Modeling Power of Mixed Integer Convex Optimization Problems And Their Effective Solution with Julia and JuMP

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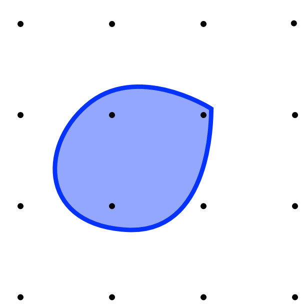
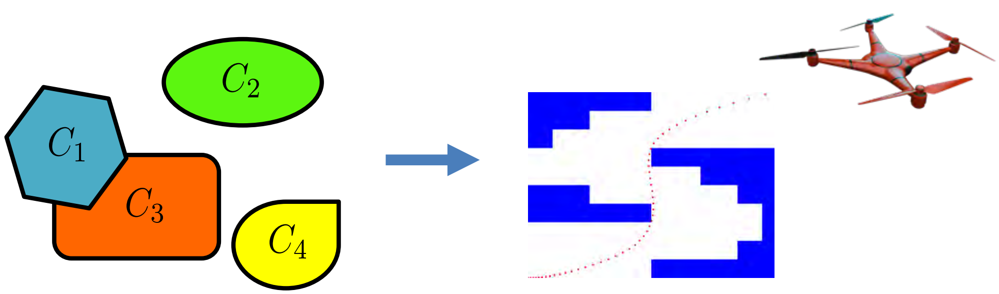
Mixed Integer Convex Optimization (MICP)

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & \\ & x \in C \\ & x_i \in \mathbb{Z} \quad i \in I \end{aligned}$$



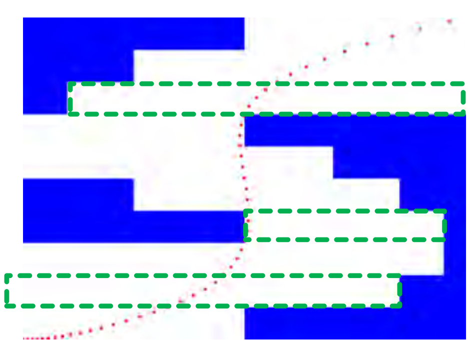
convex f and C.



http://www.gurobi.com/company/example-customers

A Mixed-Integer Infinite Dimensional Example

- Obstacle avoiding trajectory:



$(x(t), y(t))_{t\in[0,1]}$

- Step 1: discretize time into intervals
  $0 = T_1 < T_2 < \cdots < T_N$ s.t.
  $(x(t), y(t)) = p_i(t) \quad t \in [T_i, T_{i+1}]$

- Step 2: "safe polyhedrons"
  $P^r = \{x \in \mathbb{R}^2 : A^r x \le b^r\}$ s.t.
  $\forall\, i\ \exists\, r\ \ s.t.\ \ p_i(t) \in P^r \quad t \in [T_i, T_{i+1}]$

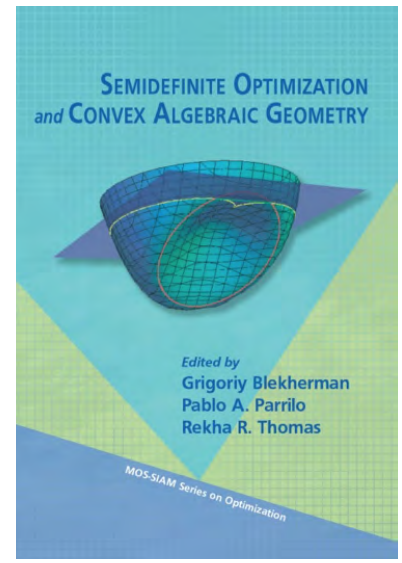
- $p_i(t) \in P^r \rightarrow q_{i,r}(t) \ge 0\ \forall t$

SOS:

$q_{i,r}(t) = \sum_j r_j^2(t)$

- Bound degree of polynomials:

Semidefinite Programming (SDP)



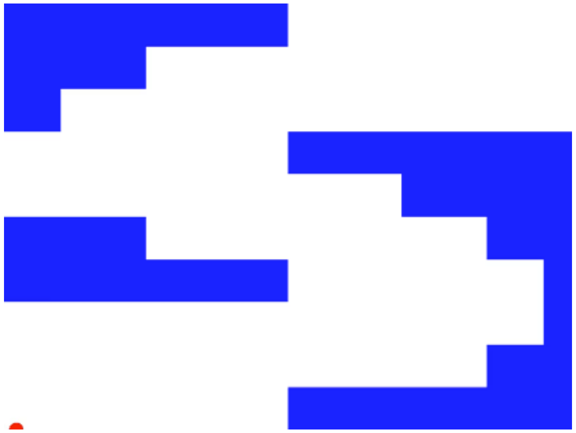
- MI-SDP solver:

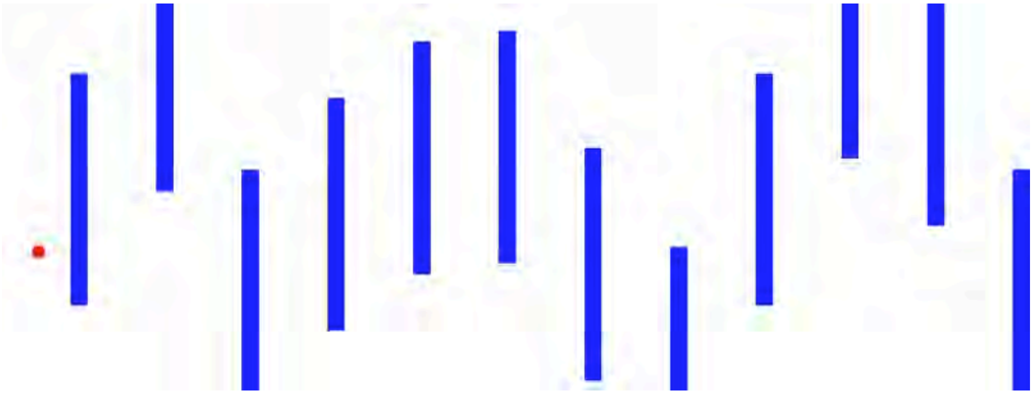
PaJaRiTo julia JuMP

Solving MI-SDP to Global Optimality?

Optimal "Smoothness" in 651 seconds

Optimal "clicks" in 80 seconds





Outline

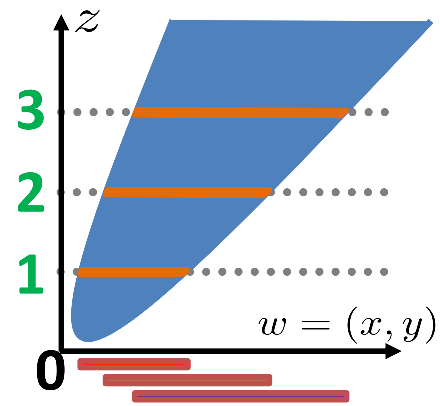
- **MICP Representability**: Characterize what can we model with MICP?
  - Joint work with M. Lubin and I. Zadik
- **Computational Solution of MICP**: Methods and Solvers based on

  - MICP solvers is joint work with Joint work with R. Bent, C. Coey, I. Dunning, J. Huchette, L. Kapelevich, M. Lubin, E. Yamangil
  - JuMP is joint work with and independent work by M. Lubin, I. Dunning, J. Huchette, B. Legat,. O. Dowson, C. Coey, C. Coffrin, J. Dias Garcia, T. Koolen, V. Nesello, F. Pacaud, R. Schwarz, I. Tahiri, U. Worsøe, ….

What can we model with MICP?

MICP /MICONV Formulations and Representability

- $S \subseteq \mathbb{R}^n$ is MICP representable (MICP-R) iff it has an MICP formulation:
  - A closed convex set $M \subseteq \mathbb{R}^{n+p+d}$
  - auxiliary continuous variables $y \in \mathbb{R}^p$
  - auxiliary integer variables $z \in \mathbb{Z}^d$



$$x \in S \quad \Leftrightarrow \quad \begin{array}{c} \exists (y, z) \in \mathbb{R}^p \times \mathbb{Z}^d \text{ s.t.} \\ (x, y, z) \in M \end{array}$$

or equivalently

$$S = \mathrm{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

MICP-R $\Longrightarrow$ Countable Union of Projections of Closed Convex Sets

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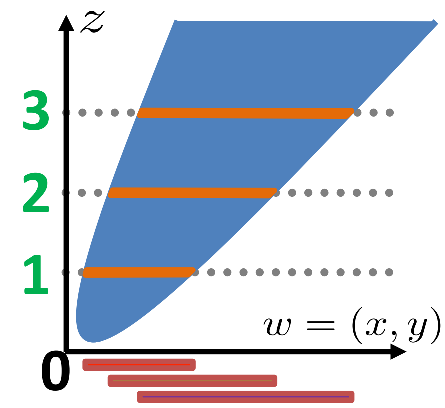


$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \operatorname{proj}_x (B_z)$$

$$I = \operatorname{proj}_z (M) \quad \text{convex}$$

$$B_z = M \cap \left(\mathbb{R}^{n+p} \times \{z\}\right)$$

closed and convex

- Simple Proposition:
  - Complement of any convex body is a countable union of projections of closed convex sets

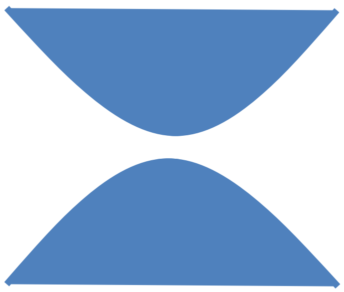
What Countable Unions are MICP-R? Jeroslow and Lowe Regularity

$$\exists \{r(i)\}_{i=1}^{t} \subseteq \mathbb{Q}^n \;\; s.t.\; S = \bigcup_{i=1}^{k} P_i + \left\{ \sum_{i=1}^{t} \lambda_i r(i) : \lambda \in \mathbb{Z}_+^t \right\}$$

- Regularity Conditions:
  - M rational polyhedron $\Longrightarrow$ P_i = rational polytopes (Jeroslow and Lowe '84):
  - $M = \{x \in \mathbb{Z}^2 : x_1 \cdot x_2 \geq \alpha\}$ $\Longrightarrow$ P_i = points (Dey & Moran '13)
  - M = Rational Polyhedron ∩ "Rational" Ellipsoidal Cylinder $\Longrightarrow$ P_i = Rational Ellipsoid ∩ Polytope (Del Pia & Poskin '16)
  - M = Compact Convex + Rational Polyhedron Cone $\Longrightarrow$ P_i = Compact Convex (Lubin, Zadik & V. 17')

What **Other** Countable Unions are MICP-R?

Two sheet hyperbola?

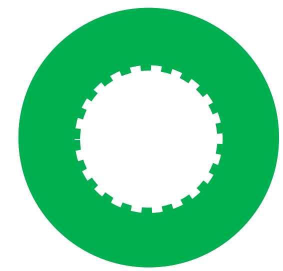


$\{x \in \mathbb{R}^2 \,:\, 1 + x_1^2 \le x_2^2\}$

Spherical shell?

$\{x \in \mathbb{R}^2 \,:\, 1 \le \|x\| \le 2\}$

"Clopen" Spherical shell?



$\{x \in \mathbb{R}^2 \,:\, 1 < \|x\| \le 2\}$

- Integer points in parabola $\{(x, x^2) : x \in \mathbb{Z}\}$?
- The set of $n \times n$ matrices with rank $\le k$?
- Dense discrete set? $\left\{\sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \,:\, x \in \mathbb{N}\right\} \subseteq [0, 1]$
- Set of prime numbers?

"God made the integers, all else is the work of man"

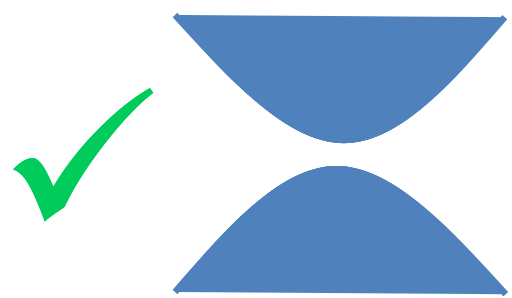
- Leopold Kronecker

0-1 MICP = Finite Union of (Closed) Convex Sets

- $T_1, \ldots, T_k$ be closed convex set. Formulation of $\boldsymbol{x} \in \bigcup_{i=1}^{k} T_i$:

$$\left(\boldsymbol{x}^i, z_i\right) \in \overline{\text{cone}}(T_i \times \{1\}) \qquad \forall i \in [k]$$
$$\left\|\boldsymbol{x}^i\right\|_2^2 \le z_i t_i. \qquad \forall i \in [k]$$
$$\sum_{i=1}^{k} \boldsymbol{x}^i = \boldsymbol{x},$$
$$\sum_{i=1}^{k} z_i = 1,$$
$$\boldsymbol{z} \in \{0,1\}^k,$$
$$t \in \mathbb{R}_+^k,$$
$$\boldsymbol{x}^i \in \mathbb{R}^n \qquad \forall i \in [k]$$

Two sheet hyperbola?



$$\left\{x \in \mathbb{R}^2 \,:\, 1 + x_1^2 \le x_2^2\right\}$$

A Simple Lemma for non-MICP Representability

- Obstruction for MICP representability of S :

$$\text{infinite } R \subseteq S \quad \text{s.t.} \quad \frac{u+v}{2} \notin S \quad \forall u, v \in R,\ u \neq v$$

- Proof: Assume for contradiction there exists M such that:

$$S = \operatorname{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

$$\begin{array}{c} (u, y_u, z_u) \in M \\ (v, y_v, z_v) \in M \end{array} \quad \Longrightarrow \quad \frac{z_u + z_v}{2} \notin \mathbb{Z}^d$$

$$z_u \equiv z_v \pmod 2 \text{ component-wise} \quad \Rightarrow \quad \frac{z_u + z_v}{2} \in \mathbb{Z}^d$$

$$\text{component-wise parity classes} = 2^d < |R| = \infty \qquad \Rightarrow\!\!\!\!\Leftarrow$$

A Simple Lemma for non-MICP Representability

- Obstruction for MICP representability of S :

$$\text{infinite } R \subseteq S \quad \text{s.t.} \quad \frac{u+v}{2} \notin S \quad \forall u, v \in R,\ u \neq v$$

Spherical shell?

✗ 

$\{x \in \mathbb{R}^2 \ : \ 1 \leq \|x\| \leq 2\}$

✗ Integer points in parabola $\{(x, x^2) : x \in \mathbb{Z}\}$?

✗ The set of $n \times n$ matrices with rank $\leq k$?

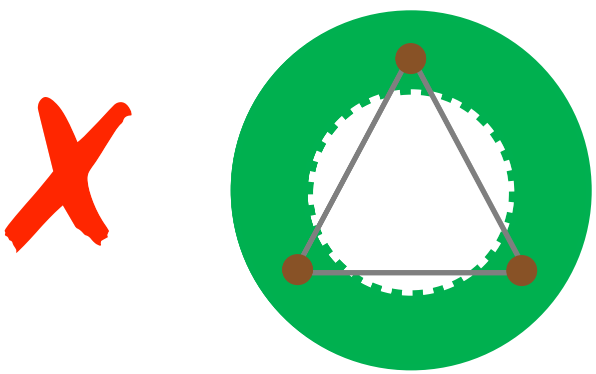
✗ Set of prime numbers?

A Simple Lemma for non-MICP Representability

$$R \subseteq S \quad \text{s.t.} \quad \frac{u+v}{2} \notin S \quad \forall u, v \in R,\ u \neq v \quad |R| = k$$

- MICP formulation of S needs $\log_2 k$ variables :

"Clopen" Spherical shell?



$$\{x \in \mathbb{R}^2 \ : \ 1 < \|x\| \leq 2\}$$

Arbitrarily large R

What About Irrationality?

- A **set** S is **periodic** if and only if:

$$\exists r \in \mathbb{R}^n \setminus \{\mathbf{0}\} \quad \text{s.t.} \quad x + \lambda r \in S \quad \forall \lambda \in \mathbb{Z}_+, \; x \in S$$

- Non-periodic MICP-R sets
  - Dense discrete set $\left\{\sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \, : \, x \in \mathbb{N}\right\} \subseteq [0,1]$

$$\|(z_1, z_1)\|_2 \leq z_2 + 1, \quad \|(z_2, z_2)\|_2 \leq 2z_1, \quad x_1 = y_1 - z_2,$$
$$\|(z_1, z_1)\|_2 \leq y_1, \qquad \|(y_1, y_1)\|_2 \leq 2z_1, \qquad z \in \mathbb{Z}^2$$

  - Set of naturals $\left\{x \in \mathbb{N} \, : \, \sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \notin \left(\varepsilon, 1 - \sqrt{2}\varepsilon\right)\right\}$

$$\|(x_1, x_1)\|_2 \leq x_2 + \varepsilon, \quad \|(x_2, x_2)\|_2 \leq 2x_1 + 2\varepsilon, \quad x \in \mathbb{Z}^2_+$$

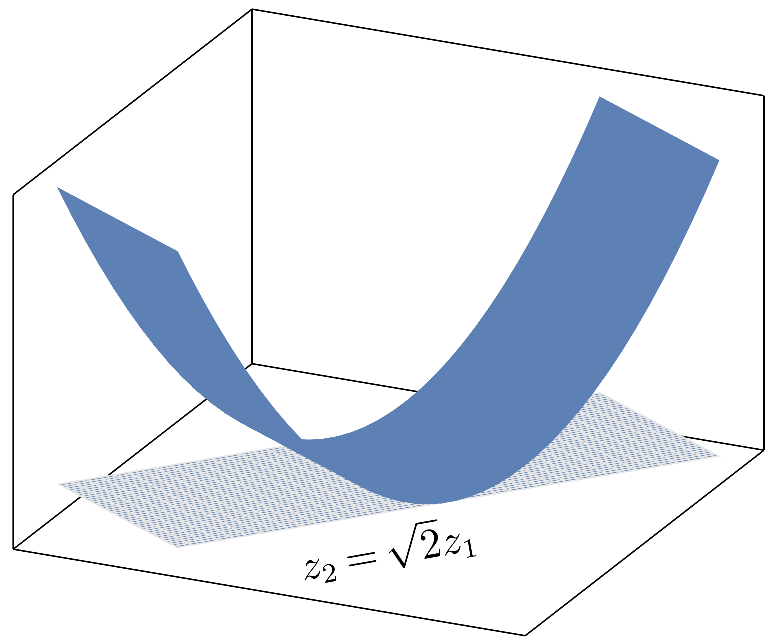
A Definition of Rational MICP-R

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \operatorname{proj}_x(B_z)$$

$$S = \operatorname{proj}_x\left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d\right)\right)$$

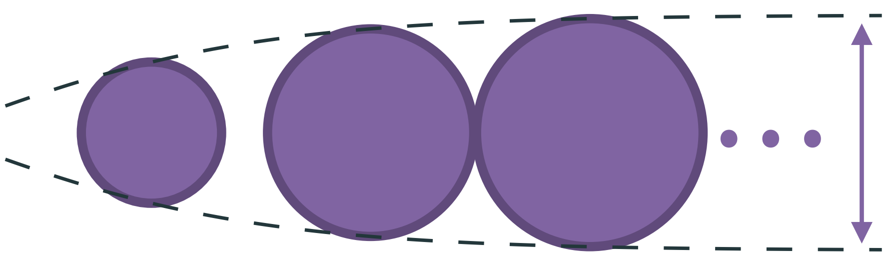
$$I = \operatorname{proj}_z(M)$$

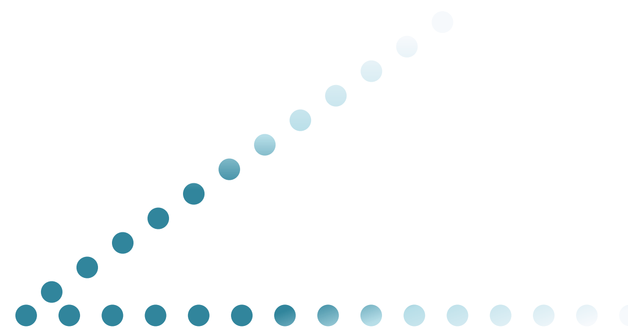
- Any rational affine mapping of index set I :
  - Is bounded, or
  - Has an integer (rational) recession direction
- Irrational directions can hide!



Properties of Rational MICP-R Sets

- A **rational MICP-R** set S is a **finite union** of **convex** or **periodic** sets if
  - S is **closed** and its **convex subsets** have **upper bounded diameter**
  - S is a **not necessarily closed discrete** set
    - **Dense discrete** and **non-periodic naturals** **NOT R-MICPR**





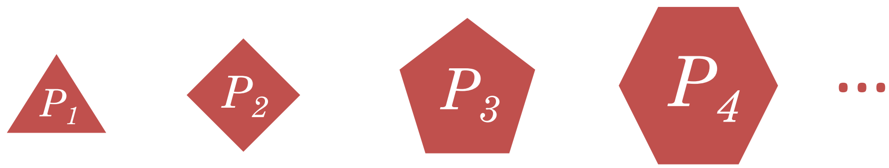
Properties of Rational MICP-R Sets

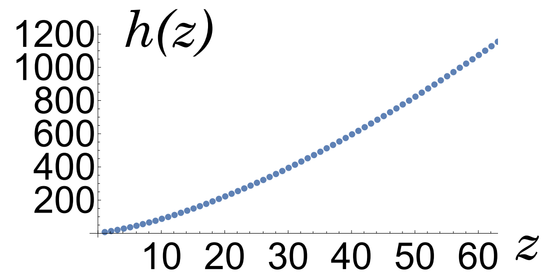
- Other consequences:
  - **Compact rational MICP-R :**
    - **finite unions** of **compact convex** sets
  - **Rational MICP-R** subsets of the **naturals** =
    - **Union** of **finite** points and **one periodic** set
    - **Union** of **finite** points and **one MILP-R** set

$$\{p_i\}_{i=1}^k \cup \{a \cdot t + b \,:\, t \in \mathbb{Z}_+\}$$

- **Rational MICP Representability**:
  - **Closed** under: **Finite Union, Cartesian Product, rational affine transformations and Minkowski sum**
  - **NOT Closed** under **Intersection**.

Rational MICP-R does Not Imply Finite Shapes





- There exists an increasing function h such that:
  - $P_z \subseteq \mathbb{R}^2$ regular $h(z)$-gon centered at $(z, 0)$
  - $P_z \cap P_{z'} = \emptyset, \quad z \neq z'$
  - $S = \bigcup_{z=1}^{\infty} P_z$ is R-MICPR and periodic

- Equal volume ⇒ Finite # of Shapes

MICP with

julia & JuMP

50+ Years of MIP = Significant Solver Speedups

- Algorithmic Improvements (**Machine Independent**):
  - CPLEX → ILOG → IBM
    - v1.2 (1991) – v11 (2007): **29,000 x** speedup
  - GUROBI OPTIMIZATION
    - v1 (2009) – v6.5 (2015): **48.7 x** speedup

  ≈ **1.9 x / year**

- Also MICP:
  - GUROBI OPTIMIZATION
    - v6.0 (2014) – v6.5 (2015) quadratic: **4.43 x**

      (V., Dunning, Huchette, Lubin, 2015)

Accessing MIP Solvers = Modelling Languages

- User-friendly algebraic modelling languages (AML):

Standalone and Fast   Based on General Language and Versatile

- Fast and Versatile, but complicated (and possibly proprietary)
  - Low-level C/C++ solver or Coin-OR interfaces & frameworks
- Best of all worlds?

21st Century Programming/Modelling Languages

- Open-source and free!
- "Floats like python/matlab, stings like C/Fortran"
- Petaflop scaling: C/C++, Fortran and Julia!
- Powerful **compiler** and **meta-programming** features

- Open-source and free!
- Modelling language, interface and software ecosystem for optimization (~20 solvers)
- **Easy to use**, **fast** and advanced
- Integrated into Julia

Large Software Stack and Vibrant Community

juliacon 2018

University College London

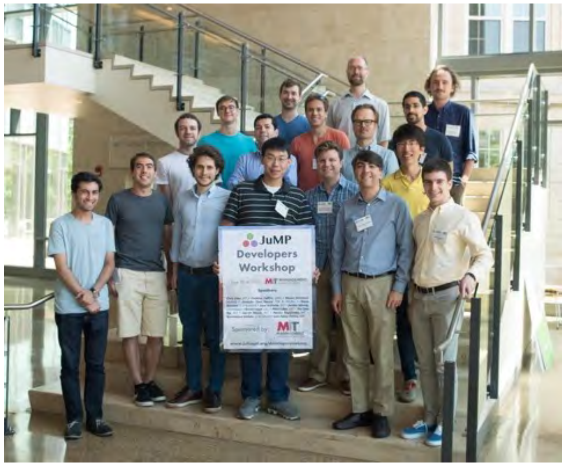
JuliaCon is coming to Baltimore

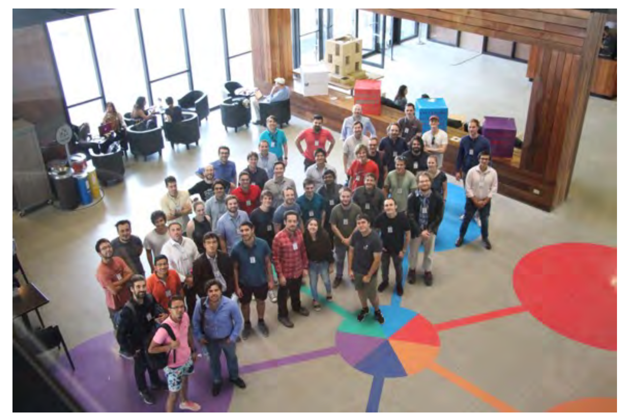
Monday 22nd to Friday 26th of July, 2019

at the University of Maryland Baltimore (UMB),

Baltimore, MD, USA

Large Software Stack and Vibrant Community





State-of-the-JuMP

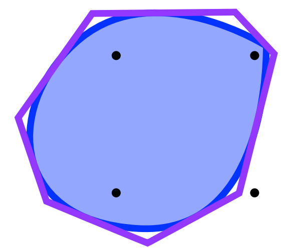
- NumFOCUS Sponsored project since 2018
- NSF funding for annual meeting until 2023 (OAC-1835443)
- Hoping to have the first 3 GSoC students under NumFOCUS umbrella in 2019
- Towards v1.0 (see Miles talk at JuMP-dev):
  - v0.19 released on February 2019:
    - ~2 year and ~30,000 lines of code
  - To-do: Documentation, usability and regressions from v0.18

NUMFOCUS
[FISCALLY SPONSORED PROJECT]
Contribute Code
Donate To JuMP

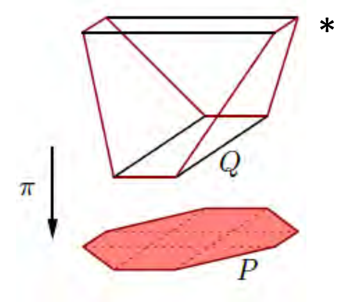
Google
Summer of Code

TOS: M. Lubin, I. Dunning & J. Huchette
Next-Gen: B. Legat & O. Dowson
Contributors: C. Coey, C. Coffrin, J. Dias Garcia, T. Koolen, V. Nesello, F. Pacaud, R. Schwarz, I. Tahiri, J. P. Vielma, U. Worsøe

Polyhedral Outer-Approximation for MICP Solvers

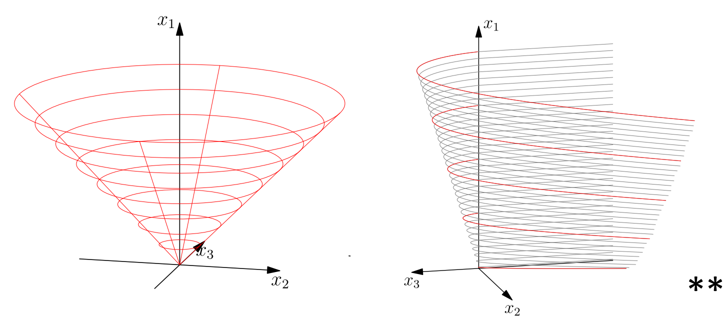


- Dynamically approximate **convex constraints** with **polyhedral** to combine **MILP** and **convex solvers**
  - Basis for most commercial & open source solvers
- Performance keys:
  1. Use power of **projection** to build **extended** or **lifted polyhedral** relaxations
  2. Exploit **geometry** and **duality** from **Conic Programming** (SOCP, SDP, etc.):



$$\min_{\boldsymbol{x}\in\mathbb{R}^N} \quad \langle \boldsymbol{c}, \boldsymbol{x} \rangle : \ \boldsymbol{b}_k - \boldsymbol{A}_k \boldsymbol{x} \in \mathcal{C}_k, \ x_i \in \mathbb{Z}$$

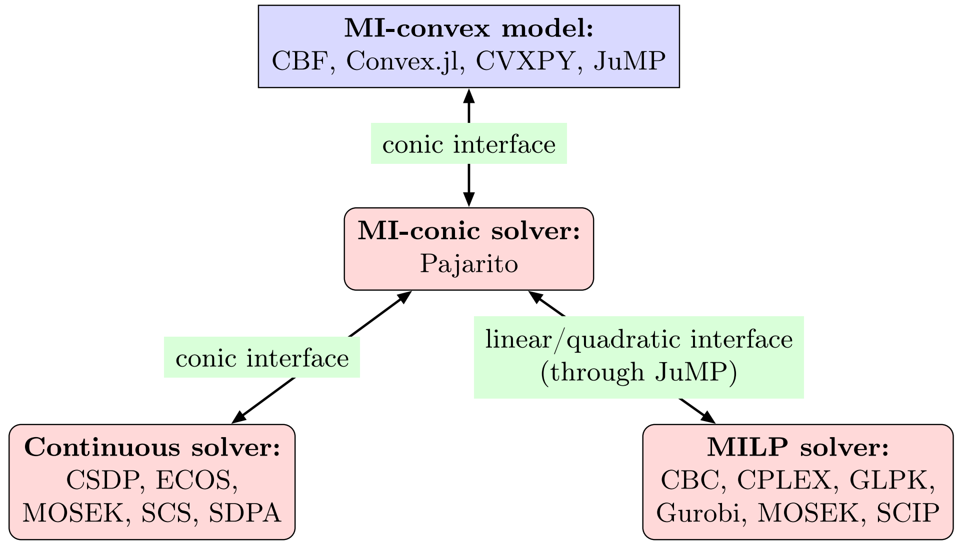
For closed convex cones $\mathcal{C}_k$



* https://rjlipton.wordpress.com, ** MOSEK Modelling Cookbook

Pajarito: A Julia-based MI**CONIC** Solver

Los Alamos
NATIONAL LABORATORY



- Solved gams01, tls5 and tls6 (MINLPLIB2)
- Fastest “open-source” MISOCP solver:
  - faster than Bonmin nearly matches SCIP
- Improves performance & reliability of CPLEX

Stability of **CONIC Interior Point** Algorithms is KEY!

- Why? Avoid non-differentiability issues? Stronger theory?
- Industry change in 2018:
  - KNITRO version 11.0 adds support for SOCP constraints
  - mosek version 9.0 deprecates nonlinear formulations

$$\begin{array}{lll} \min & f(x) & \\ \text{subject to} & g(x) & \leq \quad 0, \end{array}$$

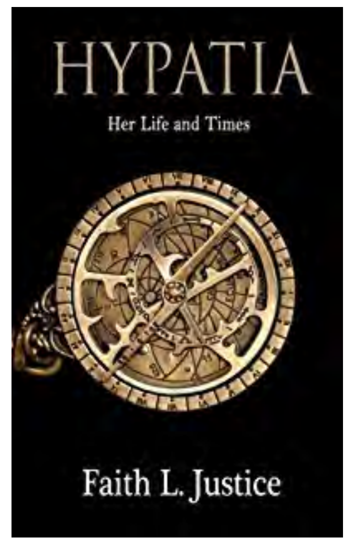
and focuses on pure conic (linear, SOCP, rotated SOCP, SDP, exp & power)

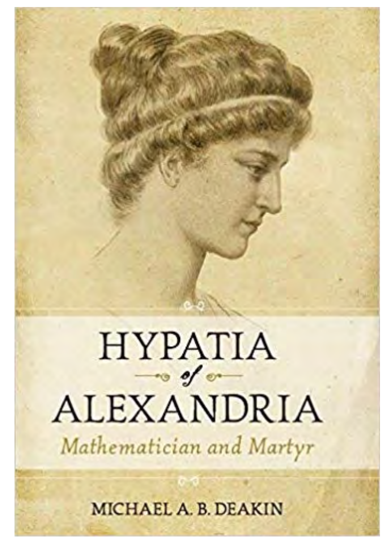
Hypatia: **Pure Julia-based** IPM Beyond **"Standard" Cones**

- A homogeneous interior-point solver for nonsymmetric cones (based on Skajaa and Ye '15, Papp and Yıldız '17, Andersen, Dahl, and Vandenberghe '04-18)
- Cones:
  - LP, SOCP, RSOCP, 3-dim exponential cone, PSD, L_∞, n-dim power cone, spectral norm, log-Det cone,…
  - Sum-of-Squares, "Matrix" Sum-of-Squares, SOCP Sum-of-Squares, …
- Customizable: "Bring your own barrier"

Chris Coey

Lea Kapelevich





Summary

- MICP can model many problems (but not all)
- How to solve MICP? Don't solve MICP, solve MICONIC
- Easy access to optimization modeling and solvers with
- Advanced solver development with Julia &