Cutting Planes and Elementary Closures for Non-linear Integer Programming

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joint work with

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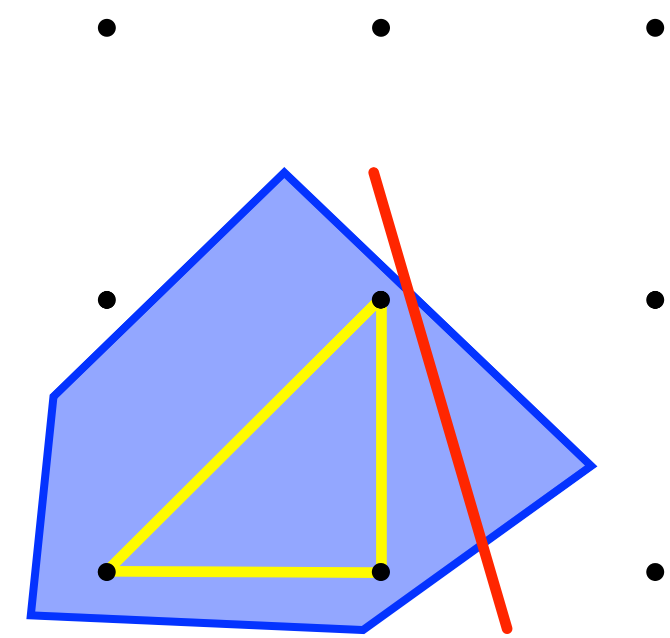
Cutting Planes for Integer Programming

- Valid Inequalities for the convex hull of integer feasible solutions.
- 50+ Years of development for Linear Integer Programming.
- Used to get tighter Linear Programming Relaxations.
- Crucial for state of the art solvers.

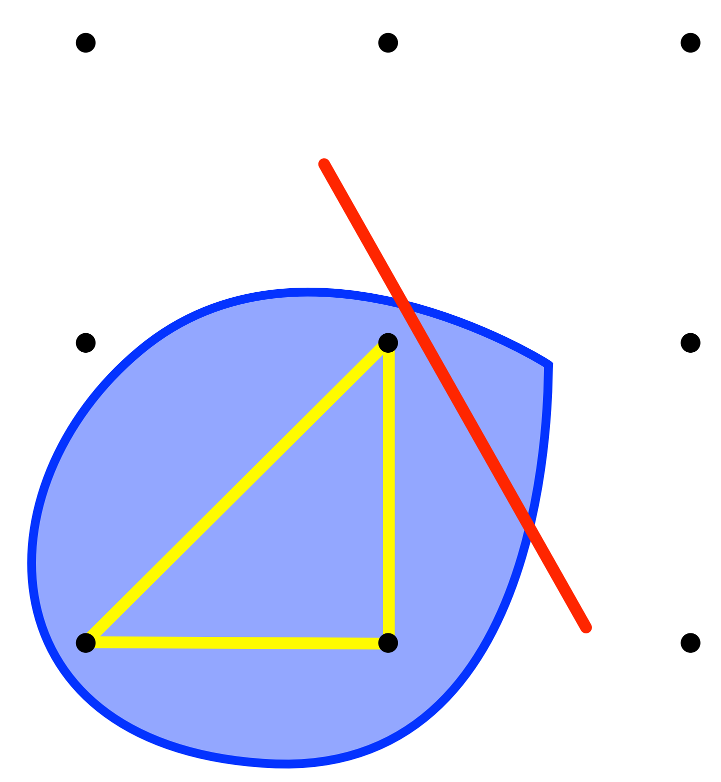
$$\begin{aligned} \max \quad & \langle c, x \rangle \\ s.t. \quad & \\ & Ax \leq b \\ & x \in \mathbb{Z}^n \end{aligned}$$



Convex Non-Linear Integer Programming

- Problems with convex continuous relaxation.
- Many applications, results and algorithms available.
- Cutting planes significantly less developed.
- Need new tools: linear results strongly rely on rationals.

$$\begin{aligned} \max \quad & \langle c, x \rangle \\ s.t. \quad & \\ & g_i(x) \leq 0,\ i \in I \\ & x \in \mathbb{Z}^n \end{aligned}$$



Two Classic Cutting Planes

- Chvátal-Gomory Cuts (Gomory 68, Chvátal 73):
  - AKA Gomory Fractional Cut
  - Simple, but yield pure cutting plane algorithm, Blossom's for Matching and Comb's for TSP.
- Split Cuts (Cook, Kannan and Shrijver 1990):
  - AKA MIG (Gomory 1960) and MIR (Nemhauser and Wolsey 1988)
  - Yield Flow Cover Cuts and modern IP solvers.

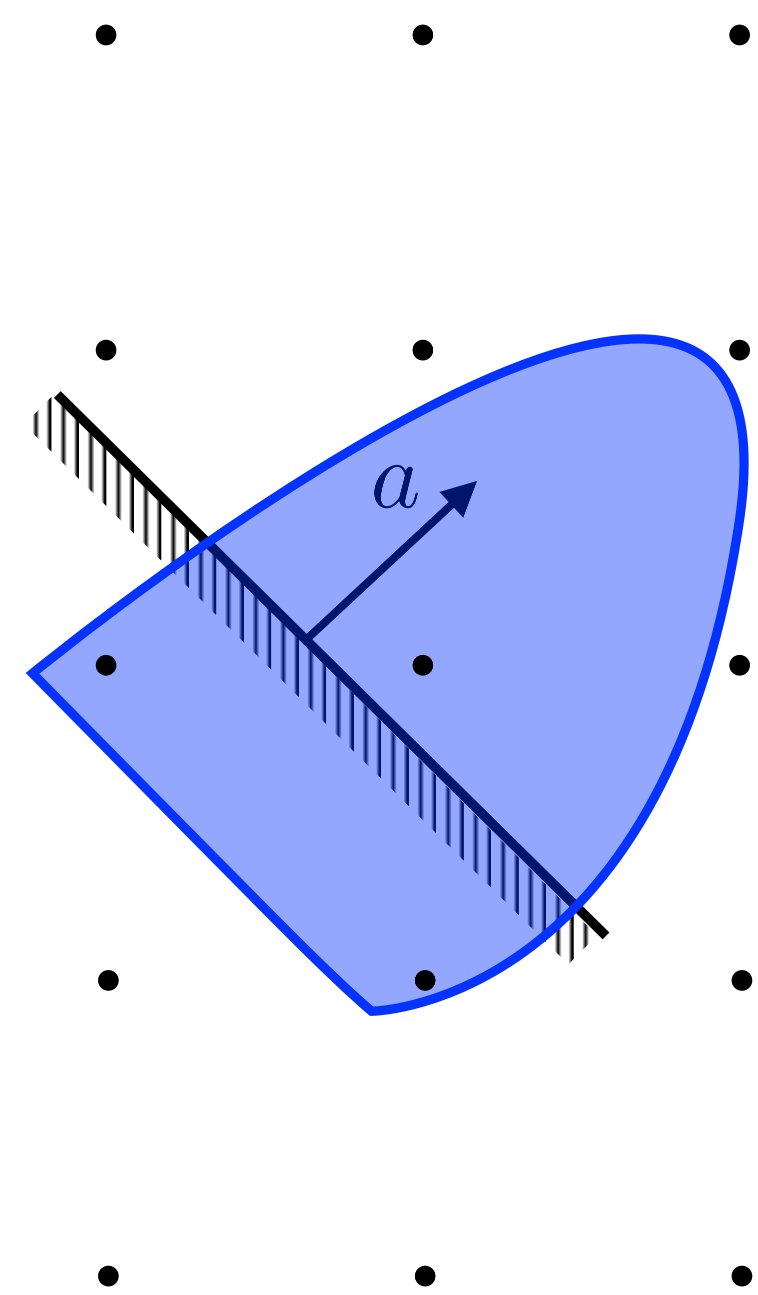
Outline

- Chvátal-Gomory Cuts for Non-Linear IP:
  - Polyhedrality of the Chvátal-Gomory Closure
- Split Cuts for Non-Linear IP:
  - Closed form Expressions.
  - Finite Generation v/s Polyhedrality of Split Closure
- Other Results and Open Questions

Chvátal-Gomory Cuts

CG Cuts and Support Function

$$\sigma_C(a) := \sup\{\langle a, x\rangle \;:\; x \in C\}$$



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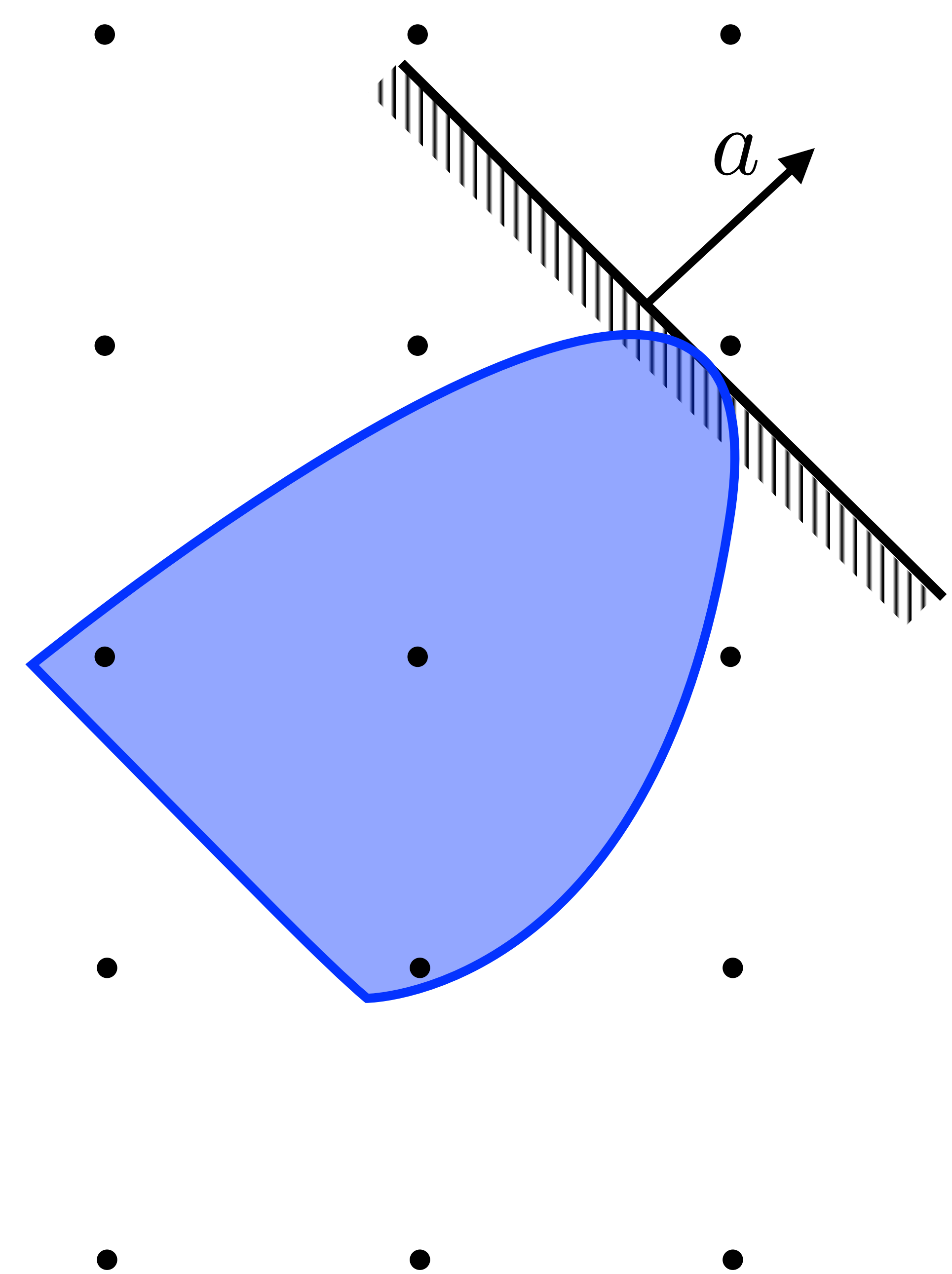

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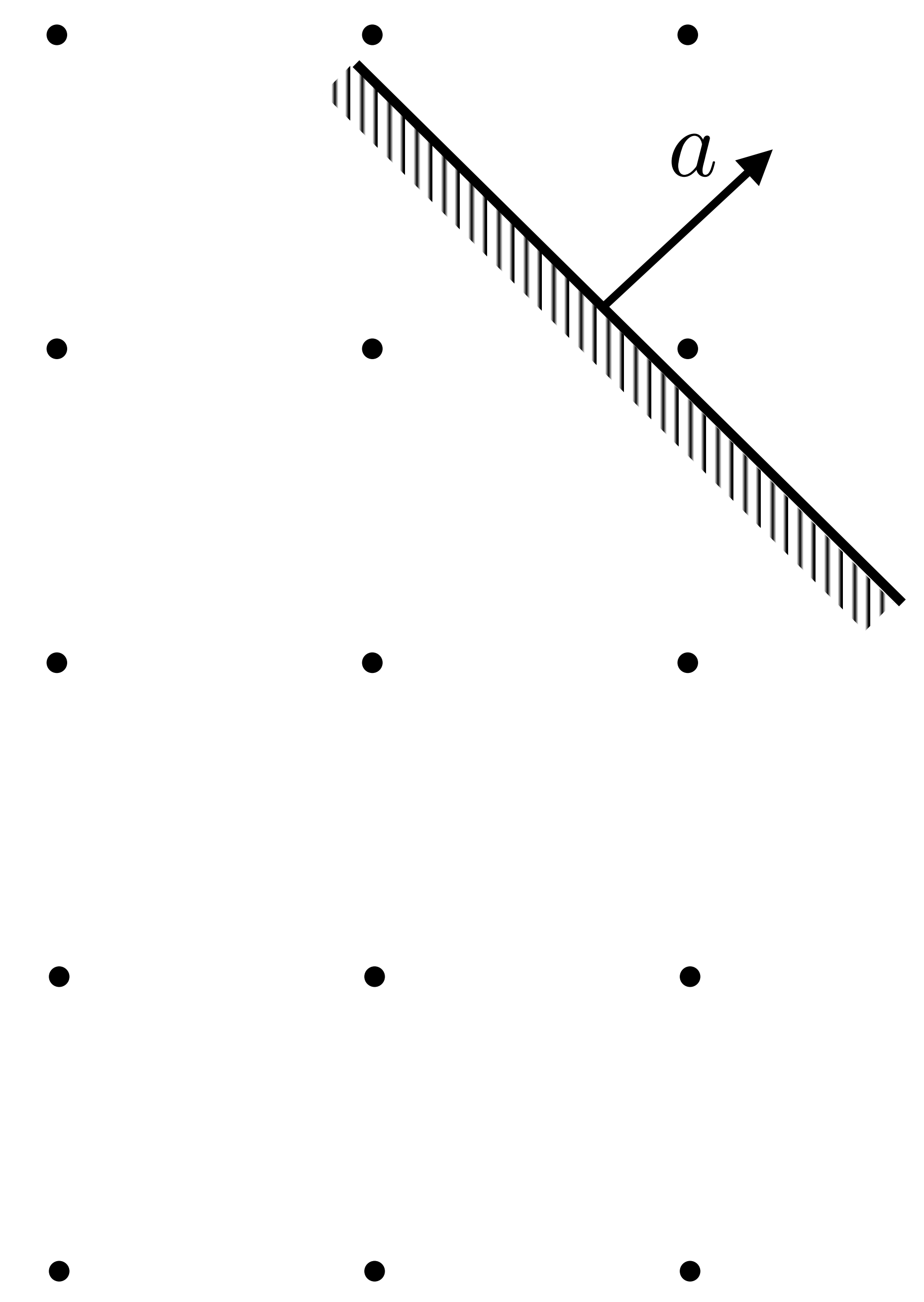

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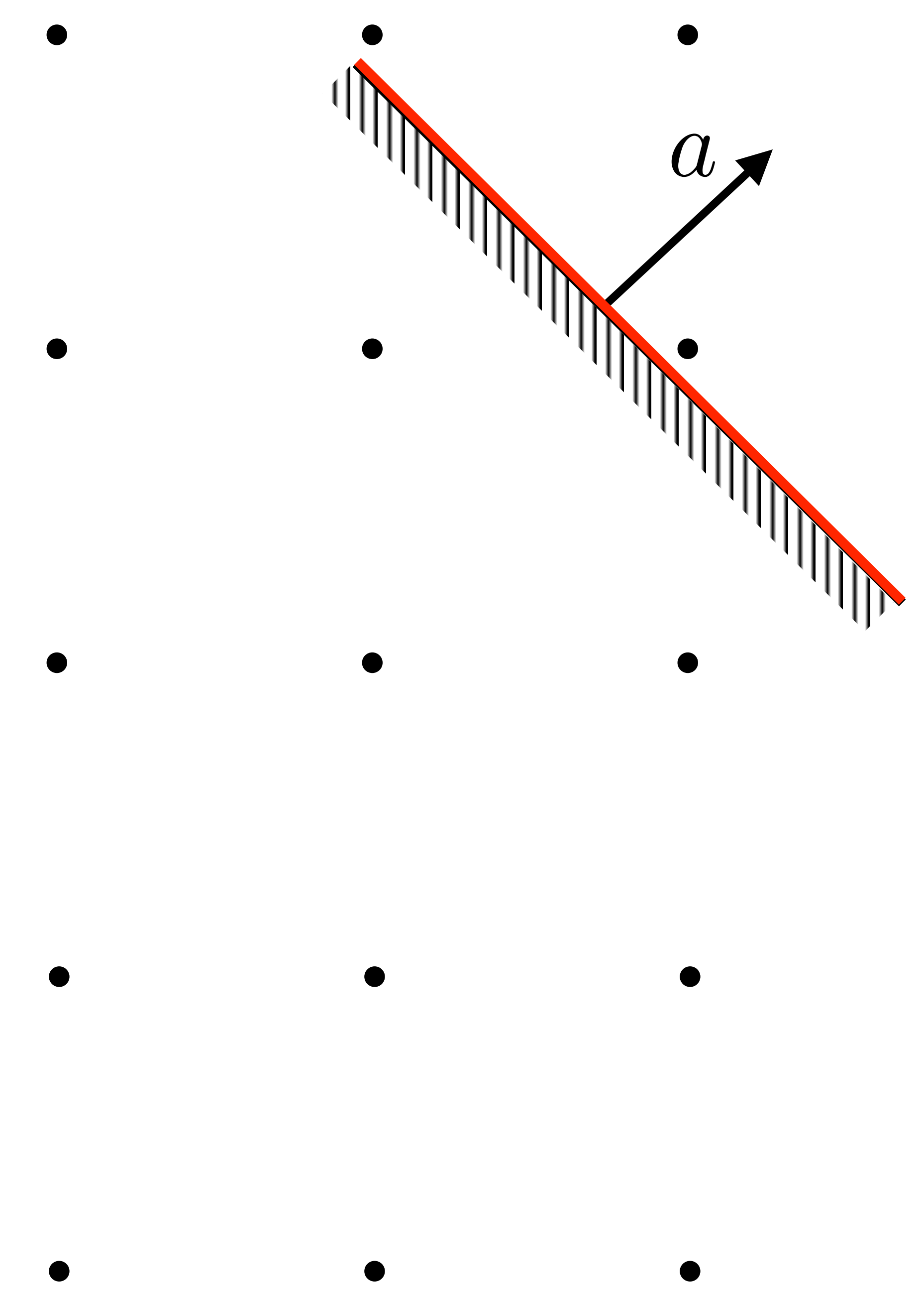
$$\text{if } a,\, x \in \mathbb{Z}^n$$



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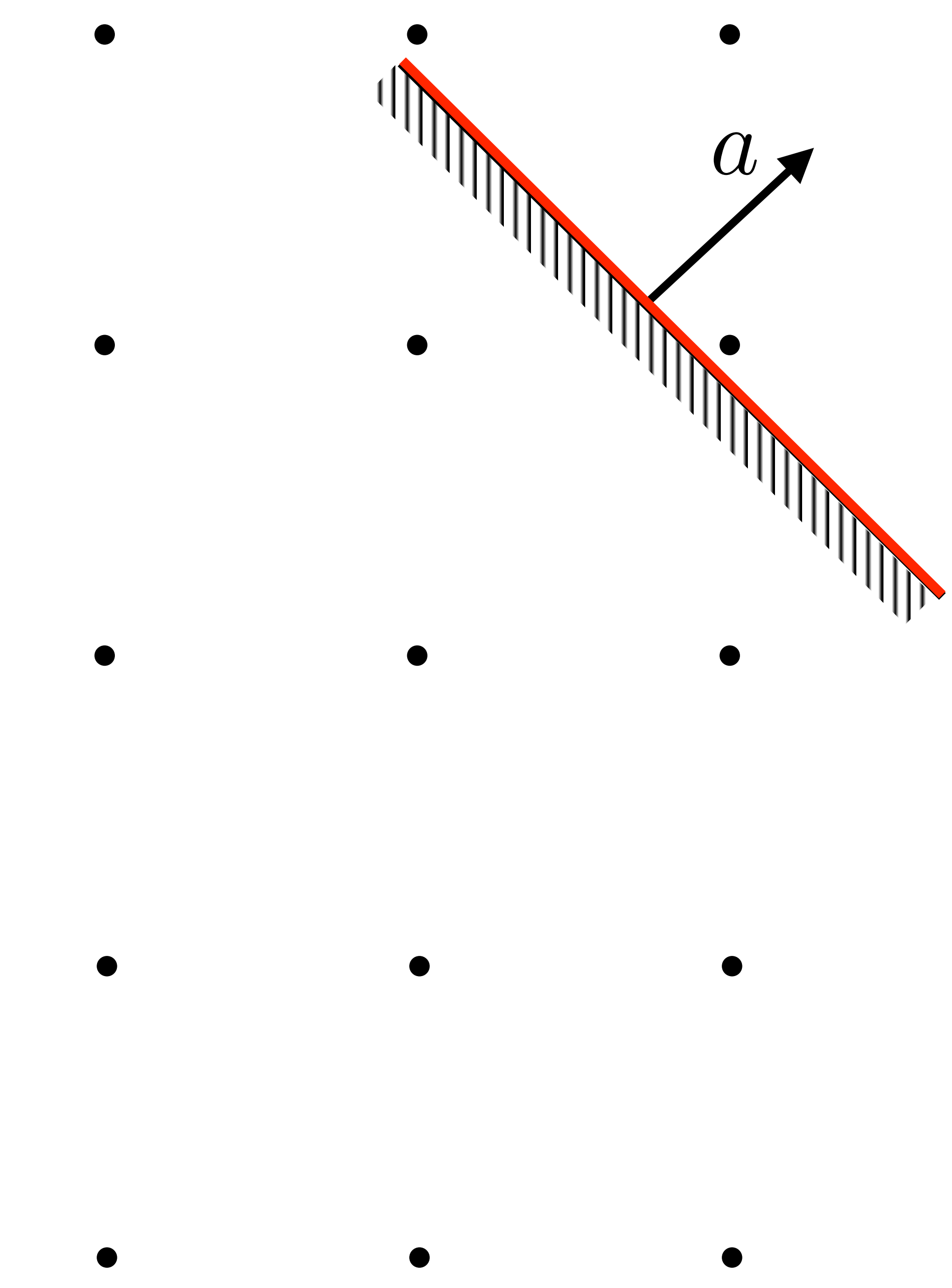
if $a,\, x \in \mathbb{Z}^n$

$$\langle a, x\rangle \leq \lfloor \sigma_C(a) \rfloor$$

Valid for $H \cap \mathbb{Z}^n$



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if a, $x \in \mathbb{Z}^n$

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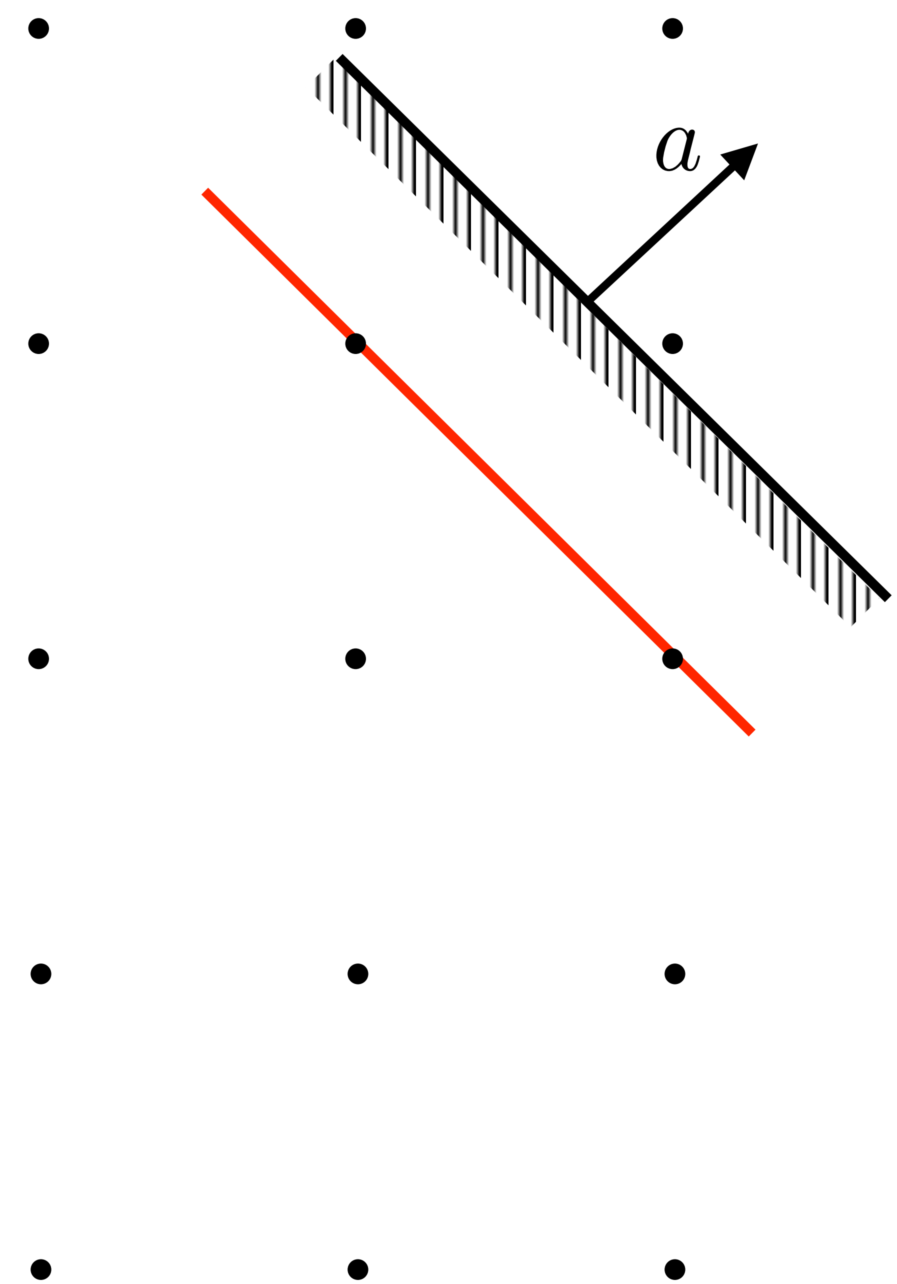

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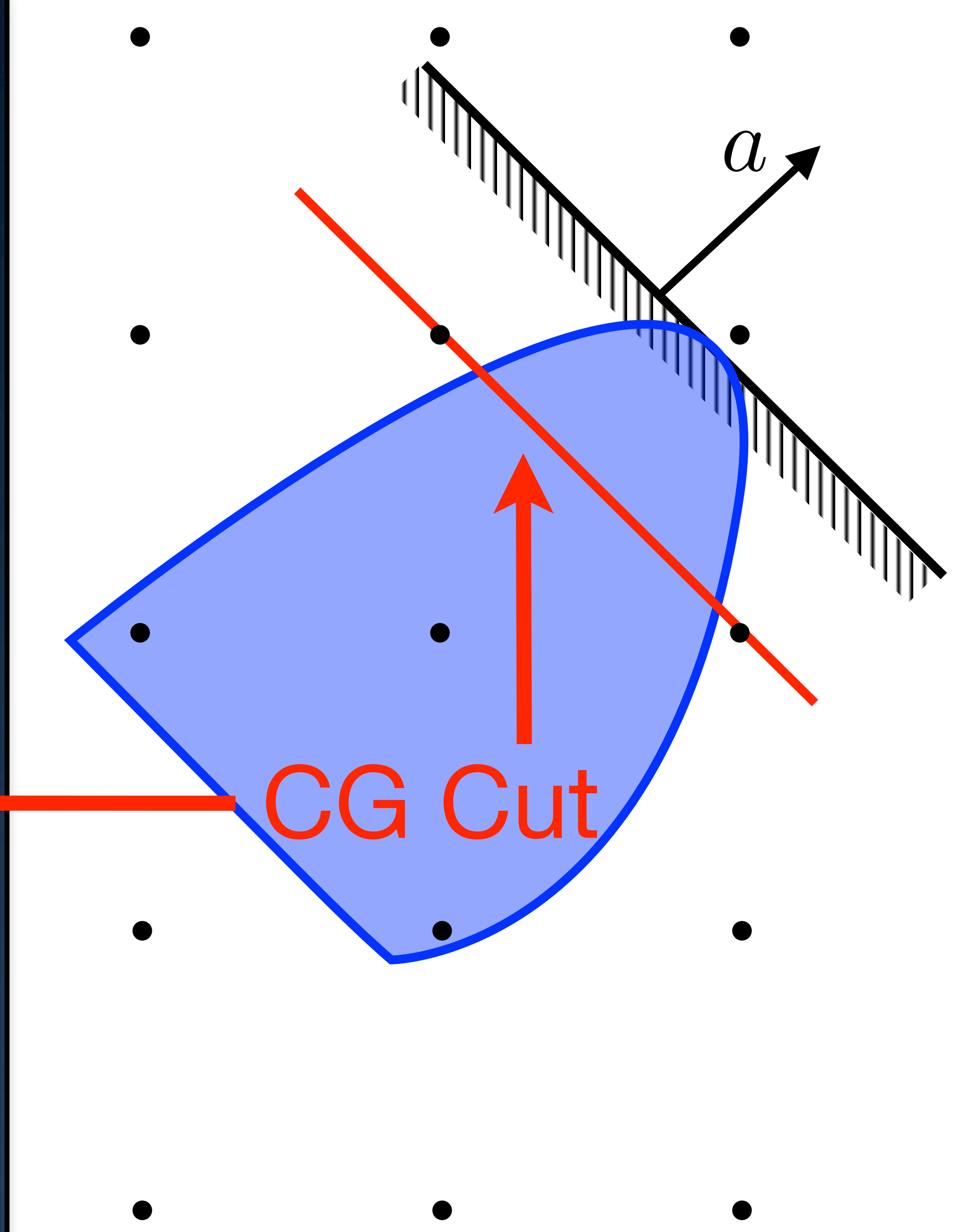
Valid for $H \cap \mathbb{Z}^n$

Valid for $C \cap \mathbb{Z}^n$



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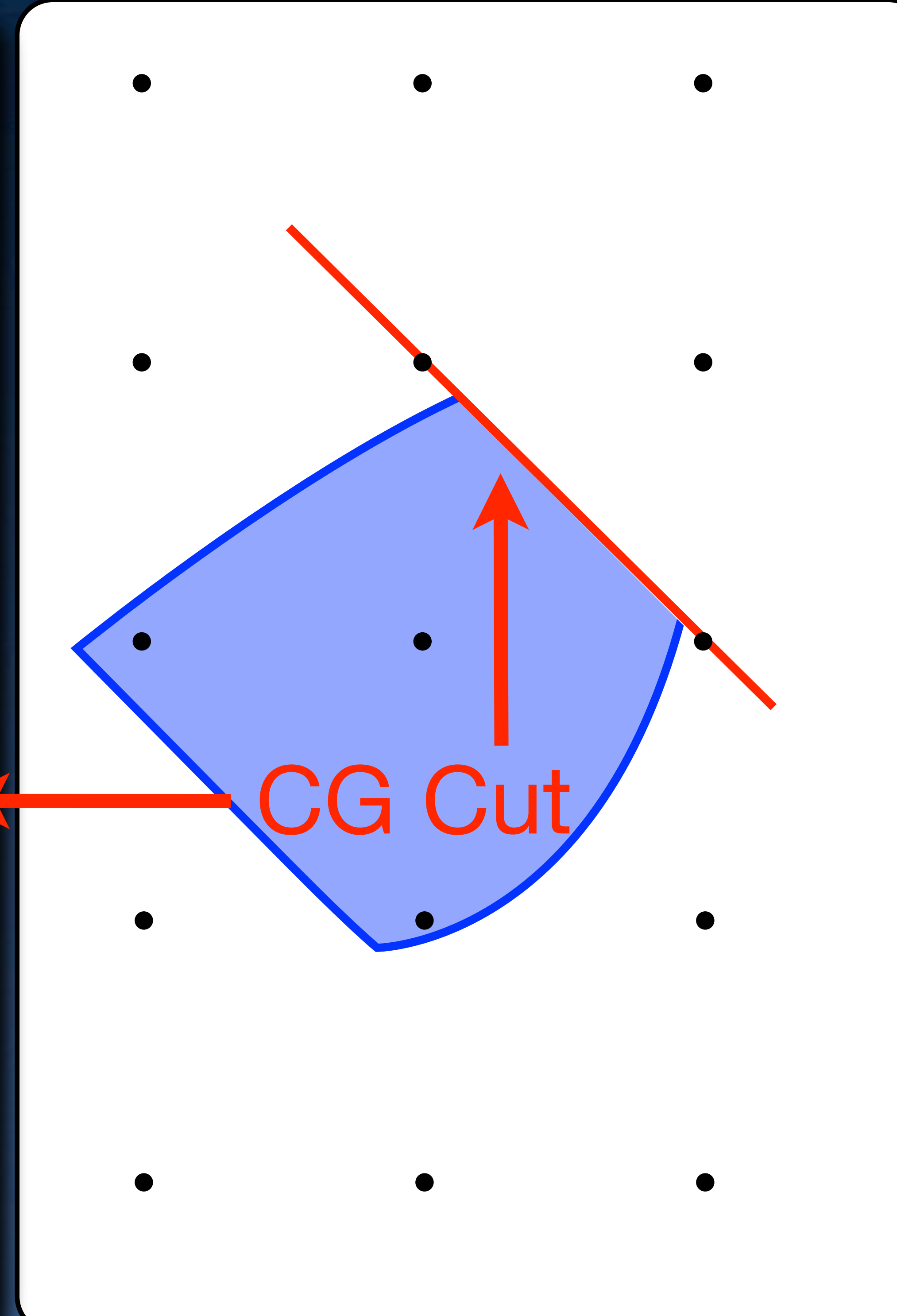
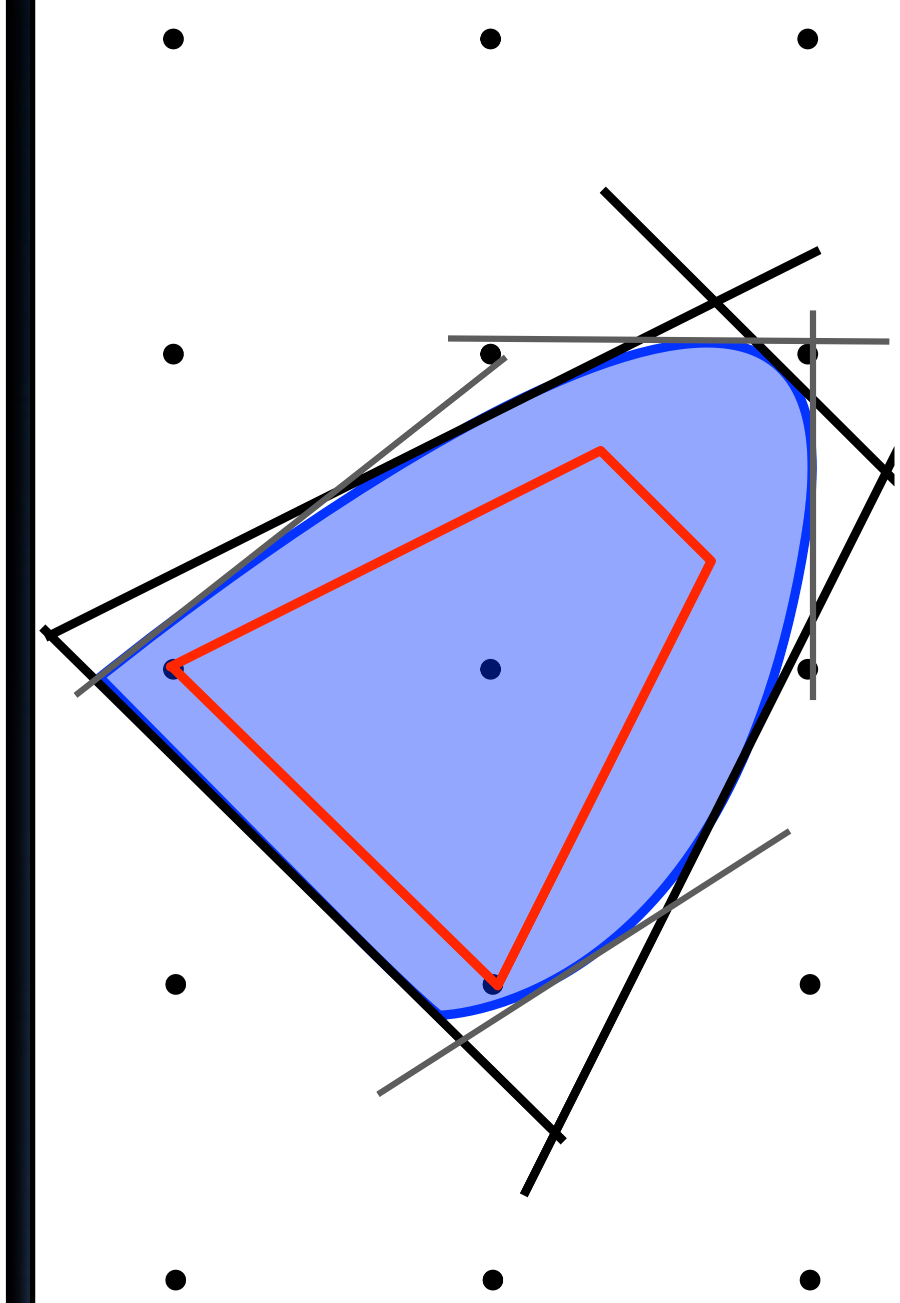


CG Closure = Add all CG Cuts

$$\mathrm{CC}(C) := \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}$$



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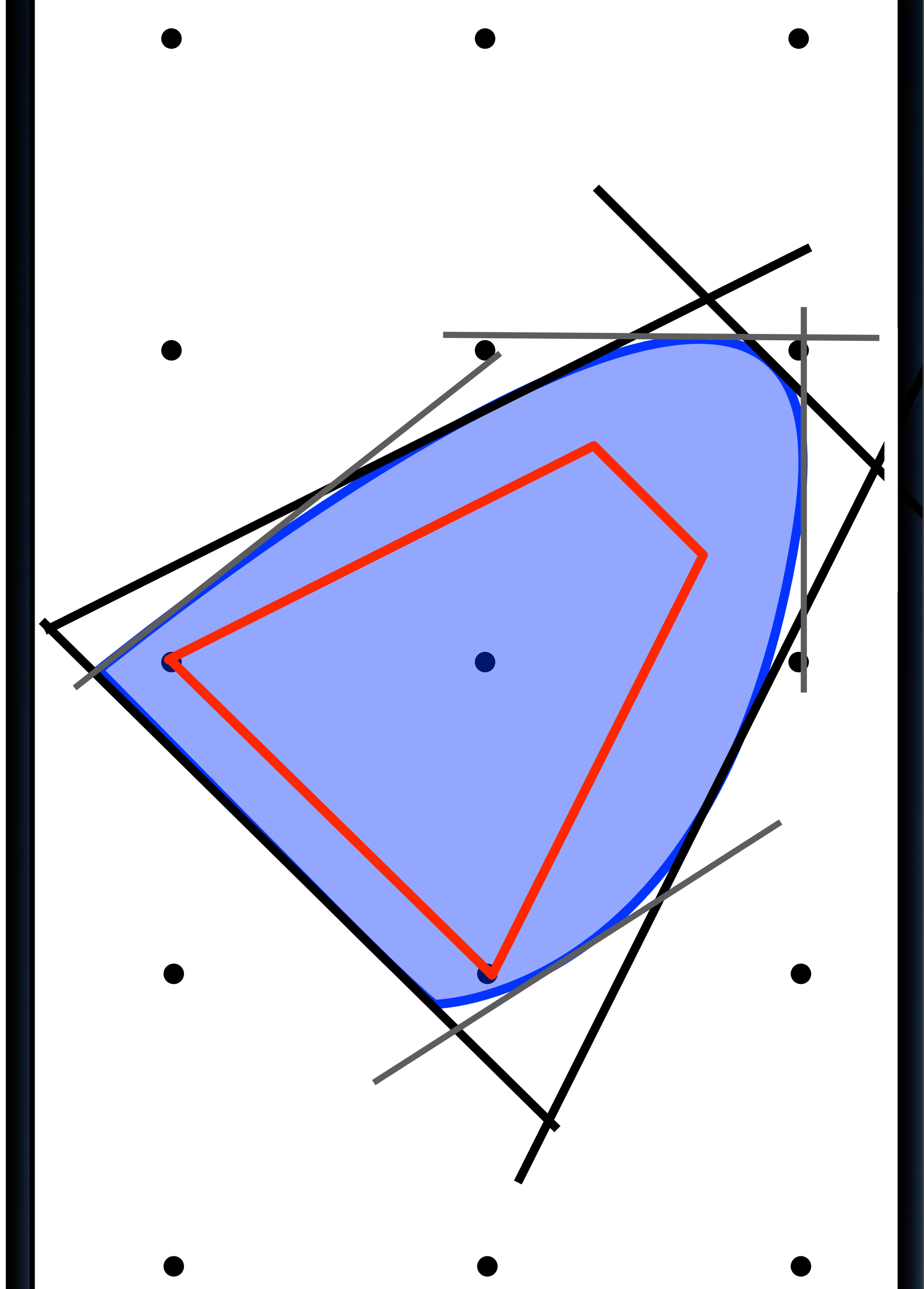
Polyhedral?



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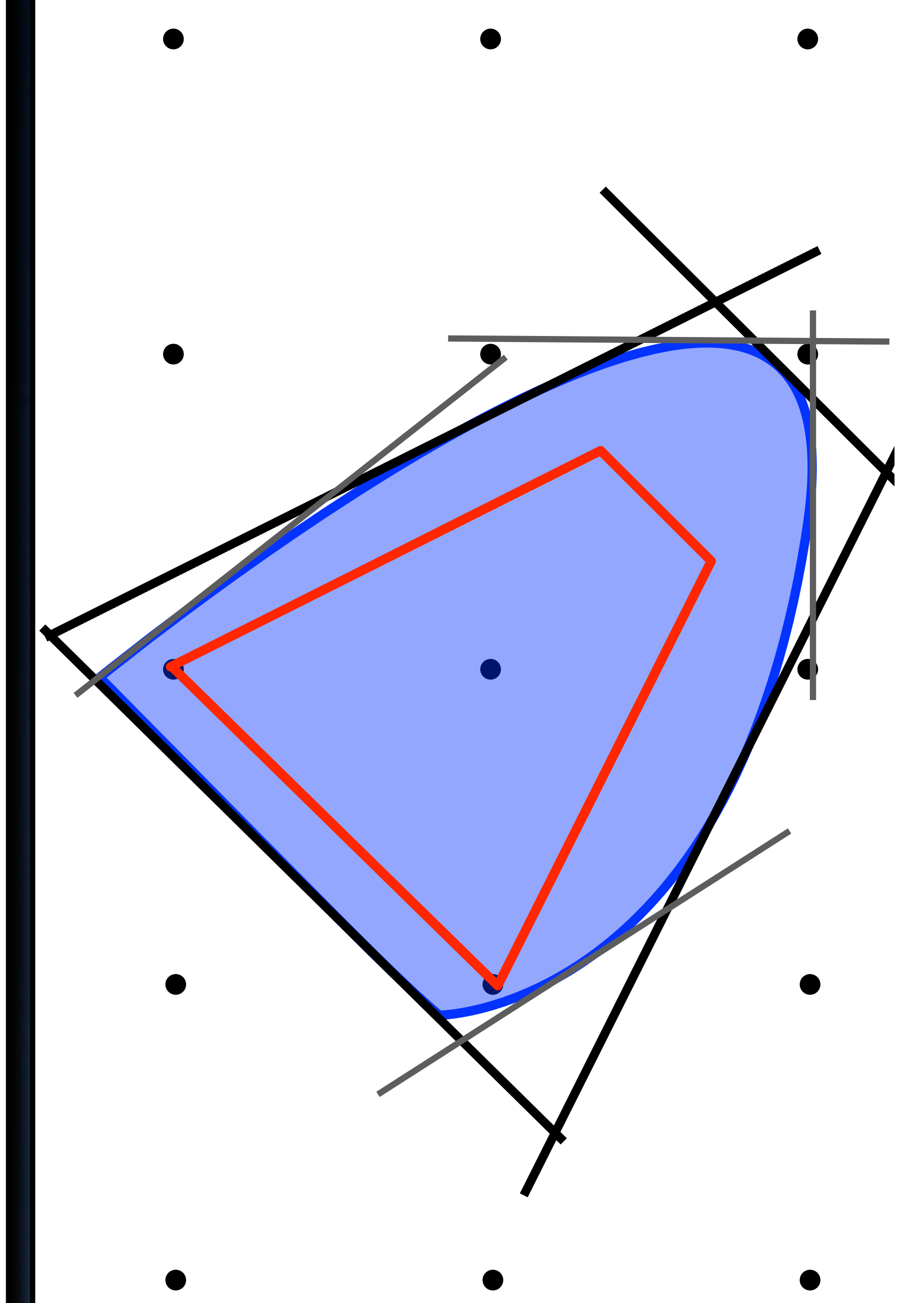

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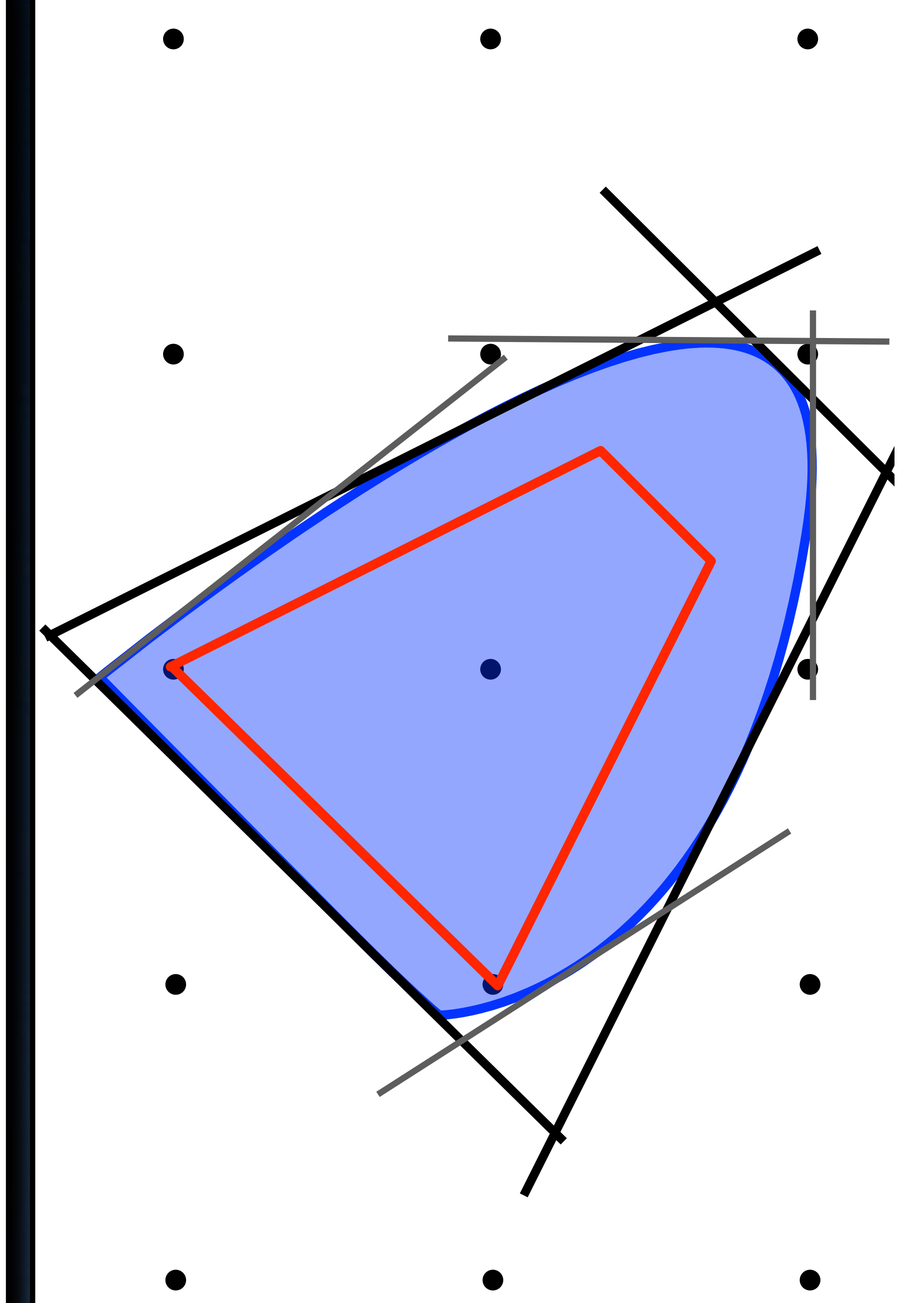
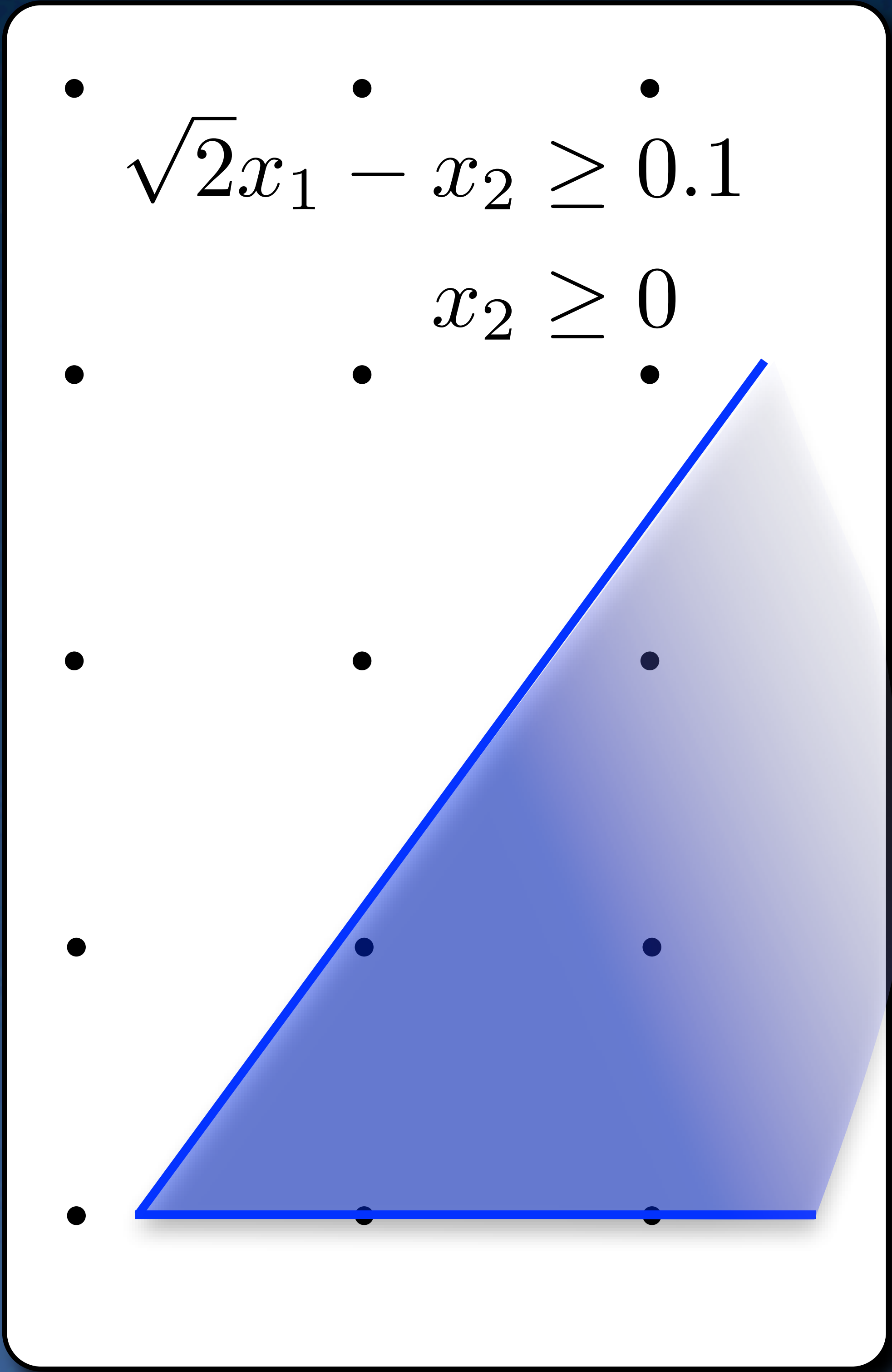


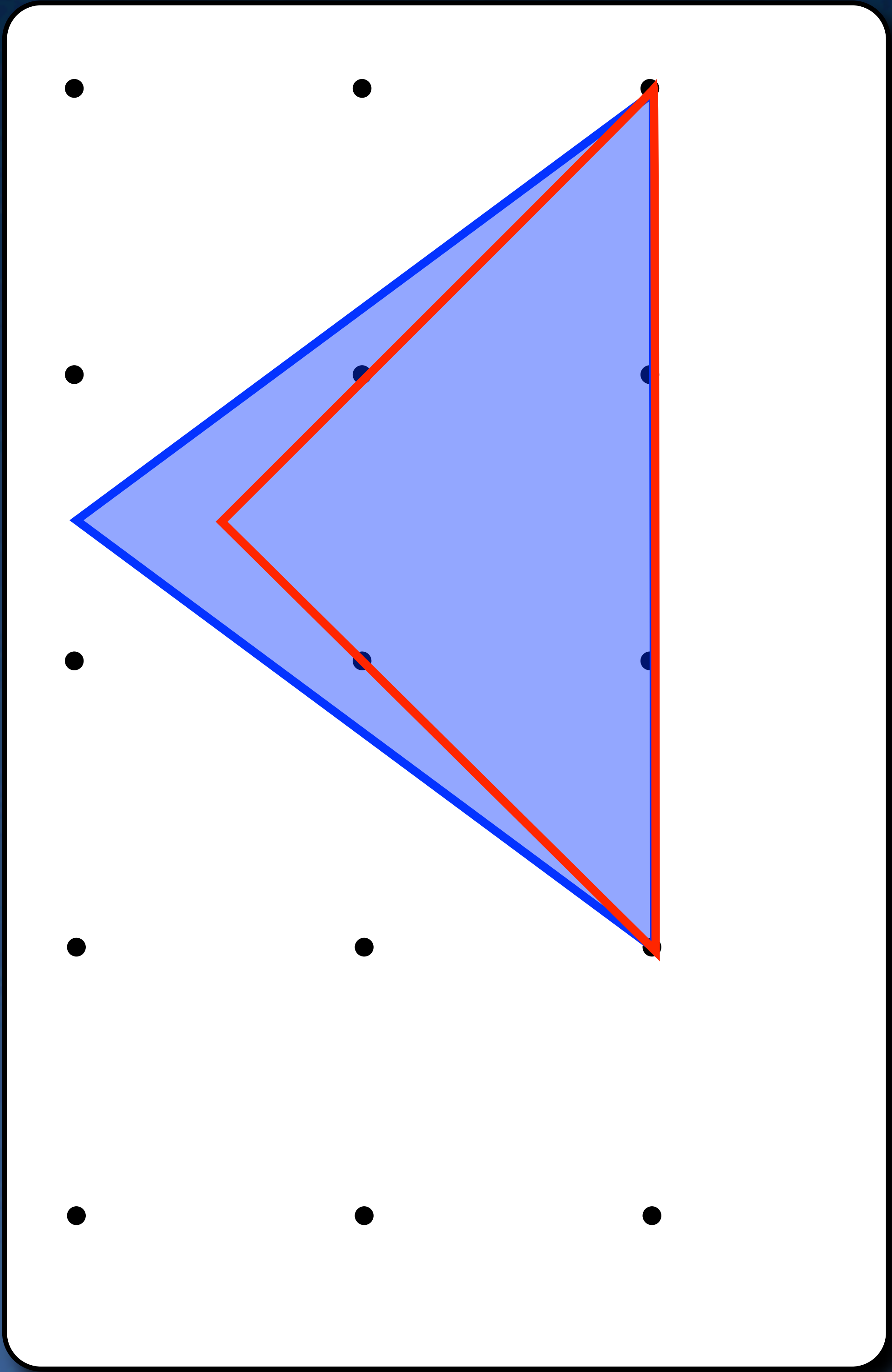
Polyhedrality of CG Closure

- Not always a polyhedron



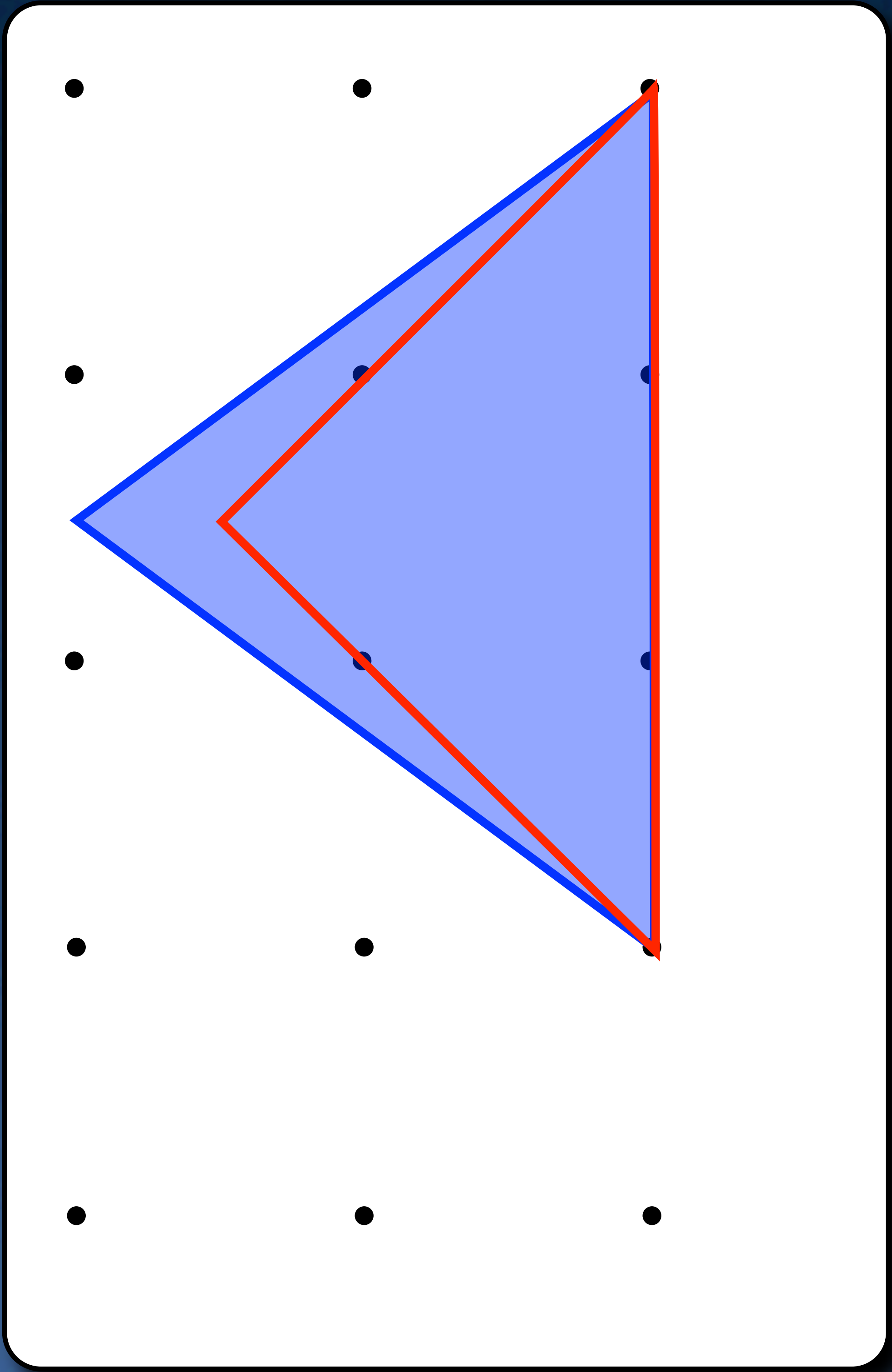
Polyhedrality of CG Closure

- Not always a polyhedron
- Shrijver 1980:
  - Theorem: If C is a rational polyhedron then $\mathrm{cc}(C)$ is too. (Constructive Proof)



Polyhedrality of CG Closure

- Not always a polyhedron
- Shrijver 1980:
  - Theorem: If C is a rational polyhedron then $\mathrm{CC}(C)$ is too. (Constructive Proof)
  - Question: What about for non-rational polytopes?



CG Closure is Finitely Generated

- Theorem (Dadush, Dey, V. 2011): If C is a compact convex set then there exists a finite $S \subseteq \mathbb{Z}^n$ such that:

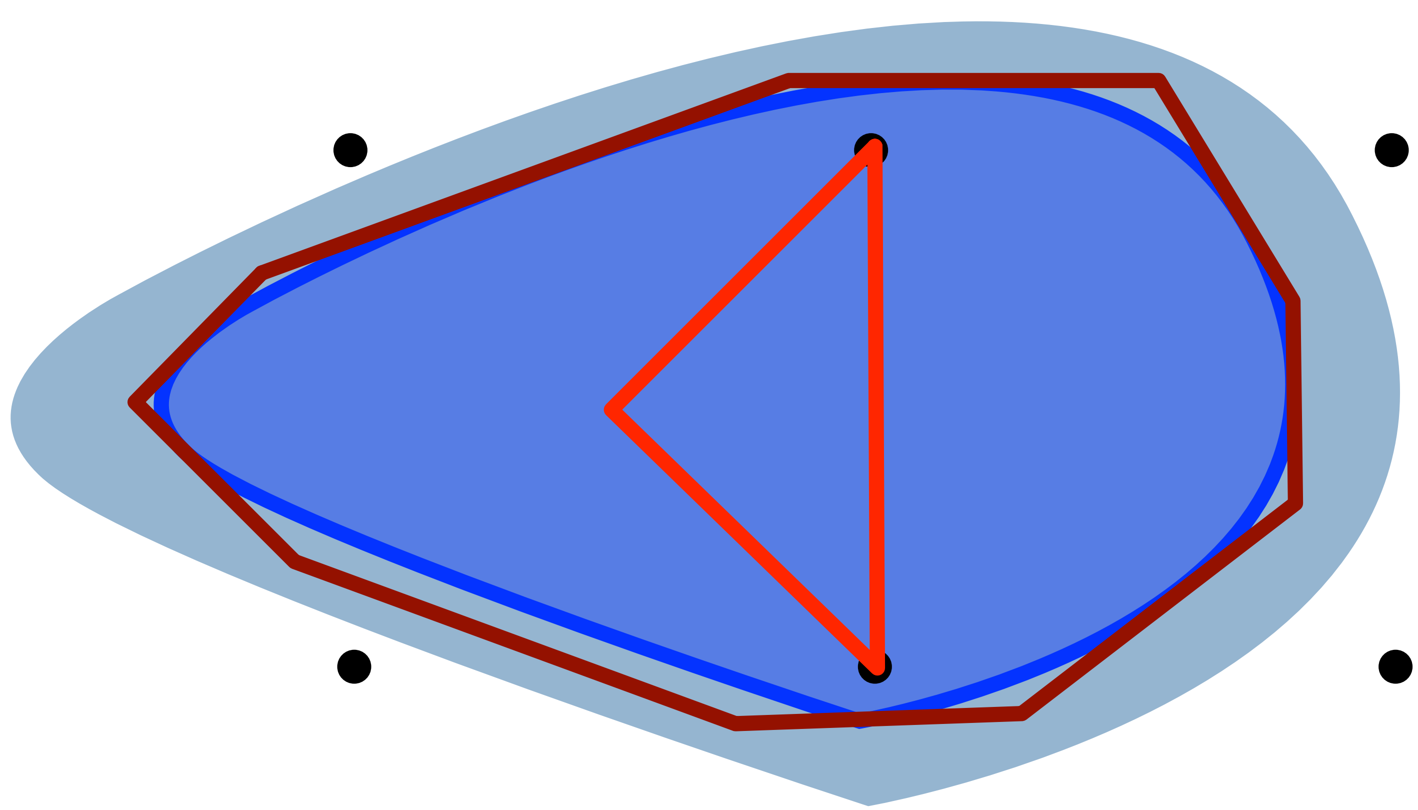
$$\mathrm{cc}(C) = \underbrace{\bigcap_{a \in S} \{x \in \mathbb{R}^n \ : \ \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}}_{\mathrm{cc}(C,S)}$$

- Corollary: $\mathrm{cc}(C)$ is a rational polytope.
- In particular answers Shrijver's question.
  - Also answered by Dunkel and Schulz 2011.

Corollary: Stability of CG Closure

$\exists \varepsilon > 0$ such that:

$$C \subseteq P \subseteq C + \varepsilon B_2 \Rightarrow \mathtt{CC}(C) = \mathtt{CC}(P)$$



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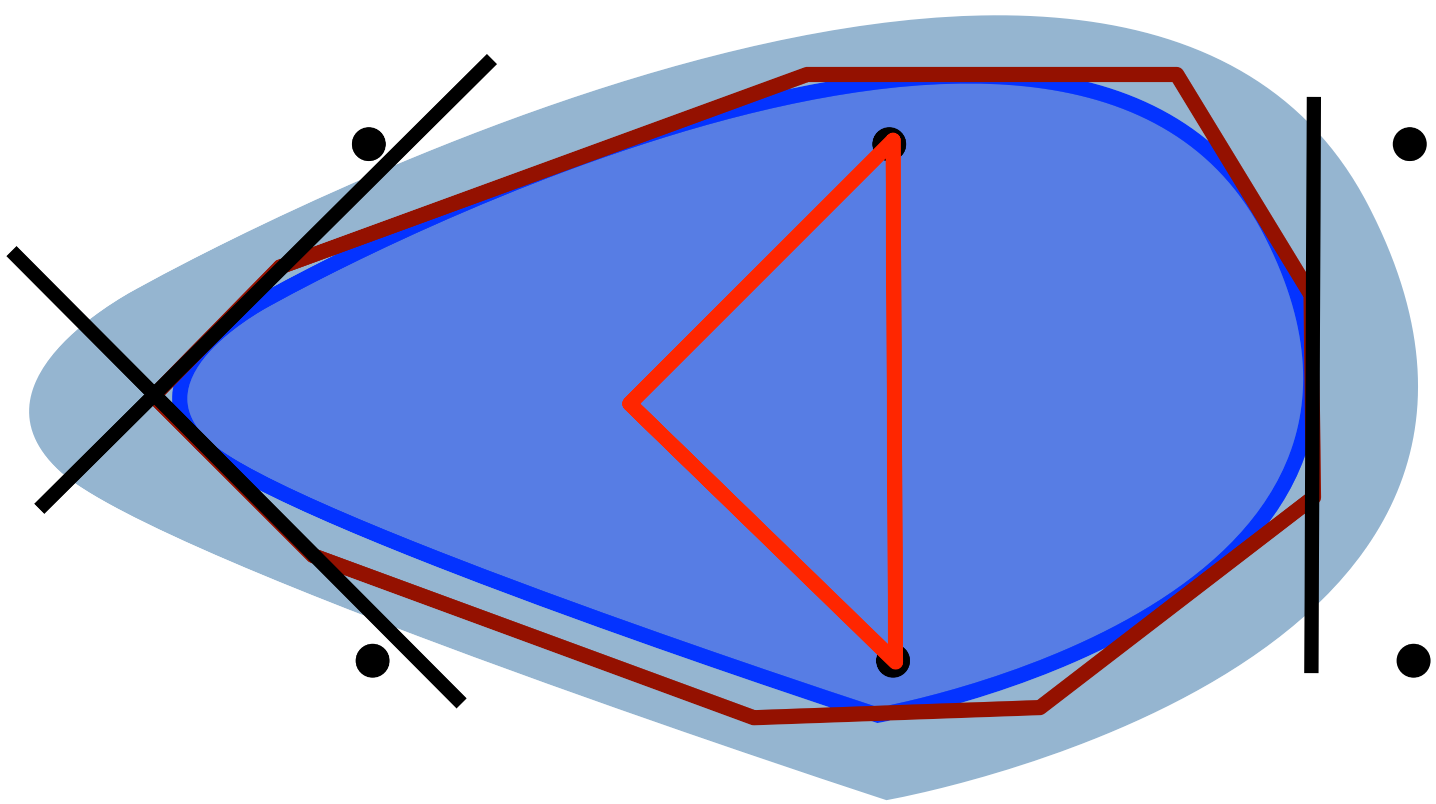


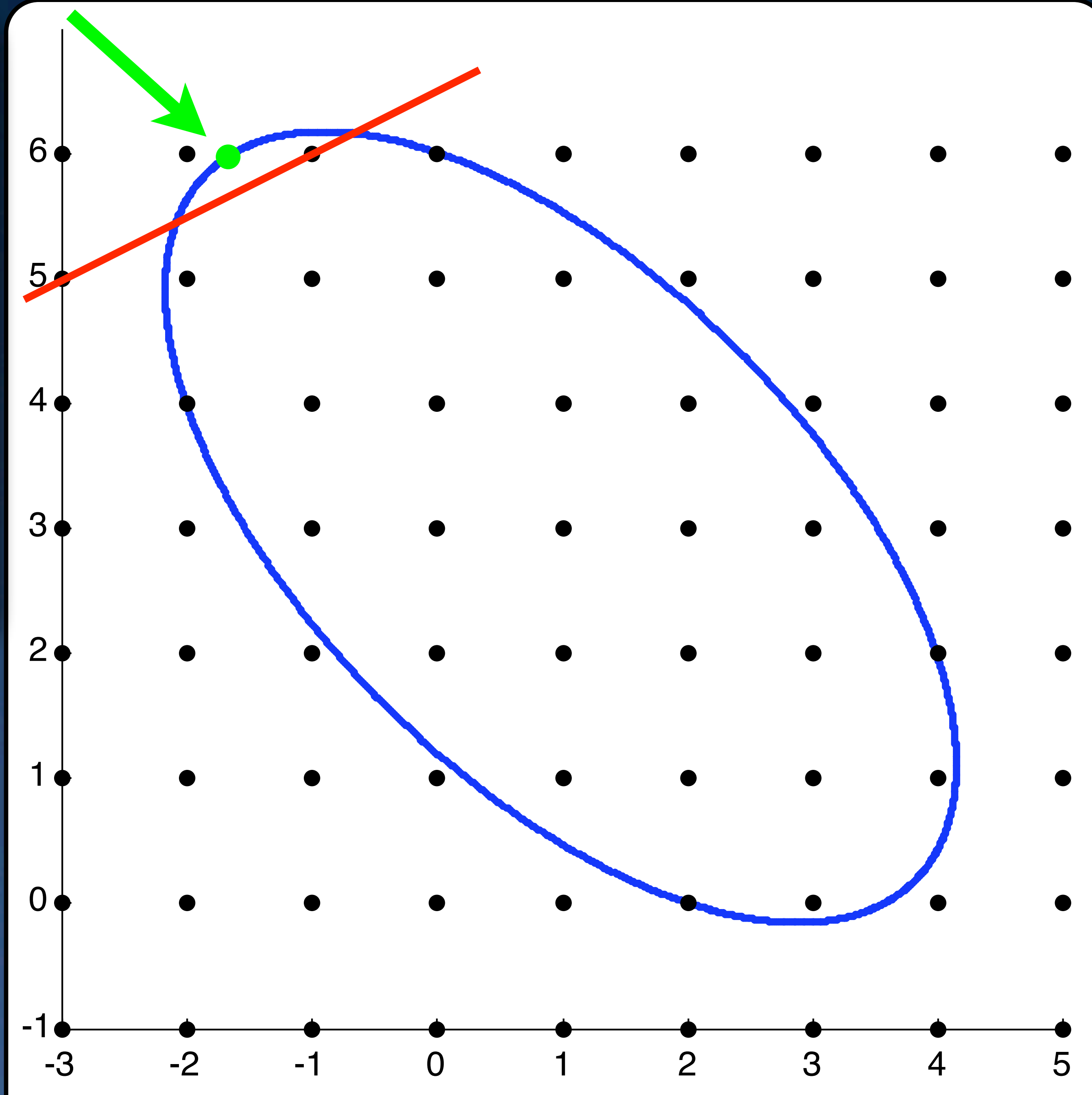
Proof Sketch of Theorem

- For strictly convex sets without integral points in boundary
- Proof Outline:
  - Step 1: Create finite S_1 s.t. $\text{cc}(C, S_1) \subseteq \text{int}(C)$.
    - Separate points in boundary
    - Compactness argument
  - Step 2: Show only missed finite number of cuts

Separate points in $\mathrm{bd}(C)$

$u \in \mathrm{bd}(C) \setminus \mathbb{Z}^n$ $\exists\, a^u \in \mathbb{Z}^n$

$$\langle a^u, u \rangle > \lfloor \sigma_C(a^u) \rfloor$$



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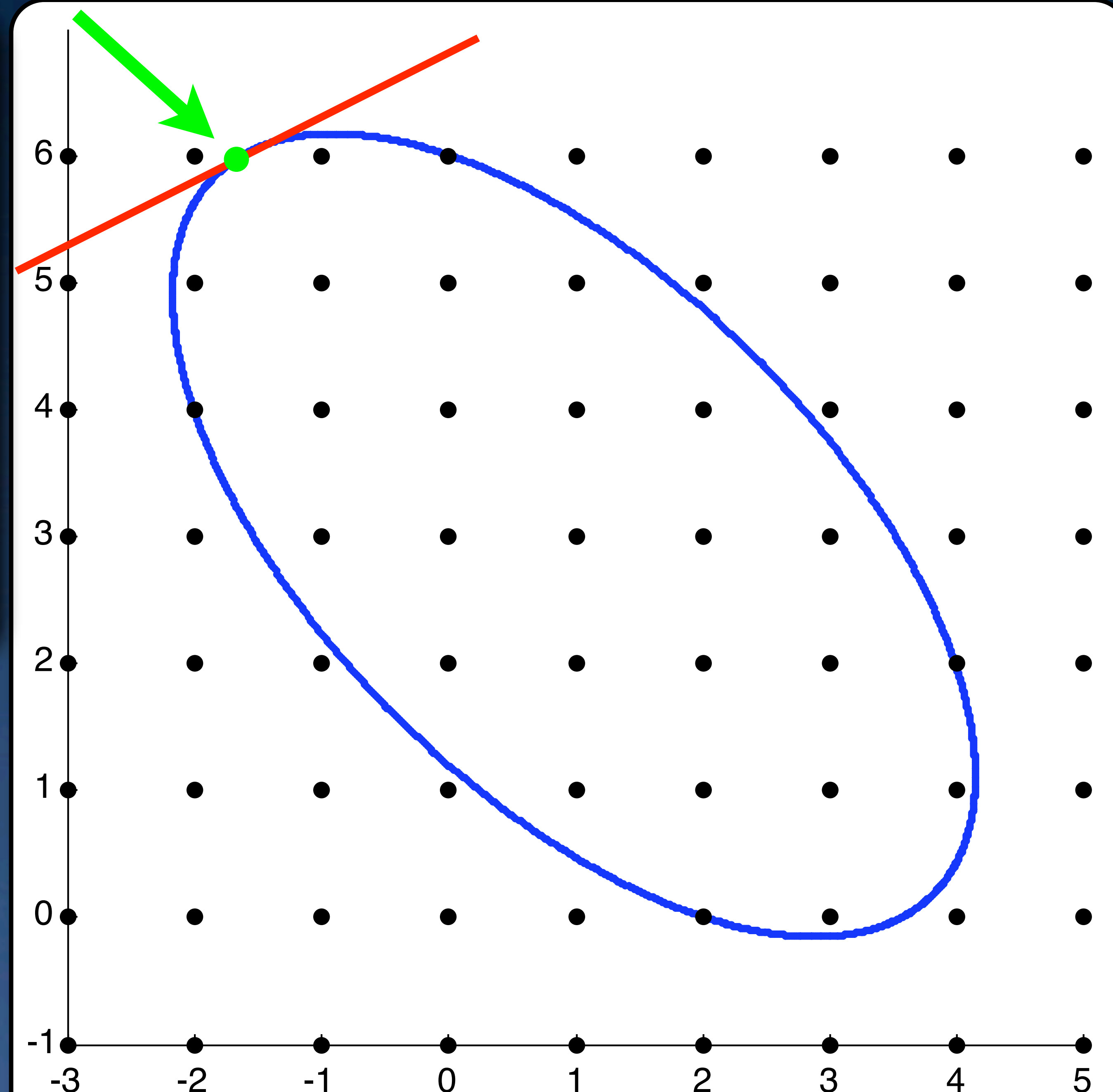

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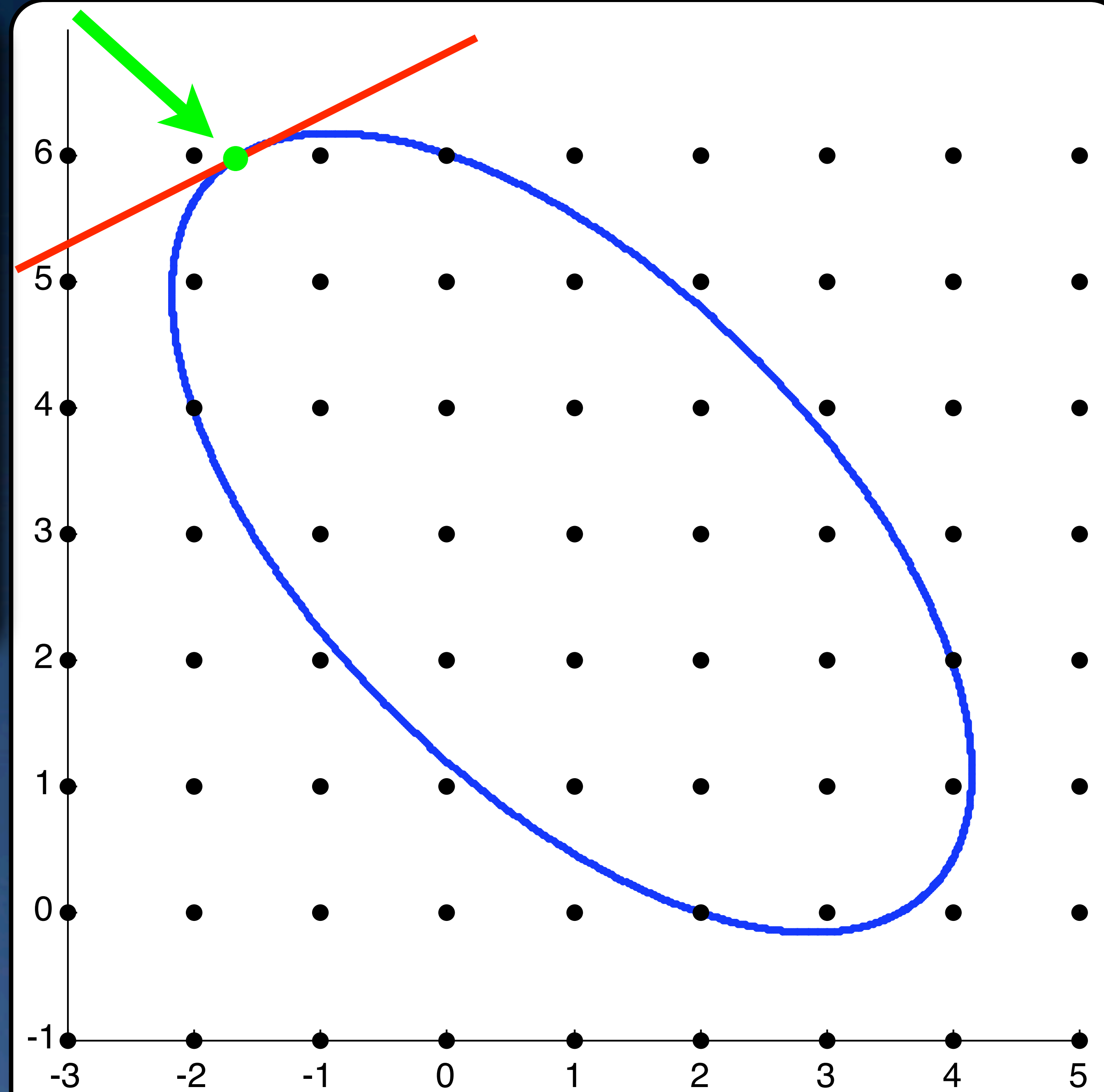

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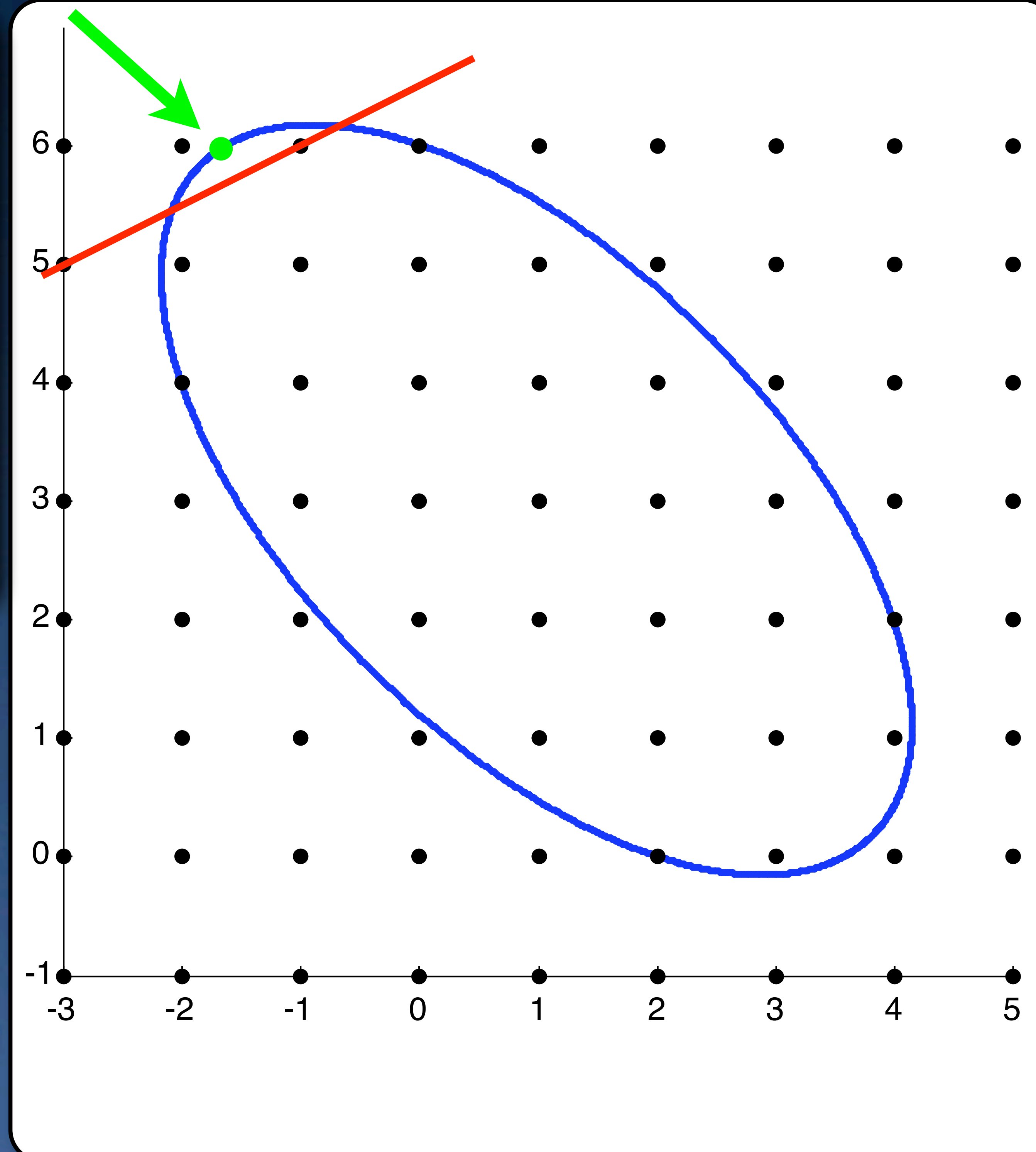

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$\lambda s(u) \notin \mathbb{Z}^n \quad \forall \lambda > 0:$

$\lambda s(u) \in \mathbb{Z}^n \Rightarrow \sigma_C(\lambda s(u)) \in \mathbb{Z}:$

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$$C = \left\{ x \in \mathbb{R}^2 \ : \ \sqrt{x_1^2 + x_2^2} \leq 1 \right\}$$

$$u = (1/2, \sqrt{3}/2)^T \in \mathrm{bd}(C)$$

$$s(u) = (1/2, \sqrt{3}/2)^T$$

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$$C = \left\{ x \in \mathbb{R}^2 \ : \ \sqrt{x_1^2 + x_2^2} \leq 5 \right\}$$

$$u = (25/13, 60/13)^T \in \operatorname{bd}(C)$$

$$s(u) = (5, 12)^T, \ \sigma_C(s(u)) = 65$$

Separate points in $\mathrm{bd}(C)$

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$$\frac{s^i}{\|s^i\|} \xrightarrow{i\to\infty} \frac{s(u)}{\|s(u)\|}$$

$$\lim_{i\to\infty} \langle s^i, u\rangle - \lfloor \sigma_C(s^i) \rfloor > 0$$

Diophantine approx. of $s(u)$

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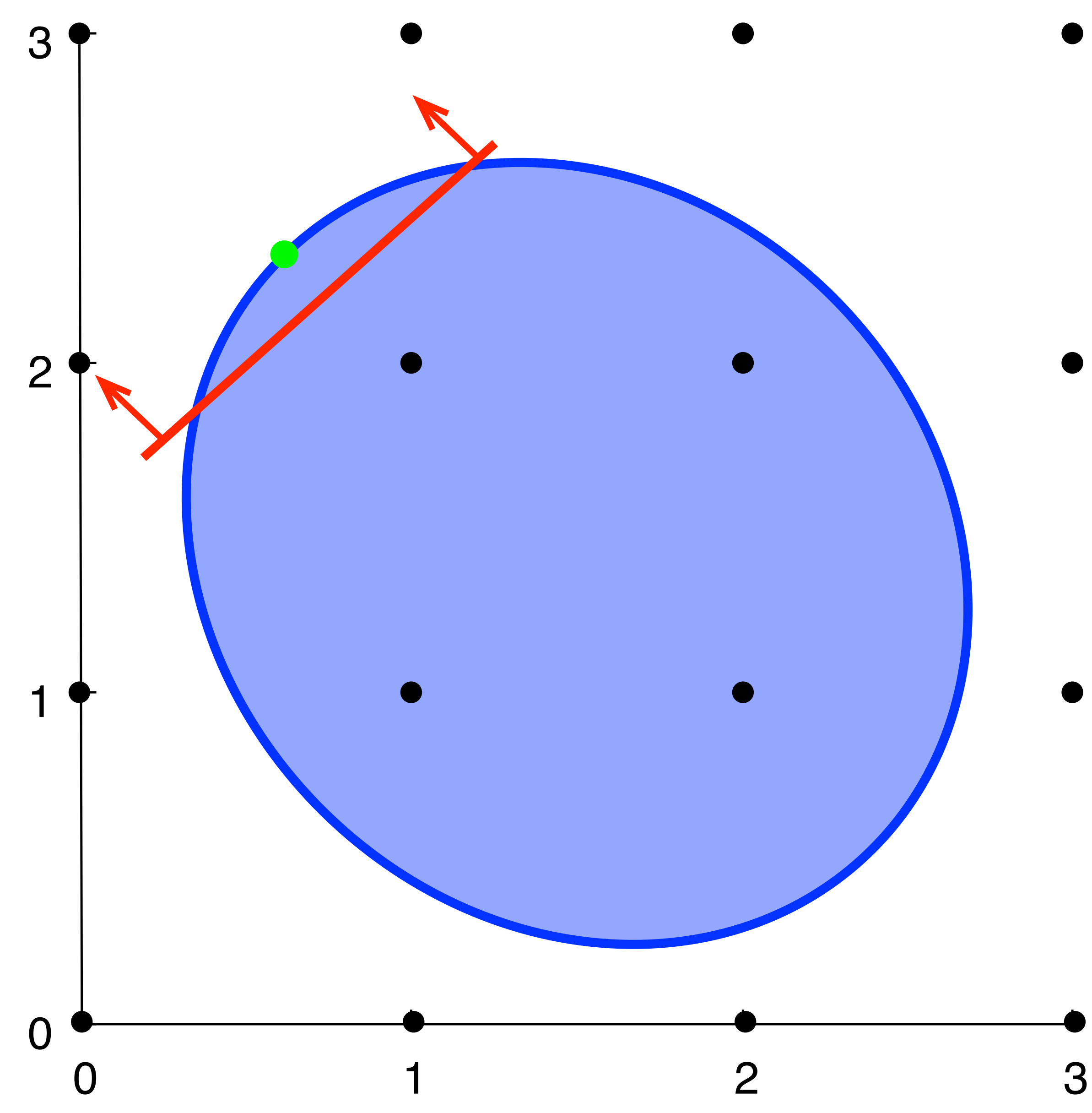
Compactness Argument

$K := \mathrm{bd}(C)$

$S_u := \{x \,:\, \langle a^u, x \rangle > \lfloor \sigma_C(a^u) \rfloor\}$



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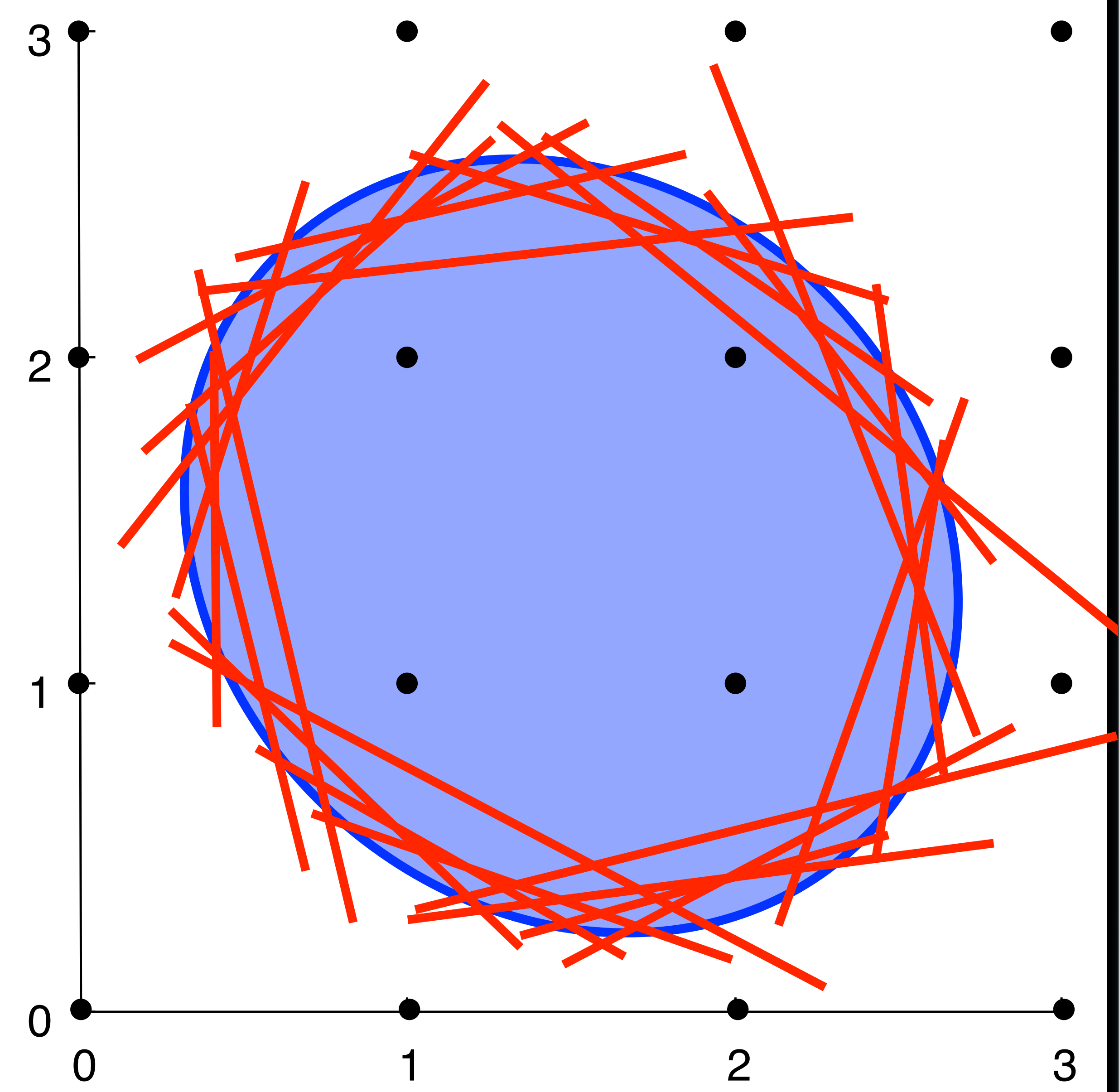
$$K \subset \bigcup_{u \in K} S_u$$



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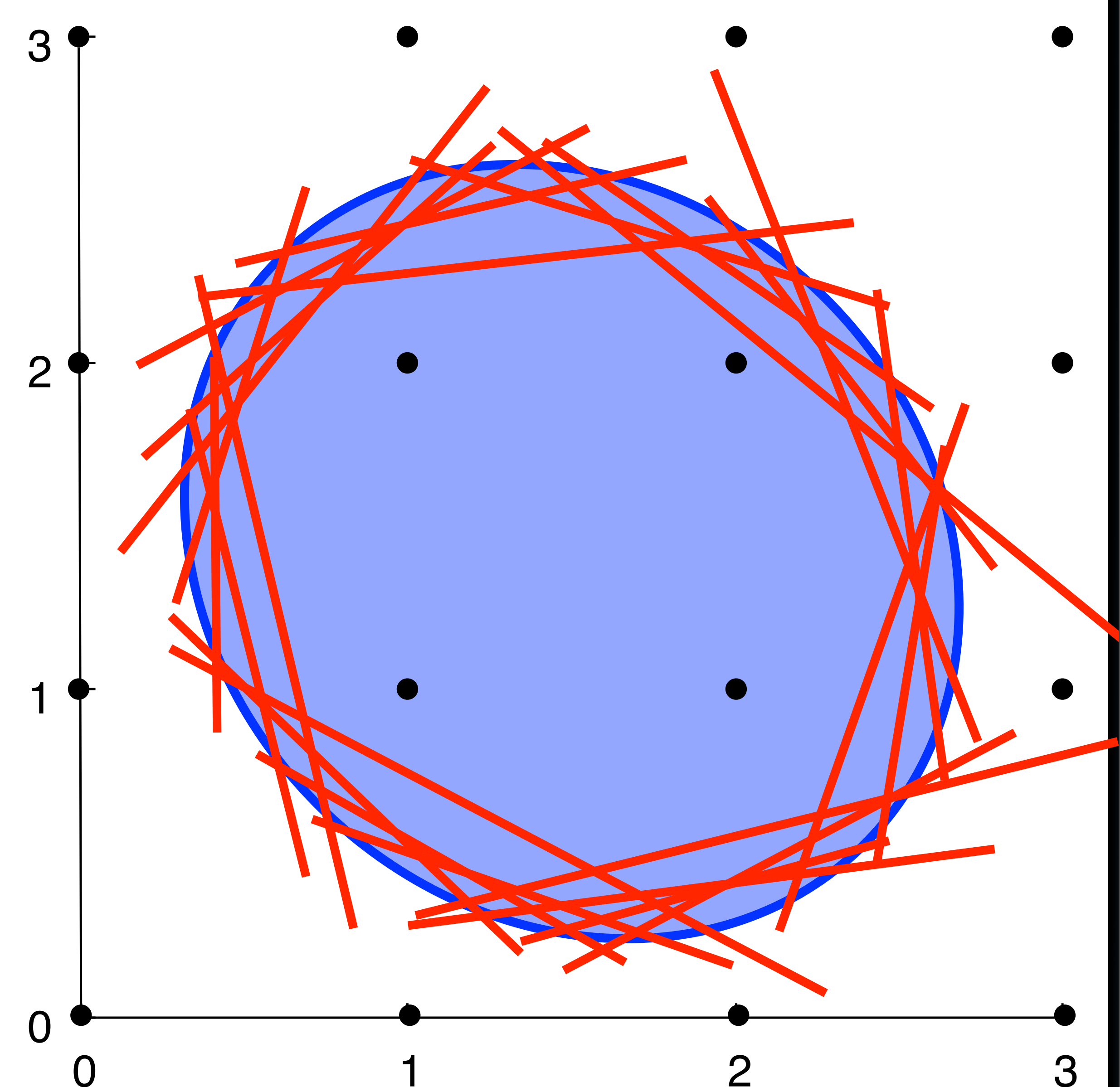

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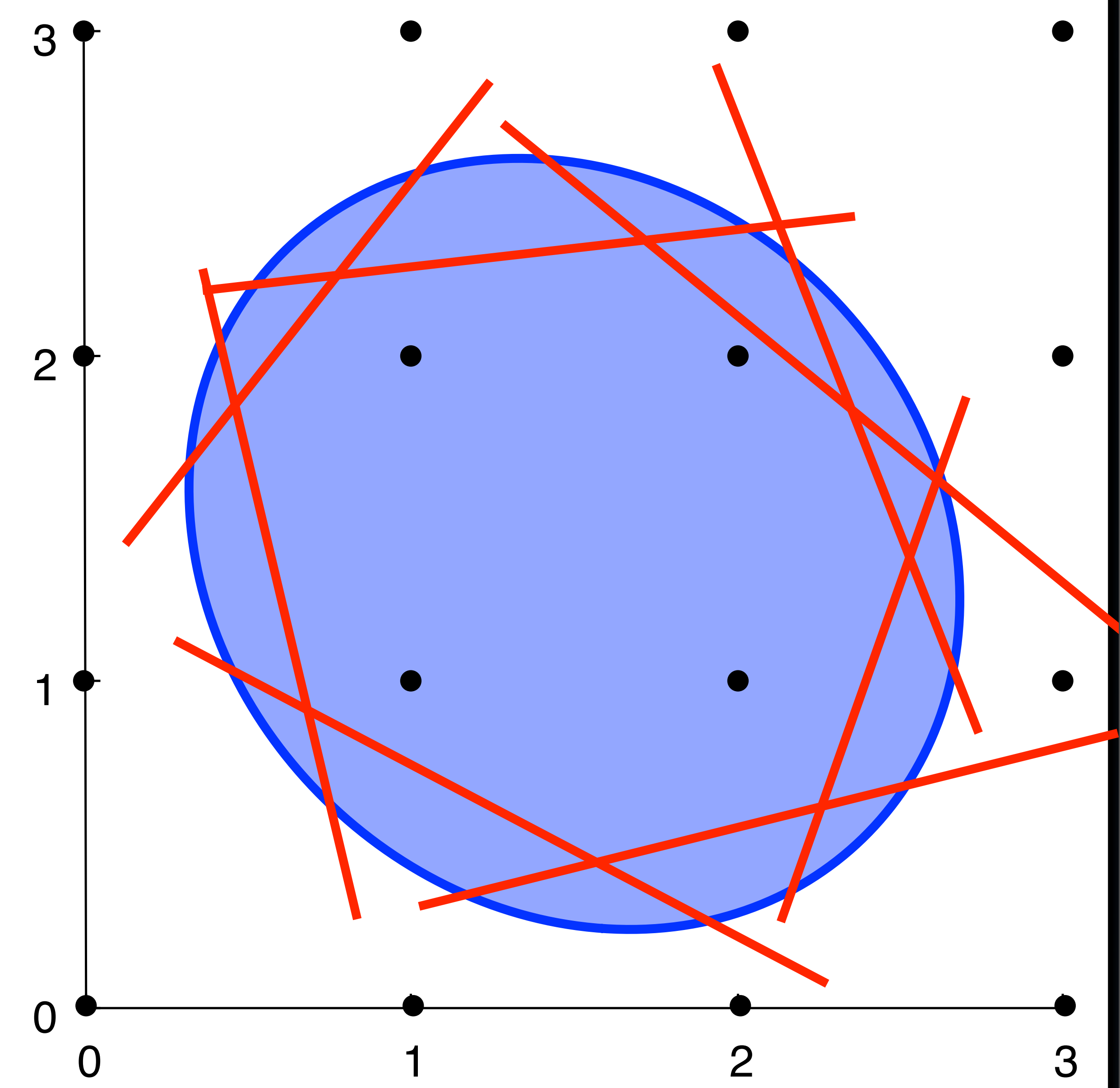
$\downarrow$

$K \subset \bigcup_{i=1}^{m} S_{u^i}$



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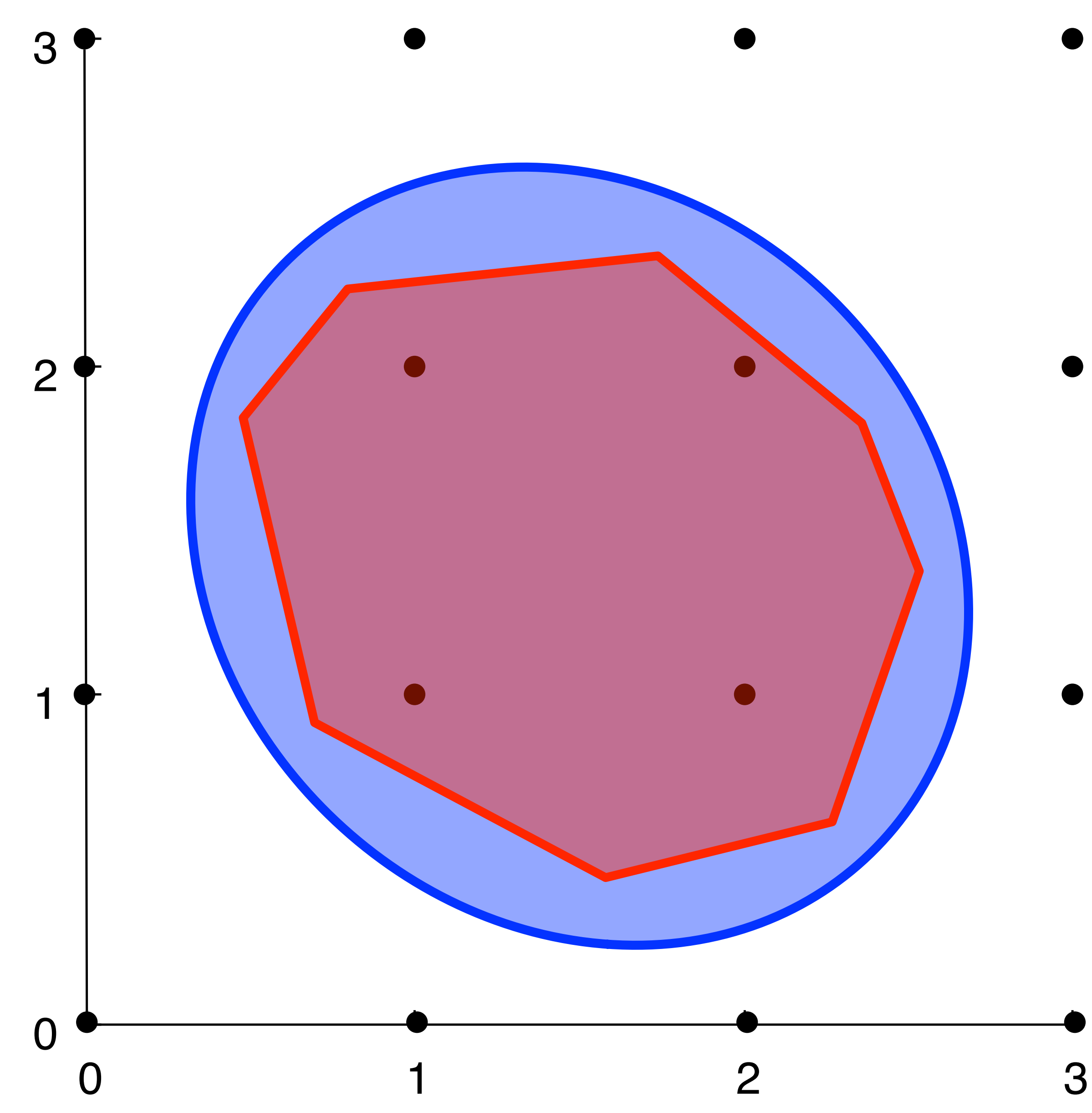
$$\Downarrow$$

$$K \subset \bigcup_{i=1}^{m} S_{u^i}$$

$$S^1 = \bigcup_{i=1}^{m} \left\{ a^{u^i} \right\}$$



Step 2 : Separate $\mathrm{cc}(S^1, C) \setminus \mathrm{cc}(C)$

$$V := \mathrm{ext}\left(\mathrm{cc}(S^1, C)\right)$$

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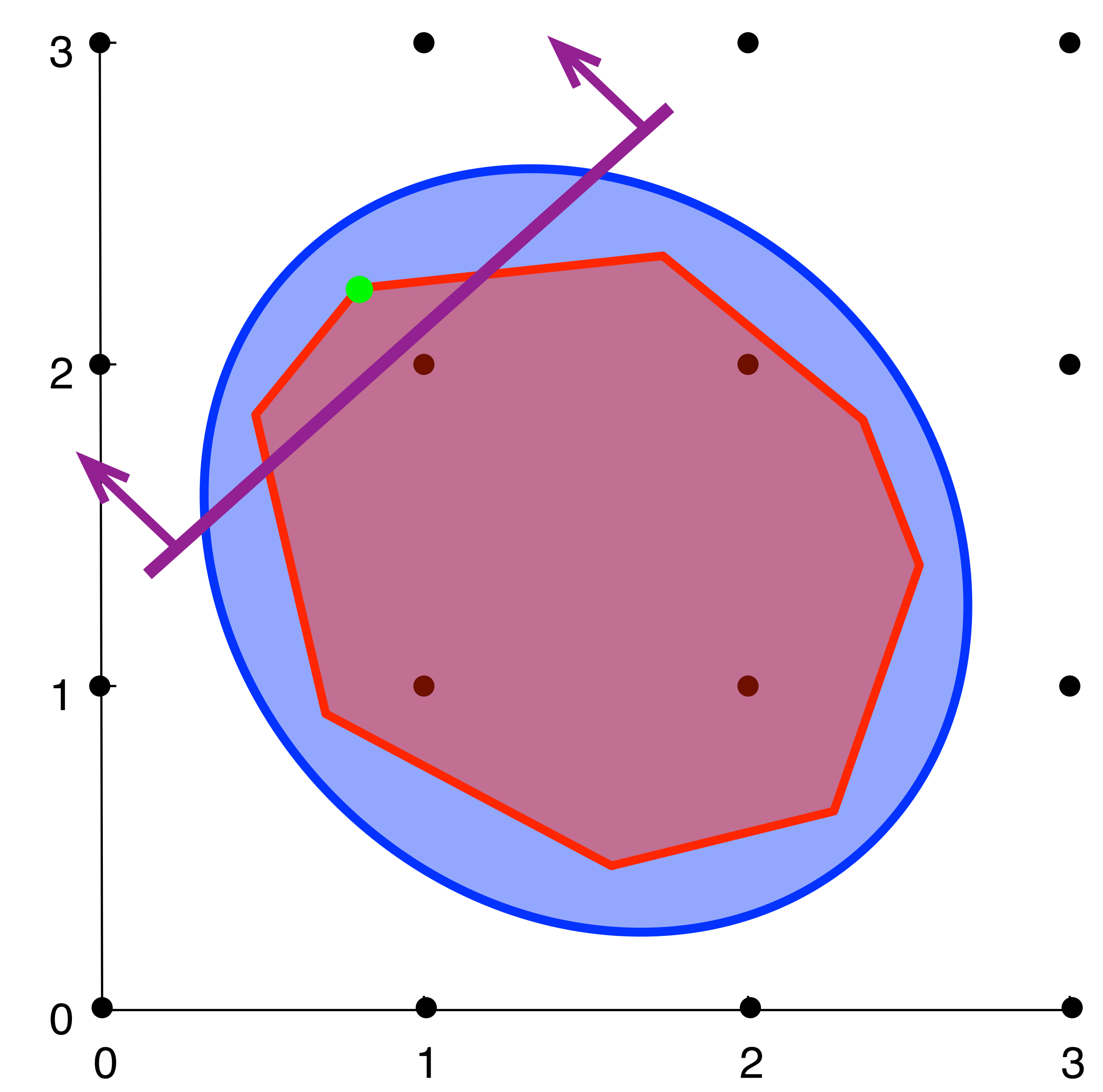

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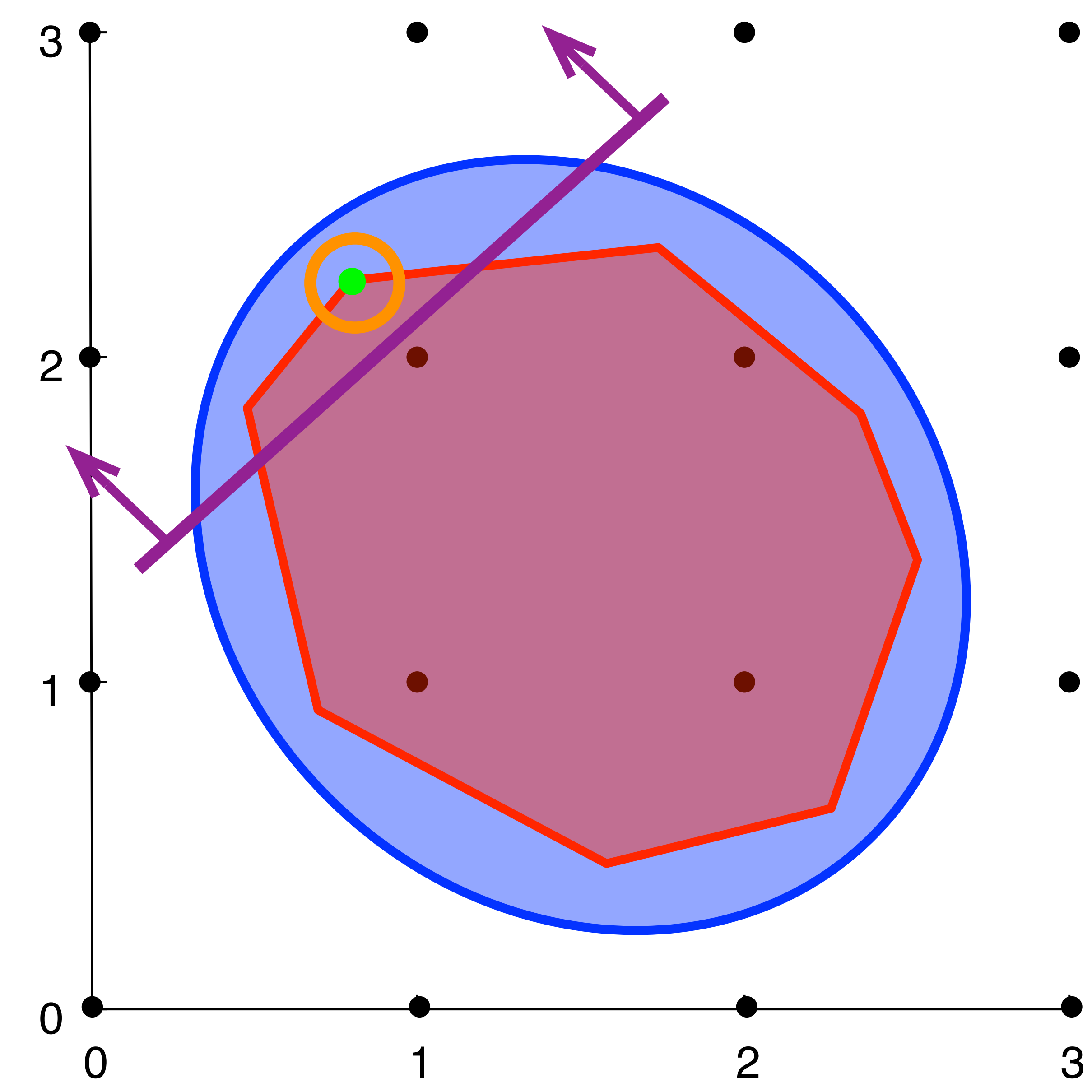
$$\exists \varepsilon > 0 \quad \varepsilon B^n + v \subset C \quad \forall v \in V$$



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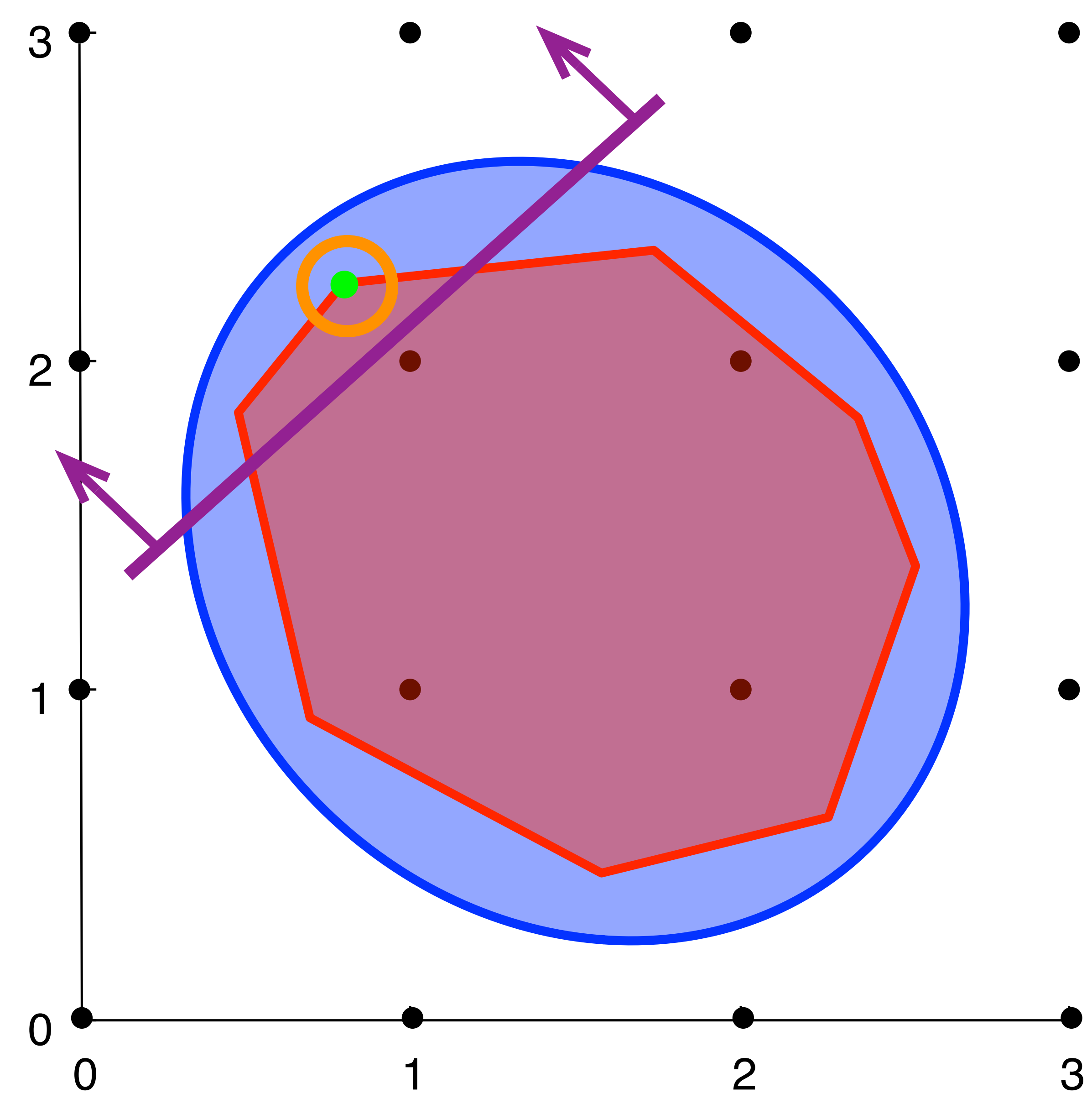
$\exists \varepsilon > 0 \quad \varepsilon B^n + v \subset C \quad \forall v \in V$

$\|a\| \geq \frac{1}{\varepsilon} \Rightarrow$

$$\begin{aligned}
\lfloor \sigma_C(a) \rfloor &\geq \sigma_C(a) - 1 \\
&\geq \sigma_{v+\varepsilon B^n}(a) - 1 \\
&= \langle v, a\rangle + \varepsilon\|a\| - 1 \\
&\geq \langle v, a\rangle
\end{aligned}$$



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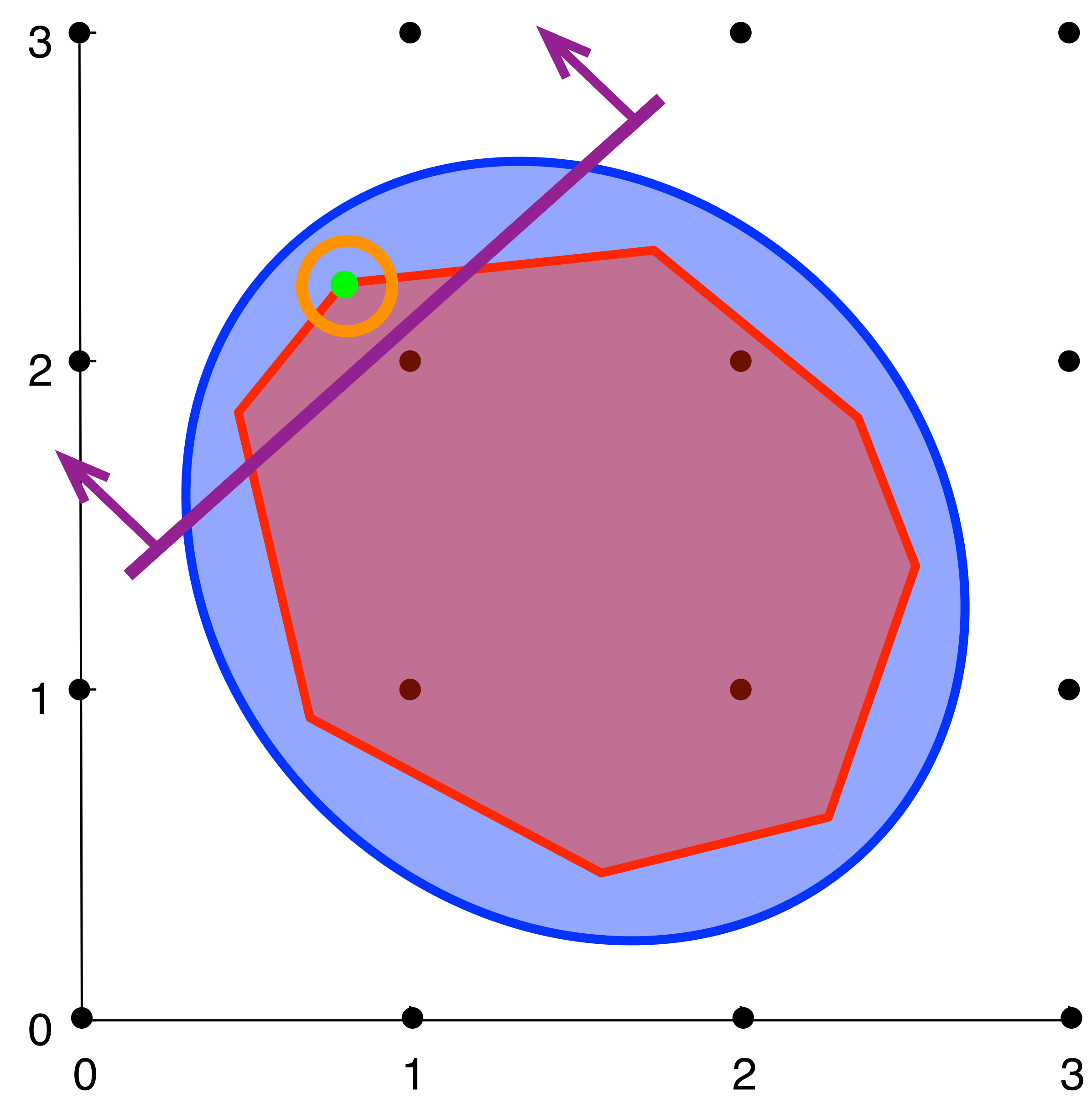
$$\exists \varepsilon > 0 \quad \varepsilon B^n + v \subset C \quad \forall v \in V$$

$$\|a\| \geq \frac{1}{\varepsilon} \Rightarrow$$

$$\begin{aligned} \lfloor \sigma_C(a) \rfloor &\geq \sigma_C(a) - 1 \\ &\geq \sigma_{v+\varepsilon B^n}(a) - 1 \\ &= \langle v, a\rangle + \varepsilon\|a\| - 1 \\ &\geq \langle v, a\rangle \end{aligned}$$

$$S^2 = (1/\varepsilon)B \cap \mathbb{Z}^n$$



Main Tool: Lifting Cuts for Faces

P polyhedron, F face of P

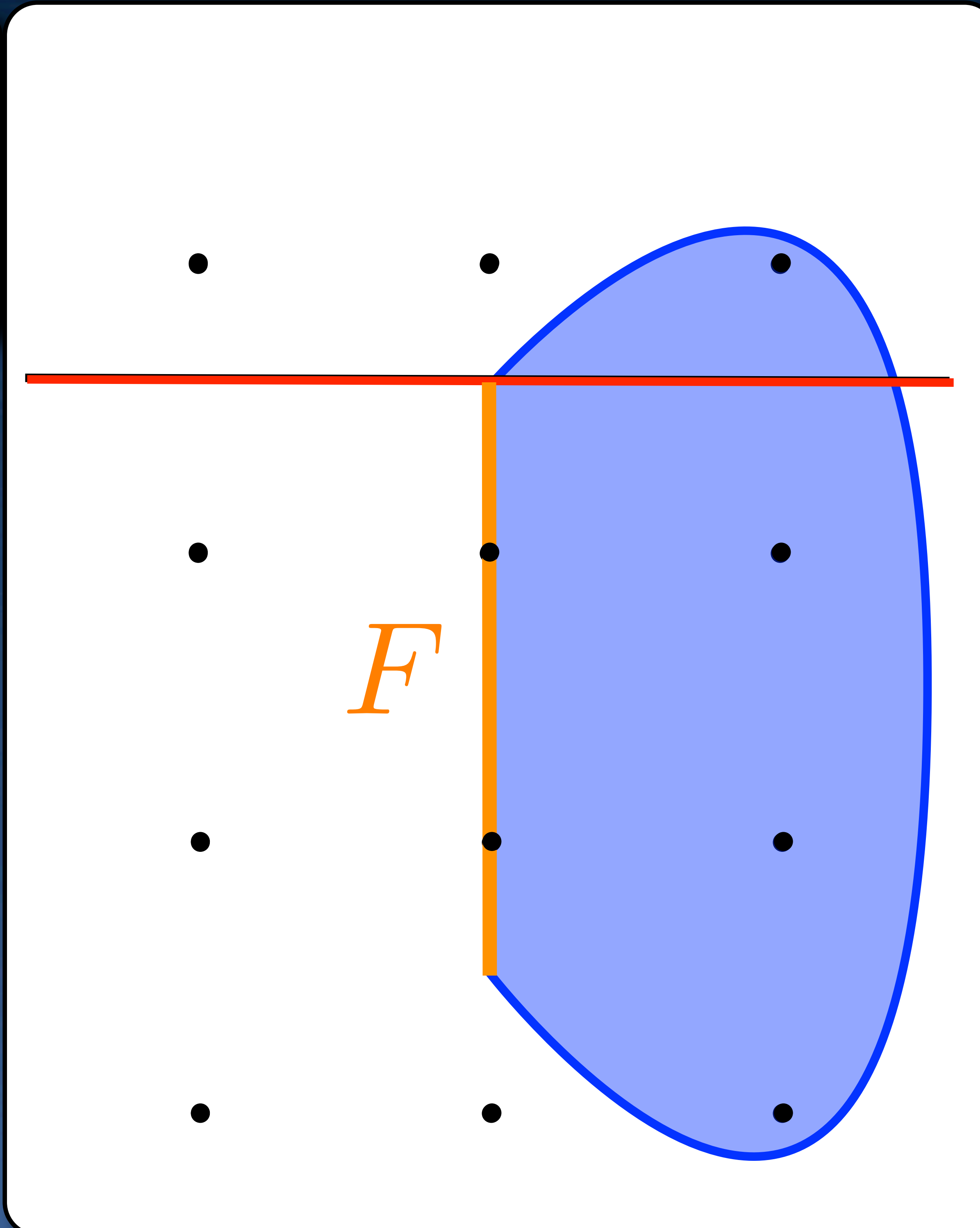
$\mathsf{CC}(F) = \mathsf{CC}(P) \cap F$ (Schrijver, '86)

(Convex Sets: Dadush, Dey, V. (2011))

- Part 1: Kill Irrationality:

  $\mathrm{aff}_I(F) := \mathrm{aff}\,(\mathrm{aff}(F) \cap \mathbb{Z}^n)$

  - Kronecker's approx.
- Part 2: Lift inside
  - Dirichlet's approx.



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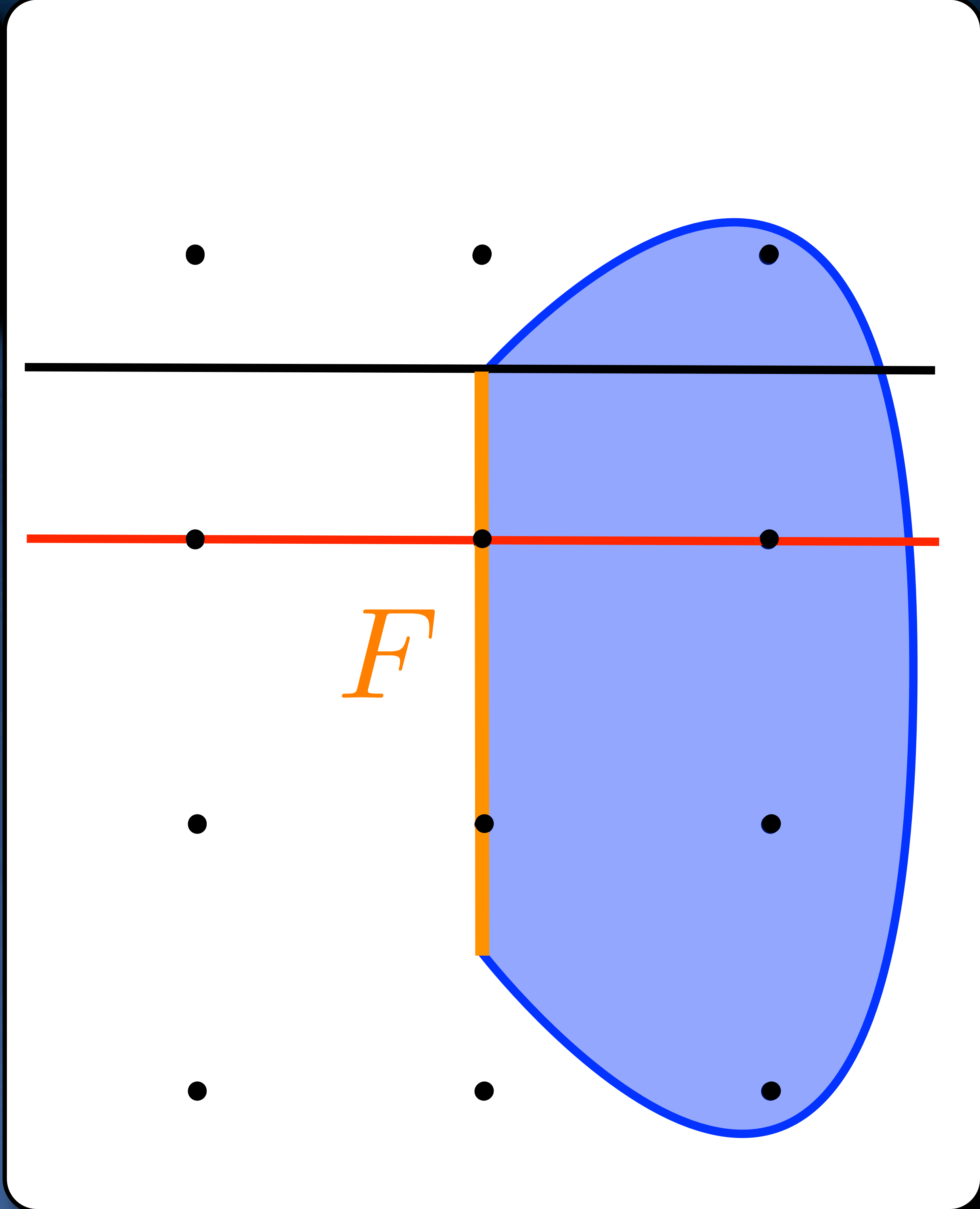

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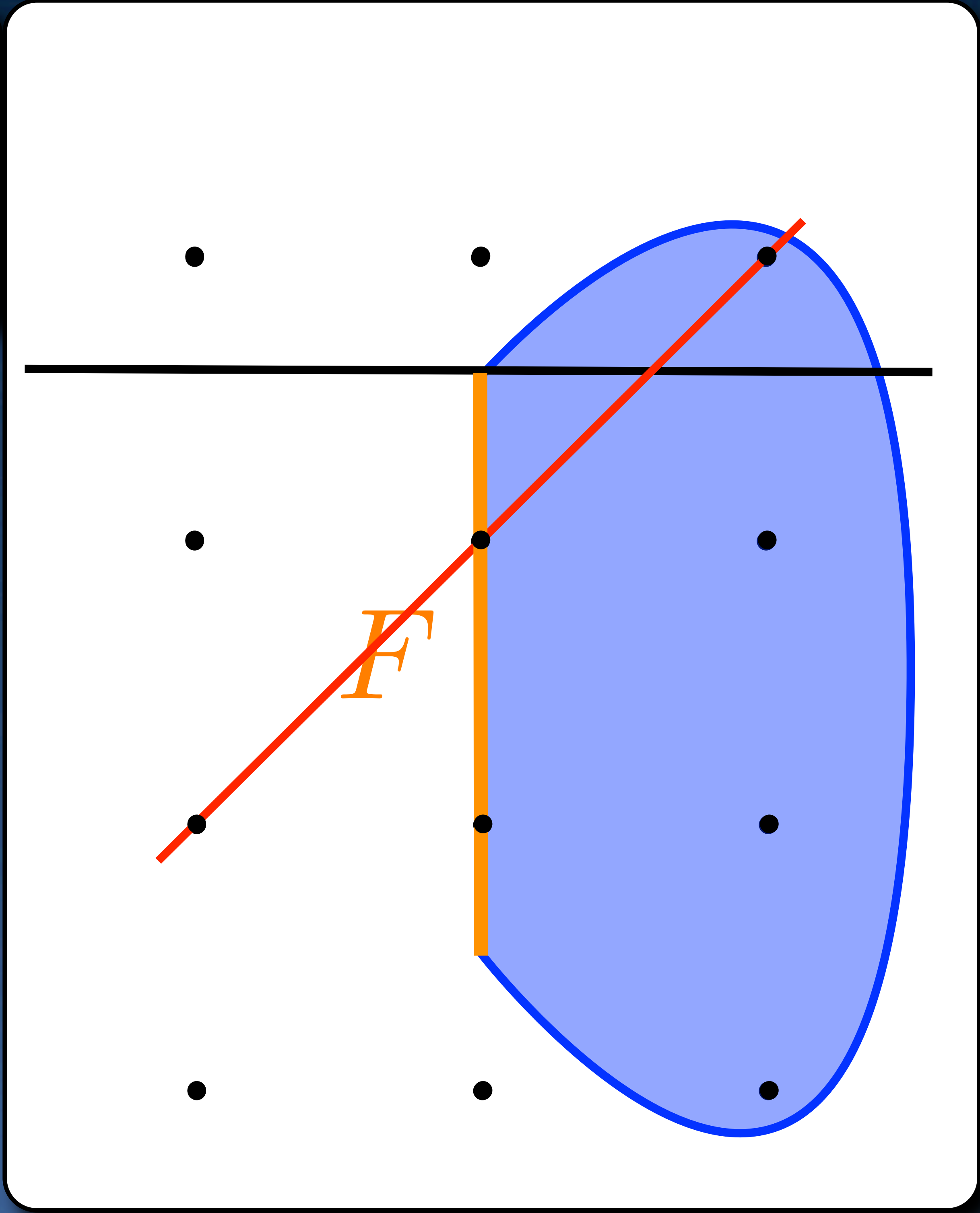

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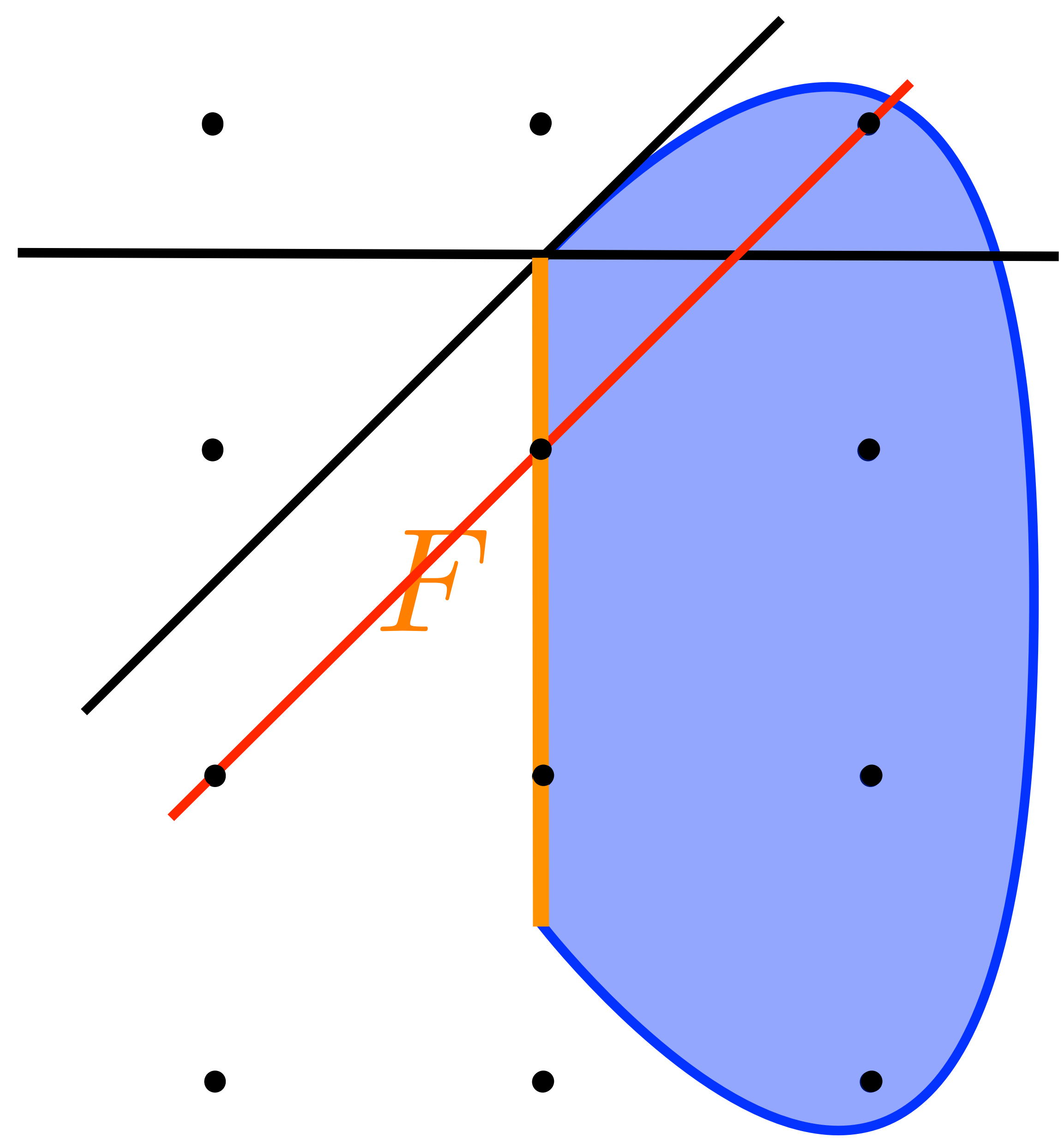
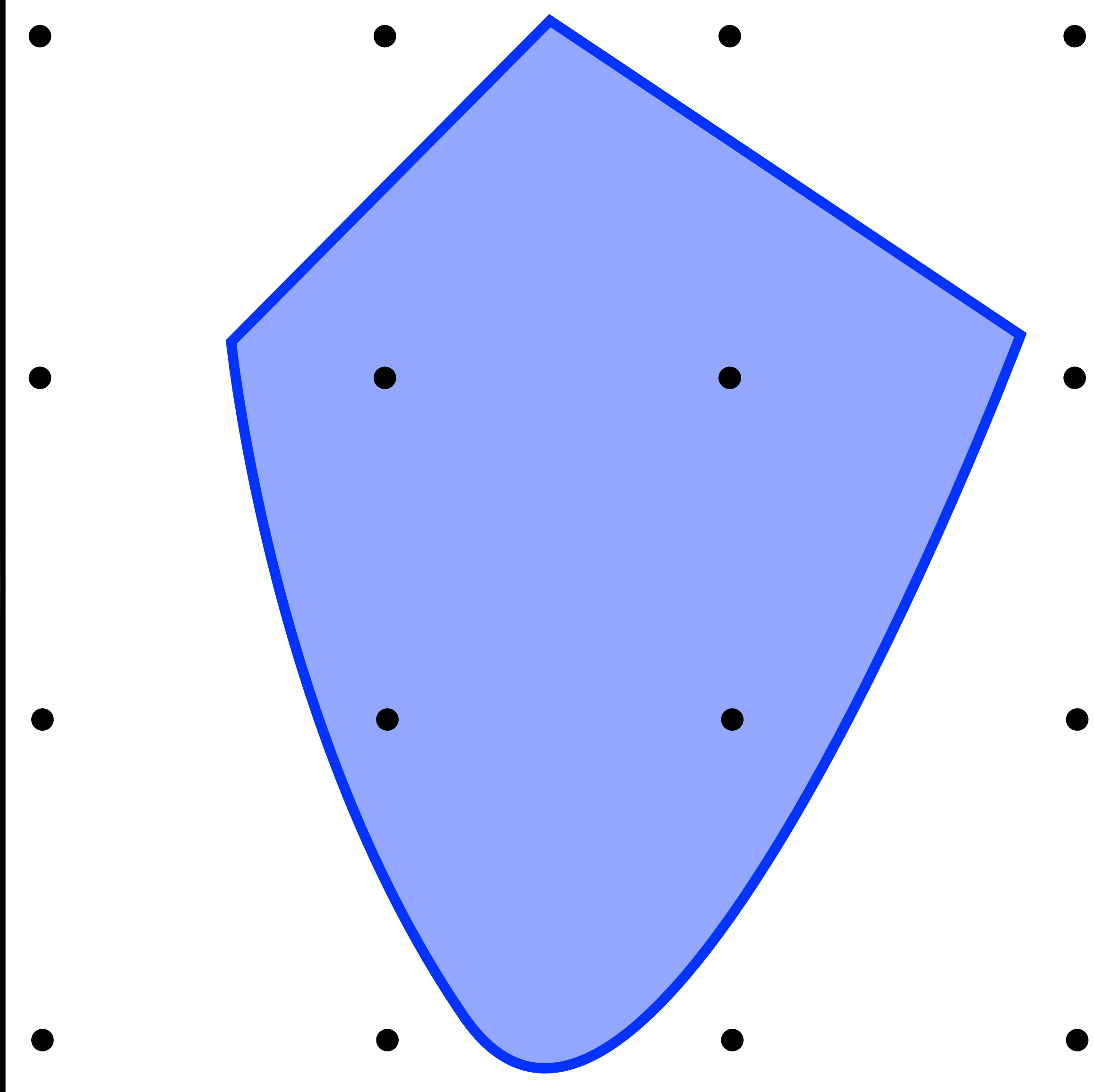


Split Cuts

Split Disjunctions and Split Cuts

$\pi \in \mathbb{Z}^n$, $\pi_0 \in \mathbb{Z}$



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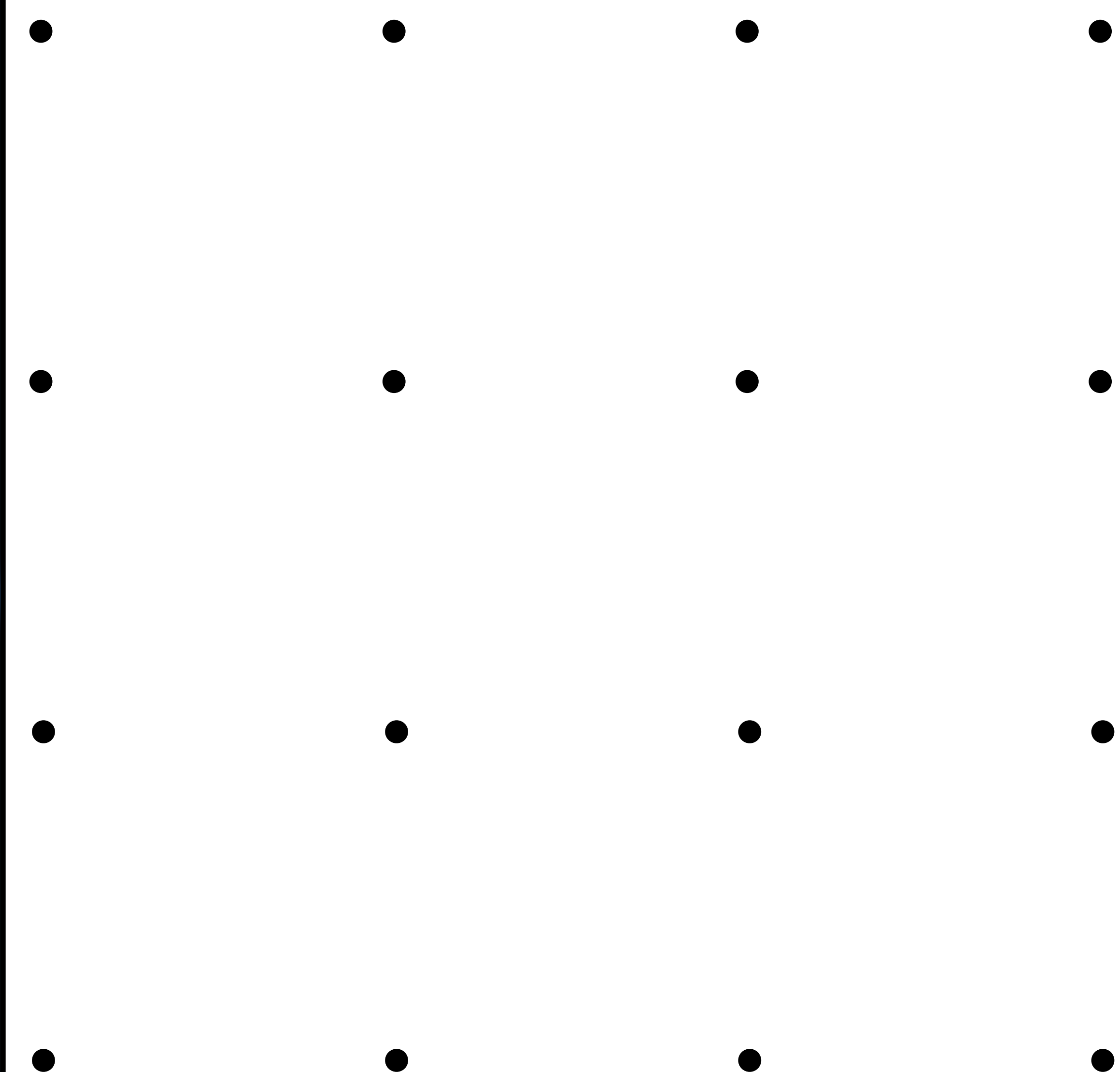


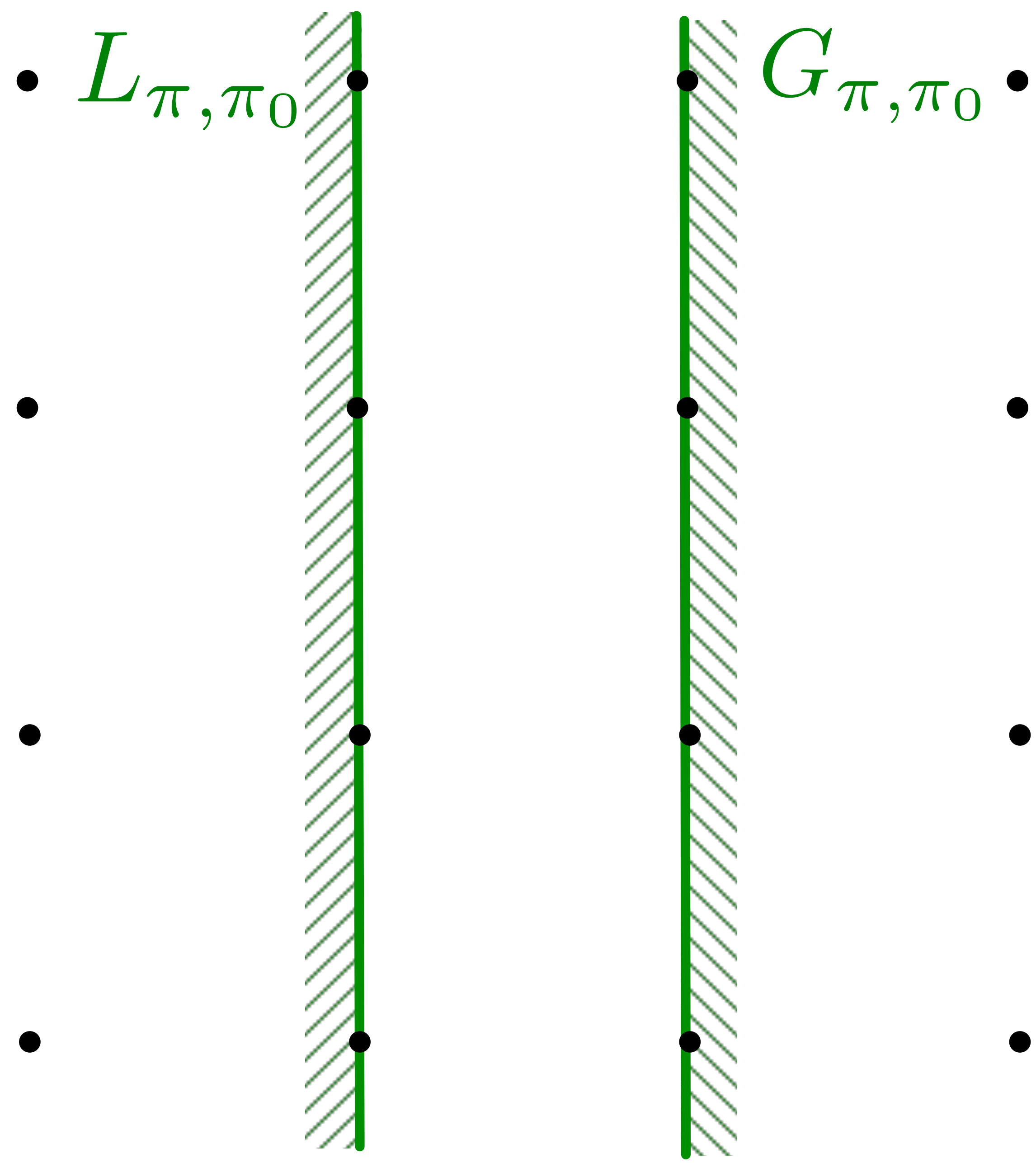
Split Disjunctions and Split Cuts

$\pi \in \mathbb{Z}^n,\ \pi_0 \in \mathbb{Z}$ Split Disjunction

$$L_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ : \ \langle \pi, x\rangle \le \pi_0\}$$

$$R_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ : \ \langle \pi, x\rangle \ge \pi_0 + 1\}$$

$$\mathbb{Z}^n \subseteq L_{\pi,\pi_0} \cup R_{\pi,\pi_0}$$



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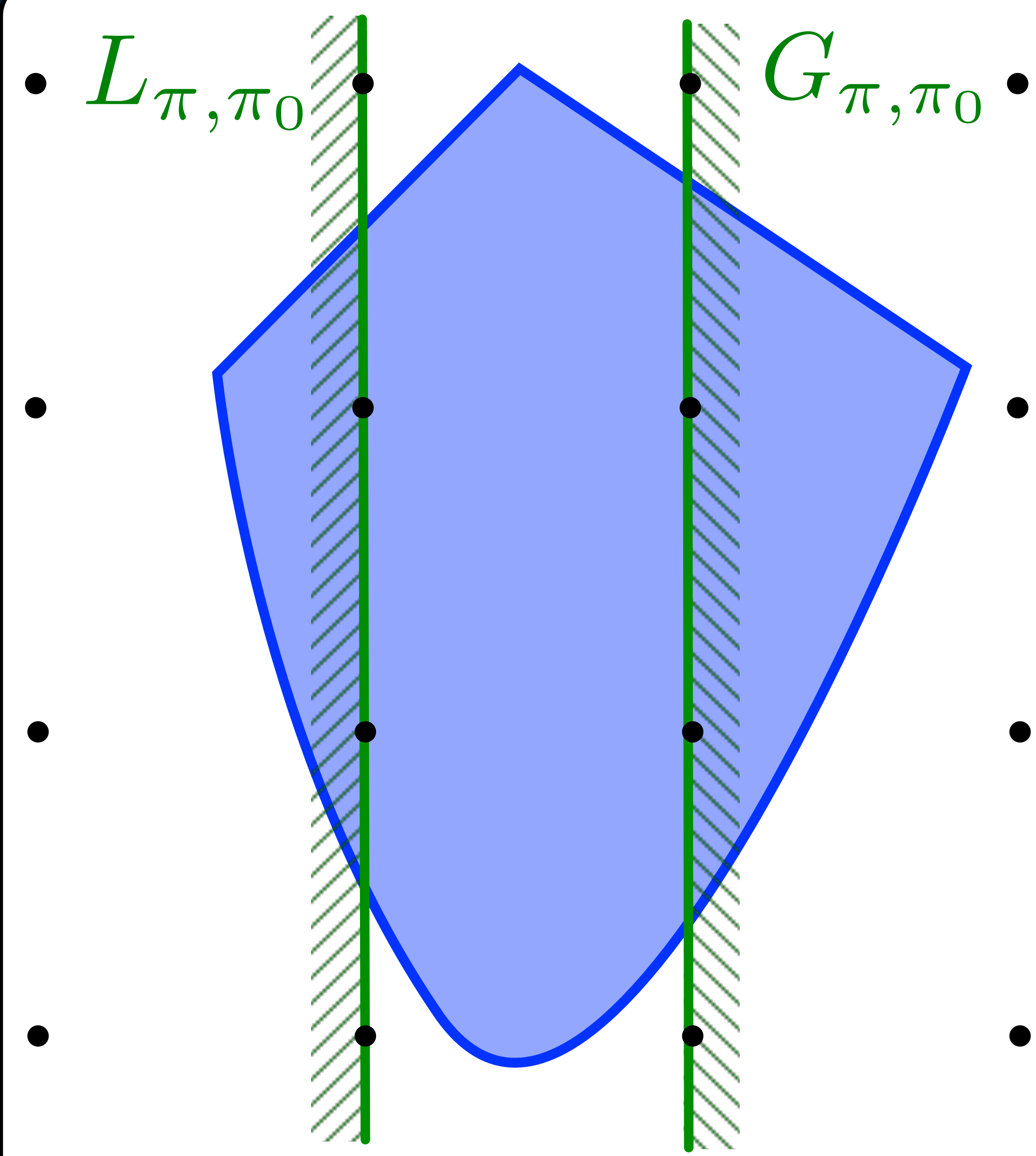

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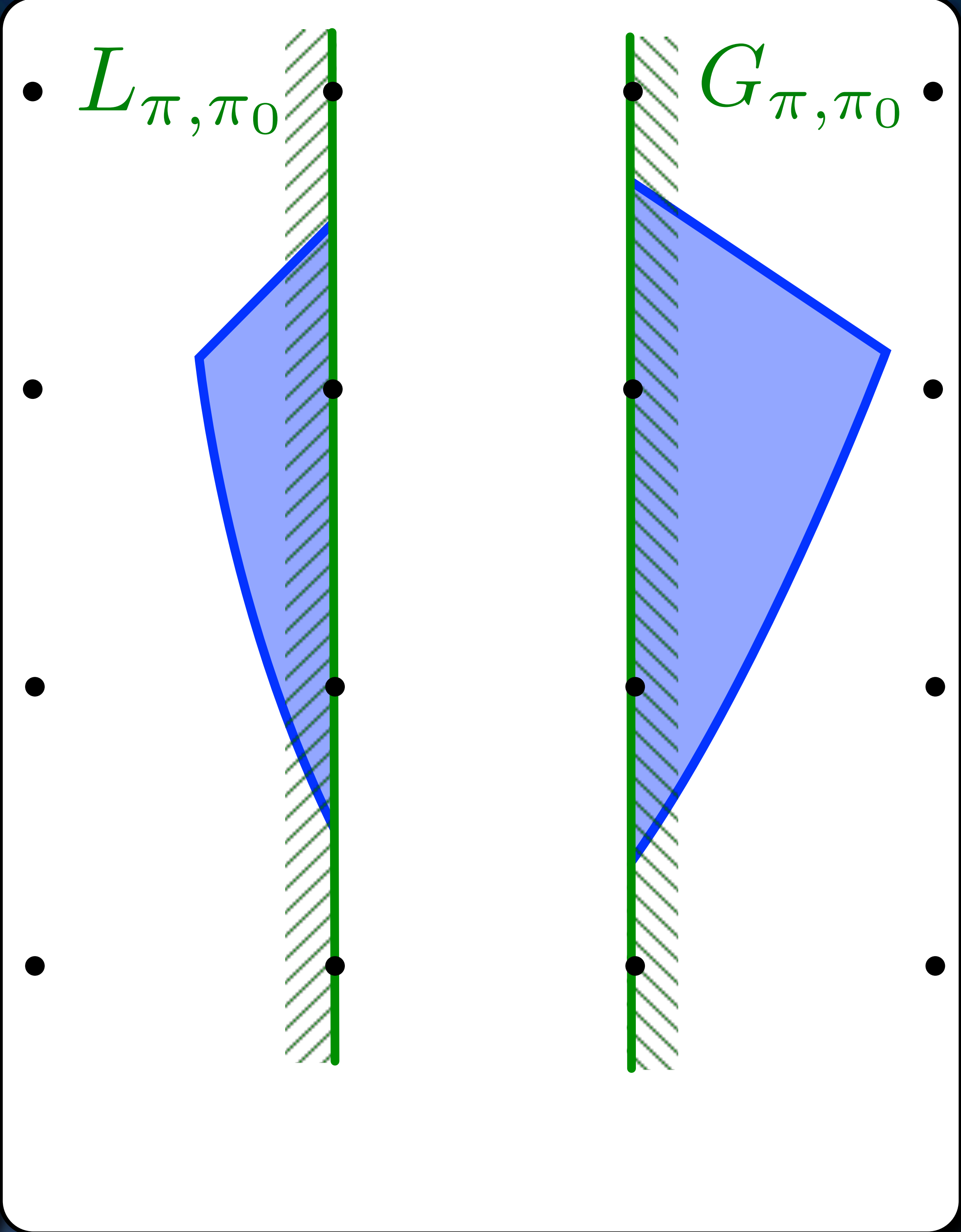

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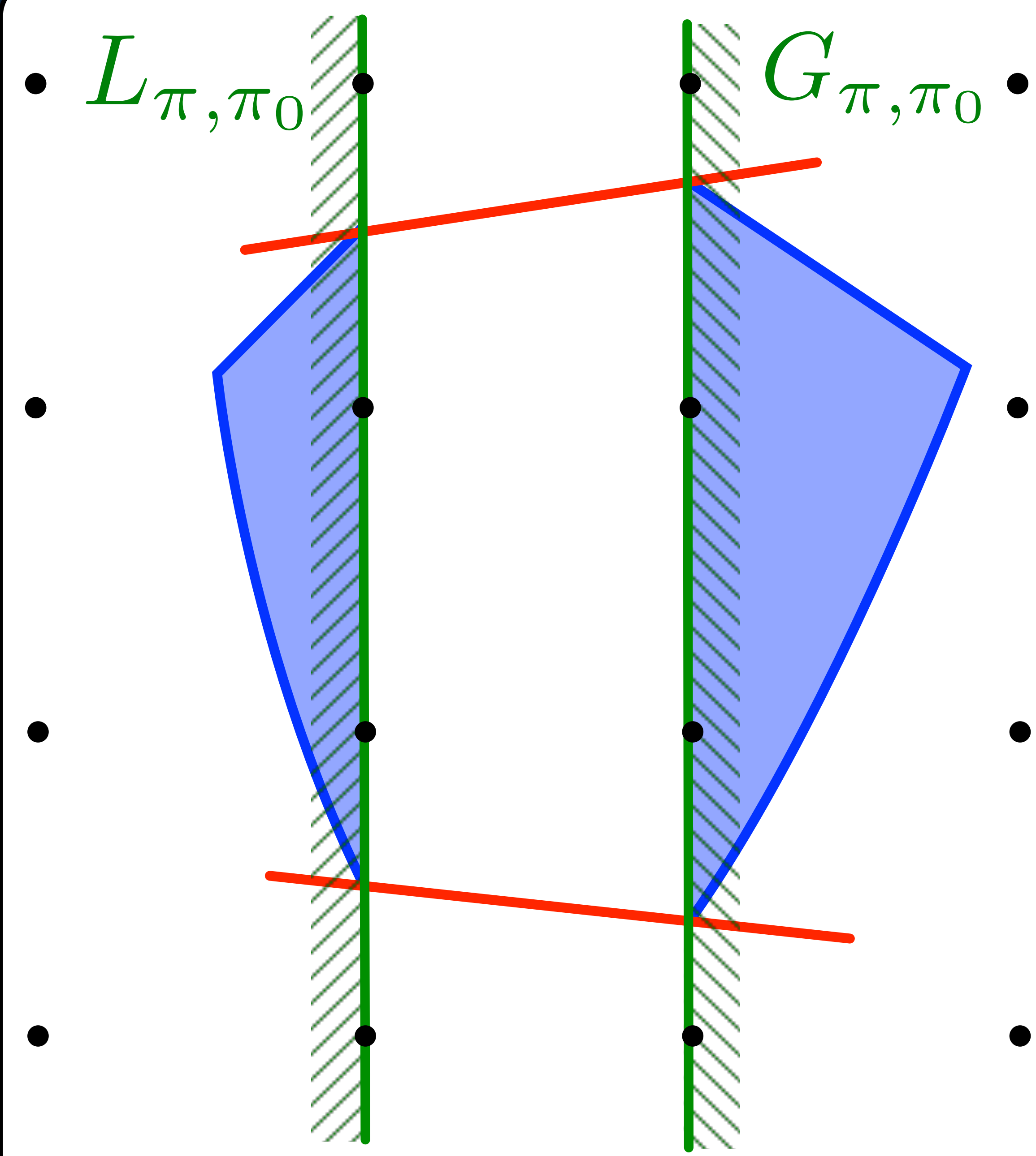

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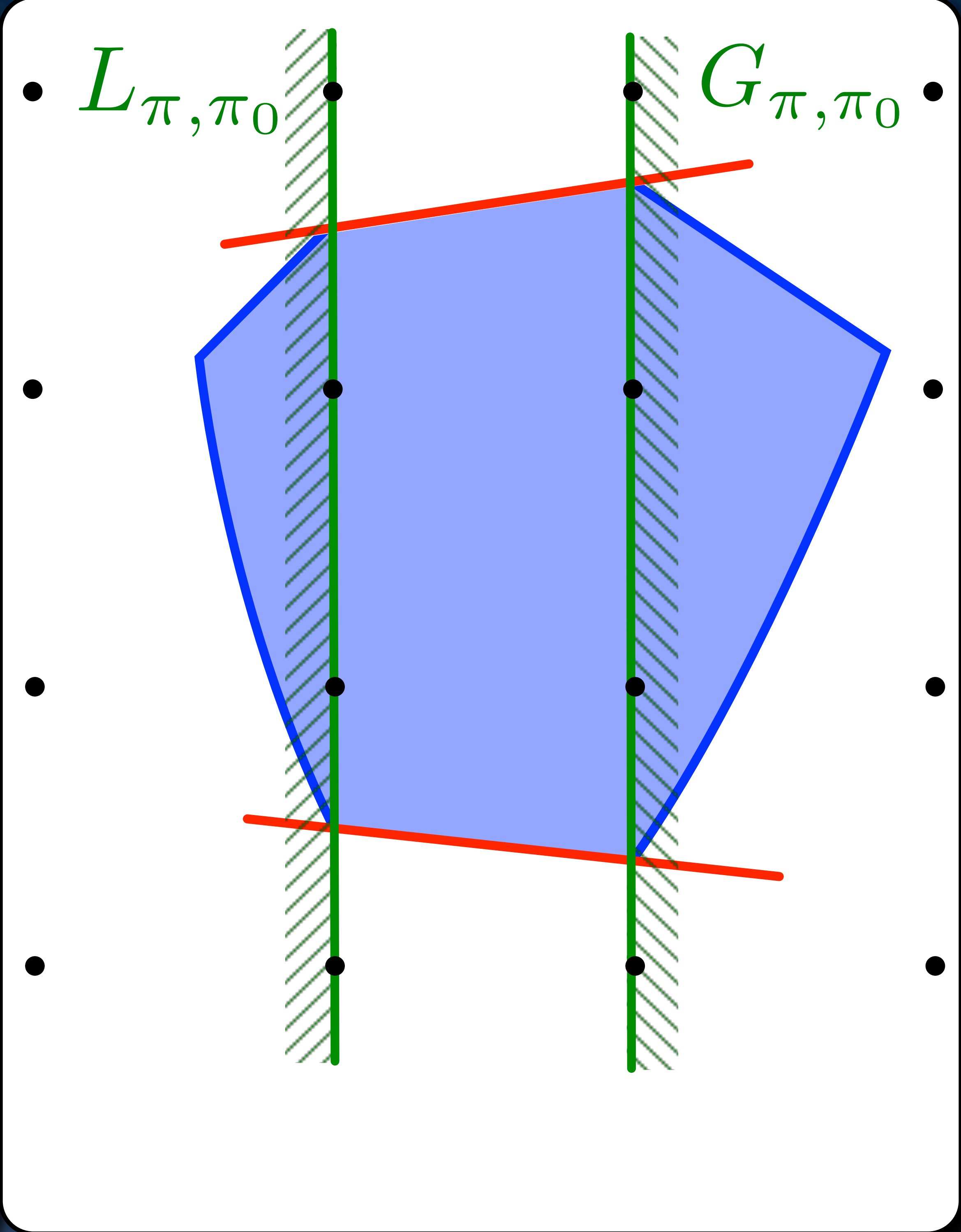

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$$||$$

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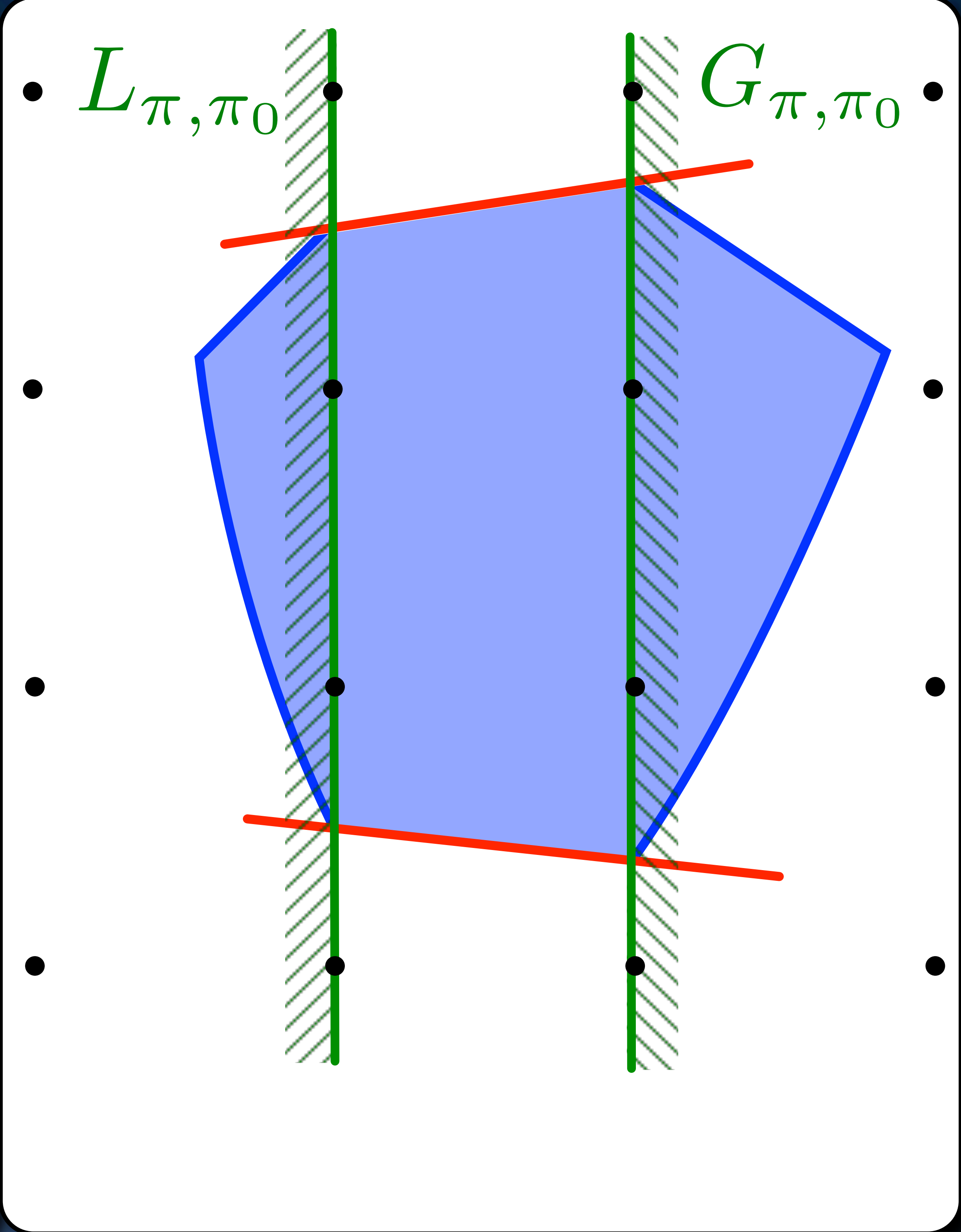

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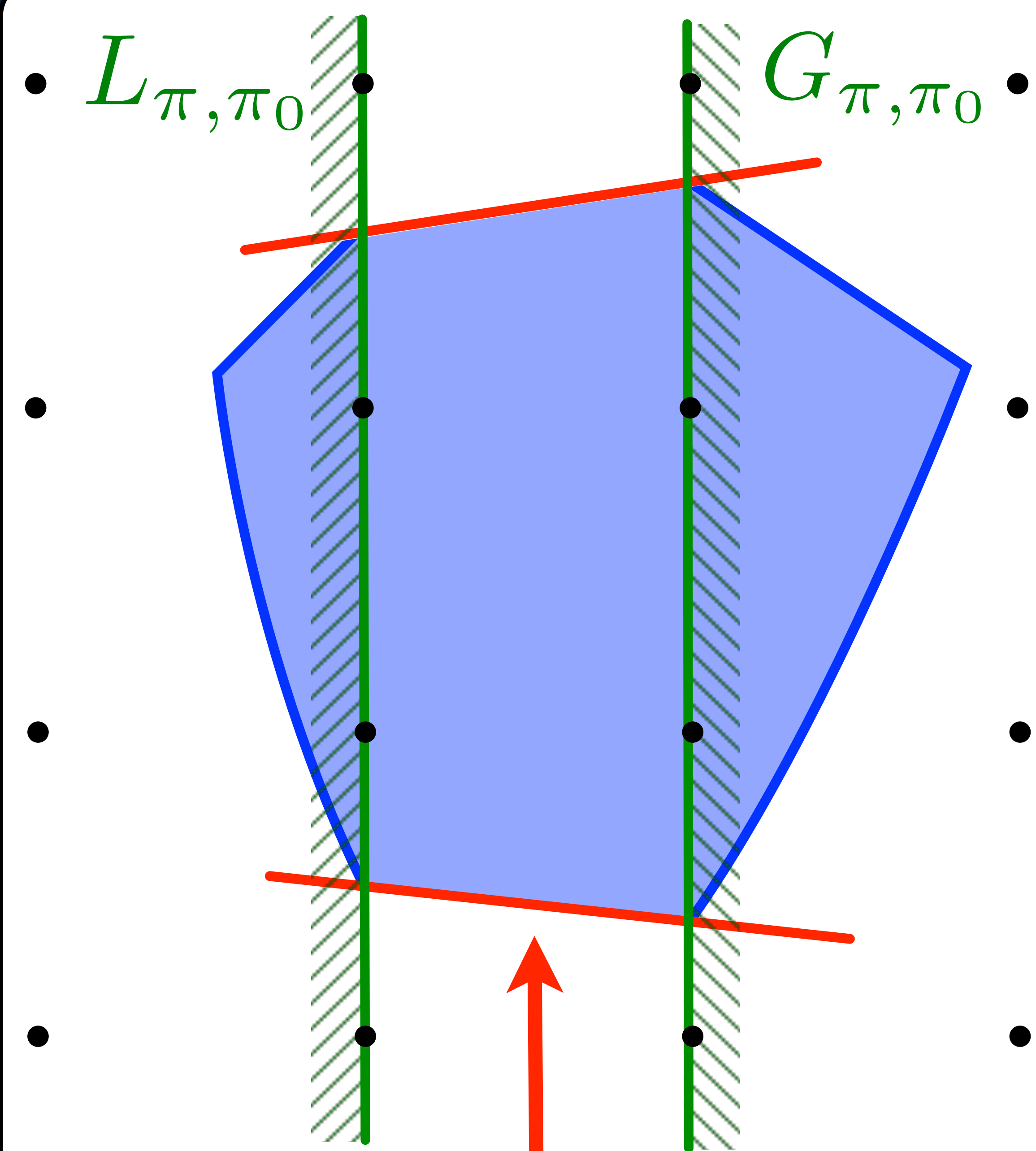
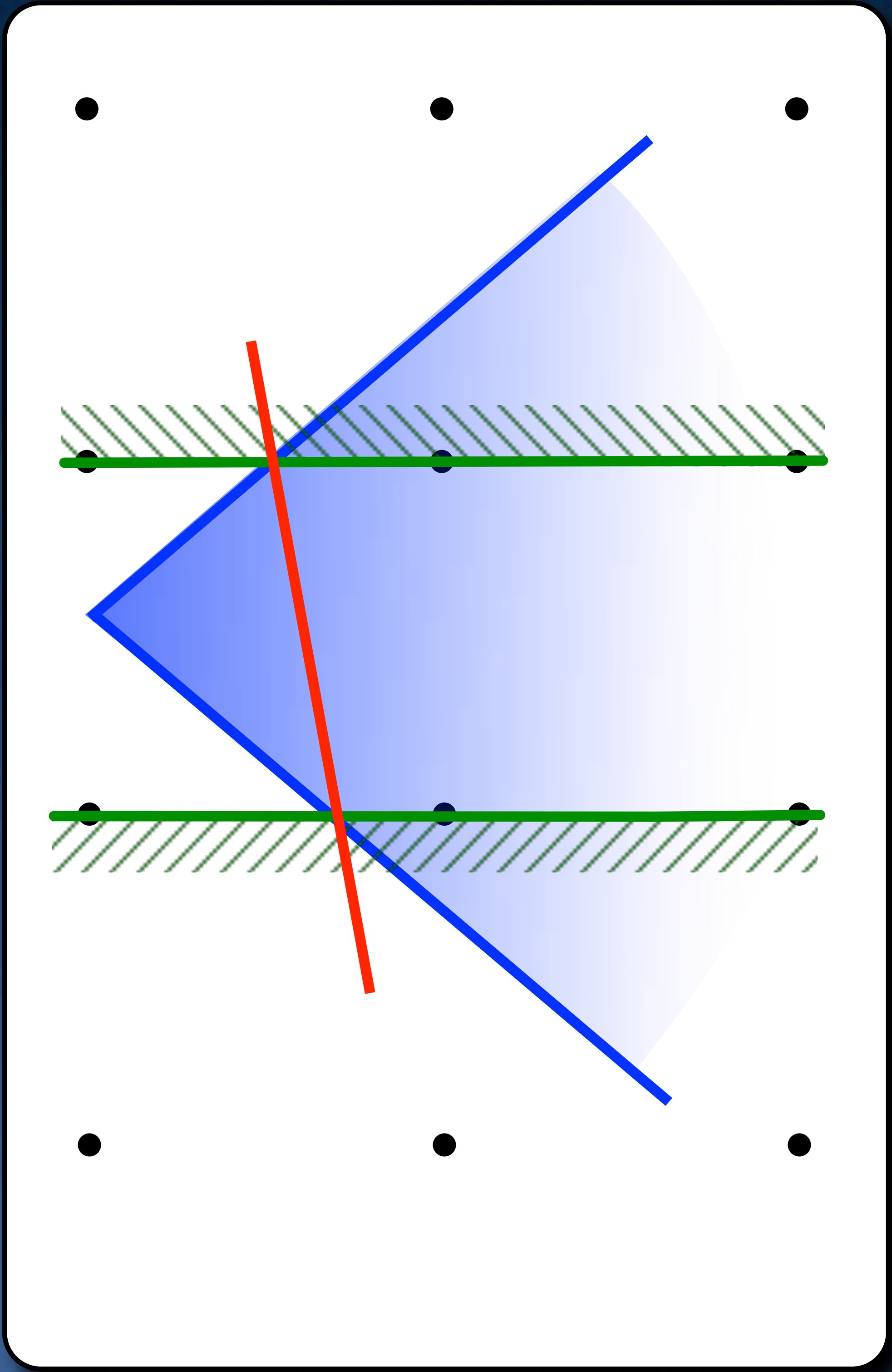


Split Cuts

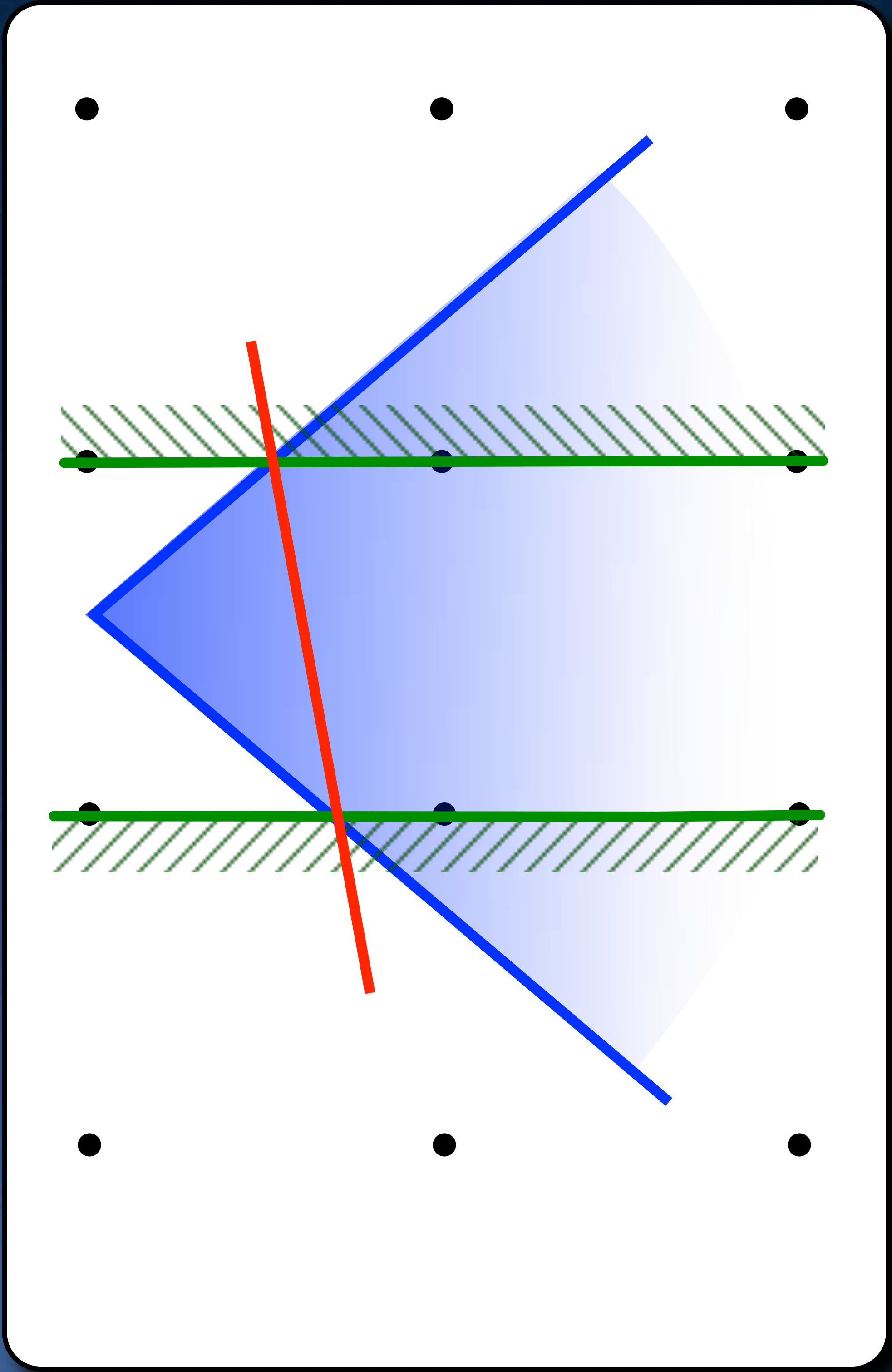
Known Facts for Rational Polyhedra

- Formulas for simplicial cones:
  - MIG (Gomory 1960) and MIR (Nemhauser and Wolsey 1988)



Known Facts for Rational Polyhedra

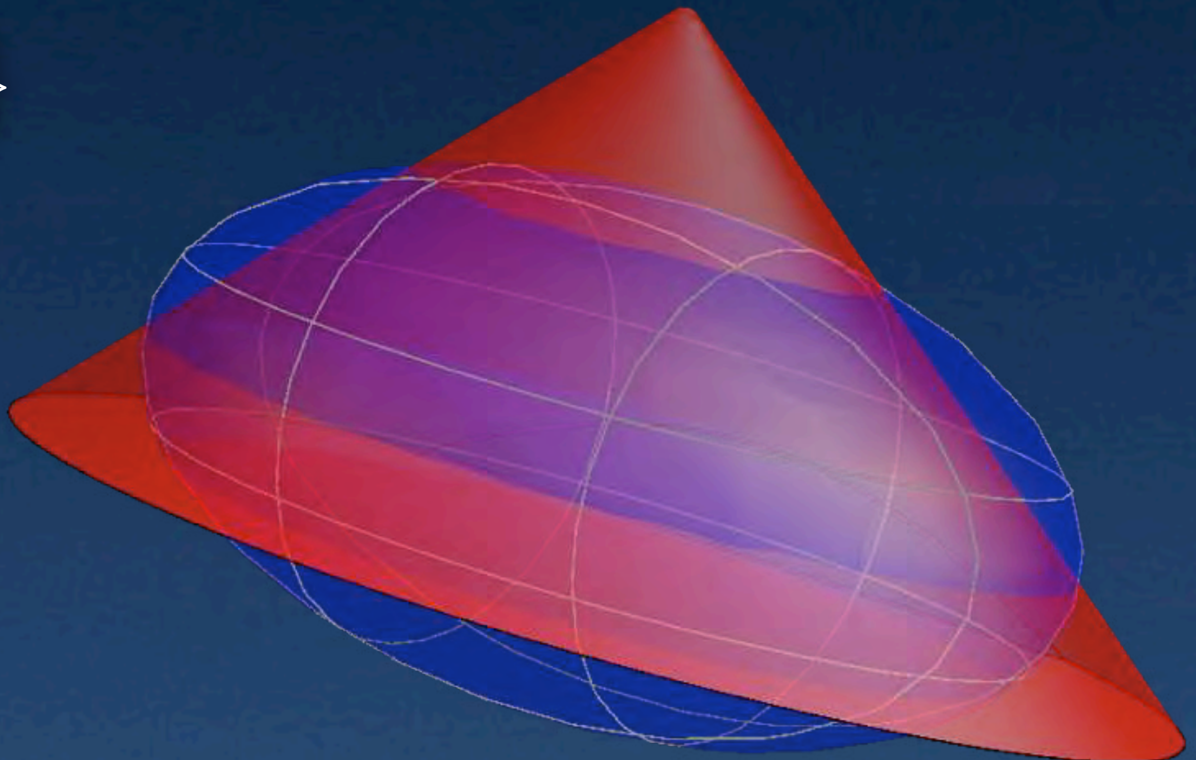
- Formulas for simplicial cones:
  - MIG (Gomory 1960) and MIR (Nemhauser and Wolsey 1988)
- Split Closure $\bigcap_{(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Z}} C^{\pi,\pi_0}$:
  - Rational Polyhedron (Cook, Kannan and Shrijver 1990)
  - Constructive Proofs:
    - Dash, Günlük and Lodi 2007; V. 2007.

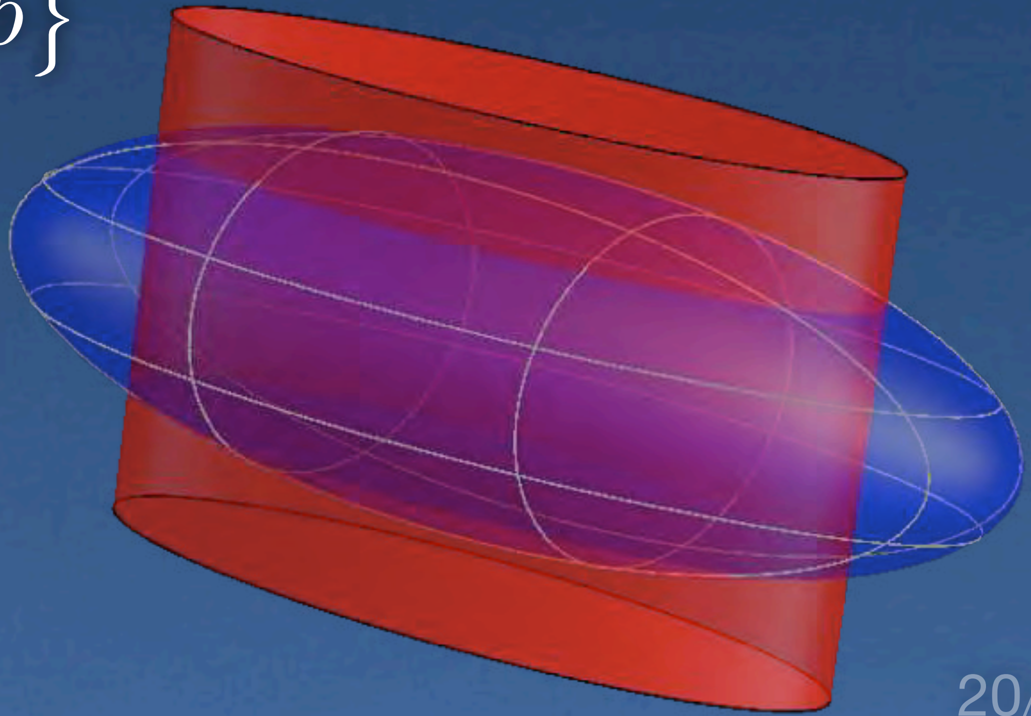


Split Cut for Ellipsoids

- $C = \{x \in \mathbb{R}^n \ : \ \|A(x-c)\|_2 \leq 1\}$
- Dadush, Dey and V. 2011:

$$C^{\pi,\pi_0} = \{x \in \mathbb{R}^n \ : \\ \|A(x-c)\|_2 \leq 1 \\ \|B(x-c)\|_2 \leq a\langle \pi, x\rangle + b\}$$





- Also see Belotti, Góez, Polik, Ralphs, Terlaky 2011

Split Cut for Quadratic Cones

- $C = \{(x,t) \in \mathbb{Z}^n \times \mathbb{R} \; : \; \|A(x-c)\|_2 \leq t\}$

- Modaresi, Kılınç, V. 2011:

$$C^{\pi,\pi_0} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \; : \; \|A(x-c)\|_2 \leq t \quad \|Bx-d\|_2 \leq t\}$$

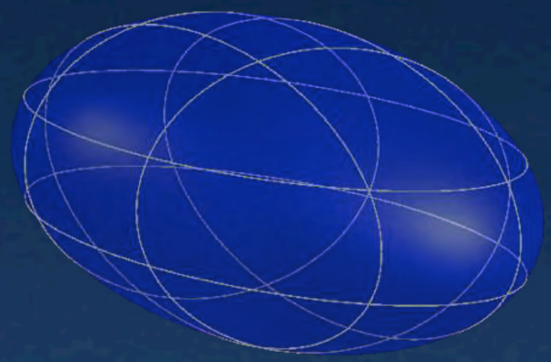
Split Closure is Finitely Generated

- Theorem (Dadush, Dey, V. 2011): If C is a strictly convex set then there exists a finite $D \subseteq \mathbb{Z}^n \times \mathbb{Z}$ such that:

$$\bigcap_{(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Z}} C^{\pi,\pi_0} = \bigcap_{(\pi,\pi_0)\in D} C^{\pi,\pi_0}$$

- Does not imply polyhedrality of split closure.
- Split Closure is not stable

Split Closure Can Be Non-Polyhedral



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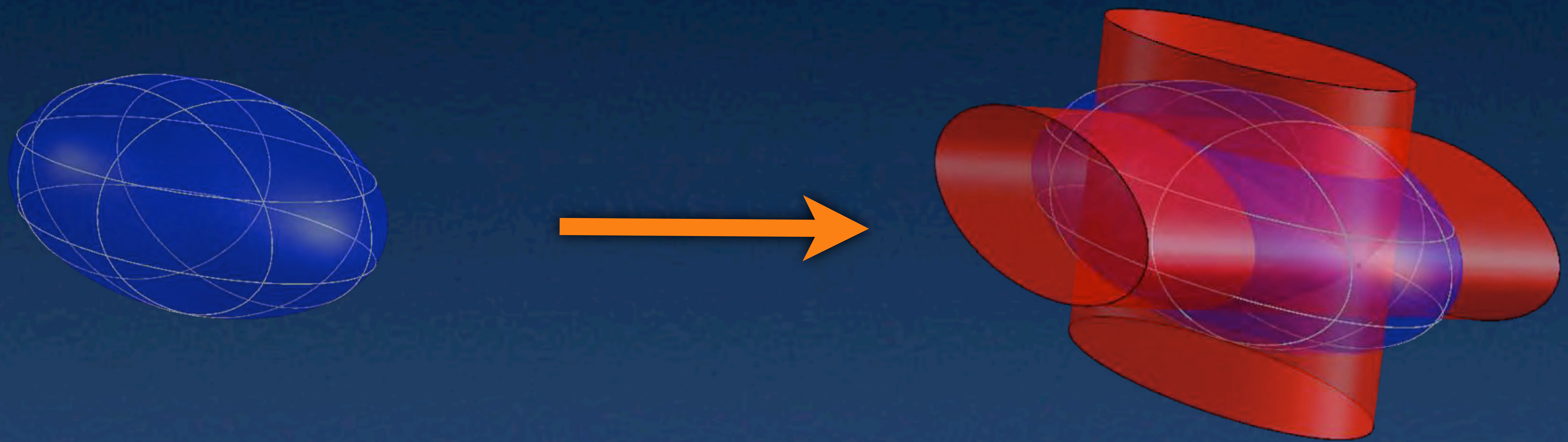

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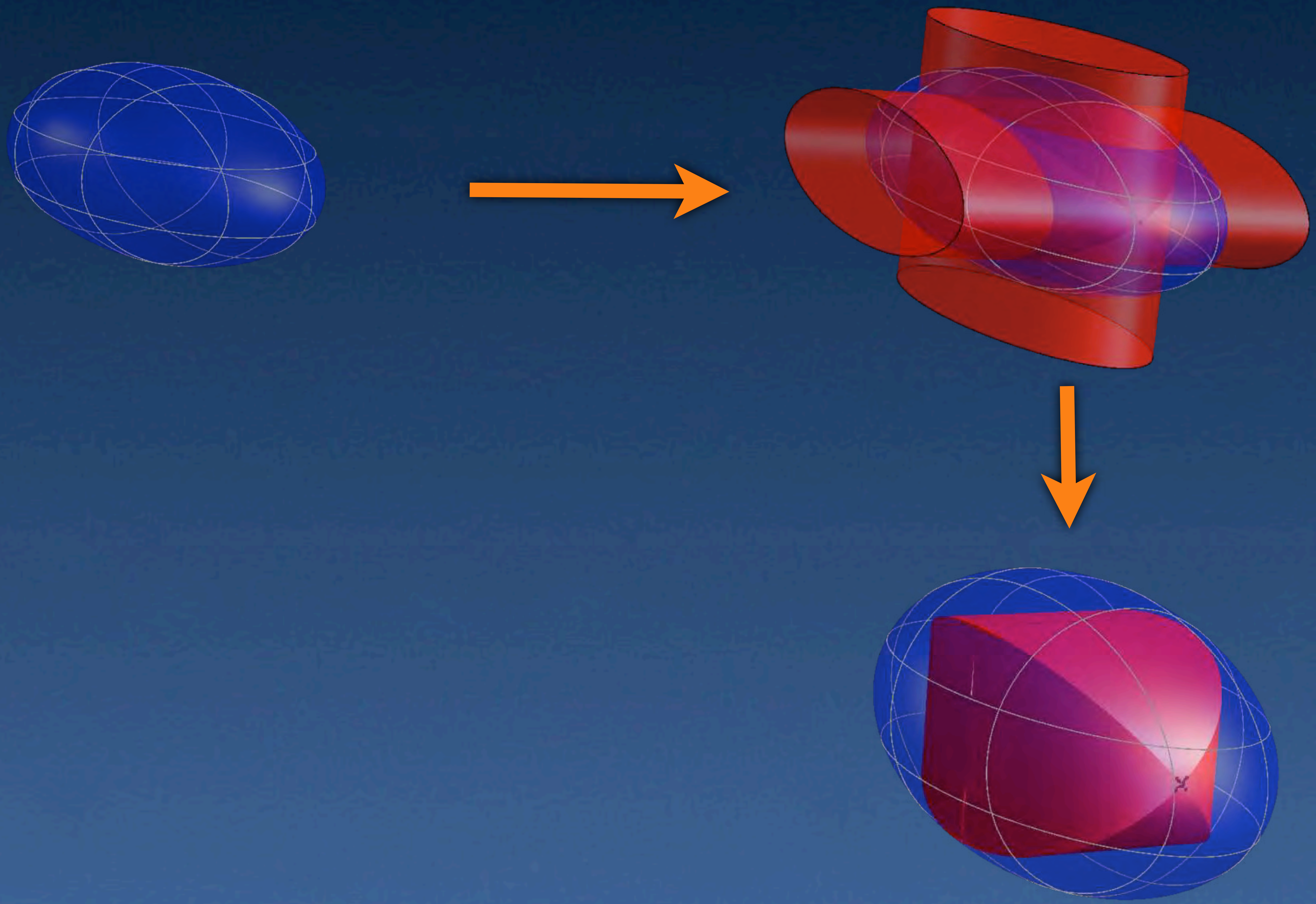

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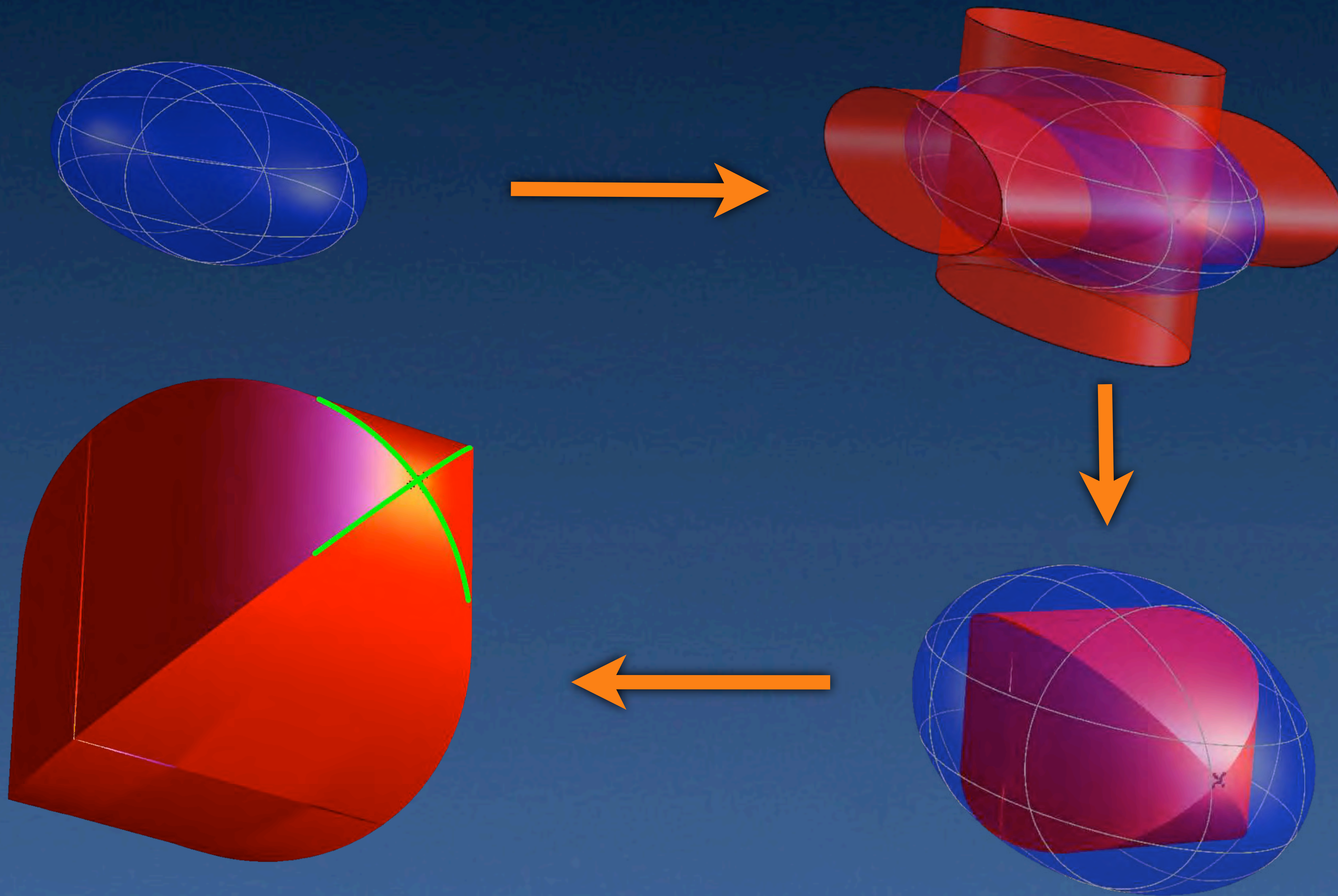

Other Results and Open Questions

- CG closure is polyhedron for a class of unbounded sets:
  - Class includes rational polyhedra = True generalization of Schrijver theorem.
- Open Questions:
  - Constructive characterization of CG closure.
  - Algorithms to separate CG/Split cuts.