The Geometry of Nonlinear Mixed Integer Programming Formulations

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Joint work with Miles Lubin and Ilias Zadik

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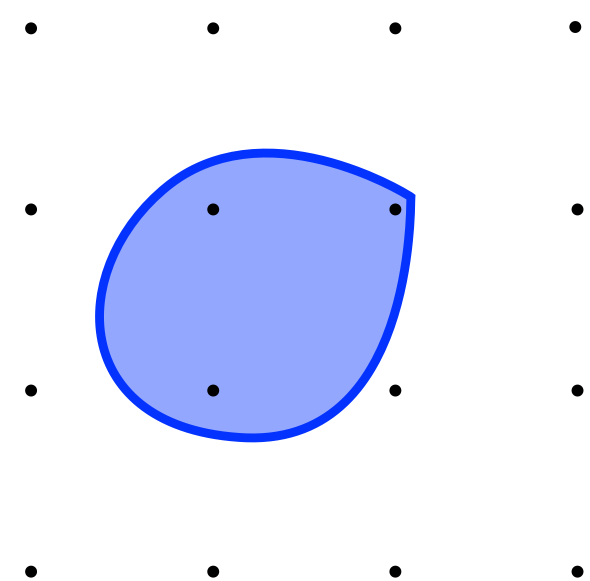
Mixed Integer Convex Optimization (MICP)

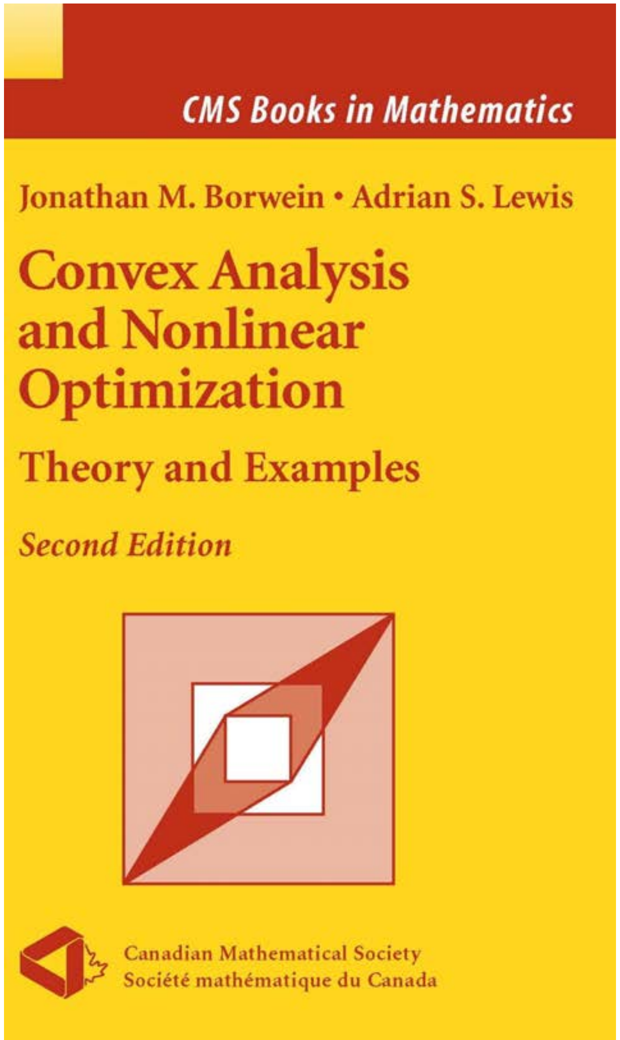
$$\min \quad f(x)$$
$$s.t.$$
$$x \in C$$
$$x_i \in \mathbb{Z} \quad i \in I$$

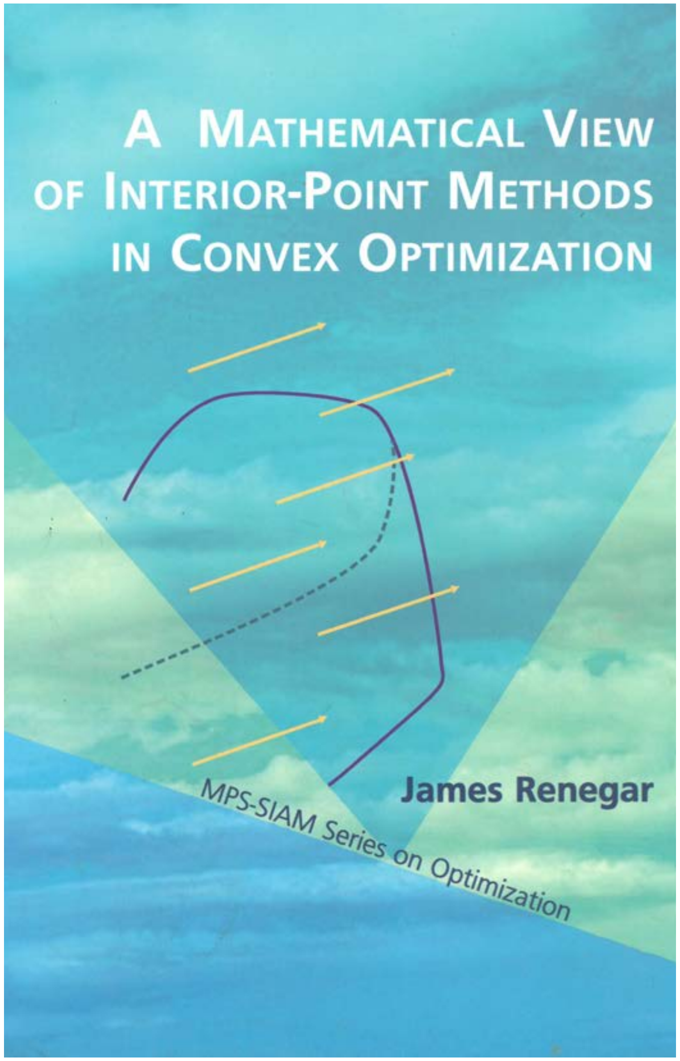
Convex f and C.

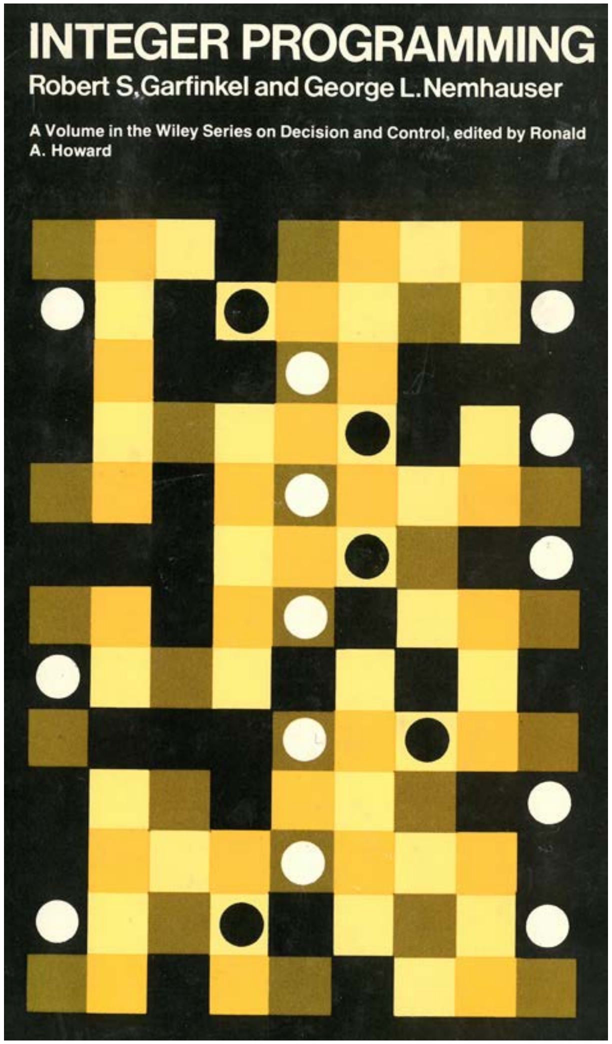


- Examples:
  - MI-Second Order Cone Programming (MISOCP)
  - MI-Semidefinite Programming (MISDP)

Standing On The Shoulders of Gigants







MI-SOCP Solvers and Applications

- Effective and improving solvers:
  - Gurobi v6.0 (2014) – v6.5 (2015) MISOCP: 4.43 x (V., Dunning, Huchette, Lubin, 2017)
- Applications:
  - Portfolio optimization, pricing, regression, experimental design, etc.

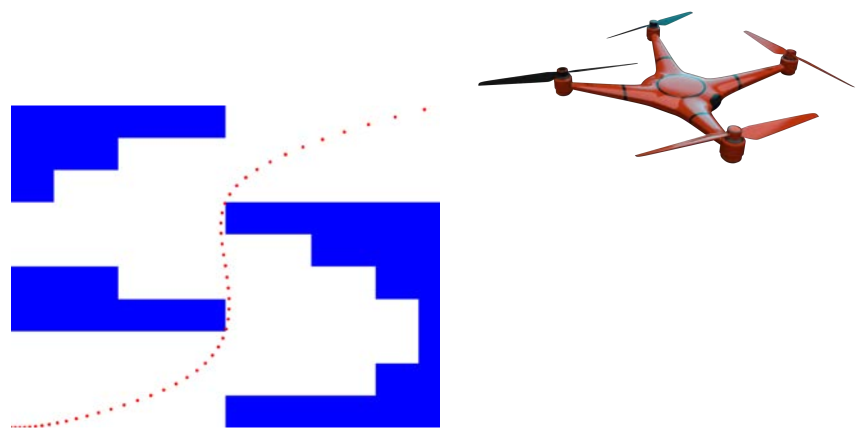
http://www.gurobi.com/company/example-customers

MI-SDP (& other cones) Solvers and Applications

- Emerging Solvers:

Coey, Bent, Lubin, V. and Yamangil '16

- Applications:
  - Collision avoidance with mixed integer sum-of-squares for optimal control of polynomial trajectories



Outline

- General Mixed Integer Convex Representability
  - What can be modeled with MICP ?
  - Joint work with Miles Lubin and Ilias Zadik
- 0-1 Mixed Integer Convex Representability
  - Unions of Convex Sets
  - Small and Strong Formulations

MICP Formulations and Representability

- A set $S \subseteq \mathbb{R}^n$ is MICP representable (MICPR) if it has an MICP formulation:
  - A closed convex set $M \subseteq \mathbb{R}^{n+p+d}$
  - auxiliary continuous variables $y \in \mathbb{R}^p$
  - auxiliary integer variables $z \in \mathbb{Z}^d$

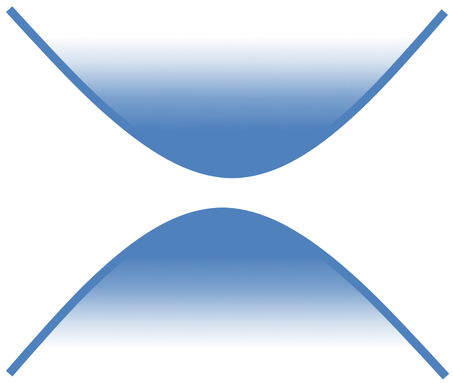
$$x \in S \quad \Leftrightarrow \quad \begin{array}{c} \exists (y, z) \in \mathbb{R}^p \times \mathbb{Z}^d \text{ s.t.} \\ (x, y, z) \in M \end{array}$$

or equivalently

$$S = \operatorname{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

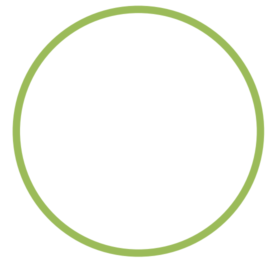
What Sets are MICP Representable (MICPR) ?

Two sheet hyperbola?



$$\{x \in \mathbb{R}^2 \; : \; 1 + x_1^2 \leq x_2^2\}$$

Spherical shell?



$$\{x \in \mathbb{R}^2 \; : \; 1 \leq \|x\| \leq 2\}$$

- Discrete subsets of the real line or natural numbers:
  - Dense discrete set? $\left\{\sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \; : \; x \in \mathbb{N}\right\} \subseteq [0,1]$
  - Set of prime numbers?

A Simple Lemma for non-MICP Representability

- Obstruction for MICP representability of S :

$$\text{infinite } R \subseteq S \quad \text{s.t.} \quad \frac{u+v}{2} \notin S \quad \forall u, v \in R,\ u \neq v$$

Proof: Assume for contradiction there exists M such that:

$$S = \operatorname{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

$$\begin{aligned} (u, y_u, z_u) &\in M \\ (v, y_v, z_v) &\in M \end{aligned} \quad \Rightarrow \quad \frac{z_u + z_v}{2} \notin \mathbb{Z}^d$$

$$z_u \equiv z_v \pmod 2 \text{ component-wise} \quad \Rightarrow \quad \frac{z_u + z_v}{2} \in \mathbb{Z}^d$$

$$\text{component-wise parity classes} = 2^d < |R| = \infty \qquad \Rightarrow\!\!\Leftarrow$$

Examples of non-MICPR Sets From Lemma

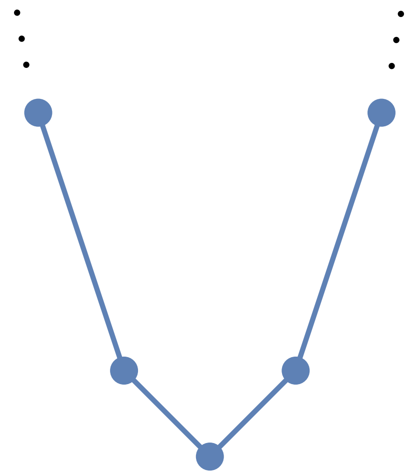
✗ **Spherical shell** $\{x \in \mathbb{R}^2 \,:\, 1 \leq \|x\| \leq 2\}$

✗ **Set of prime numbers**

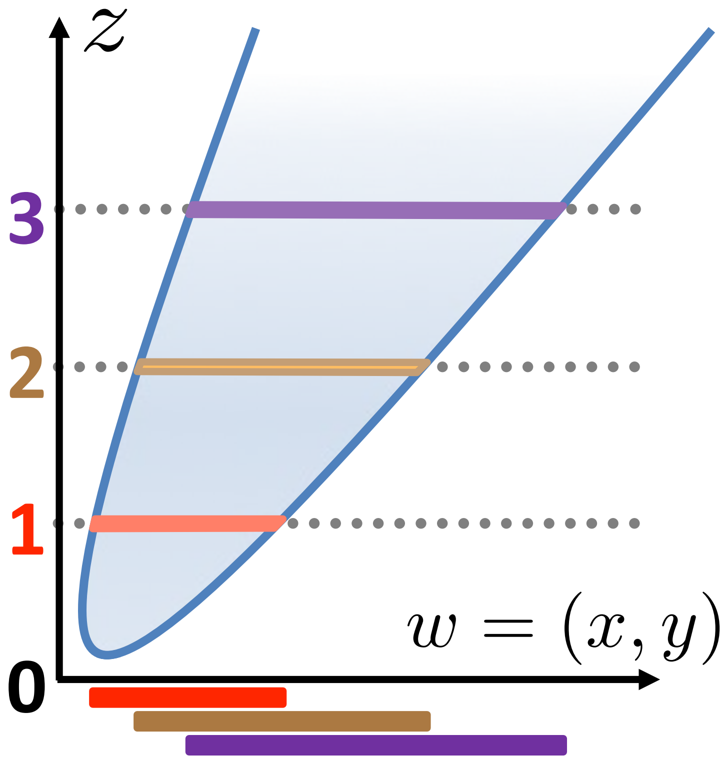
- However **prime numbers** has a **non-convex** polynomial **integer** programming formulation

✗ Set of Matrices of rank at most k

✗ Piecewise linear interpolation of x^2 at all integers



MICPR = Convex Sets Indexed by Integers in Convex



closed convex M

$$S = \operatorname{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

$$\Updownarrow$$

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \operatorname{proj}_x \left(B_z \right)$$

I convex and $B_z : \mathbb{R}^d \rightrightarrows \mathbb{R}^{n+p}$ closed and convex:

- $\left(w_m \in B_{z_m}, \quad (w_m, z_m) \underset{m}{\longrightarrow} (w, z) \right) \Rightarrow w \in B_z$
- $\lambda B_z + (1 - \lambda) B_{z'} \subseteq B_{\lambda z + (1-\lambda) z'}$

MI-Linear Programming: Rational Polyhedral M

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \operatorname{proj}_x (B_z) = \bigcup_{z \in I \cap \mathbb{Z}^d} P_z$$

- P_z = rational polyhedra with the same recession cone
- Representation simplifies to (Jeroslow and Lowe '84):
  - $S = \bigcup_{i=1}^{k} P_i + \left\{ \sum_{i=1}^{t} \lambda_i r^i \;:\; \lambda \in \mathbb{Z}_+^t \right\}$
  - P_i = rational polytopes
  - Very regular infinite union
- Bounded or 0-1 MILPR / MICPR = Bounded I = Finite union
  - MILPR: of polyhedra with the same recession cone
  - MICPR: of non-polyhedral convex sets …

Extra from MICP 1: Non-Polyhedral Unions

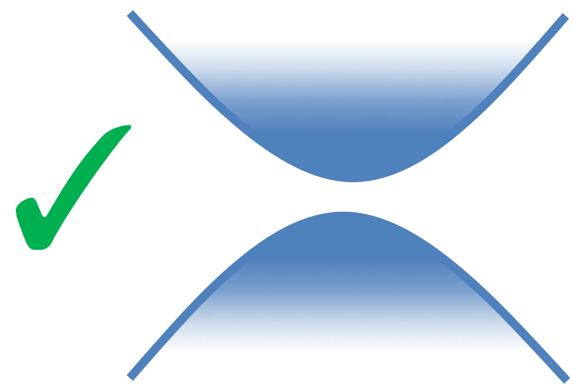
$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \text{proj}_x \left(B_z \right)$$

1. Unions of Non-Polyhedral sets

Plus Projection:

2. Unions of non-closed sets
3. Unions of convex sets with different recession cones

Two sheet hyperbola?



$$\{x \in \mathbb{R}^2 \ : \ 1 + x_1^2 \leq x_2^2\}$$

$$B_z = \left\{ (x, y) \in \mathbb{R}^{n+1} \ : \ x \in C_z, \quad \|x\|_2^2 \leq y \right\}$$

Extra from MICP 2: Non-Polyhedral Index Set

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \operatorname{proj}_x (B_z)$$

"God made the integers, all else is the work of man"

- Leopold Kronecker

- Integers + **non-rational** unbounded **ray** = Trouble !

✓ Dense discrete set $\left\{ \sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \;:\; x \in \mathbb{N} \right\} \subseteq [0,1]$

$$\|(z_1, z_1)\|_2 \le z_2 + 1, \quad \|(z_2, z_2)\|_2 \le 2z_1, \quad x_1 = y_2 - z_2,$$
$$\|(z_1, z_1)\|_2 \le y_1, \quad \|(y_1, y_1)\|_2 \le 2z_1, \quad z \in \mathbb{Z}^2$$

One (Somewhat Extreme) Way to Add Regularity

$$S = \operatorname{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right), \quad I = \operatorname{proj}_z \left(M \right)$$

- $M = B + K$:
  - B compact convex set
  - K rational polyhedral cone
- Then
  - $S = \bigcup_{i=1}^{k} C_i + \left\{ \sum_{i=1}^{t} \lambda_i r^i \; : \; \lambda \in \mathbb{Z}^t_+ \right\}$
  - C_i = compact convex sets
- Less extreme, but still well behaved "**Rational** MICPR":
  - Any **rational** affine mapping of index set I Is bounded, or has a **rational** recession direction

Bounded or 0-1 MICP Formulations for Unions of Convex Sets

A Classical Strong Formulation for $\bigcup_{i=1}^{k} C_i$

$$C_i = \left\{ x \in \mathbb{R}^n \; : \; A^i x \preceq_i b^i \right\}, \quad C_i^\infty = C_j^\infty$$

$$A^i x^i \preceq_i b^i z_i, \quad \forall i \in [k]$$

$$\sum_{i=1}^{k} x^i = x,$$

$$\sum_{i=1}^{k} z_i = 1, \quad z \in \{0,1\}^k$$

$$x,\; x^i \in \mathbb{R}^n, \quad \forall i \in [k]$$

- Auxiliary continuous variables are copies of original variables
  - $y = \left(x^i\right)_{i=1}^{k}$
- "Ideal" Formulation Strength:
  - Extreme points of continuous relaxation satisfy **integrality constraints** on z
  - **Variable copies** crucial here, but **slow down computations** (usually worse than Big-M)

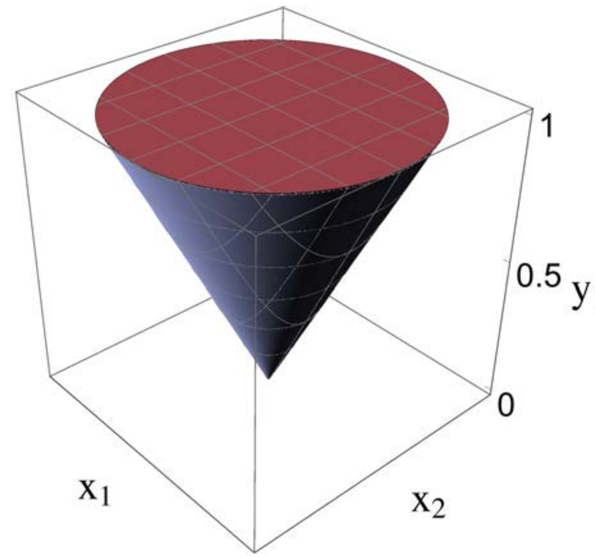
- Balas, Jeroslow and Lowe (Polyhedral), Ben-tal, Nemirovski, Helton, Nie (Conic)

Generic Geometric Formulation = Gauge Functions

- For C such that $\mathbf{0} \in \operatorname{int}(C)$ let:

$$\gamma_C(x) := \inf\{\lambda > 0 \ : \ x \in \lambda C\}$$

$$\operatorname{epi}(\gamma_C) = \operatorname{cone}(C \times \{1\})$$



- If $b^i \in C_i$ then ideal formulation:

$$\gamma_{C^i - \{b^i\}}\left(x^i - z_i b^i\right) \leq z_i \qquad \forall i \in [k]$$

$$\sum_{i=1}^{k} x^i = x$$

$$\sum_{i=1}^{k} z_i = 1$$

$$z \in \{0,1\}^k$$

$$x, \ x^i \in \mathbb{R}^n \qquad \forall i \in [k]$$

Simple Ideal Formulation without Variable Copies

- Unions of (nearly) Homothetic Closed Convex Sets (V. 17):

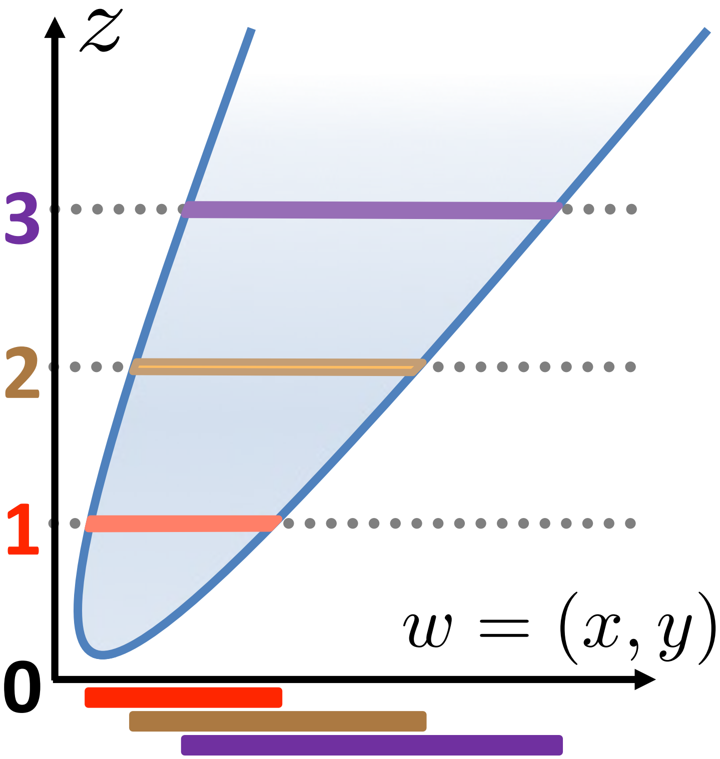
$$C_i = \lambda_i C + b^i + C^\infty$$

$$\gamma_C\left(x - \sum_{i=1}^n z_i b^i\right) \leq \sum_{i=1}^n \lambda_i z_i$$

$$\sum_{i=1}^n z_i = 1,\ z \in \{0,1\}^n$$

≈ to polyhedral results from Balas ‘85, Jeroslow ’88 and Blair ‘90

Embedding Formulation Construction



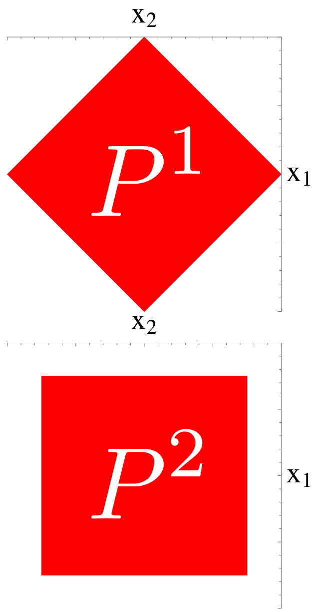
$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \overset{C_z}{\cancel{\operatorname{proj}_x (B_z)}}$$

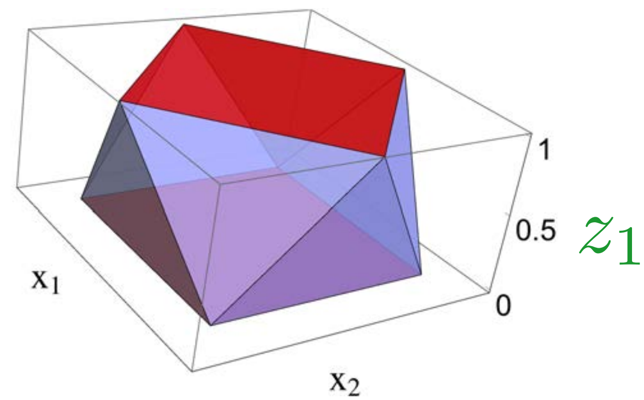
$$M = \overset{\text{conv}}{\cancel{\overline{\text{conv}}}} \left(\bigcup_{z \in I \cap \mathbb{Z}^d} \overset{C_z}{\cancel{B_z}} \times \{z\} \right)$$

I convex and $B_z : \mathbb{R}^d \rightrightarrows \mathbb{R}^{n+p}$ closed and convex:

- $C_z^\infty = C_{z'}^\infty$ • $\cancel{\left(w_m \in B_{z_m},\ (w_m, z_m) \to_m (w, z)\right) \Rightarrow w \in B_z}$
- $I \cap \mathbb{Z}^d \subseteq \{0,1\}^d$ • $\cancel{\lambda B_z + (1-\lambda) B_{z'} \subseteq B_{\lambda z + (1-\lambda) z'}}$

Embedding Formulation = Automatically **Ideal**





$$\text{conv}\left(\bigcup_{i=1}^{k} P^i \times \{h^i\}\right)$$

- Originally for Polyhedra (V. '17)
  - Small size with careful choice of **encoding** $\{h^i\}_{i=1}^{k} \subseteq \{0,1\}^d$
- Extensions to **general integers**, practical construction techniques, computations, applications and software tools:
  - Huchette and V. '17a,b,c; Huchette, Dey and V. '17

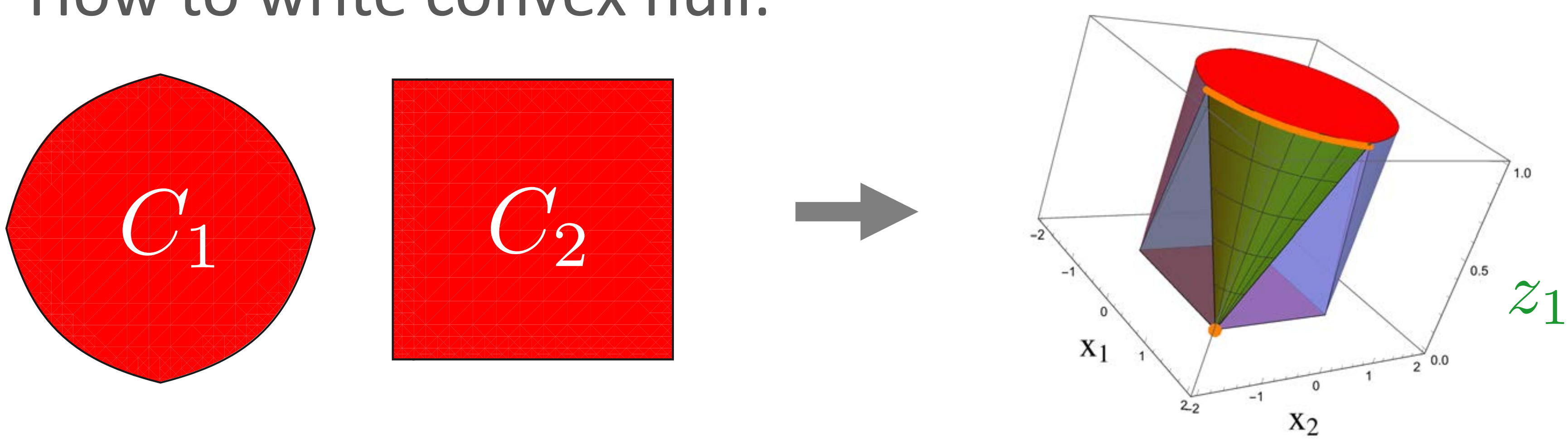
Focus for Non-Polyhedral Embedding Formulations

- Unary encoding: $\{h^i\}_{i=1}^k = \{e^i\}_{i=1}^k \subseteq \{0,1\}^k$
  - Related to Cayley Embedding for Minkoski sums
  - Homothetic formulation

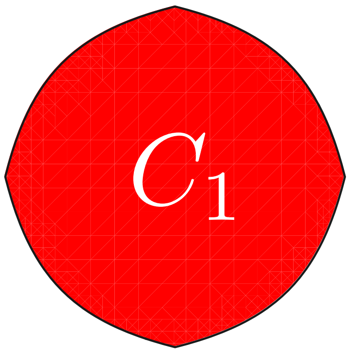
$$\gamma_C\left(x - \sum_{i=1}^n z_i b^i\right) \le \sum_{i=1}^n \lambda_i z_i$$

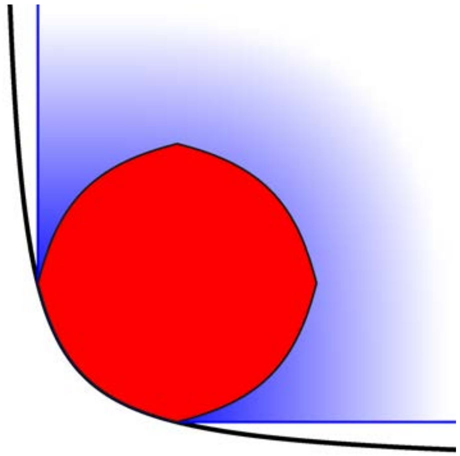
$$\sum_{i=1}^n z_i = 1, \quad z \in \{0,1\}^n$$

- How to write convex hull:



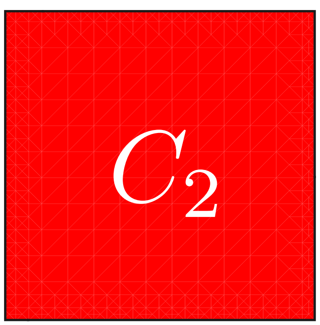
Sticking Homothetic Formulations Together

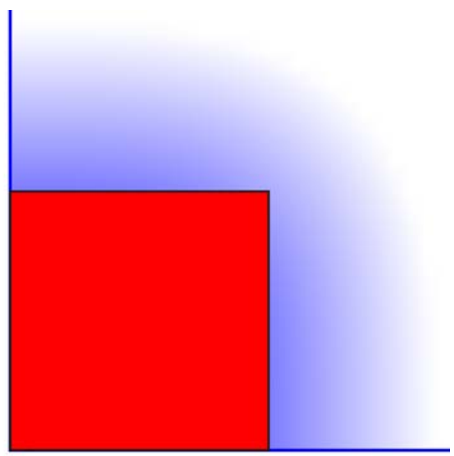




Valid, but not ideal!

Combine 4 homothetic formulations

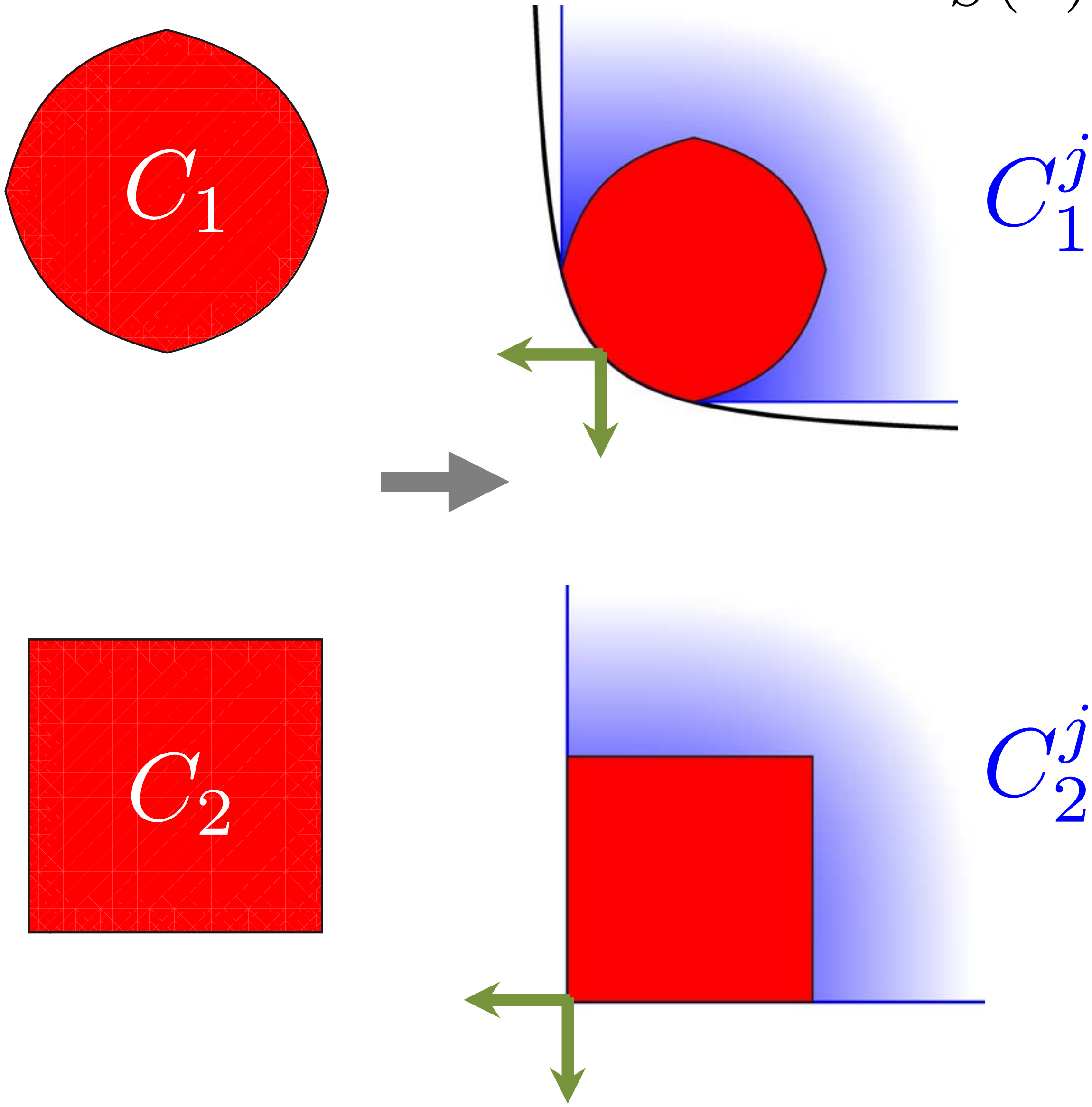




Right relaxations yield ideal formulation

Sufficient Conditions For Ideal Formulation

$$\sigma_S(u) := \sup\{u \cdot x \;:\; x \in S\}$$



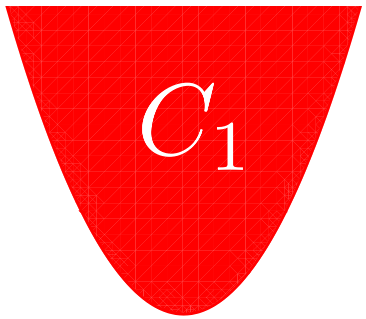
$$\forall u \in \mathbb{R}^n \quad \exists j$$

$$s.t.$$

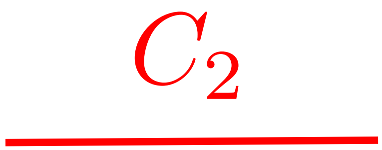
$$\sigma_{C_i}(u) = \sigma_{C_i^j}(u)$$

$$\forall i \in \{1, 2\}$$

May Need to "Find" Homothetic Constraints



$x_1^2 \le x_2 \le 1$



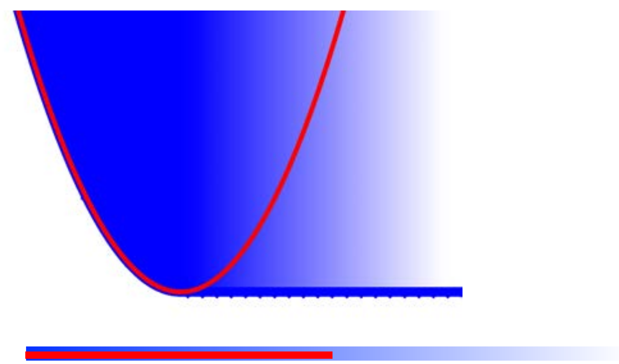
$[-1, 1] \times 0$

$C_1 + (\mathbb{R}_+ \times \{0\})$:

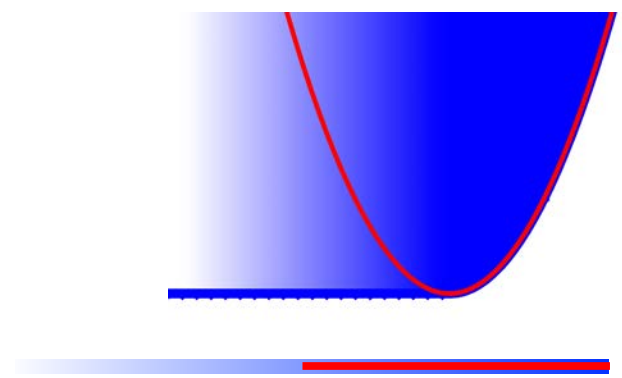
$$(\max\{x_1, 0\})^2 \le x_2 \le 1$$

Similar to Bestuzheva et al. '16 who divide sets in two.

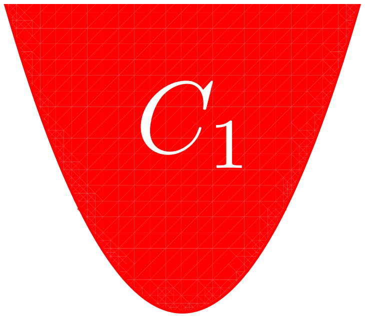
$C_i + (\mathbb{R}_+ \times \{0\})$



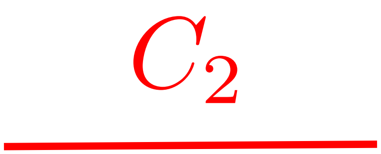
$C_i + (\mathbb{R}_- \times \{0\})$



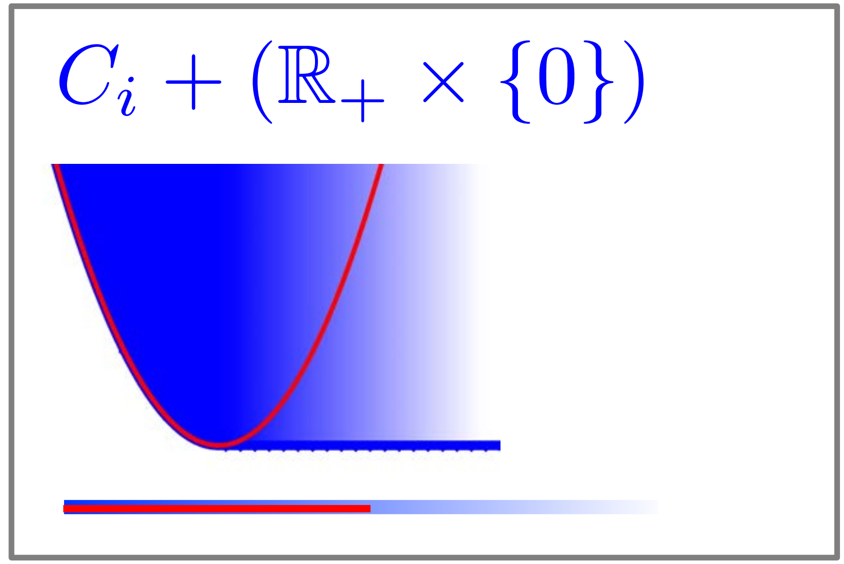
Algebraic Representation Issues



$$x_1^2 \leq x_2 \leq 1$$

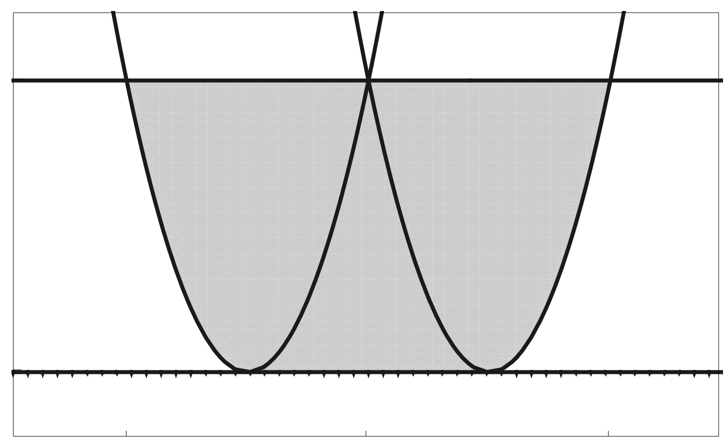


$$[-1, 1] \times 0$$



$$C_1 + (\mathbb{R}_+ \times \{0\}) : \ (\max\{x_1, 0\})^2 \leq x_2 \leq 1$$

- Non-basic semi-algebraic set contained in formulation.
- Finite polynomial inequalities requires max or auxiliary vars.



Summary

- General mixed integer convex representability (MICPR):
  - Infinite union of convex sets with specialc structure
  - More results/questions on regularity (arXiv:1706.05135)
- Bounded MICPR = Finite unions of Convex Sets
  - Variable copies = strong (ideal), but slow computation
  - Copies can be removed, but possibly at a prize
  - More on the paper (arXiv:1704.03954):
    - MIP-solver compatible formulations = gauge calculus.
    - More examples: generalizations and size reductions
    - Conditions for piecewise formulations to be ideal