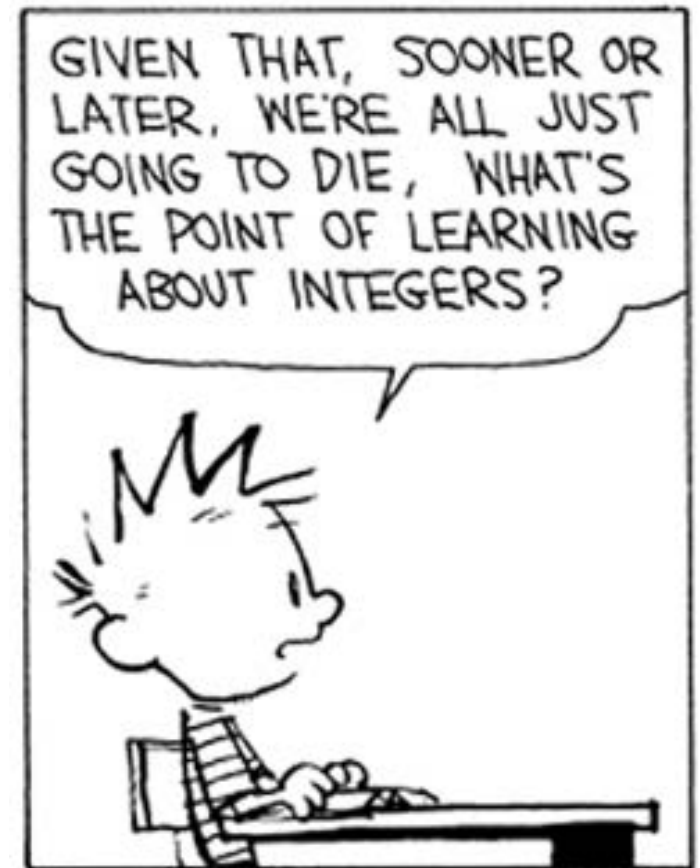# Mixed-Integer Convex (MICP) Representability

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INFORMS Optimization Society Conference
Denver, Colorado. March, 2018.

MICP Formulations and Representability

- A set $S \subseteq \mathbb{R}^n$ is MICP representable (MICPR) if it has an MICP formulation:

  – A closed convex set $M \subseteq \mathbb{R}^{n+p+d}$

  – auxiliary continuous variables $y \in \mathbb{R}^p$

  – auxiliary integer variables $z \in \mathbb{Z}^d$

$$x \in S \quad \Leftrightarrow \quad \begin{array}{l} \exists (y, z) \in \mathbb{R}^p \times \mathbb{Z}^d \text{ s.t.} \\ (x, y, z) \in M \end{array}$$

or equivalently

$$S = \text{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

What Sets are MICP Representable (MICPR) ?

Two sheet hyperbola?

Spherical shell?

$$\left\{ x \in \mathbb{R}^2 \ : \ 1 + x_1^2 \le x_2^2 \right\}$$

$$\left\{ x \in \mathbb{R}^2 \ : \ 1 \le \|x\| \le 2 \right\}$$

- Discrete subsets of the real line or natural numbers:

  – Dense discrete set? $\left\{ \sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \ : \ x \in \mathbb{N} \right\} \subseteq [0, 1]$

  – Set of prime numbers?

"God made the integers, all else is the work of man"

- Leopold Kronecker

A Simple Lemma for non-MICP Representability

- Obstruction for MICP representability of S :

$$\text{infinite } R \subseteq S \quad \text{s.t.} \quad \frac{u + v}{2} \notin S \quad \forall u, v \in R, u \neq v$$

✘ **Spherical shell** $\left\{ x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2 \right\}$
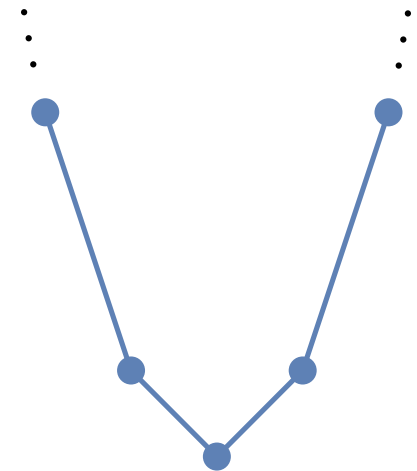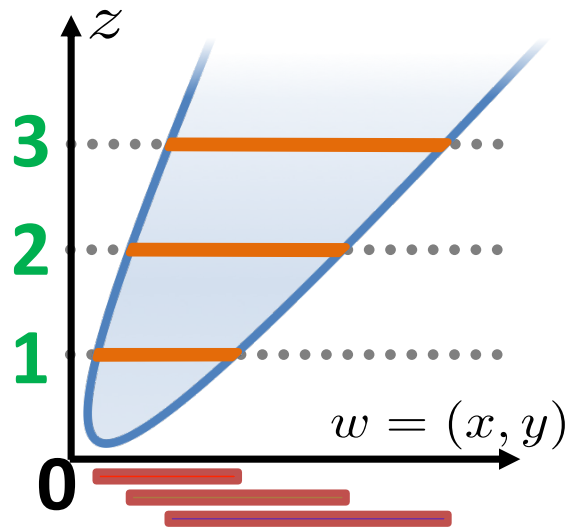
✘ **Set of prime numbers**

- Does have **non-convex** polynomial MIP

✘ Set of Matrices of rank at most k

✘ Piecewise linear interpolation of x^2 at all integers

MICPR = Convex Sets Indexed by Integers in Convex

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \text{proj}_x \left(B_z \right)$$
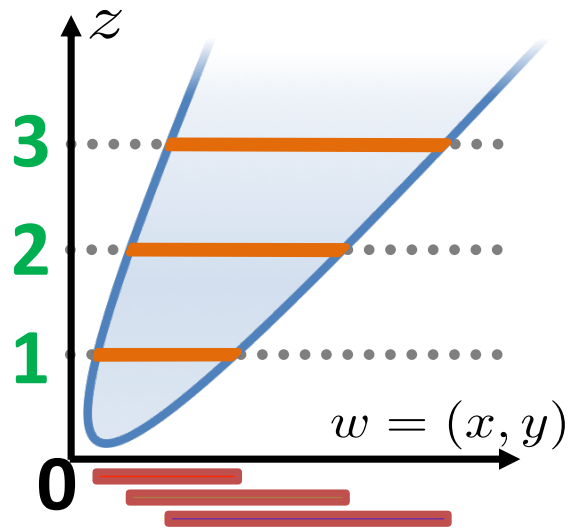
$$I = \text{proj}_z \left(M \right) \text{ convex}$$

- For rational polyhedral M (Jeroslow and Lowe '84):

$$- \quad S = \bigcup_{i=1}^{k} P_i + \left\{ \sum_{i=1}^{t} \lambda_i r^i \; : \; \lambda \in \mathbb{Z}_+^t \right\}$$

1: Rational polyhedra with the same recession cone

2: Finite # of shapes + periodic translations

MICPR = Convex Sets Indexed by Integers in Convex

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \operatorname{proj}_x \left(B_z \right)$$

$$I = \operatorname{proj}_z \left(M \right) \text{ convex}$$

- Extensions $S = \bigcup_{i=1}^{k} P_i + \left\{ \sum_{i=1}^{t} \lambda_i r^i \; : \; \lambda \in \mathbb{Z}_+^t \right\}$

  - $M = \{ x \in \mathbb{Z}^2 \; : \; x_1 \cdot x_2 \geq \alpha \} \implies P_i$ = points (Dey & Moran '13)

  - M = Rational Polyhedron ∩ "Rational" Ellipsoidal Cylinder $\implies$ P_i = Rational Ellipsoid ∩ Polytope (Del Pia & Poskin '16)

  - M = Compact Convex + Rational Polyhedron Cone $\implies$ P_i = Compact Convex (Lubin, Zadik & V. 17')

MICPR = Convex Sets Indexed by Integers in Convex
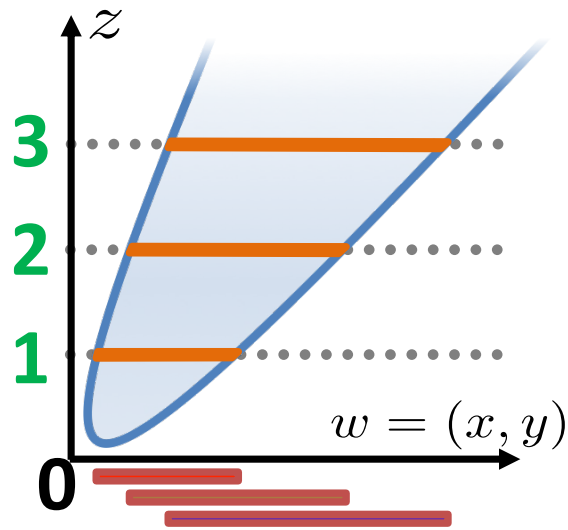
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1: Rational polyhedra with the <u>same recession cone</u>

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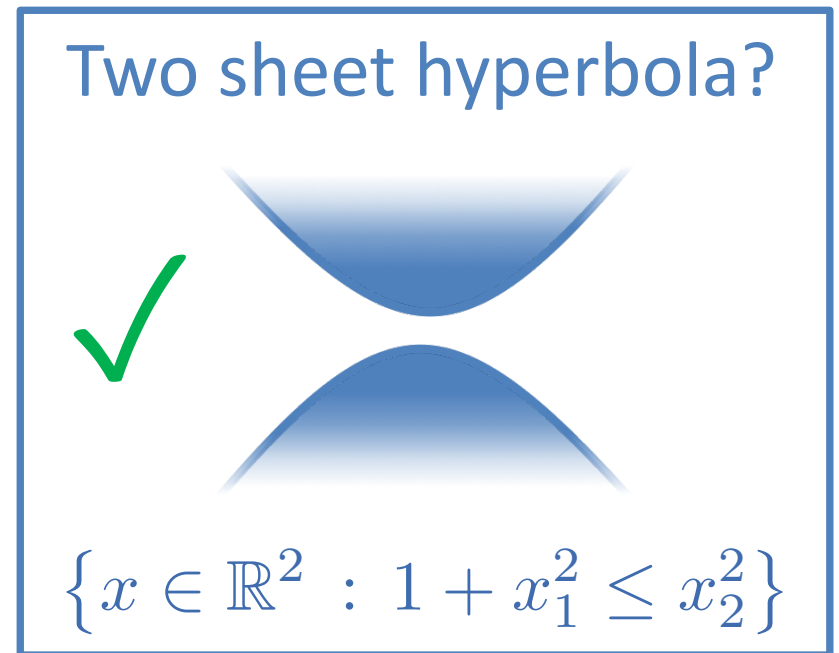
Extra from MICP 1: Non-Polyhedral Unions

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \text{proj}_x \left(B_z \right)$$

1. Unions of **Non-Polyhedral sets**

Plus Projection:

2. Unions of non-closed sets
3. Unions of convex sets with different recession cones

Two sheet hyperbola?

✓

$$\{x \in \mathbb{R}^2 \ : \ 1 + x_1^2 \leq x_2^2\}$$

$$\left\{ \left\{ (x,t) \ : \ x \in S_i, \quad \|x\|_2^2 \leq t \right\} \right\}_{i=1}^{k} \quad \text{have the same recession cone}$$

Extra from MICP 2: Non-Polyhedral I

- An **infinite set** S is **periodic** if and only if:

$$\exists r \in \mathbb{R}^n \quad \forall \lambda \in \mathbb{Z}_+, \ x \in S \quad x + \lambda r \in S$$

- Non-periodic MICPR sets

    – Dense discrete set $\left\{ \sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor : x \in \mathbb{N} \right\} \subseteq [0,1]$

$$\|(z_1, z_1)\|_2 \leq z_2 + 1, \quad \|(z_2, z_2)\|_2 \leq 2z_1, \quad x_1 = y_1 - z_2,$$

$$\|(z_1, z_1)\|_2 \leq y_1, \qquad \|(y_1, y_1)\|_2 \leq 2z_1, \qquad z \in \mathbb{Z}^2$$

    – Set of naturals $\left\{ x \in \mathbb{N} : \sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \notin \left(\varepsilon, 1 - \sqrt{2}\varepsilon \right) \right\}$

$$\|(x_1, x_1)\|_2 \leq x_2 + \varepsilon,$$

$$\|(x_2, x_2)\|_2 \leq 2x_1 + 2\varepsilon, \quad x \in \mathbb{Z}_+^2$$

A Definition Rational MICPR (R-MICPR)

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \text{proj}_x (B_z) \qquad S = \text{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

$$I = \text{proj}_z (M)$$

- Any rational affine mapping of index set I :

  – Is bounded, or

  – Has an integer (rational) recession direction

- Irrational directions can hide!

  – R-MICPR $\not\Leftrightarrow$ $\text{span}(\text{rec}(I))$ and/or $\text{aff}(I)$ = rational space

$$\left(z_1 + \sqrt{2} z_2 \right)^2 \leq z_3 \qquad \text{span}(\text{rec}(I)) = \text{span}(\{\mathbf{e}_3\})$$

$$\left(z_2 - \sqrt{2} z_1 \right)^2 \leq 1 \qquad \text{rec}(\text{proj}_{z_1, z_2}(I)) = \text{span}(\{(1, \sqrt{2})\})$$

Properties of Rational MICPR (R-MICPR)

- For **compact** S :
  - **Finite unions of compact** convex sets
- For S **infinite unions of "uniformly bounded" closed convex sets** :
  - **Finite union of periodic**
  - **Dense discrete** and **non-periodic naturals** NOT R-MICPR
- Rational MICP Representability:
  - **Closed** under: **Finite Union, Cartesian Product and Minkowski sum**
  - **NOT Closed** under **intersection**.

R-MICPR does Not Imply Finite Shapes

- There exists increasing functions h such that:
    - $P_z \subseteq \mathbb{R}^2$ regular $h(z)$-gon centered at $(z, 0)$
    - $P_z \cap P_{z'} = \emptyset, \quad z \neq z'$
    - $S = \bigcup_{z=1}^{\infty} P_z$ is R-MICPR and periodic

- Equal volume $\Rightarrow$ Finite # of Shapes

Summary

- General mixed integer convex representability (MICPR):
  - Infinite union of convex w. different recession cones
  - With special structure (e.g. Primes are not MICPR)
  - Infinite structure can be irregular (irrational rays)
    - Can be caused by hided rays for non- "thin" sets
- Rational MICPR
  - Regularity recovered forcing rational unboundedness
- Equal volume $\Rightarrow$ Finite # of Shapes

R-MICPR: Periodicity for Natural #s

- An **infinite set** of naturals S is **periodic** if and only if:

$$- \quad S = \bigcup_{i=1}^{k} \{s_i\} + \text{intcone}\left(\{r\}\right)$$

  - It is **rational MILP representable**

- A subset S of the naturals is **R-MICPR** if and only if:

  - It is the **union** of a **finite** and an **infinite periodic** set

A Simple Lemma for non-MICP Representability

- Obstruction for MICP representability of S :

$$\text{infinite } R \subseteq S \quad \text{s.t.} \quad \frac{u+v}{2} \notin S \quad \forall u, v \in R, \ u \neq v$$

Proof: Assume for contradiction there exists M such that:

$$S = \text{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

$$\begin{array}{c} (u, y_u, z_u) \in M \\ (v, y_v, z_v) \in M \end{array} \implies \frac{z_u + z_v}{2} \notin \mathbb{Z}^d$$

$$z_u \equiv z_v \ (\text{mod } 2) \text{ component-wise} \implies \frac{z_u + z_v}{2} \in \mathbb{Z}^d$$

$$\text{component-wise parity classes} = 2^d < |R| = \infty \quad \Rightarrow\!\!\!\ast\!\!\!\Leftarrow$$