Mixed-integer Convex Representability

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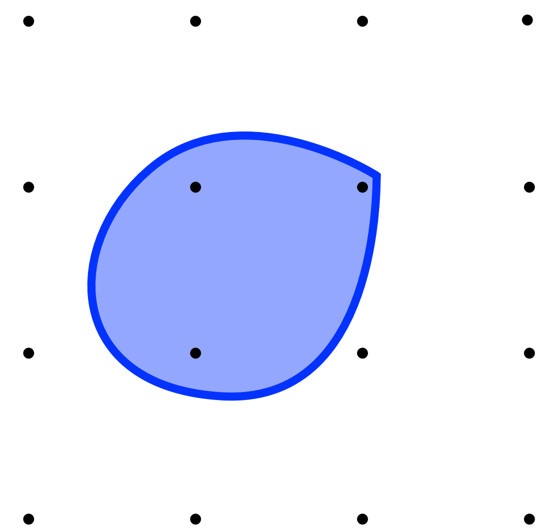
Mixed-Integer Convex Optimization (MICONV)

$$\min \quad f(x)$$

$$s.t.$$

$$x \in C$$

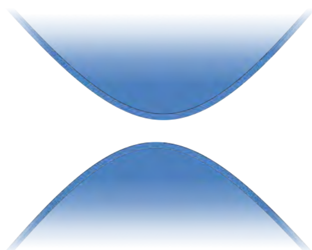
$$x_i \in \mathbb{Z} \quad i \in I$$



convex f and C.

What Can MICONV Model?

Two sheet hyperbola?



$$\{x \in \mathbb{R}^2 \;:\; 1 + x_1^2 \leq x_2^2\}$$

Spherical shell?

$$\{x \in \mathbb{R}^2 \;:\; 1 \leq \|x\| \leq 2\}$$

- Integer points in parabola $\{(x, x^2) : x \in \mathbb{Z}\}$?
- The set of $n \times n$ matrices with rank $\leq k$?
- Set of prime numbers?

MICONV Can Model Any Finite Union of (Closed) Convex Sets

- Let $T_1, \dots, T_k$ be closed convex set. A MICONV formulation of $\boldsymbol{x} \in \bigcup_{i=1}^{k} T_i$:

$$
\begin{aligned}
(\boldsymbol{x}^i, z_i) &\in \overline{\text{cone}}(T_i \times \{1\}) && \forall i \in \{1, \dots, k\} \\
\left\|\boldsymbol{x}^i\right\|_2^2 &\leq z_i t_i. && \forall i \in \{1, \dots, k\} \\
\sum_{i=1}^{k} \boldsymbol{x}^i &= \boldsymbol{x}, \\
\sum_{i=1}^{k} z_i &= 1, \\
\boldsymbol{z} &\in \{0,1\}^k, \\
t &\in \mathbb{R}_+^k, \\
\boldsymbol{x}^i &\in \mathbb{R}^n && \forall i \in \{1, \dots, k\}
\end{aligned}
$$

What Can MICONV Model?

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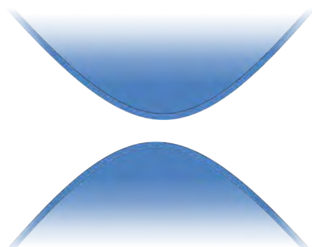

Simple MICONV Formulation

$$\{x \in \mathbb{R}^2 \;:\; 1 + x_1^2 \leq x_2^2\}$$

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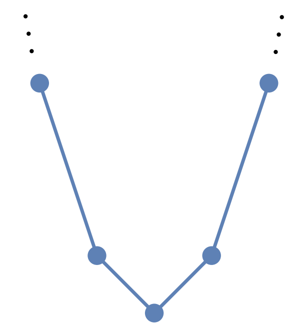


A Simple Obstruction for MICONV Formulations

- S cannot have a MICONV formulation if there exists:
  - There exist infinite $R \subseteq S \quad \text{s.t.}$

$$\frac{u+v}{2} \notin S \quad \forall u, v \in R,\ u \neq v$$

✗ **Spherical shell** $\{x \in \mathbb{R}^2 \ : \ 1 \leq \|x\| \leq 2\}$

What Can MICONV Model?

Two sheet hyperbola?

✓ 

$\{x \in \mathbb{R}^2 \;:\; 1 + x_1^2 \leq x_2^2\}$

Spherical shell?

✗ 

$\{x \in \mathbb{R}^2 \;:\; 1 \leq \|x\| \leq 2\}$

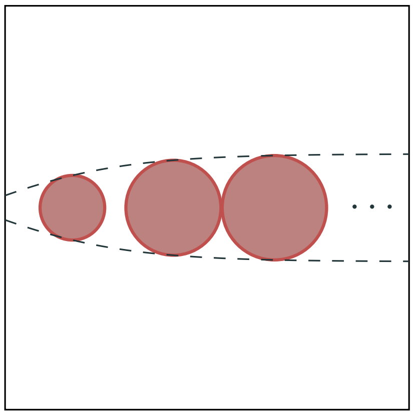
✗ Integer points in parabola $\{(x, x^2) : x \in \mathbb{Z}\}$?

✗ The set of $n \times n$ matrices with rank $\leq k$?

✗ Set of prime numbers?

Does have non-convex polynomial MIP formulation

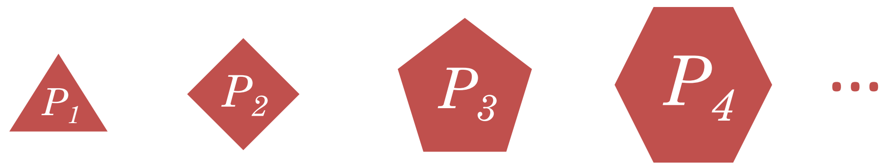
MICONV = __**Structured**__ ***Countably Infinite*** Unions of Convex Sets



$$\sqrt{(x_1 - 2z)^2 + x_2^2} \leq 1 - 1/z,$$
$$z \geq 1, \quad z \in \mathbb{Z}$$

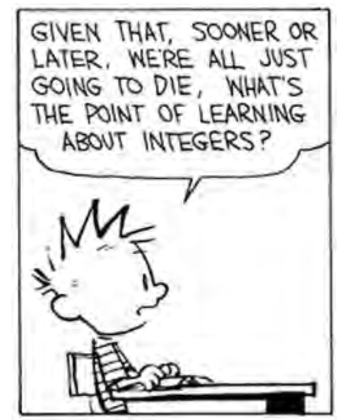
Unbounded Integer Variables

- Can be “strange” unions, e.g. :
  - Infinite number of shapes



- Can be REALLY strange:
  - Dense discrete set

$$\left\{ \sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \; : \; x \in \mathbb{N} \right\} \subseteq [0, 1]$$



$$\|(z_1, z_1)\|_2 \leq z_2 + 1, \quad \|(z_2, z_2)\|_2 \leq 2z_1, \quad x_1 = y_1 - z_2,$$
$$\|(z_1, z_1)\|_2 \leq y_1, \quad \|(y_1, y_1)\|_2 \leq 2z_1, \quad z \in \mathbb{Z}^2$$

Questions?

A Simple Lemma for non-MICP Representability

- Obstruction for MICP representability of S :

$$\text{infinite } R \subseteq S \quad \text{s.t.} \quad \frac{u+v}{2} \notin S \quad \forall u, v \in R,\, u \neq v$$

Proof: Assume for contradiction there exists M such that:

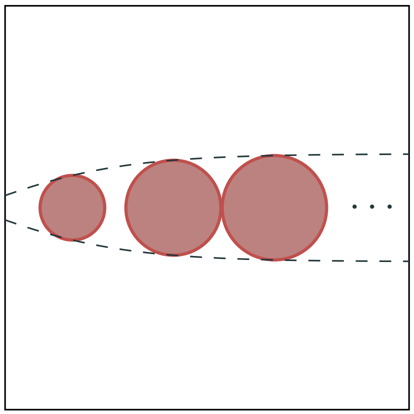
$$S = \operatorname{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

$$\begin{matrix} (u, y_u, z_u) \in M \\ (v, y_v, z_v) \in M \end{matrix} \quad \Rightarrow \quad \frac{z_u + z_v}{2} \notin \mathbb{Z}^d$$

$$z_u \equiv z_v \pmod 2 \text{ component-wise} \quad \Rightarrow \quad \frac{z_u + z_v}{2} \in \mathbb{Z}^d$$

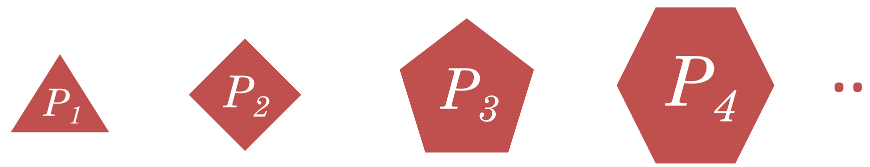
$$\text{component-wise parity classes} = 2^d < |R| = \infty \qquad \Rightarrow\!\Leftarrow$$

MICONV = **<u>Structured</u>** ***Countably Infinite*** Unions of Convex Sets



$$\sqrt{(x_1 - 2z)^2 + x_2^2} \leq 1 - 1/z,$$
$$z \geq 1, \quad z \in \mathbb{Z}$$

- Can be "strange" unions, e.g. :
  - Infinite number of shapes



- There exist an increasing function h such that:
  - $P_z \subseteq \mathbb{R}^2$ regular $h(z)$-gon
  - $S = \bigcup_{z=1}^{\infty} P_z$ has MICONV formulation

- Equal volume ⇒ Finite # of Shapes

Sets with MICONV Formulations can be REALLY "Strange"

- Dense discrete set $\left\{\sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \,:\, x \in \mathbb{N}\right\} \subseteq [0,1]$

$$\|(z_1, z_1)\|_2 \le z_2 + 1, \quad \|(z_2, z_2)\|_2 \le 2z_1, \quad x_1 = y_1 - z_2,$$
$$\|(z_1, z_1)\|_2 \le y_1, \quad \|(y_1, y_1)\|_2 \le 2z_1, \quad z \in \mathbb{Z}^2$$

- Non-Periodic Set of naturals $\left\{x \in \mathbb{N} \,:\, \sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \notin \left(\varepsilon, 1 - \sqrt{2}\varepsilon\right)\right\}$

$$\|(x_1, x_1)\|_2 \le x_2 + \varepsilon,$$
$$\|(x_2, x_2)\|_2 \le 2x_1 + 2\varepsilon, \quad x \in \mathbb{Z}^2_+$$

"God made the integers, all else is the work of man"

- Leopold Kronecker