# Mixed-Integer Convex (MICP) Representability


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# MICP Formulations and Representability

- A set $S \subseteq \mathbb{R}^n$ is MICP representable (MICPR) if it has an MICP formulation:
  - A closed convex set $M \subseteq \mathbb{R}^{n+p+d}$
  - auxiliary continuous variables $y \in \mathbb{R}^p$
  - auxiliary integer variables $z \in \mathbb{Z}^d$

$$x \in S \quad \Leftrightarrow \quad \begin{array}{l} \exists (y, z) \in \mathbb{R}^p \times \mathbb{Z}^d \text{ s.t.} \\ (x, y, z) \in M \end{array}$$

or equivalently

$$S = \operatorname{proj}_x \left( M \cap \left( \mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

# What Sets are MICP Representable (MICPR) ?

Two sheet hyperbola?

$$\{x \in \mathbb{R}^2 \ : \ 1 + x_1^2 \leq x_2^2\}$$

Spherical shell?

$$\{x \in \mathbb{R}^2 \ : \ 1 \leq \|x\| \leq 2\}$$

- Discrete subsets of the real line or natural numbers:
  - Dense discrete set? $\left\{\sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \ : \ x \in \mathbb{N}\right\} \subseteq [0,1]$
  - Set of prime numbers?

"God made the integers, all else is the work of man"

- Leopold Kronecker

# A Simple Lemma for non-MICP Representability

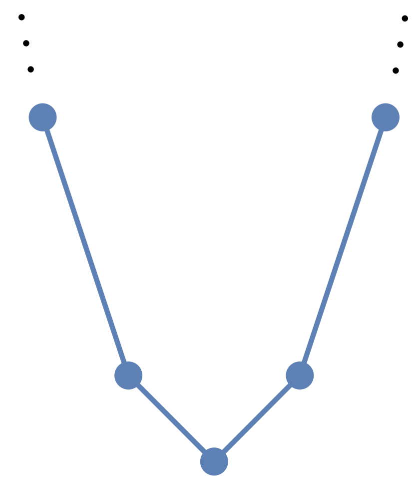
- Obstruction for MICP representability of $S$ :

$$\text{infinite } R \subseteq S \quad \text{s.t.} \quad \frac{u+v}{2} \notin S \quad \forall u, v \in R,\ u \neq v$$

✗ **Spherical shell** $\{x \in \mathbb{R}^2 \ : \ 1 \leq \|x\| \leq 2\}$


✗ **Set of prime numbers**

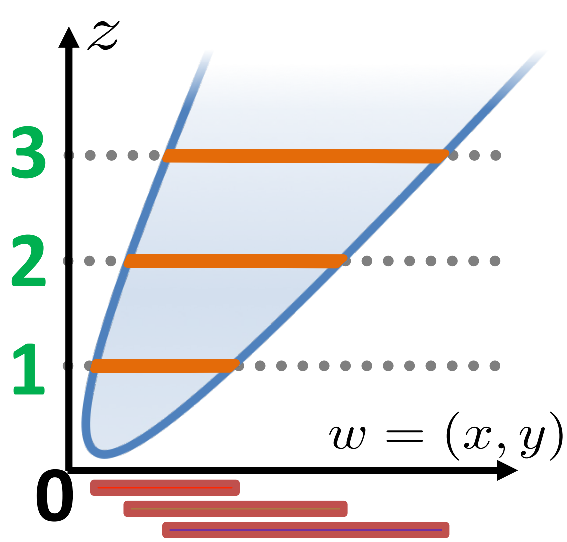
- Does have **non-convex** polynomial MIP



✗ Set of Matrices of rank at most $k$

✗ Piecewise linear interpolation of $x^2$ at all integers

# MICPR = Convex Sets Indexed by Integers in Convex




$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \operatorname{proj}_x (B_z)$$

$$I = \operatorname{proj}_z (M) \quad \text{convex}$$

- For rational polyhedral $M$ (Jeroslow and Lowe '84):
  - $S = \bigcup_{i=1}^{k} P_i + \left\{ \sum_{i=1}^{t} \lambda_i r^i \; : \; \lambda \in \mathbb{Z}_+^t \right\}$

1: Rational polyhedra with the same recession cone

2: Finite # of shapes + periodic translations

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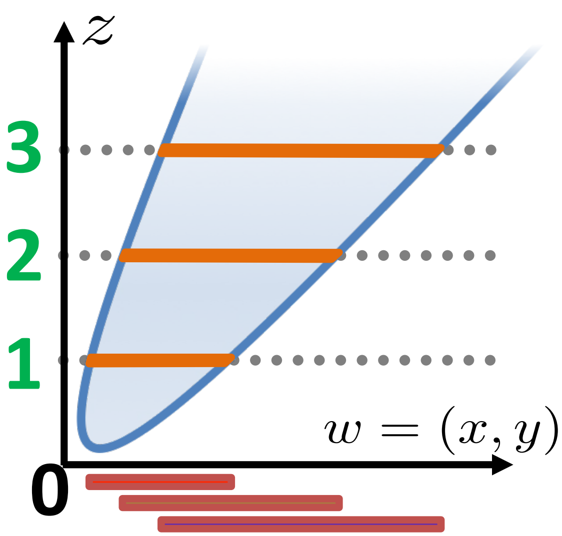


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- Extensions $S = \bigcup_{i=1}^{k} P_i + \left\{ \sum_{i=1}^{t} \lambda_i r^i \,:\, \lambda \in \mathbb{Z}_+^t \right\}$
  - $M = \{x \in \mathbb{Z}^2 : x_1 \cdot x_2 \geq \alpha\} \implies P_i$ = points (Dey & Moran '13)
  - $M$ = Rational Polyhedron ∩ "Rational" Ellipsoidal Cylinder $\implies$ $P_i$ = Rational Ellipsoid ∩ Polytope (Del Pia & Poskin '16)
  - $M$ = Compact Convex + Rational Polyhedron Cone $\implies$ $P_i$ = Compact Convex (Lubin, Zadik & V. 17')

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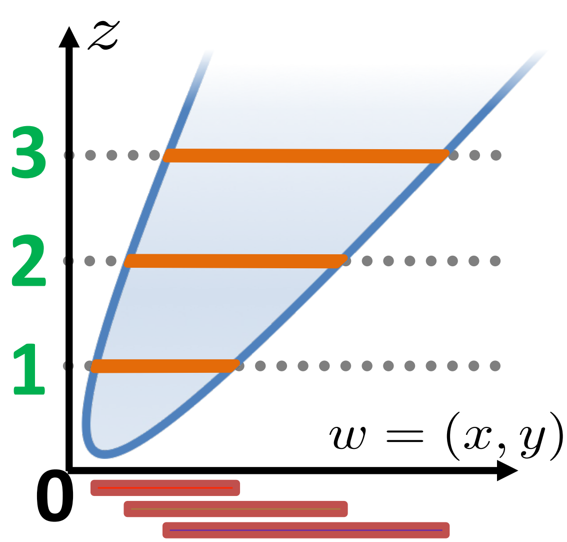


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1: Rational polyhedra with the same recession cone

2: Finite # of shapes + periodic translations

# Extra from MICP 1: Non-Polyhedral Unions

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \operatorname{proj}_x \left( B_z \right)$$

1. Unions of **Non-Polyhedral sets**

Plus Projection:

2. Unions of non-closed sets
3. Unions of convex sets with different recession cones

Two sheet hyperbola?

$$\{x \in \mathbb{R}^2 \; : \; 1 + x_1^2 \leq x_2^2\}$$

$$\left\{\left\{(x,t) \; : \; x \in S_i, \quad \|x\|_2^2 \leq t\right\}\right\}_{i=1}^{k} \text{ have the same recession cone}$$

# Extra from MICP 2: Non-Polyhedral $I$

- An **infinite set** $S$ is **periodic** if and only if:

$$\exists r \in \mathbb{R}^n \quad \forall \lambda \in \mathbb{Z}_+, \; x \in S \quad x + \lambda r \in S$$

- Non-periodic MICPR sets
  - Dense discrete set $\left\{\sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \; : \; x \in \mathbb{N}\right\} \subseteq [0,1]$

$$\|(z_1, z_1)\|_2 \leq z_2 + 1, \quad \|(z_2, z_2)\|_2 \leq 2z_1, \quad x_1 = y_1 - z_2,$$
$$\|(z_1, z_1)\|_2 \leq y_1, \qquad \|(y_1, y_1)\|_2 \leq 2z_1, \qquad z \in \mathbb{Z}^2$$

  - Set of naturals $\left\{x \in \mathbb{N} \; : \; \sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \notin \left(\varepsilon, 1 - \sqrt{2}\varepsilon\right)\right\}$

$$\|(x_1, x_1)\|_2 \leq x_2 + \varepsilon,$$
$$\|(x_2, x_2)\|_2 \leq 2x_1 + 2\varepsilon, \quad x \in \mathbb{Z}^2_+$$

# A Definition Rational MICPR (R-MICPR)

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \operatorname{proj}_x (B_z) \qquad S = \operatorname{proj}_x \left( M \cap \left( \mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

$$I = \operatorname{proj}_z (M)$$

- Any rational affine mapping of index set $I$ :
  - Is bounded, or
  - Has an integer (rational) recession direction
- Irrational directions can hide!
  - R-MICPR $\not\Leftrightarrow$ $\operatorname{span}(\operatorname{rec}(I))$ and/or $\operatorname{aff}(I)$ = rational space

$$\left(z_1 + \sqrt{2} z_2\right)^2 \leq z_3 \qquad \operatorname{span}(\operatorname{rec}(I)) = \operatorname{span}(\{\mathbf{e}_3\})$$

$$\left(z_2 - \sqrt{2} z_1\right)^2 \leq 1 \qquad \operatorname{rec}(\operatorname{proj}_{z_1, z_2}(I)) = \operatorname{span}(\{(1, \sqrt{2})\})$$

# Properties of Rational MICPR (R-MICPR)

- For **compact** $S$ :
  - **Finite unions of compact** convex sets
- For $S$ **infinite unions of "uniformly bounded" closed convex sets** :
  - **Finite union of periodic**
  - **Dense discrete** and **non-periodic naturals** NOT R-MICPR
- Rational MICP Representability:
  - **Closed** under: **Finite Union, Cartesian Product and Minkowski sum**
  - **NOT Closed** under **intersection**.

# R-MICPR does Not Imply Finite Shapes


- There exists increasing functions $h$ such that:
  - $P_z \subseteq \mathbb{R}^2$ regular $h(z)$-gon centered at $(z, 0)$
  - $P_z \cap P_{z'} = \emptyset, \quad z \neq z'$
  - $S = \bigcup_{z=1}^{\infty} P_z$ is R-MICPR and periodic

- Equal volume ⇒ Finite # of Shapes

# Summary

- General mixed integer convex representability (MICPR):
  - Infinite union of convex  w. different recession cones
  - With special structure (e.g. Primes are not MICPR)
  - Infinite structure can be irregular (irrational rays)
    - Can be caused by hided rays for non- “thin” sets
- Rational MICPR
  - Regularity recovered forcing rational unboundedness
- Equal volume $\Rightarrow$ Finite # of Shapes

# R-MICPR: Periodicity for Natural #s

- An **infinite set** of naturals $S$ is **periodic** if and only if:
  - $S = \bigcup_{i=1}^{k} \{s_i\} + \text{intcone}(\{r\})$
  - It is **rational MILP representable**
- A subset $S$ of the naturals is **R-MICPR** if and only if:
  - It is the **union** of a **finite** and an **infinite periodic** set

# A Simple Lemma for non-MICP Representability

- Obstruction for MICP representability of $S$ :

$$\text{infinite } R \subseteq S \quad \text{s.t.} \quad \frac{u+v}{2} \notin S \quad \forall u, v \in R,\ u \neq v$$

Proof: Assume for contradiction there exists $M$ such that:

$$S = \operatorname{proj}_x \left( M \cap \left( \mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

$$\begin{matrix} (u, y_u, z_u) \in M \\ (v, y_v, z_v) \in M \end{matrix} \quad \Rightarrow \quad \frac{z_u + z_v}{2} \notin \mathbb{Z}^d$$

$$z_u \equiv z_v \pmod 2 \text{ component-wise} \quad \Rightarrow \quad \frac{z_u + z_v}{2} \in \mathbb{Z}^d$$

$$\text{component-wise parity classes} = 2^d < |R| = \infty \qquad \Rightarrow\!\Leftarrow$$