen: Or (The Unexpected Virtue of Discipline)

Extended Formulations: Perspective "v/s" Cones

- e.g. Ceria and Soares '99

$$C_i = \{x \in \mathbb{R}^d \; : \; f_i(x) \le 0\}$$

$$\tilde{f}(x,y) = \begin{cases} y f(x/y) & \text{if } y > 0 \\ \lim_{\alpha \downarrow 0} \alpha f(x' - x + x/\alpha) & \text{if } y = 0 \\ +\infty & \text{if } y < 0 \end{cases}$$

$$\boxed{\begin{aligned} \tilde{f}_i\left(x^i, y_i\right) \le 0 \qquad & \forall i \in [n] \\ \sum_{i=1}^n x^i = x & \\ \sum_{i=1}^n y_i = 1 & \\ y \in \{0,1\}^n & \\ x,\; x^i \in \mathbb{R}^d \qquad & \forall i \in [n] \end{aligned}}$$

- e.g. Ben-tal and Nemirovski '01, Helton and Nie '09

$$C_i = \left\{ x \in \mathbb{R}^d \; : \; \begin{array}{c} \exists u \in \mathbb{R}^{p_i} \text{ s.t.} \\ A^i x + D^i u - b \in K^i \end{array} \right\}$$

K^i closed convex cone

$$\boxed{\begin{aligned} A^i x^i + D^i u^i - b y_i \in K^i \qquad & \forall i \in [n] \\ \sum_{i=1}^n x^i = x & \\ \sum_{i=1}^n y_i = 1 & \\ y \in \{0,1\}^n & \\ x,\; x^i \in \mathbb{R}^d \qquad & \forall i \in [n] \\ u^i \in \mathbb{R}^{p_i} \qquad & \forall i \in [n] \end{aligned}}$$

- Both formulations are ideal (extreme points of continuous relaxation satisfy integrality constraints)

Cones Can Mitigate Unintended Numerical Issues

- Let $C_i = \left\{x \in \mathbb{R}^2 \;:\; f_i\left(x\right) \leq 0\right\}$ where $f_i(x) = x_1^2 - x_2 - 1$

$$\tilde{f}_i(x,y) = \begin{cases} y(x_1\boxed{/y)}^2 - x_2 - y & \text{if } y > 0 \\ -x_2 & \text{if } y = x_1 = 0 \\ +\infty & \text{if } o.w. \end{cases}$$

- Conic (SOCP) representation

$$C_i = \left\{x \in \mathbb{R}^2 \;:\; \sqrt{x_2^2 + 4x_1^2} \leq 2 + x_2\right\}$$

$$\sqrt{\left(x_2^i\right)^2 + 4\left(x_1^i\right)^2} \leq 2y_i + x_2$$

Conic = Really Extended

- Conic representation = additional auxiliary variables

$$C_i = \left\{ x \in \mathbb{R}^d \ : \ \begin{array}{r} \exists \boxed{u} \in \mathbb{R}^{p_i} \ \text{ s.t.} \\ A^i x + D^i \boxed{u} - b \in K^i \end{array} \right\}$$

- Bad for NLP solvers, but good for polyhedral approximations in MINLP solvers:
  - Separable = (Tawarmalani and Sahinidis '05, Hijazi et al. '12)
  - SOCP = (V. et al. '15) = If SOCP representable use MI-SOCP solver!
  - General MINLP? = Disciplined Convex Programming (DCP) (Grant et al. 06): Implemented in CVX
    - Systematic way to prove convexity
    - Yields extended conic representation (MINLPLib2 -1)

Solving Mixed Integer Disciplined Convex Programs

- Pajarito solver!
  - Lubin, Yamangil, Bent and V. '15
  - SLay10M (it / time):
    - Bonmim = 69 it / 1,379 s
    - Hijazi et al. = 23 it / 14 s
    - Pajarito = 5 it / 12 s (automatic separability)
  - Solved gams01 from MINLPLIB2 (prev 91% GAP)
  - Solved tls5-6 (prev 25%-29% GAP) = Just SOCP + Gurobi!
  - ~200 lines of Julia code

Miles Lubin and Emre Yamangil

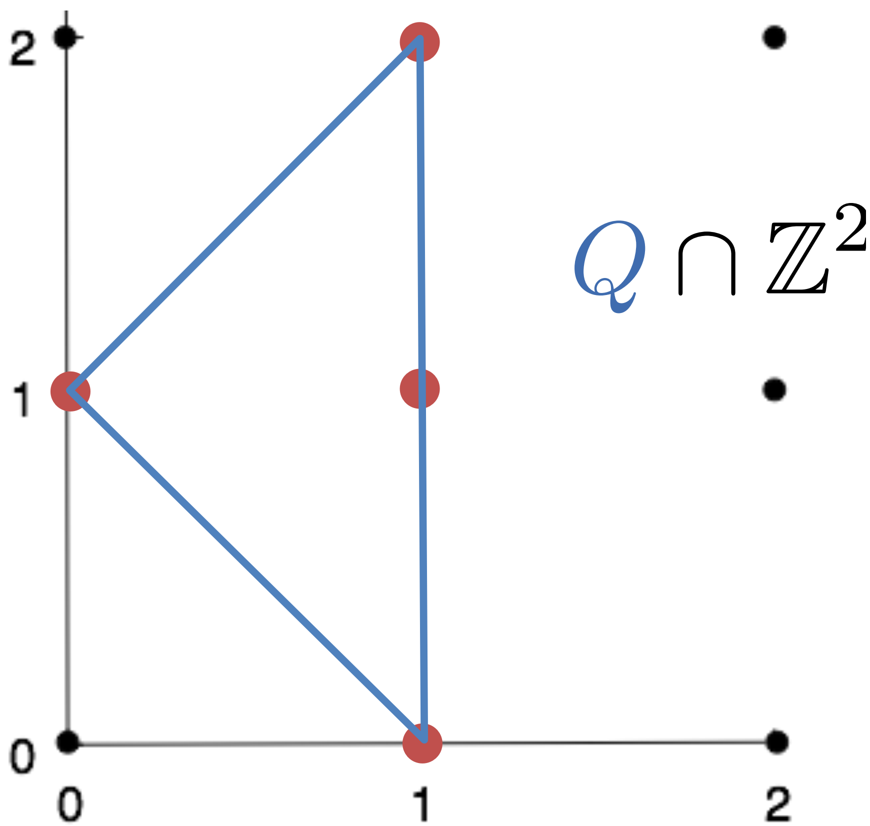
Strong Non-Extended Formulations:

Minkowski Sums, Good or Evil

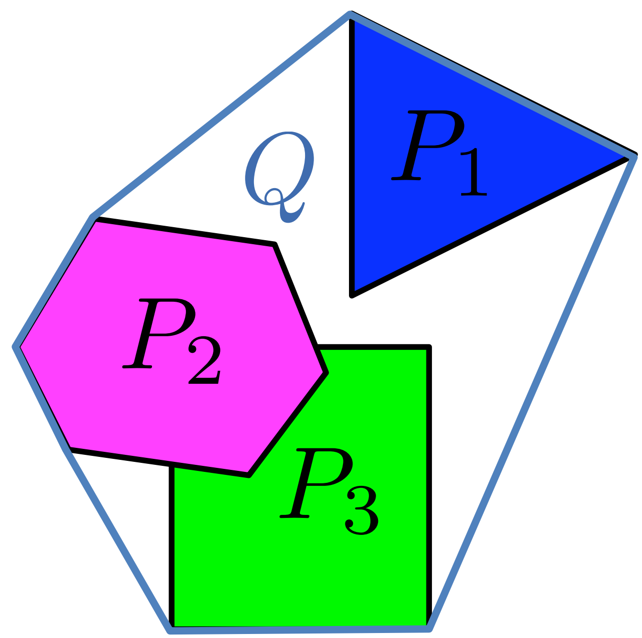
Constructing Non-extended Ideal Formulations

- Pure Integer :

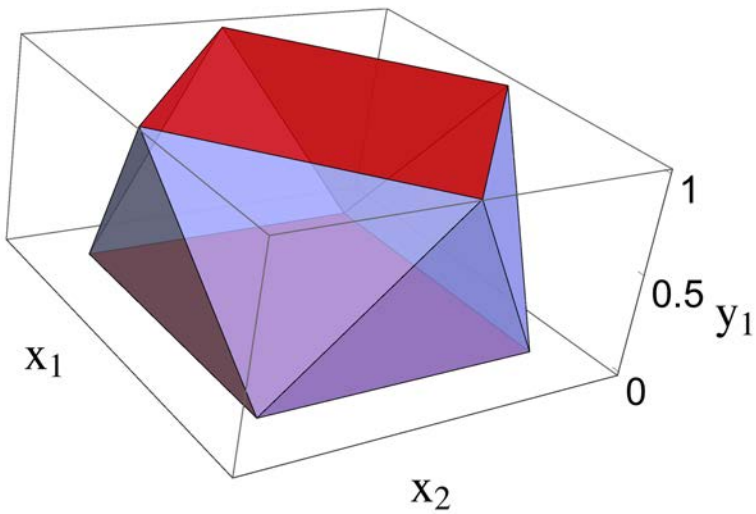
$$Q := \operatorname{conv}\left(\{p^i\}_{i=1}^n\right)$$

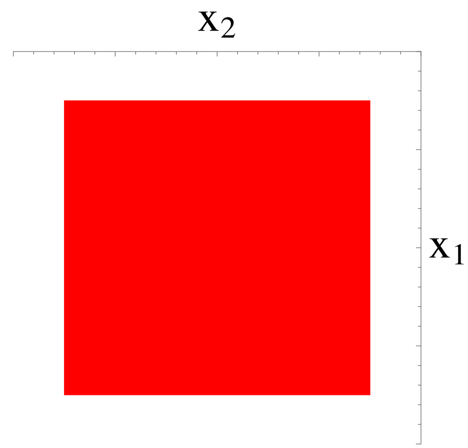


- Mixed Integer:

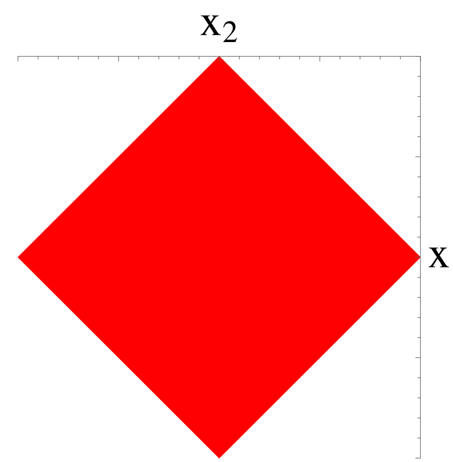


Embedding Formulation = Ideal non-Extended





P_1



P_2

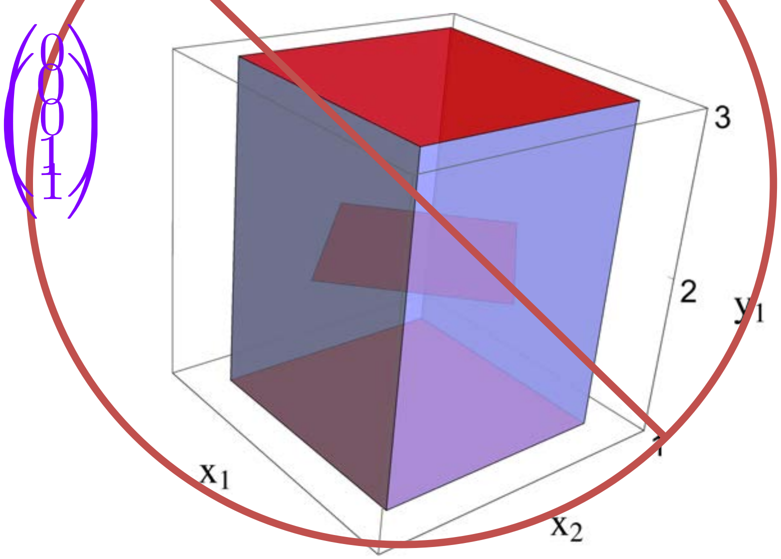
$$Q\left(H\right) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \left\{h^i\right\}\right)$$

$$(x, y) \in Q \cap \left(\mathbb{R}^d \times \mathbb{Z}^k\right) \quad \Leftrightarrow \quad y = h^i \ \wedge \ x \in P_i$$

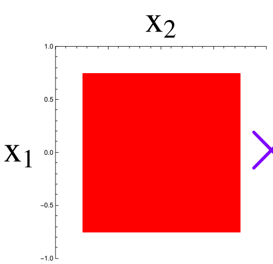
$$\operatorname{ext}(Q) \subseteq \mathbb{R}^d \times \mathbb{Z}^k \qquad H := \left\{h^i\right\}_{i=1}^{n} \subseteq \{0, 1\}^k, \quad h^i \neq h^j$$

Alternative Encodings

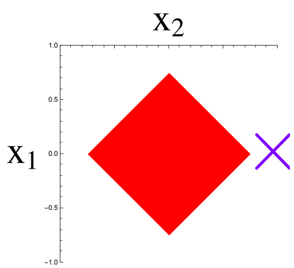
$$\bigcup_{i=1}^{n} P_i \times \{i\}$$



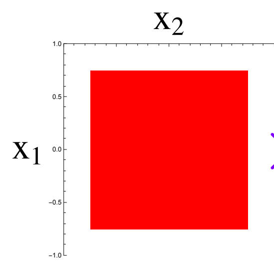
- "Only" use 0-1 encodings



$\times \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$ P_1



$\times \begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}$ P_2



$\times \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}$ P_3

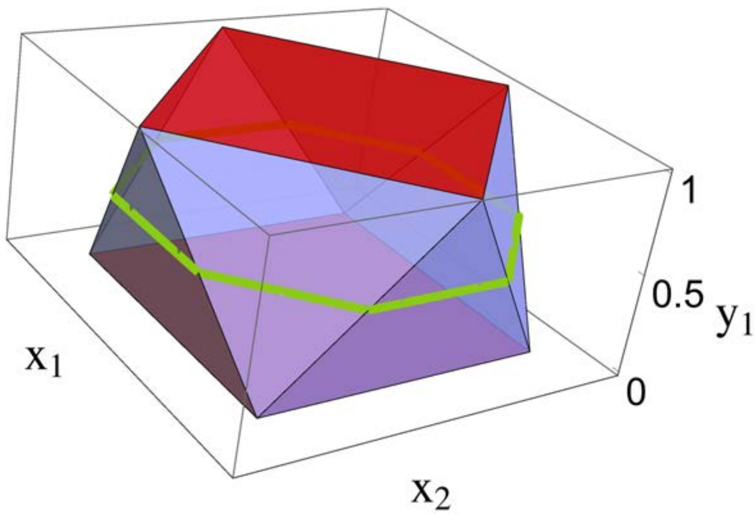
- Options for 0-1 encodings:
  - Traditional or **Unary** encoding

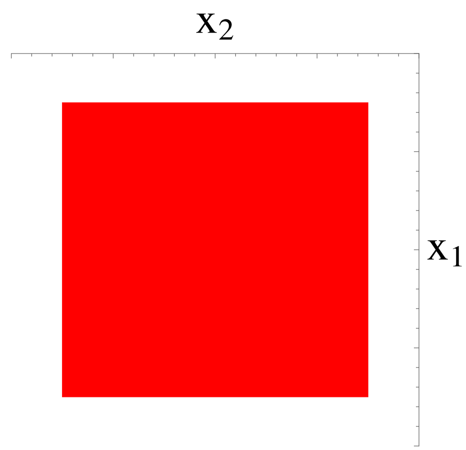
$$H = \left\{ y \in \{0,1\}^n \; : \; \sum_{i=1}^{n} y_i = 1 \right\}$$
$$= \{\mathbf{e}^i\}_{i=1}^{n}$$

$$\mathbf{e}^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

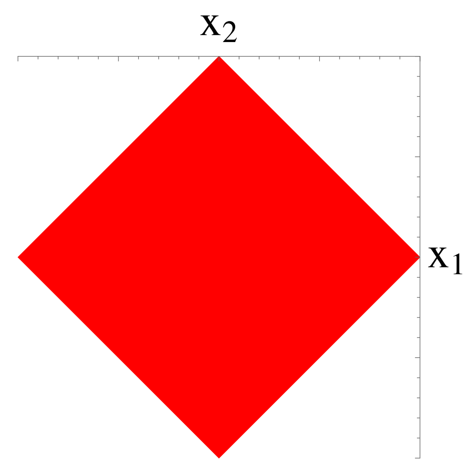
  - **Binary** encodings: $H \equiv \{0,1\}^{\log_2 n}$
  - Others (e.g. **incremental** encoding $\equiv$ unary)

Unary Encoding, Minkowski Sum and Cayley Trick



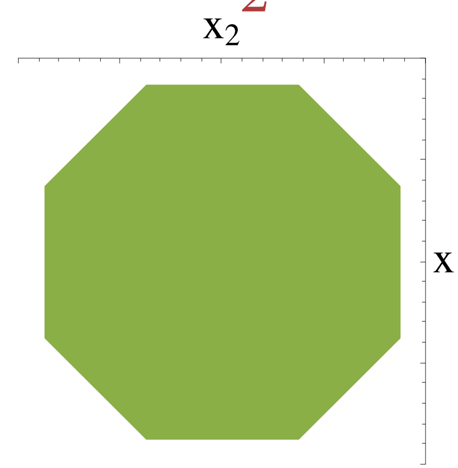


P_1



P_2

$$Q \cap \left(\mathbb{R}^2 \times \{0.5\}\right) \equiv P_1 + P_2 =$$



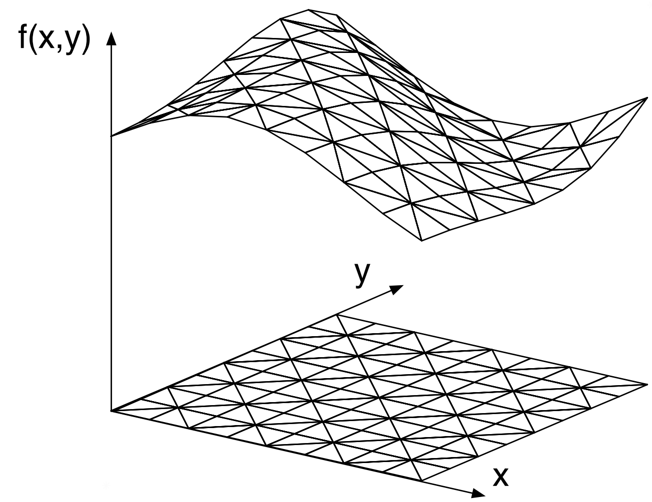
For traditional or unary encoding:

$$Q\,(H) \cap \left(\mathbb{R}^d \times \left\{\tfrac{1}{n}\sum_{i=1}^{n} \mathbf{e}^i\right\}\right) \equiv \sum_{i=1}^{n} P_i$$

Encoding Selection Matters: Evil Minkowski Sum

- Size of unary formulation is:
  (Lee and Wilson ’01)

$$\underbrace{\binom{2\sqrt{n/2}}{\sqrt{n/2}}}_{\text{General Inequalities}} + \underbrace{\left(\sqrt{n/2}+1\right)^2}_{\text{Variable Bounds}}$$



- Size of one binary formulation:
  (V. and Nemhauser ’08)

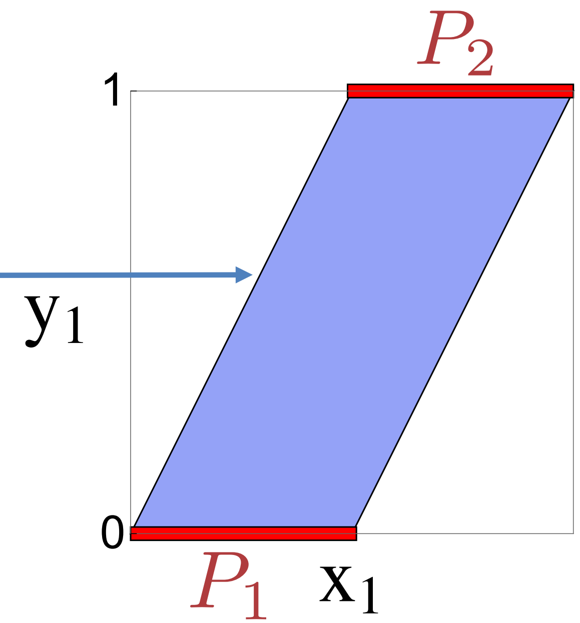
$$4\log_2\sqrt{n/2}+2+\left(\sqrt{n/2}+1\right)^2$$

- Right embedding = significant computational advantage over alternatives (Extended, Big-M, etc.)

Complexity of Family of Polyhedra $\mathcal{P} := \{P_i\}_{i=1}^{n}$

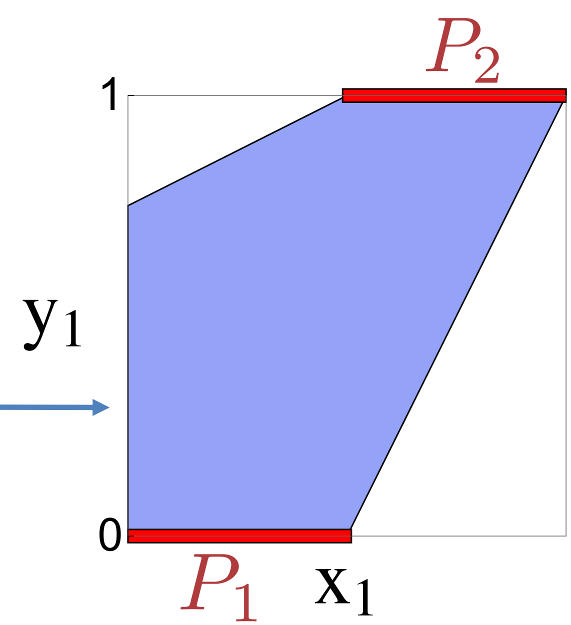
- Embedding complexity = smallest **ideal** formulation

$$\mathrm{mc}\left(\mathcal{P}\right) := \min_{H} \left\{\mathrm{size}\left(Q\left(H\right)\right)\right\}$$



- Relaxation complexity = smallest formulation

$$\mathrm{rc}\left(\mathcal{P}\right) := \min_{Q,H} \left\{\mathrm{size}(Q)\right\}$$



Complexity Results

- Lower and Upper bounds for special structures:
  - e.g. for Special Order Sets of Type 2 (SOS2) on n variables
    - Embedding complexity (ideal)

$$2\lceil \log_2 n \rceil \longleftarrow \text{General Inequalities}$$

$$n+1 \leq \ldots \leq n+1+2\lceil \log_2 n \rceil \longleftarrow \text{Total}$$

    - Relaxation complexity (non-ideal)

$$2 \leq \ldots \leq 4 \longleftarrow \text{General Inequalities}$$

$$2 \leq \ldots \leq 5+2n \longleftarrow \text{Total}$$

- Relation to other complexity measures

$$\mathrm{hc}\left(\mathcal{P}\right) := \mathrm{size}\left(\mathrm{conv}\left(\bigcup\nolimits_{i=1}^{n} P_i\right)\right)$$

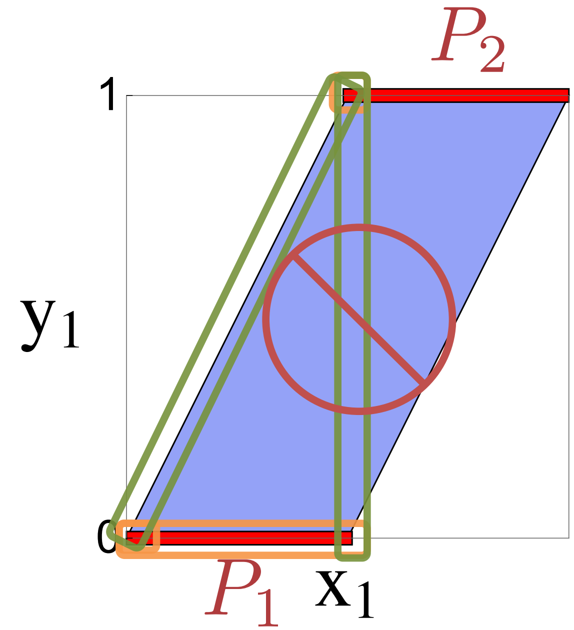
$$\mathrm{xc}\left(\mathcal{P}\right) := \min_R \left\{\mathrm{size}\left(R\right) \,:\, \mathrm{proj}_x\left(R\right) = \mathrm{conv}\left(\bigcup\nolimits_{i=1}^{n} P_i\right)\right\}$$

- Still open questions (see V. 2015)

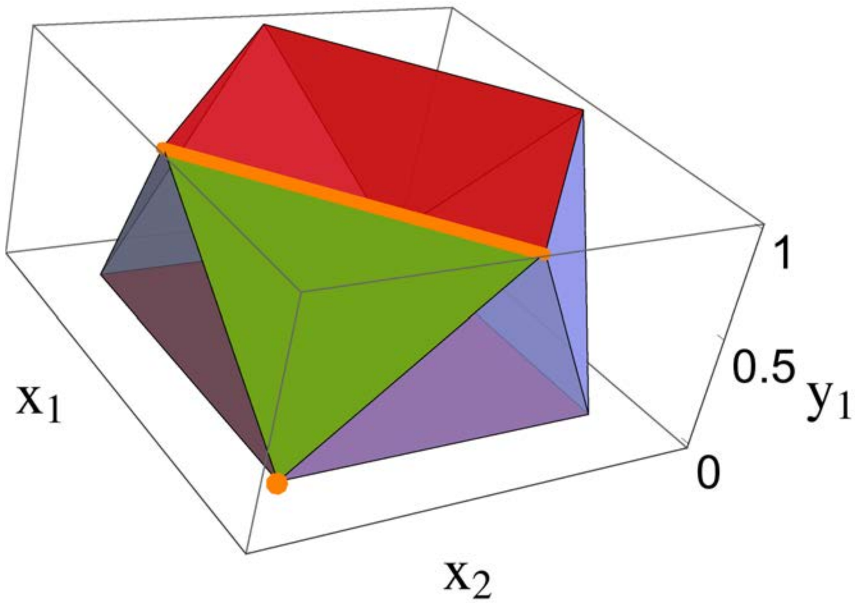
Faces for Unary Encoding: Good Minkowski Sum

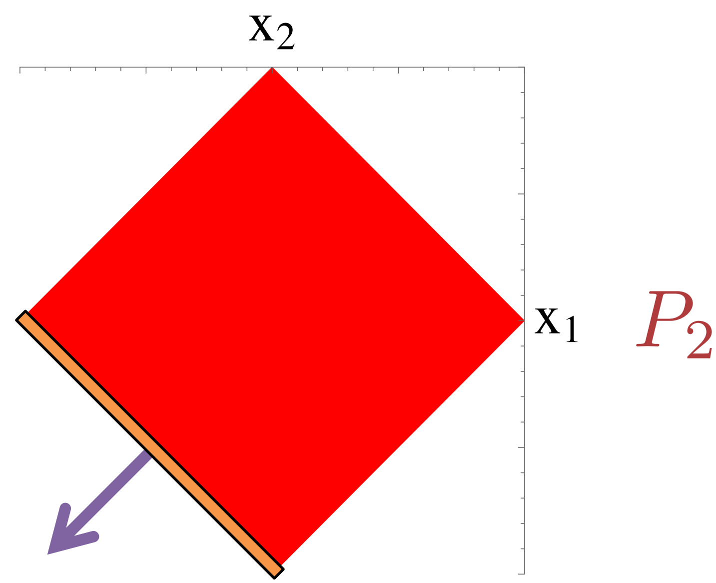
- Two types of facets (or faces):
  - $P_1 \times \{0\} \equiv y_i \geq 0$
  - $\mathrm{conv}\left((F_1 \times 0) \cup (F_2 \times 1)\right)$

    F_i proper face of P_i
  - Not all combinations of faces
  - Which ones are valid?
    - Minkowski to the rescue!



Valid Combinations = Common Normals



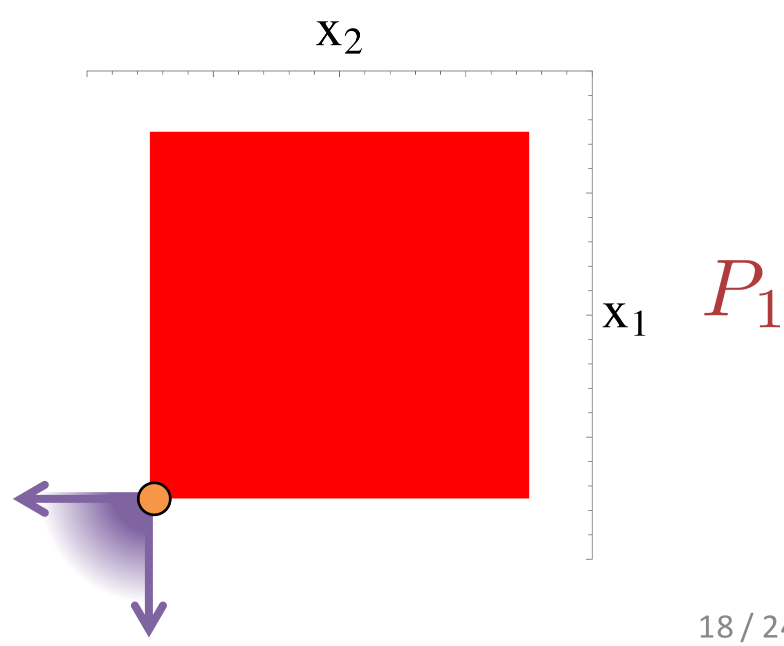


$$N(F_1) \cap N(F_2) \neq \emptyset$$

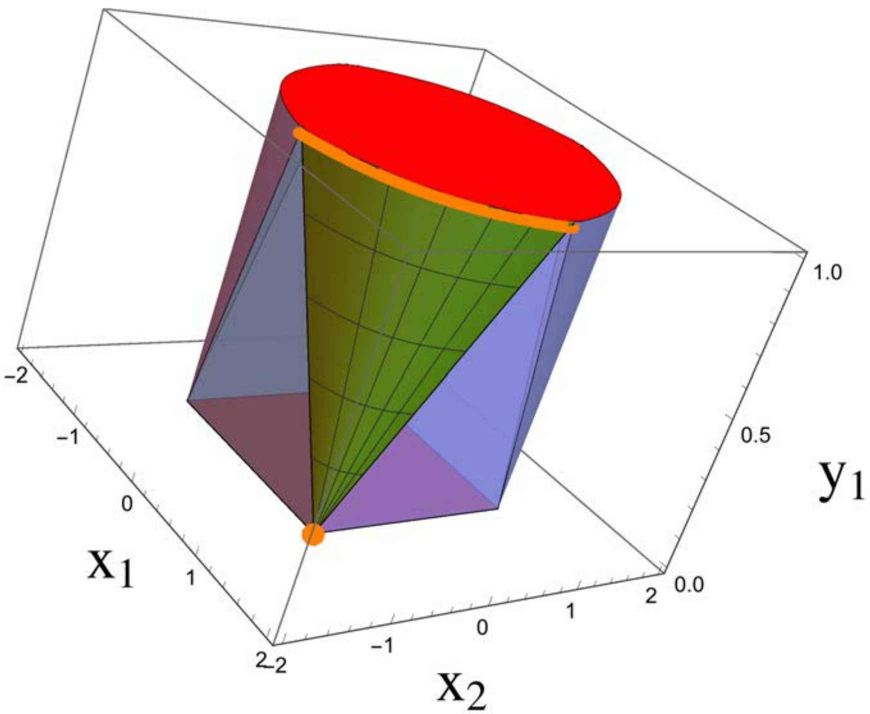
$$\Updownarrow$$

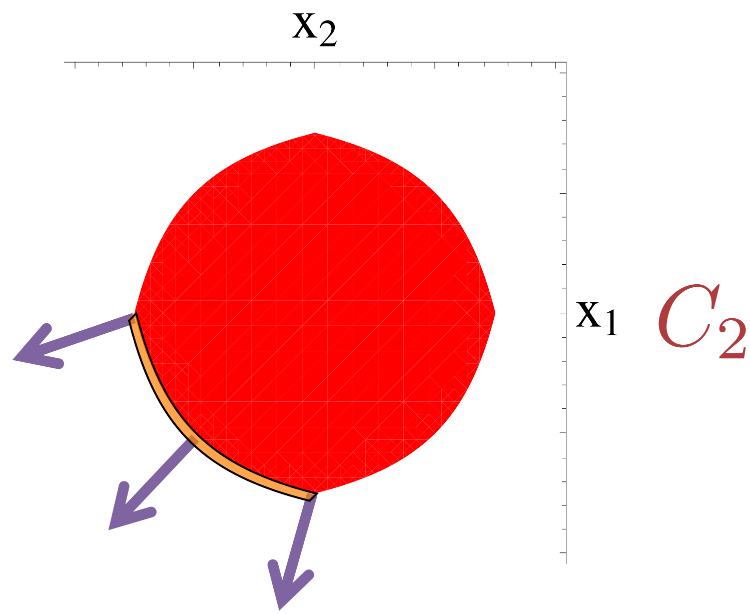
$$\operatorname{conv}\left((F_1 \times 0) \cup (F_2 \times 1)\right)$$

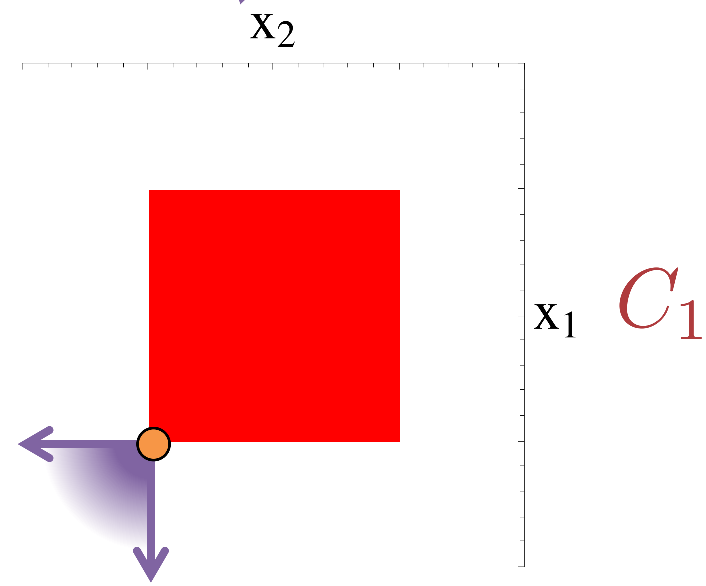
is face of $Q(H)$



Unary Embedding for Unions of Convex Sets



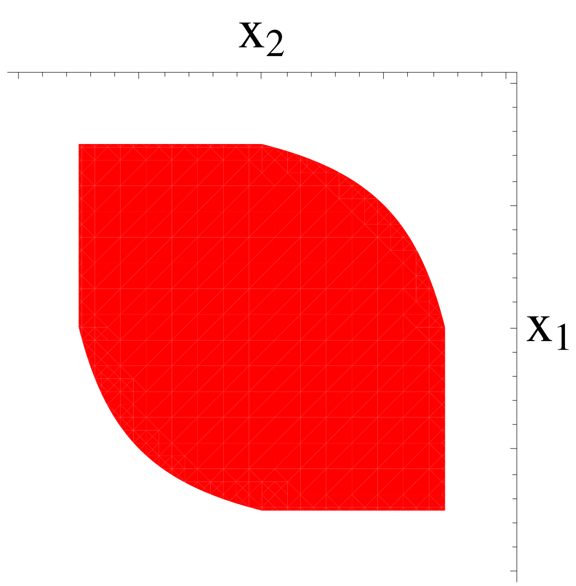




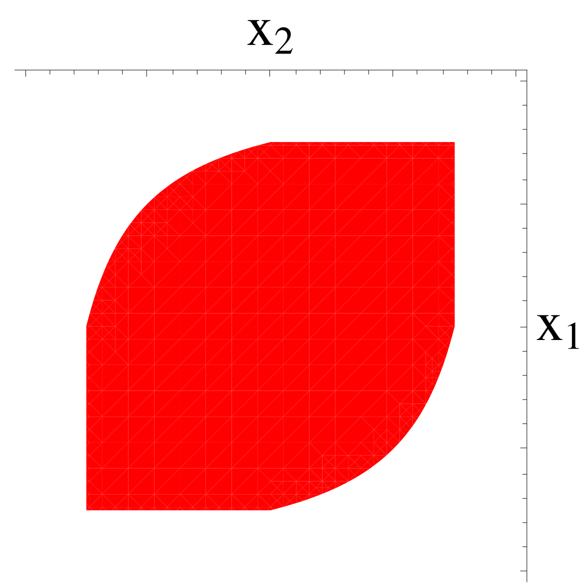
- Description of boundary of $Q(H)$ is easy if "normals condition" yields convex hull of 1 nonlinear constraint and point(s)

Easy to Recover and Generalize Existing Results

- Isotone function results from Hijazi et al. ‘12 and Bonami et al. ’15 (n=1, 2):
  - $C_i = \{x \in \mathbb{R}^d \;:\; l^i \leq x \leq u^i, \quad f_i(x) \leq 0\}$
- Can generalize to n ≥ 3 and two functions per set:



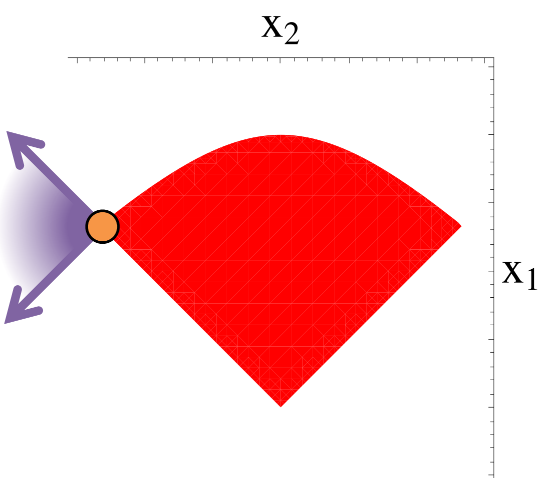
C_1



C_2

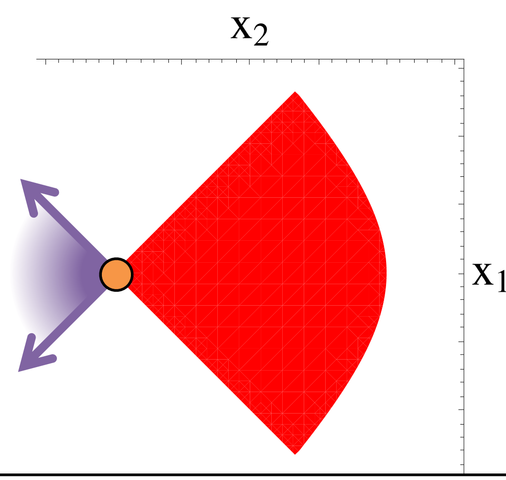
- Other special cases (previous slide)

Also “Non-isotone” Results: Pizza Slices



$C_1:$

$$\sqrt{1+x_1^2} \le 2 - x_2$$
$$x_1 - x_2 \le 1$$
$$-x_2 - x_1 \le 1$$



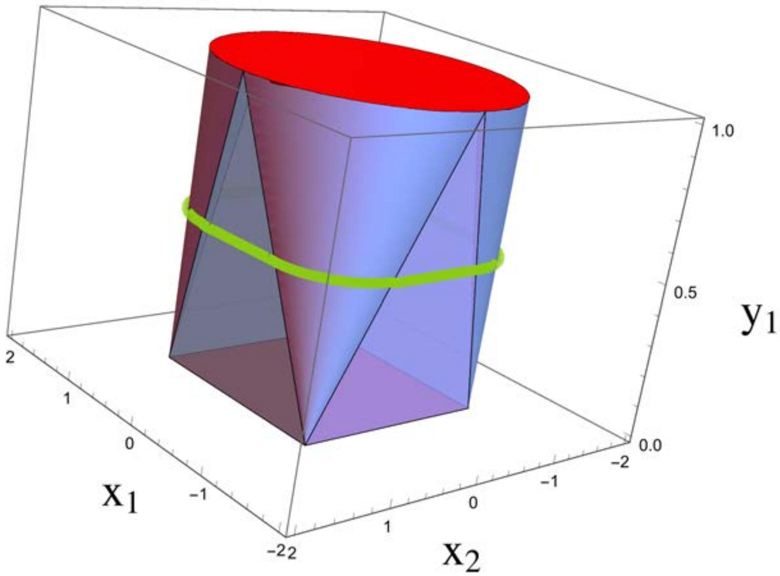
$C_2:$

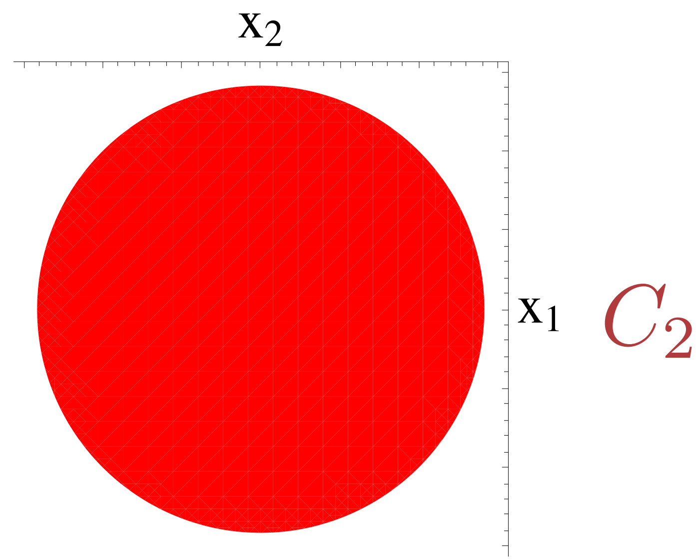
$$\sqrt{1+x_2^2} \le 2 - x_1$$
$$x_2 - x_1 \le 1$$
$$-x_2 - x_1 \le 1$$

$$\sqrt{y_4^2 + \left(x_2 - \frac{1}{3}y_1 + \frac{1}{3}y_3\right)^2} \le 2y_4 + x_1 + \frac{4}{3}y_1 + y_2 + \frac{4}{3}y_3$$

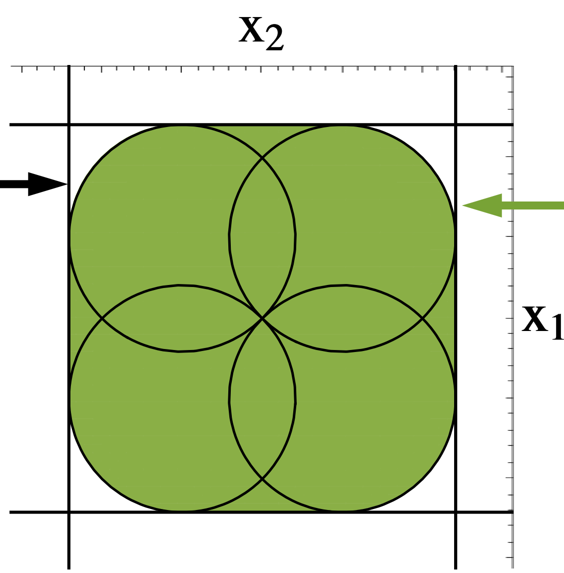
$\operatorname{conv}\left(\bigcup_{i=1}^{4}\left(C_i \times \{\mathbf{e}^i\}\right)\right)$ = 4 conic + 4 linear inequalities

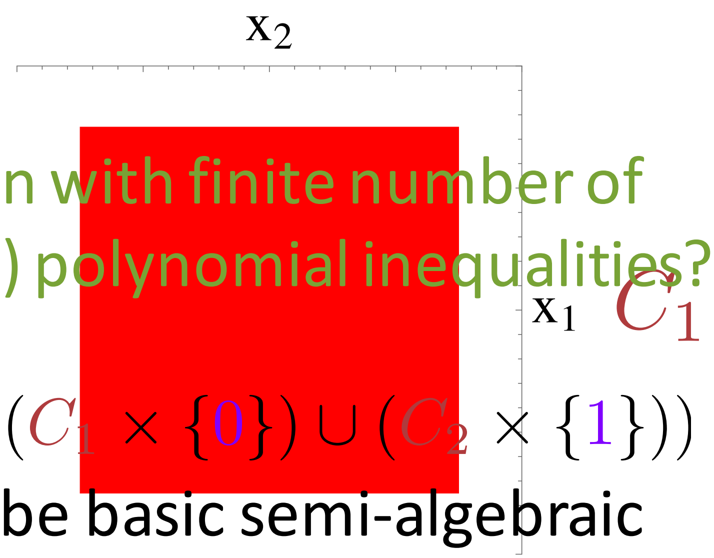
Bad Example: Representability Issues





Zariski closure of boundary?





Description with finite number of (quadratic) polynomial inequalities?

$$Q := \operatorname{conv}\left((C_1 \times \{0\}) \cup (C_2 \times \{1\})\right)$$

can fail to be basic semi-algebraic

Final Positive Results

- Unions of Homothetic Convex Bodies $C_i = \lambda_i C + b^i$ (all extreme points exposed)

$$\operatorname{conv}\left(\bigcup_{i=1}^n \left(C_i \times \{\mathbf{e}^i\}\right)\right) =$$

$$\gamma_C\left(x - \sum_{i=1}^n y_i b^i\right) \leq \sum_{i=1}^n \lambda_i y_i$$

$$\sum_{i=1}^n y_i = 1$$

$$y \geq 0 \qquad \forall i \in [n]$$

$$\gamma_C(x) := \inf\{\lambda > 0 \;:\; x \in \lambda C\}$$

- Generalizes polyhedral results from Balas '85, Jeroslow '88 and Blair '90

Summary

- Extended formulations
  - Really extended formulations through SOCP or DCP
  - Pajarito = MIDCP / MINLP extended polyhedral solver
- Embedding formulations = systematic procedure for ideal non-extended formulations
  - Polyhedral case = Formulations and complexity
  - Non-Polyhedral 1 = Simplified proofs, extensions and new formulations
  - Non-Polyhedral 2 = Representability issues