Modeling Disjunctive Constraints with a Logarithmic Number of Binary Variables and Constraints

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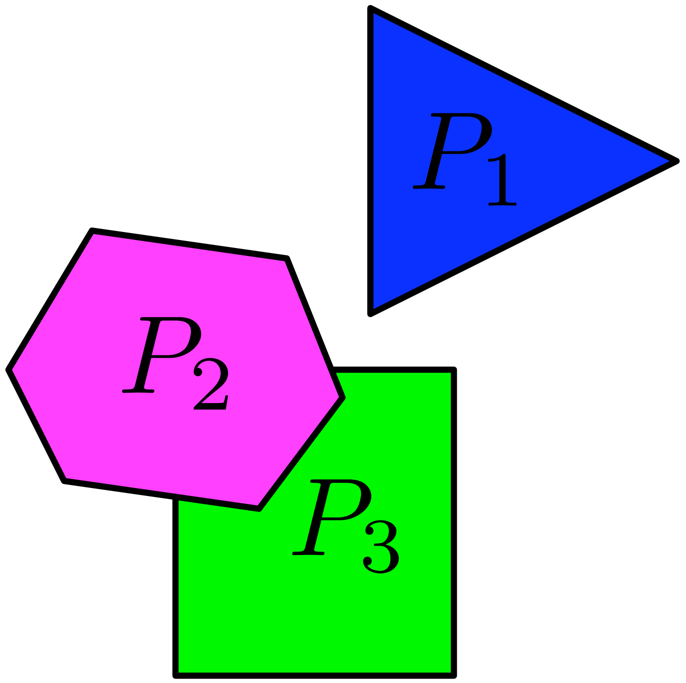
Outline

- Introduction: MIP models for disjunctive constraints.
- Smaller MIPs for SOS1, SOS2, piecewise linear functions
- Computational Results.

MIPs for Disjunctive Constraints/Set

$$x \in \bigcup_{i=1}^{m} P_i \subset \mathbb{R}^n$$

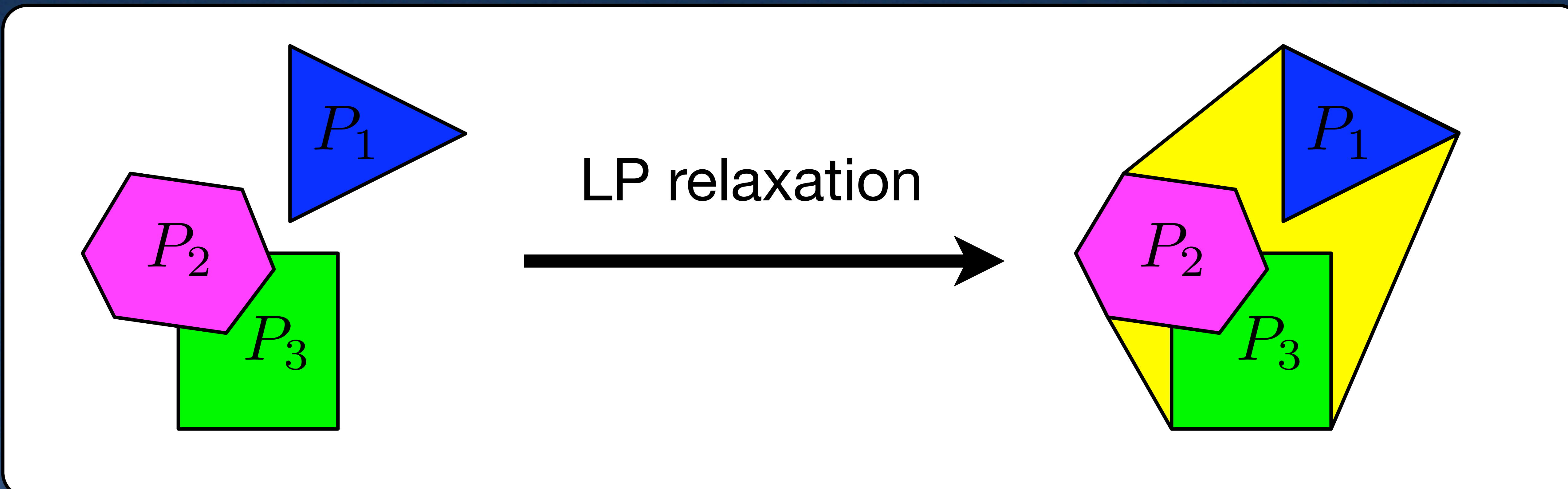
$$P_i := \text{polytope}$$



- 0-1 Mixed Integer Programming (MIP) formulation.
- Can use auxiliary variables besides 0-1 variables.
- Want strong but small formulations.

Strong MIPs : Two Levels

- ***Sharpness***: Projection of LP relaxation onto original variables equals $\operatorname{conv}\left(\bigcup_{i=1}^{m} P_i\right)$.



- ***Locally Ideal*** (stronger): LP has integral extreme points.

A Standard MIP Formulation

$$x \in \bigcup_{i=1}^{m} P_i \subset \mathbb{R}^n$$

$$P_i := \{x \in \mathbb{R}^n \ : \ A^i x \leq b^i\}$$

$$A^i \in \mathbb{R}^{r \times n},\ b^i \in \mathbb{R}^{r_i}$$

$$\longrightarrow$$

$$A^i x^i \leq b^i y_i \quad \forall i$$

$$\sum_{i=1}^{m} x^i = x$$

$$\sum_{i=1}^{m} y_i = 1$$

$$y \in \{0,1\}^M$$

- Sharp and Locally Ideal.
- $\Theta(nm)$ extra vars and $\Theta\left(n + rm\right)$ constraints.

Special Disjunctive Constraint

$$x \in \bigcup_{i=1}^{m} P(F_i) \qquad P(F_i) := \{x \in \Delta^n \,:\, x_j \leq 0 \quad \forall j \in F_i\}$$

$$\Delta^n := \left\{x \in [0,1]^n \,:\, \sum\nolimits_{i=1}^{n} x_i = 1\right\}$$

- SOS1: $m = n, \quad F_i = \{1, \ldots, n\} \setminus \{i\}$.
- SOS2: $m = n - 1, \quad F_i = \{1, \ldots, n\} \setminus \{i, i+1\}$.
- Continuous Piecewise Linear Functions.
- Standard formulation has $\Theta(nm)$ extra vars and constraints $\Theta\left(n + m\right)$.

Eliminate copies of x and stay Sharp?

$$\sum_{i=1}^{m} x^i = x, \quad Ax^i \le b^i y_i \quad \forall i \qquad \longrightarrow \qquad Ax \le \sum_{i=1}^{m} b^i y_i$$

$$\sum_{i=1}^{m} y_i = 1, \quad y \in \{0,1\}^m \qquad\qquad \sum_{i=1}^{m} y_i = 1, \quad y \in \{0,1\}^m$$

- Works for special case (Balas, Blair and Jeroslow).
- $\Theta(m)$ extra vars and $\Theta(n)$ constraints.

$$\sum_{j=1}^{n} x_j = 1, \; x \ge 0, \; x_j \le \sum_{i : j \notin F_i} y_i, \; \sum_{i=1}^{m} y_j = 1, \quad y \in \{0,1\}^m$$

Rewrite disjunction = reduce binaries.

$$\lambda \in \bigcup_{i=1}^{m} P(F_i) = \bigcap_{k=1}^{\lceil \log_2 m \rceil} \left(P(F_1^k) \cup P(F_2^k) \right) \qquad F_1^k \cap F_2^k = \emptyset$$

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$$\begin{array}{llll} \lambda \in \Delta^n & \lambda_j \le y_1^k & \forall j \in F_2^k & \\ & \lambda_j \le y_2^k & \forall j \in F_1^k & y_1^k + y_2^k = 1 \\ & \lambda_j \le y_1^k + y_2^k & \forall j \notin F_1^k \cup F_2^k & y^k \in \{0,1\}^2 \end{array}$$

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$$\lambda \in \Delta^n$$

$$\begin{aligned} &\lambda_j \le y_1^k && \forall j \in F_2^k && \\ &\lambda_j \le y_2^k && \forall j \in F_1^k && y_1^k + y_2^k = 1 \\ &\cancel{\lambda_j \le y_1^k + y_2^k} && \cancel{\forall j \notin F_1^k \cup F_2^k} && y^k \in \{0,1\}^2 \end{aligned}$$

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$\lambda \in \Delta^n$

$$\sum_{j \in F_2^k} \lambda_j \leq y_1^k$$

$$\sum_{j \in F_1^k} \lambda_j \leq y_2^k$$

$$y_1^k + y_2^k = 1$$

$$y^k \in \{0,1\}^2$$

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$$\sum\nolimits_{j \in F_1^k} \lambda_j \leq y_2^k \qquad y_1^k + y_2^k = 1$$

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$$y_1^k + y_2^k = 1$$

$$\sum_{j \in F_1^k} \lambda_j \le y_2^k \qquad y^k \in \{0,1\}^2$$

- $O(\log_2 m)$ extra vars/constraints and locally ideal!
- Vielma and Nemhauser 08/09, Vielma et al. 09.

Rewrite = Independent Branching

- Special Branching Scheme (e.g. SOS2 branch):
  - Both sides implemented by fixing vars to zero.
  - Levels are independent.
- Formulation: 1 binary for each dichotomy.
- For SOS1/SOS2 and Univariate/Multivariate Continuous/Discontinuous Piecewise Linear Functions.

SOS2: Non-zero = two adjacent vars.

- Standard Branching: $\bigvee$ $x_j = 0 \; \forall j < k$ / $x_j = 0 \; \forall j > k$

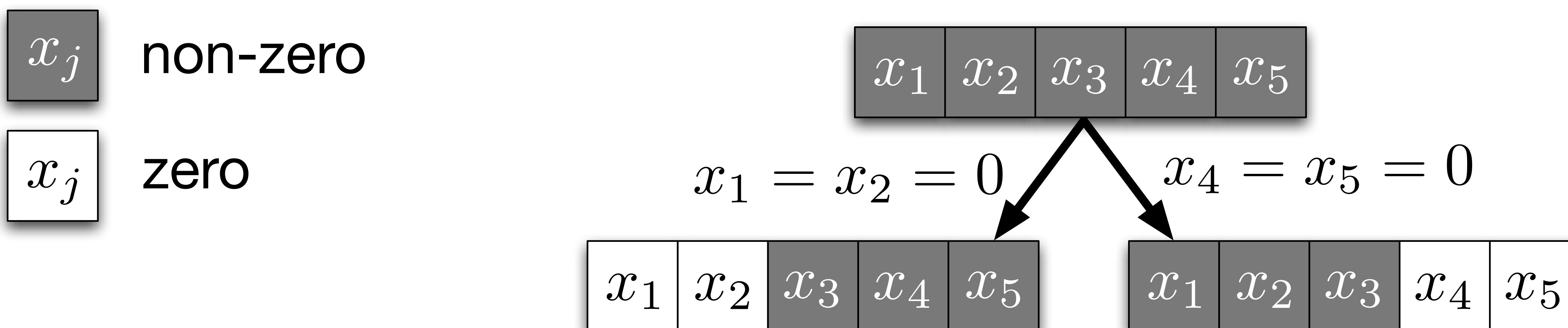
$\boxed{x_j}$ non-zero

$\boxed{x_j}$ zero

x_1	x_2	x_3	x_4	x_5

SOS2: Non-zero = two adjacent vars.

- Standard Branching: $\bigvee \begin{cases} x_j = 0 \ \forall j < k \\ x_j = 0 \ \forall j > k \end{cases}$



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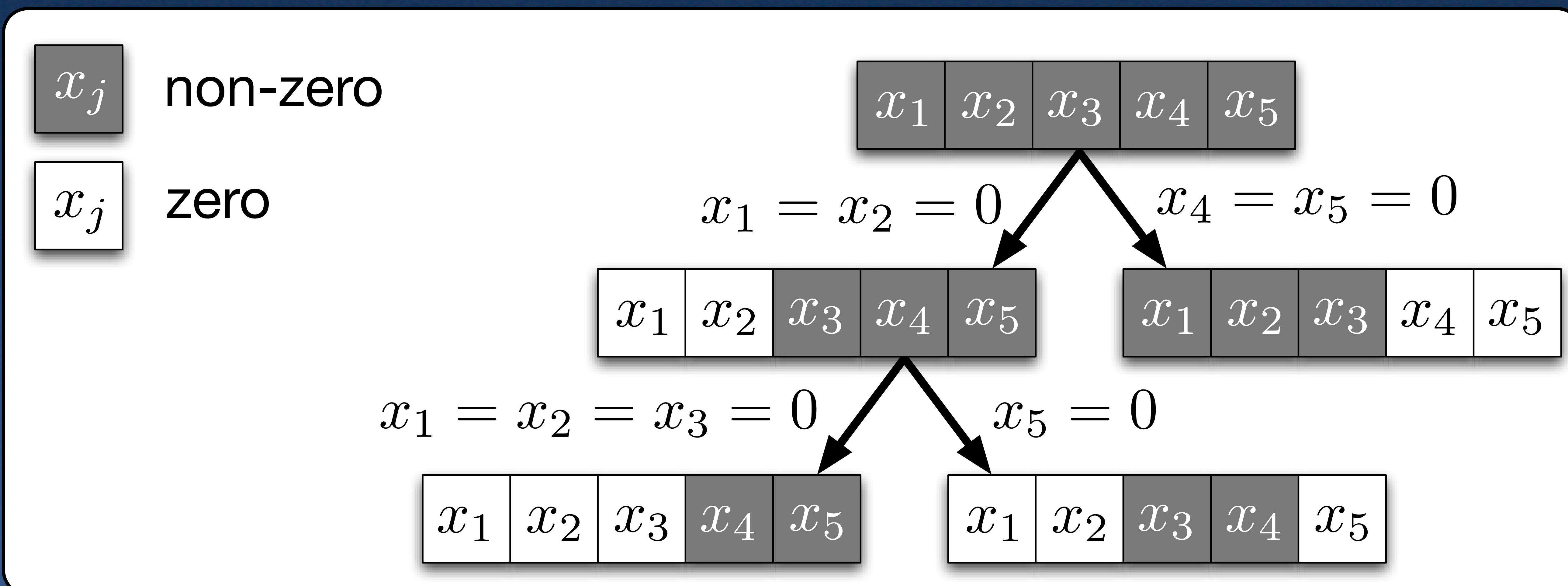

SOS2: Non-zero = two adjacent vars.

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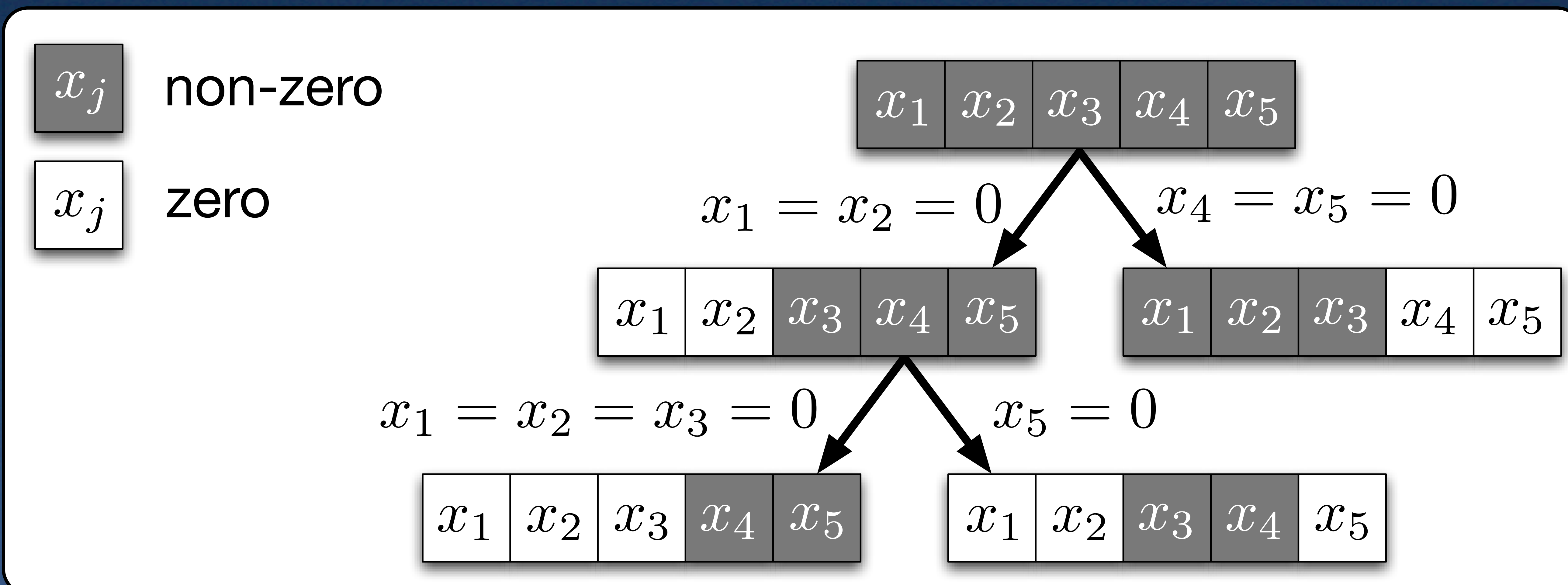
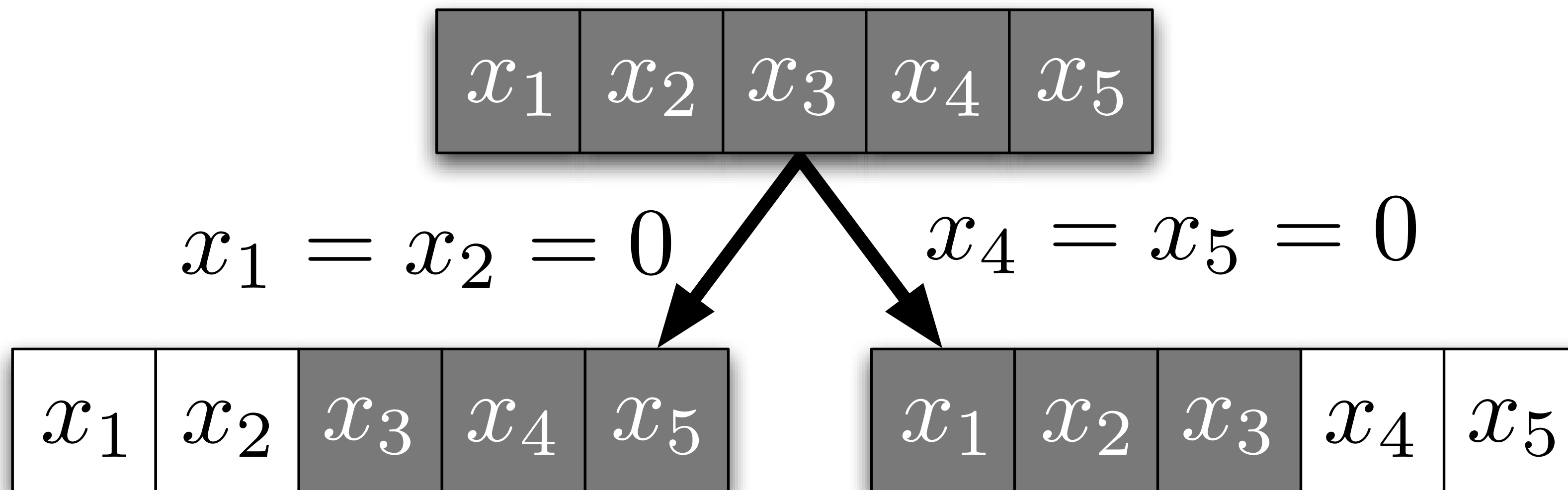


- Total independent dichotomies = m := # vars.

Independent Branching for SOS2

x_j non-zero

x_j zero



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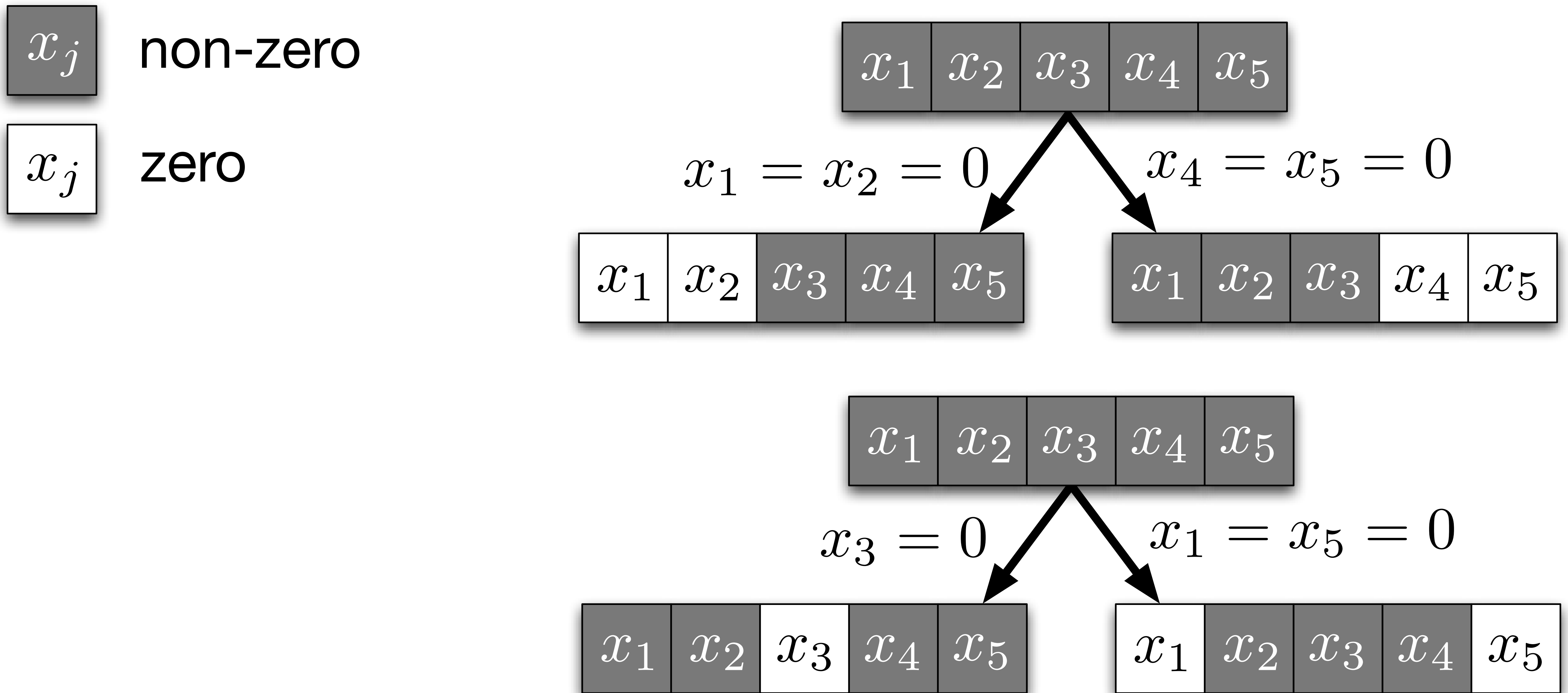

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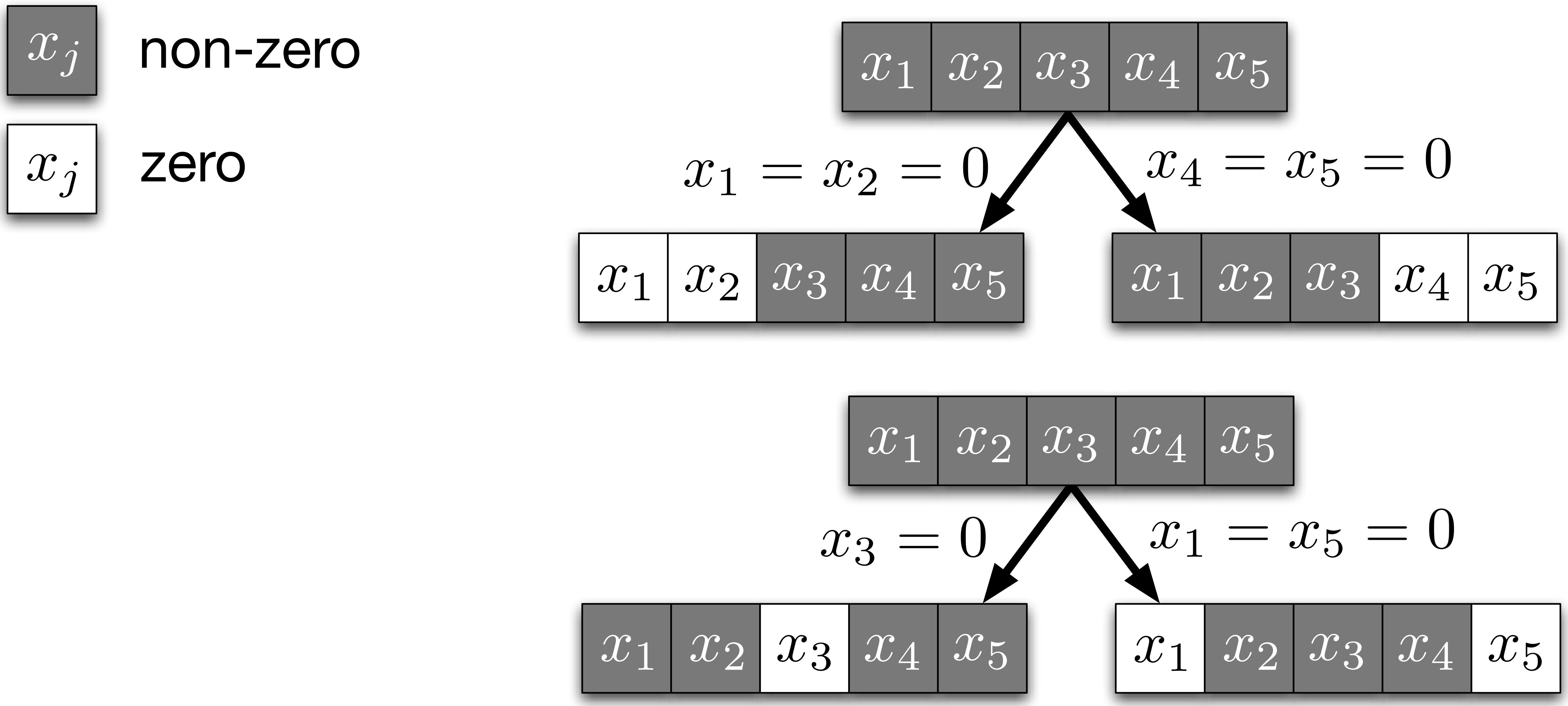
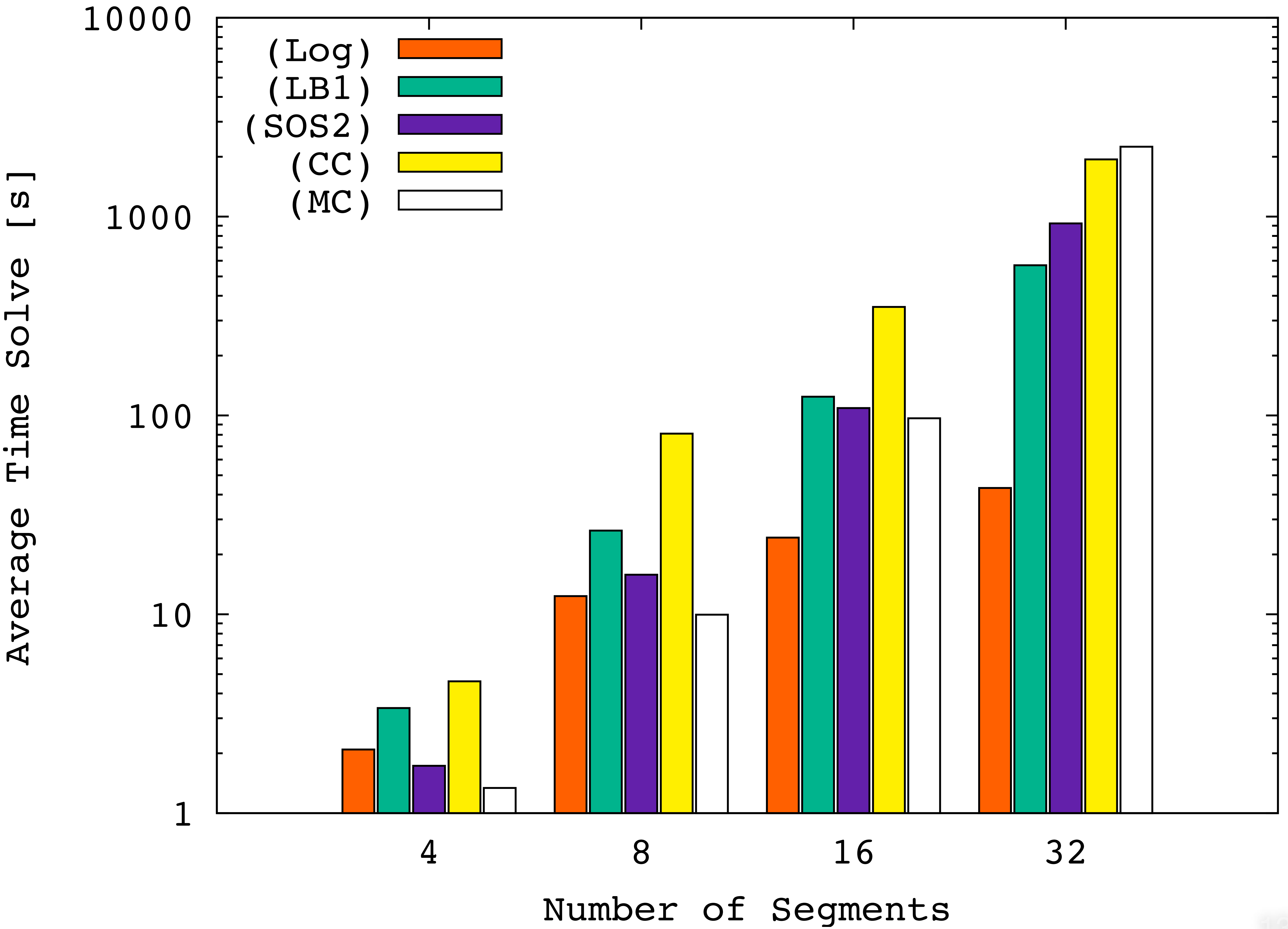


- Based on Gray Codes = More than one choice.

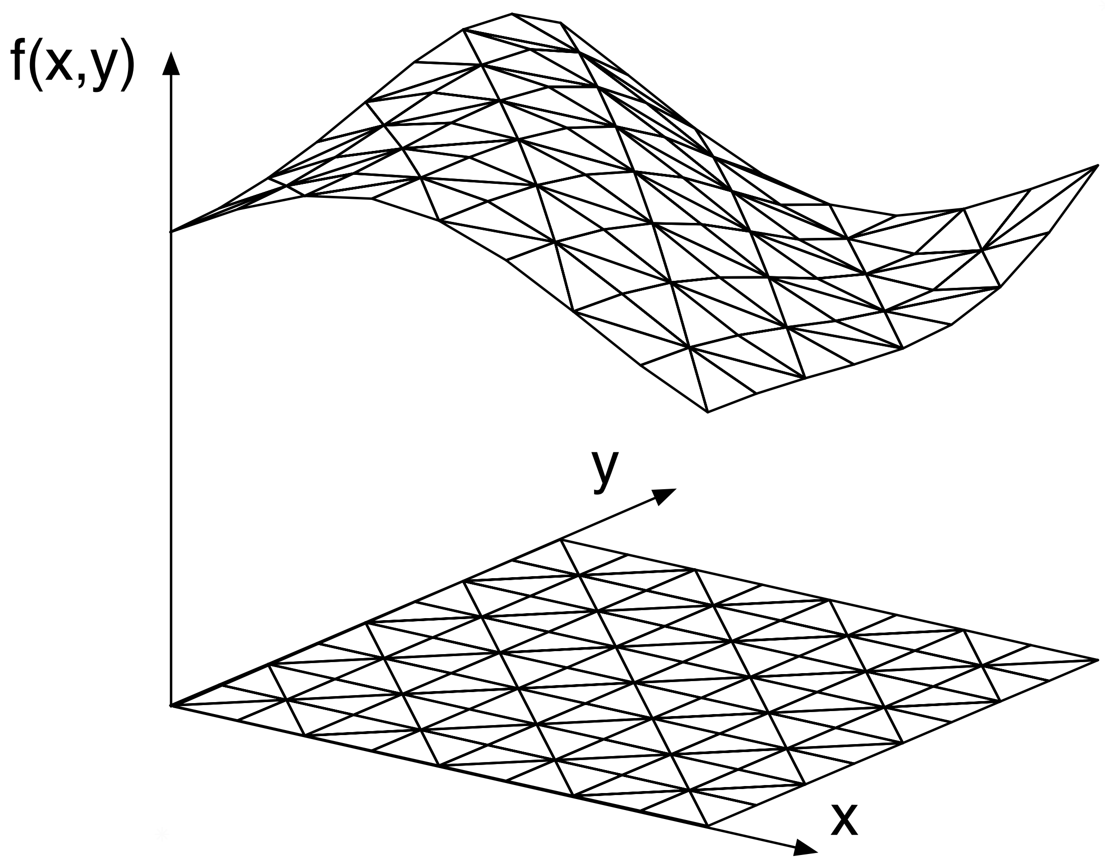
Computational Experiments

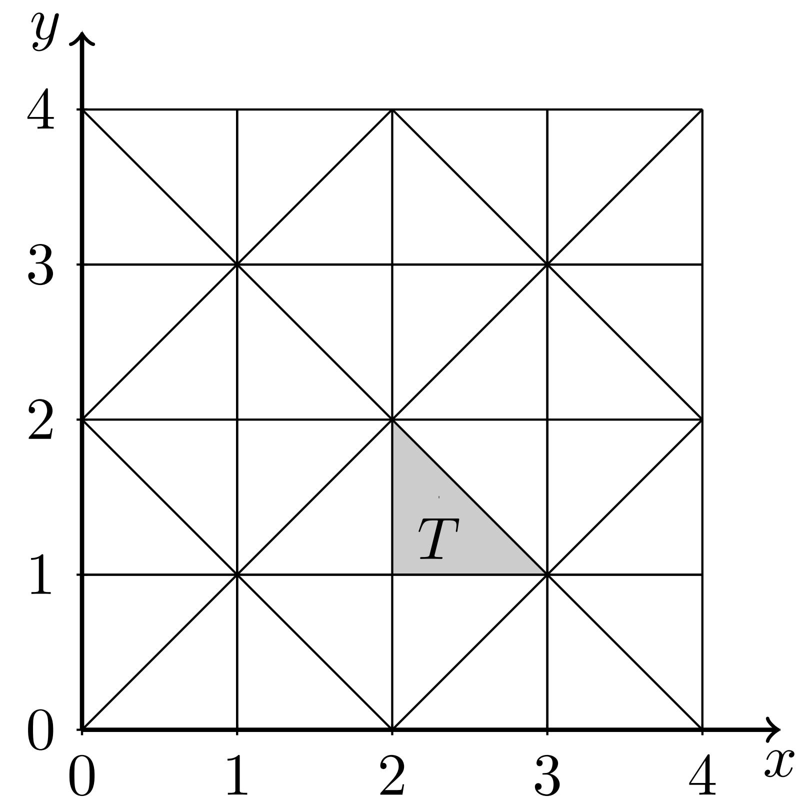
- Instances: Transp. probs. w. piecewise linear cost.
- CPLEX 11, 2.4GHz Xeon with 2GB of RAM.
  - Log: Log size Ind. Branch. for SOS2.
  - LB1: Linear size Ind. Br. for SOS2 (Shields, 07)
  - SOS2: CPLEX 11 specialized SOS2 branch.
  - CC: Standard formulation for SOS2.
  - MC: Non-SOS2 formulation for piecewise linear.

Computational Experiments



Multivariate Piecewise Linear Functions





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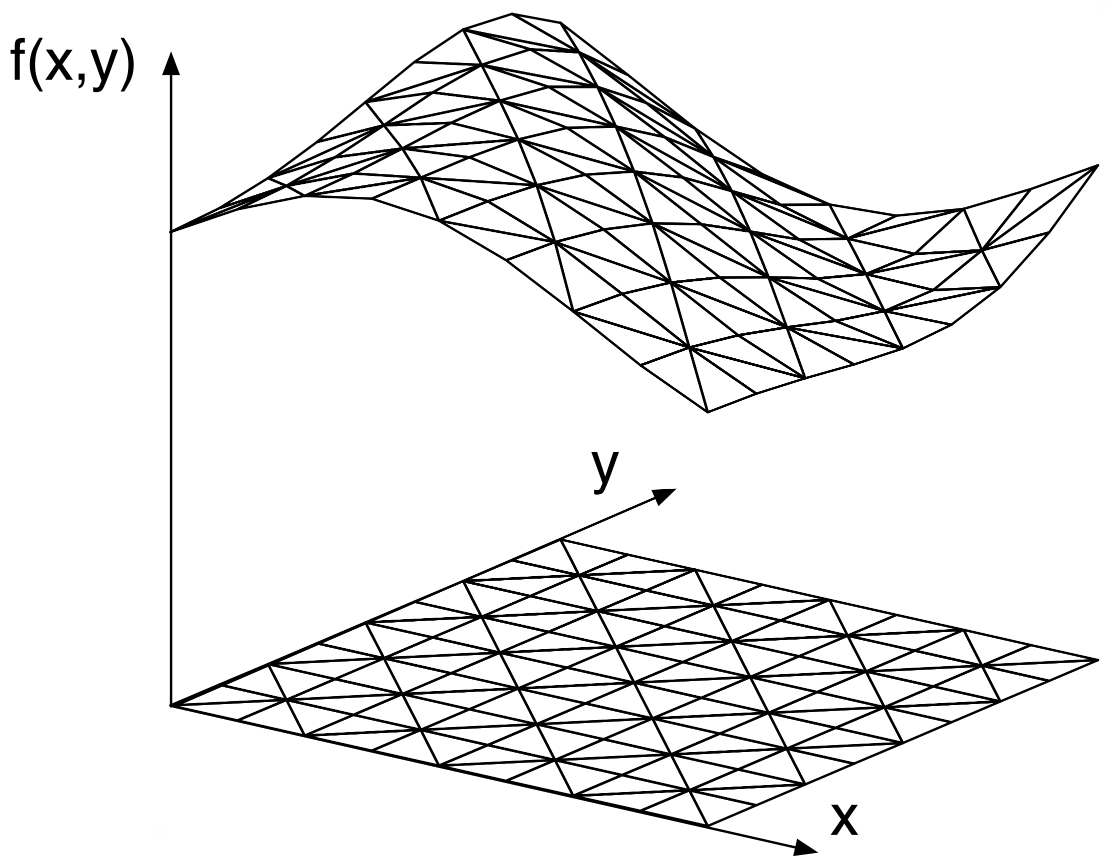
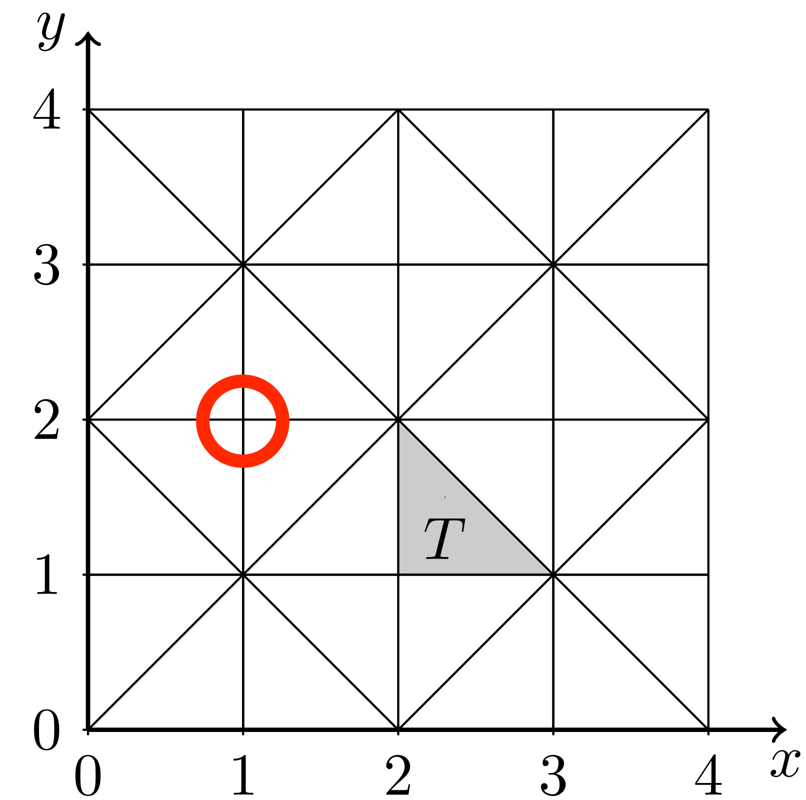
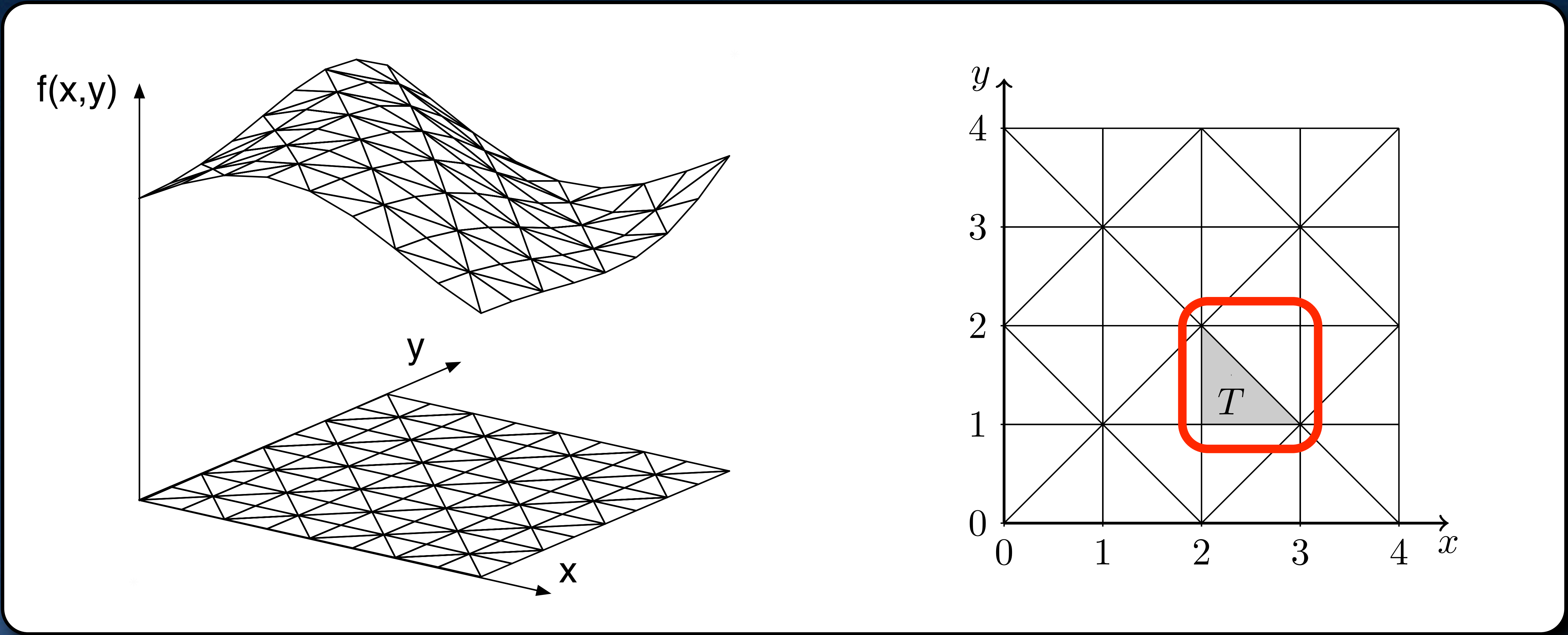




- Variables = Vertices.

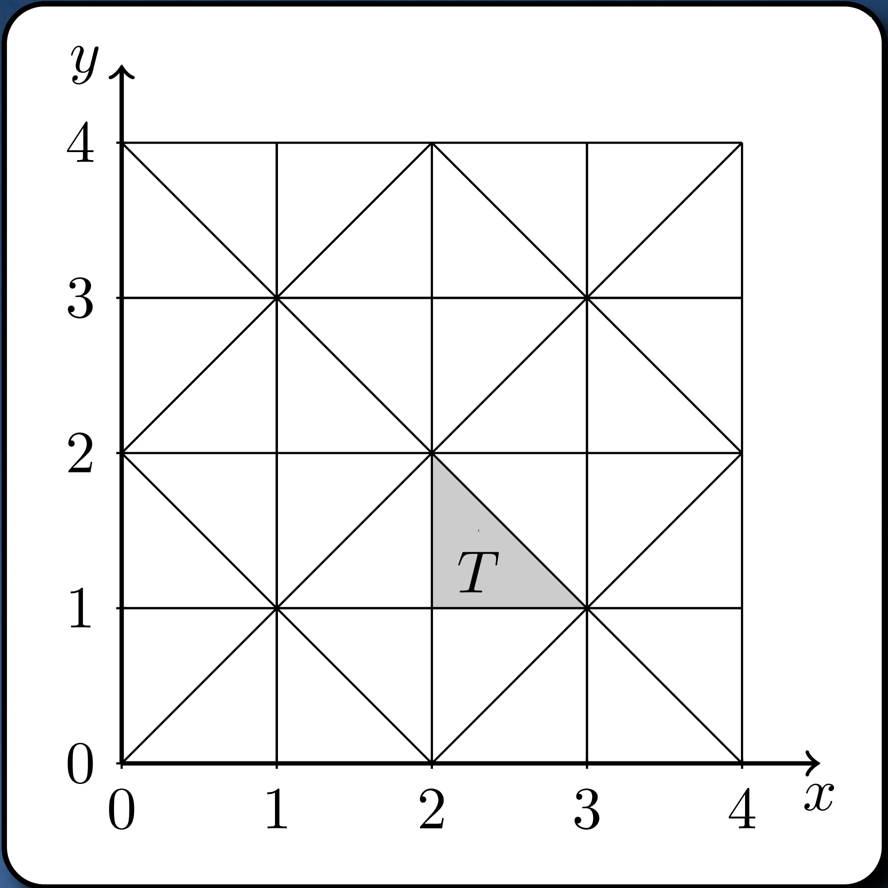
Multivariate Piecewise Linear Functions



- Variables = Vertices.
- Allowed non-zero variables = Vertices of a triangle.

Independent Branching PWL Function

- Select Triangle by forbidding vertices.
- 2 stages:
  - Select Square by SOS2 on each variable.
  - Select 1 triangle from each square.



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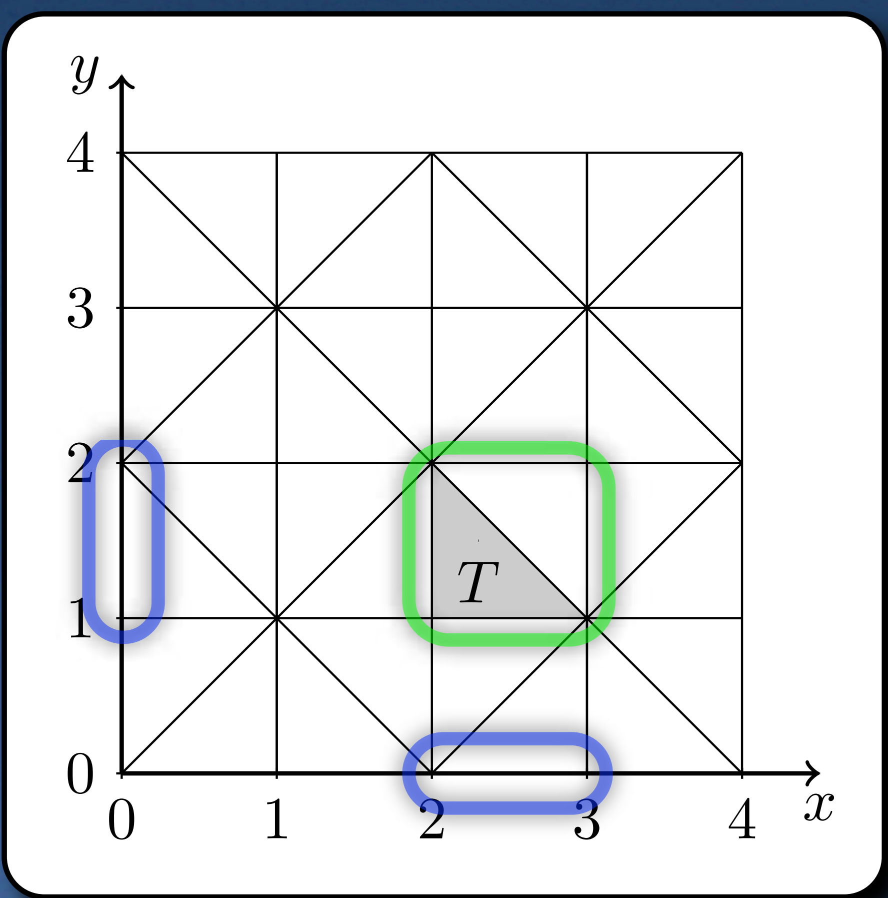

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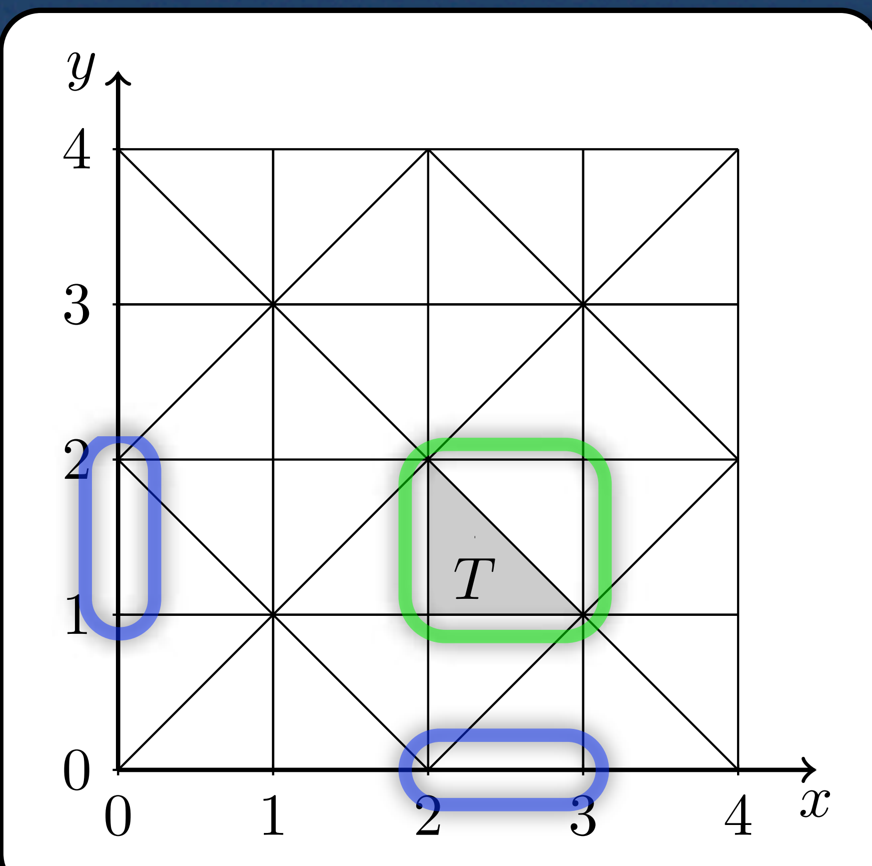
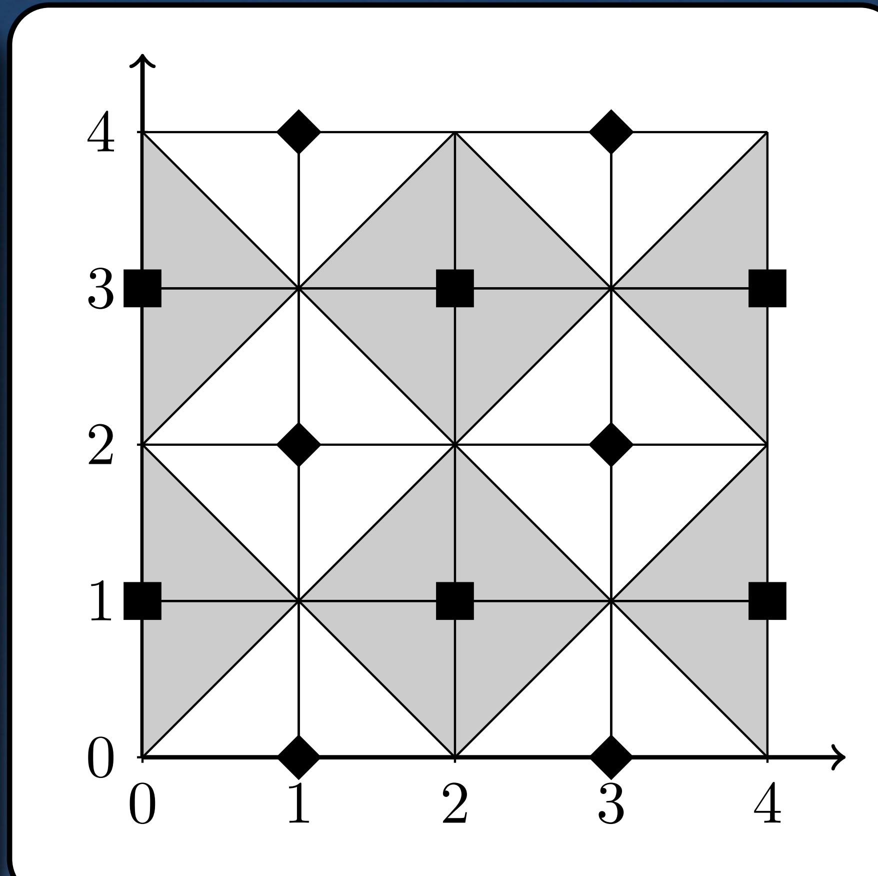
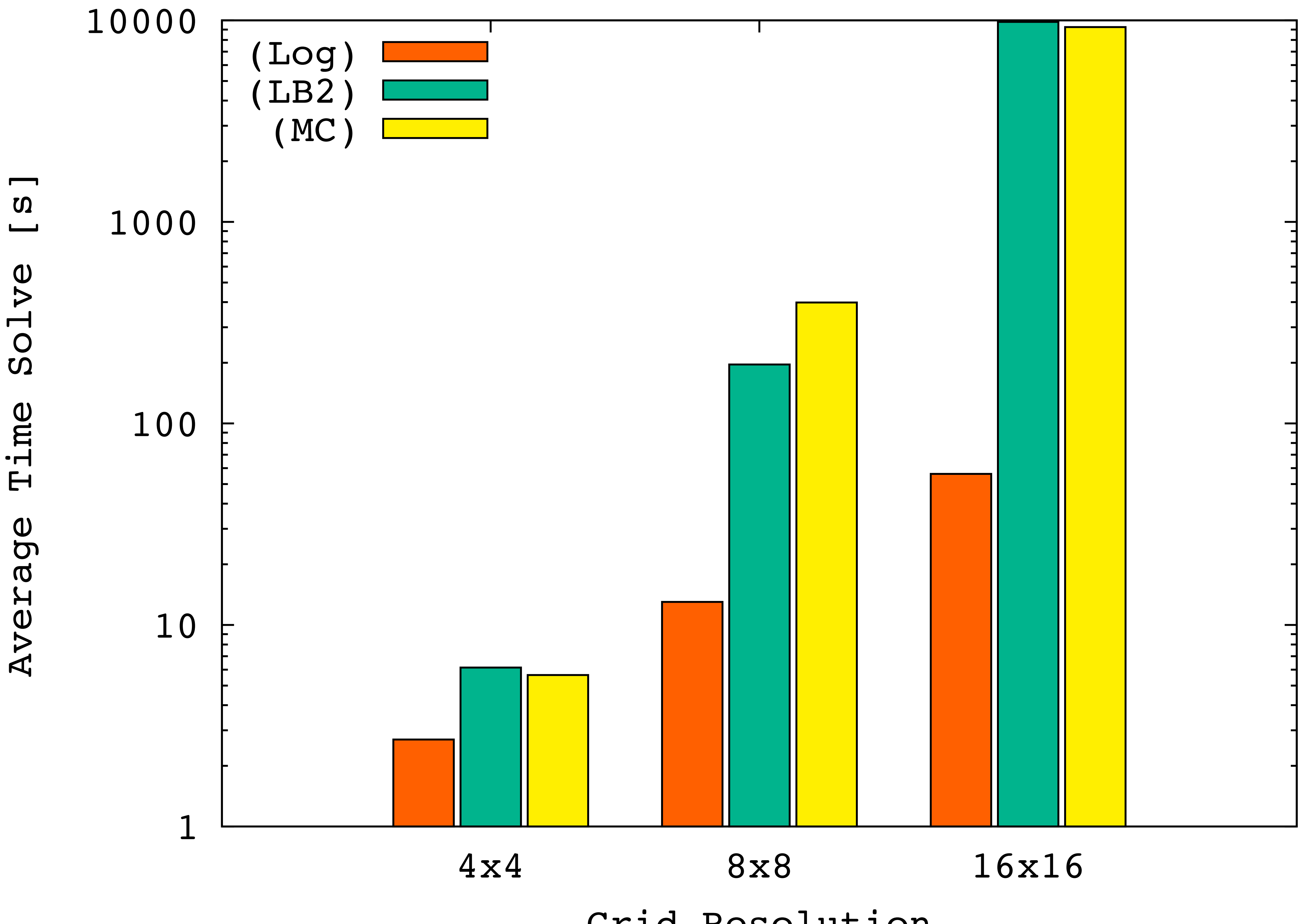




$$\bar{L} = \{(r,s) \in J : \; r \text{ even and } s \text{ odd}\} = \{\text{square vertices}\}$$

$$\bar{R} = \{(r,s) \in J : \; r \text{ odd and } s \text{ even}\} = \{\text{diamond vertices}\}$$


Average Time Solve [s]
10000
1000
100
10
1
(Log)
(LB2)
(MC)
4x4
8x8
16x16
Grid Resolution