Modeling Disjunctive Constraints with a Logarithmic Number of Binary Variables and Constraints

Juan Pablo Vielma George L. Nemhauser

H. Milton Stewart School of Industrial and Systems Engineering
Georgia Institute of Technology

IPCO 2008 – Bertinoro, Italy

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Outline

Constraints: Only some subsets of variables can be non-zero at the same time

- SOS1: $\lambda \in [0,1]^n$ such that at most one λ_j is non-zero.
- SOS2: $(\lambda_j)_{j=0}^n \in [0,1]^{n+1}$ such that at most two λ_j's are non-zero. Two non-zero λ_j's must be adjacent:

$\surd$ $(0, 1, \frac{1}{2}, 0, 0)$ X $(0, 1, 0, \frac{1}{2}, 0)$

- In general, for finite set J and finite family $\{S_i\}_{i\in I} \subset J$

$$\lambda \in \bigcup_{i\in I} Q(S_i) \subset [0,1]^J$$

where $Q(S_i) = \left\{\lambda \in [0,1]^J \ : \ \lambda_j \leq 0 \, \forall\, j \notin S_i\right\}$.

- For "simplicity" we restrict to the simplex $\Delta^J := \{\lambda \in \mathbb{R}_+^J \ : \ \sum_{j\in J} \lambda_j \leq 1\}$.
- Standard MIP models have $|I|$ binaries and $|J|$ extra constraints.

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- For "simplicity" we restrict to the simplex $\Delta^J := \{\lambda \in \mathbb{R}^J_+ : \sum_{j \in J} \lambda_j \leq 1\}$.
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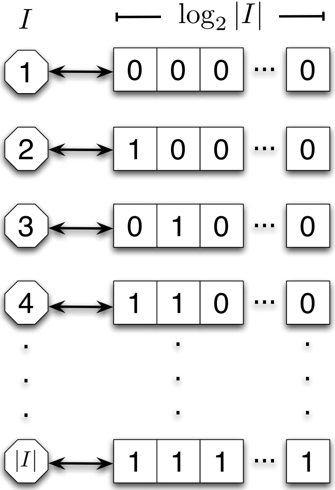
One-to-One correspondence between elements of I and vectors in $\{0,1\}^{\log_2 |I|}$



$|I| = 2^k$

- In general, an injective function:

$$B : I \to \{0,1\}^{\lceil \log_2 |I| \rceil}$$

- Easy to get a formulation with $\lceil \log_2 |I| \rceil$ binary variables and $|I|$ extra constraints (e.g. Ibaraki 1976).

Log number of binary variables and extra constraints for SOS1 over $\lambda \in \Delta^J \subset \mathbb{R}^4_+$, $(I = J = \{1, \ldots, 4\})$

i	S_i		$B(i)$	
1	$\{1\}$	$\longleftrightarrow$	0	0
2	$\{2\}$	$\longleftrightarrow$	1	0
3	$\{3\}$	$\longleftrightarrow$	0	1
4	$\{4\}$	$\longleftrightarrow$	1	1
			x_1	x_2

$x_1, x_2 \in \{0, 1\}$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

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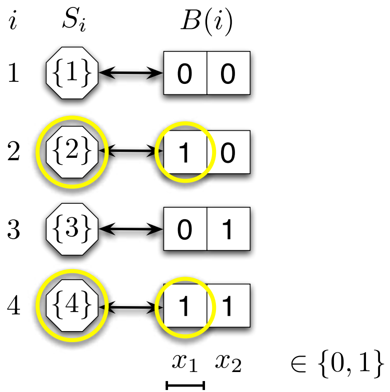

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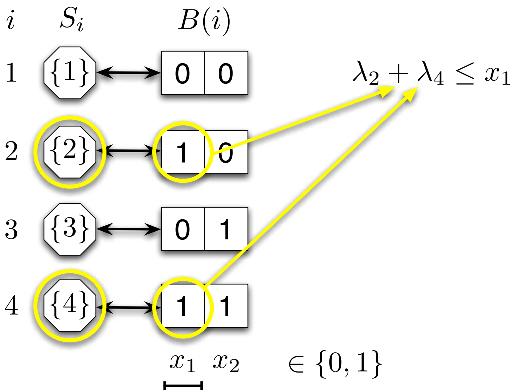

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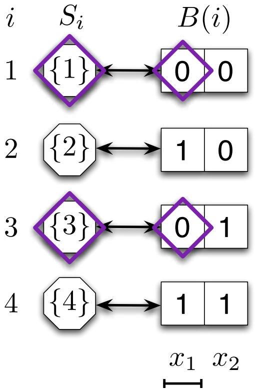

$$\lambda_2 + \lambda_4 \le x_1$$

$\in \{0, 1\}$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \le 1, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$$

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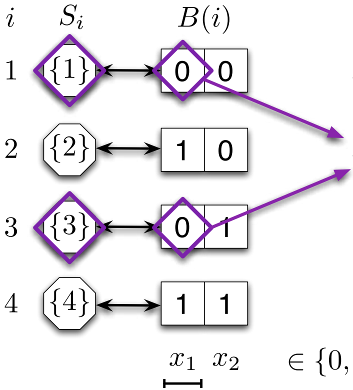

$$\lambda_2 + \lambda_4 \leq x_1$$

$$\lambda_1 + \lambda_3 \leq (1 - x_1)$$

$\in \{0, 1\}$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

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4	$\{4\}$	$\longleftrightarrow$	1 1	$\lambda_3 + \lambda_4 \leq x_2$

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$$x_1 \quad x_2 \qquad \in \{0, 1\}$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

- In general $\lceil \log_2 |I| \rceil$ binaries and $2\lceil \log_2 |I| \rceil$ extra constraints.

Logarithmic Model for SOS2 over $(\lambda_j)_{j=0}^4 \in \Delta^J \subset \mathbb{R}_+^5$

- $J = \{0, \ldots, 4\}$, $I = \{1, \ldots, 4\}$.

i	S_i		$B(i)$	
1	$\{0, 1\}$	$\longleftrightarrow$	0	0
2	$\{1, 2\}$	$\longleftrightarrow$	1	0
3	$\{2, 3\}$	$\longleftrightarrow$	0	1
4	$\{3, 4\}$	$\longleftrightarrow$	1	1
			x_1	x_2

$x_1 \ x_2 \quad \in \{0, 1\}$

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

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			x_1	x_2

$\le x_2$

$x_1 \; x_2 \in \{0,1\}$

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$$\begin{array}{cccccc}
i & S_i & & B(i) & & \\
1 & \{0,1\} & \longleftrightarrow & \boxed{0} & \boxed{0} & \\
2 & \{1,2\} & \longleftrightarrow & \boxed{1} & \boxed{0} & \\
3 & \{2,3\} & \longleftrightarrow & \boxed{0} & \boxed{1} & \\
4 & \{3,\boxed{4}\} & \longleftrightarrow & \boxed{1} & \boxed{1} & \lambda_4 \le x_2 \\
 & & & x_1 & x_2 & \in \{0,1\}
\end{array}$$

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \le 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$$

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			x_1	x_2

$$\lambda_3 + \lambda_4 \leq x_2$$

$$x_1 \quad x_2 \quad \in \{0, 1\}$$

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

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- $J = \{0, \ldots, 4\}$, $I = \{1, \ldots, 4\}$.

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$x_1 \; x_2 \quad \in \{0,1\}$

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- λ_2 does not show in any constraint!

Logarithmic Model for SOS2 over $(\lambda_j)_{j=0}^4 \in \Delta^J \subset \mathbb{R}^5_+$

- $J = \{0,\ldots,4\}$, $I = \{1,\ldots,4\}$.

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$$\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4 \le 1, \quad \lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4 \ge 0$$

- First Option: Add $\lambda_2 \le x_1 + x_2, \quad \lambda_2 \le 2 - x_1 - x_2$.

Logarithmic Model for SOS2 over $(\lambda_j)_{j=0}^4 \in \Delta^J \subset \mathbb{R}^5_+$

- $J = \{0, \ldots, 4\}$, $I = \{1, \ldots, 4\}$.

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- Second Option: Modify $B(i)$.

Logarithmic Model for SOS2 over $(\lambda_j)_{j=0}^4 \in \Delta^J \subset \mathbb{R}_+^5$

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			x_1	x_2

$x_1 \; x_2 \in \{0,1\}$

$$\lambda_4 \le x_1$$

$$\lambda_0 \le (1 - x_1)$$

$$\lambda_0 + \lambda_1 \le (1 - x_2)$$

$$\lambda_3 + \lambda_4 \le x_2$$

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \le 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$$

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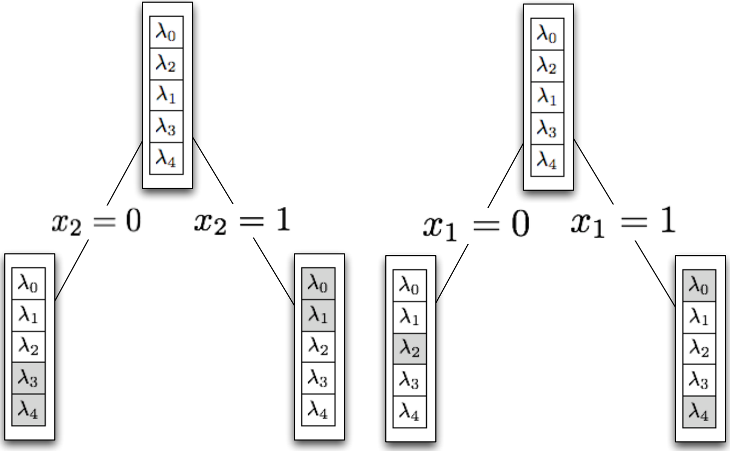
			$x_1 \ x_2$	$\in \{0,1\}$

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

- Condition: $B(i)$ and $B(i+1)$ only differ in one component (Gray codes).

Logarithmic Model and Independent Branching



$$\lambda_0 + \lambda_1 \leq (1 - x_2)$$

$$\lambda_3 + \lambda_4 \leq x_2$$

$$\lambda_2 \leq x_1$$

$$\lambda_0 + \lambda_4 \leq (1 - x_1)$$

Independent Branching Scheme for $\lambda \in \bigcup_{i\in I} Q(S_i)$

- Independent Branching: $L_k, R_k \subset J$ s.t.

$$\bigcup_{i\in I} Q(S_i) = \bigcap_{k=1}^{d} \left(Q(L_k) \cup Q(R_k)\right)$$

$$\left(Q(S_i) = \left\{\lambda \in \Delta^J \, : \, \lambda_j \leq 0 \, \forall \, j \notin S_i\right\}\right)$$

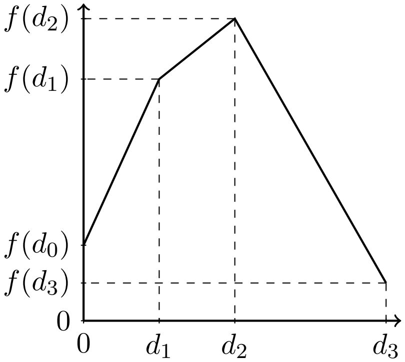
- Formulation: $\lambda \in \Delta^J$ plus $\forall k \in \{1, \ldots, d\}$

$$\sum_{j \notin L_k} \lambda_j \leq x_k, \quad \sum_{j \notin R_k} \lambda_j \leq (1 - x_k), \quad x_k \in \{0, 1\}$$

- Independent branchings for SOS1 and SOS2 have “depth” $d = \lceil \log_2 |I| \rceil$.

Application: Piecewiselinear Functions

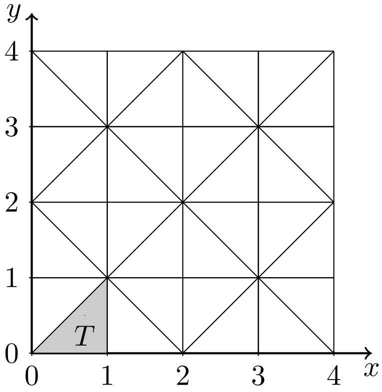
- Single variable: SOS2 on $\lambda \in \Delta^J$ for $J = \{0, \ldots, K\}$.



$K = 3$

Application: Piecewiselinear Functions

- Single variable: SOS2 on $\lambda \in \Delta^J$ for $J = \{0, \ldots, K\}$.
- Extension for $f(x, y) : [0, K]^2 \to \mathbb{R}$ (Lee and Wilson 01, Martin et. al 06)



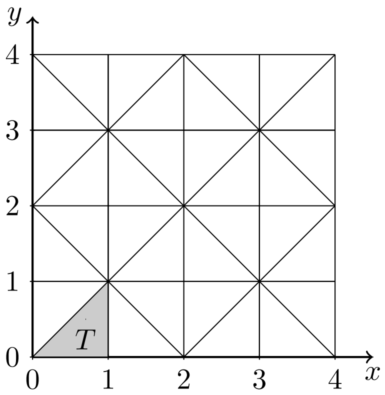
$K = 4$

$$\lambda \in \Delta^J$$

$$\lambda \in \bigcup_{i \in I} Q(S_i)$$

- $J = \{0, \ldots, K\}^2 = \{\text{vertices}\}$.
- $I = \{\text{triangles}\}$,
  $S_i = \{\text{vertices of triangle } i\}$
  $(S_T = \{(0,0), (1,0), (1,1)\})$.

Independent Branching for Two Variable Functions



$K = 4$

- Select a triangle by forbidding the use of vertices ($J = \{\text{vertices}\}$):

$$\sum_{j \notin L_k} \lambda_j \leq x_k$$

$$\sum_{j \notin R_k} \lambda_j \leq (1 - x_k)$$

$$x_k \in \{0, 1\}$$

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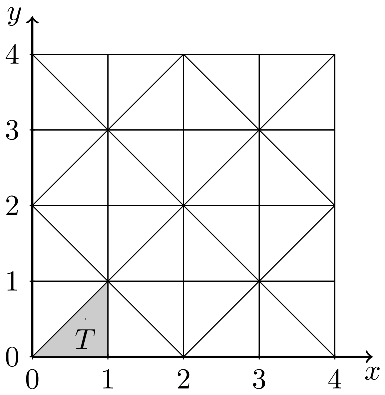

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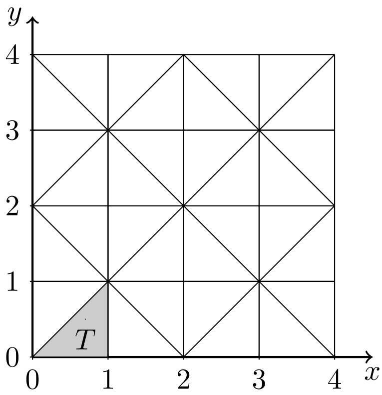

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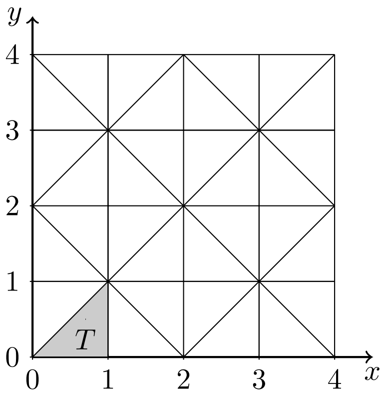
$$x_k \in \{0, 1\}$$

- $\overline{L}_k = J \setminus L_k$, $\overline{R}_k = J \setminus R_k$.

Independent Branching for Two Variable Functions



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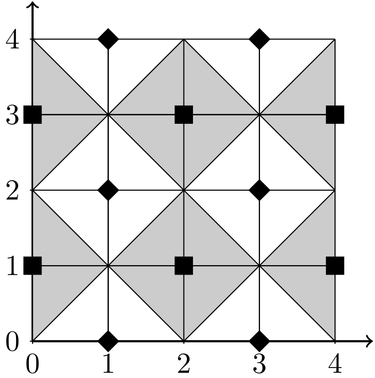
- $\overline{L}_k = J \setminus L_k$, $\overline{R}_k = J \setminus R_k$.
- Two phases:
  1. Square selection: SOS2 for each component. (Tomlin 81 and Martin et. al. 06)
  2. Triangle selection.

Triangle Selecting Independent Branching: Select one of the two triangles in each square



- Forbid white triangles in one branch and grey triangles in the other.

$$\bar{L} = \{(r,s) \in J \;:\; r \text{ even and } s \text{ odd}\}$$
$$= \{\text{square vertices}\}$$
$$\bar{R} = \{(r,s) \in J \;:\; r \text{ odd and } s \text{ even}\}$$
$$= \{\text{diamond vertices}\}$$

- Depth of independent branching is $\lceil \log_2 \mathcal{T} \rceil$ for $\mathcal{T}$ = total # of triangles.

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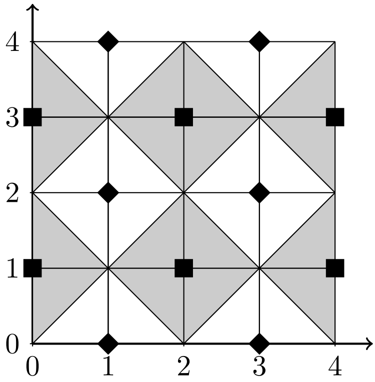

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- Depth of independent branching is $\lceil \log_2 \mathcal{T} \rceil$ for $\mathcal{T}$ = total # of triangles.

Computational Experiments (Instances)

- Single Variable:
  - 10×10 transportation problems.
  - Minimize $\sum_{e \in E} f_e(x_e)$. $x_e =$ flow in arc e.
  - $f_e(x_e)$ non-decreasing continuous concave piecewiselinear.
  - Number of segments where $f_e(x_e)$ is linear: $K = \{4, 8, 16, 32\}$.
- Two Variables:
  - 5×5 two-commodity transportation problems.
  - Minimize $\sum_{e \in E} f_e(x_e^1, x_e^2)$. $x_e^i =$ flow of commodity i in arc e.
  - $f_e(x_e^1, x_e^2)$ interpolation on grid of $g\left(\left\|\left(x_e^1, x_e^2\right)\right\|_2\right)$.
    g non-decreasing continuous concave piecewiselinear.
  - Interpolation grid resolution: 4×4, 8×8 and 16×16.
- 100 instances for each K or grid resolution.

Computational Experiments (Solver and Formulations)

- Solver and Machine Stats:
  - CPLEX 11.
  - Dual 2.4GHz Xeon Linux workstation with 2GB of RAM.
  - Time Limit of 10,000 seconds.
- Formulations:

(Log) Logarithmic formulation.

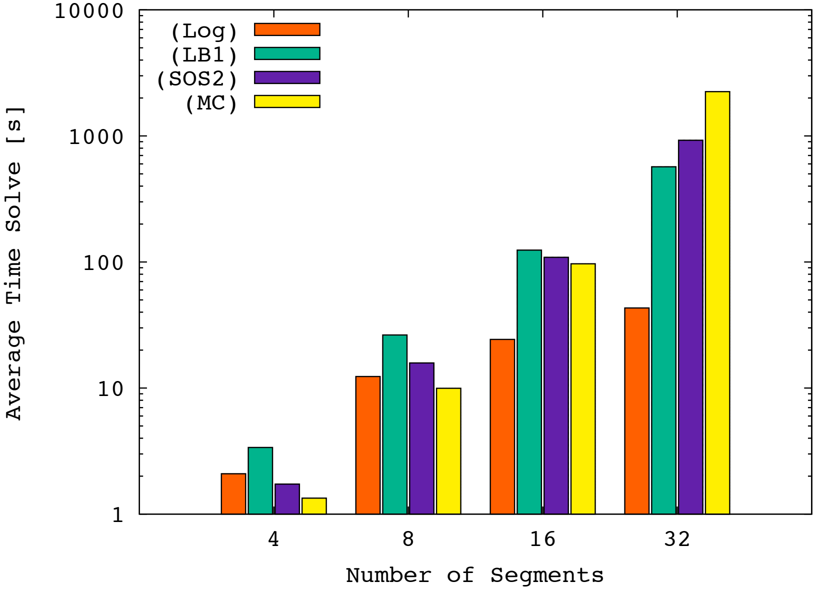
(LB1) Independent branching formulations of linear depth (Shields 2007). Only for single variable.

(LB2) Independent branching formulations of linear depth (Martin et. al. 2006).

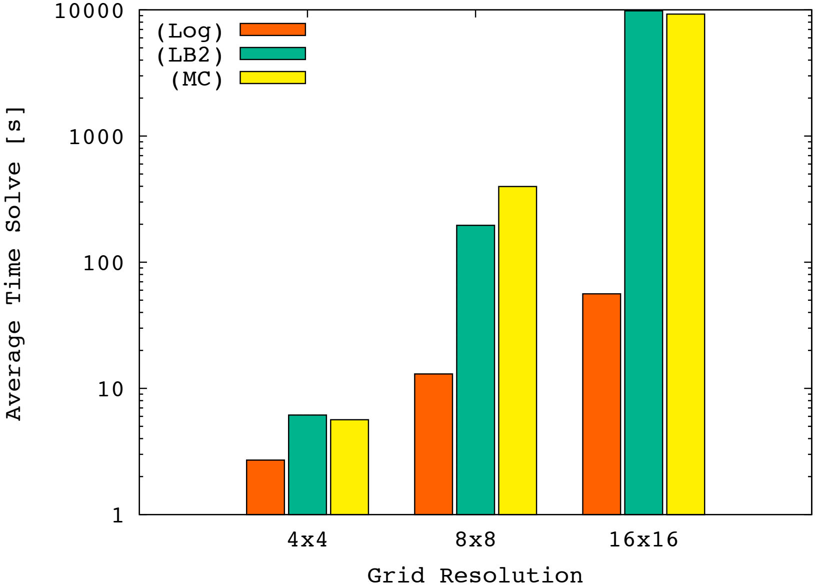
(SOS2) SOS2 based formulation. Only for single variable.

(MC) Multiple choice formulation (Jeroslow and Lowe 1984, Balakrishnan and Graves 1989, Croxton et. al 2003).

Average Solve Times for One Variable Functions



Average Solve Times for Two Variable Functions



Advantage of Independent Branching Formulations

- Independent branching formulations effectively turn CPLEX's binary branching into a specialized branching scheme (e.g. SOS2 branching).
- Independent branching formulations are "as tight as possible":
  - Projection of LP relaxation into λ variables is

$$\operatorname{conv}\left(\bigcup_{i\in I} Q(S_i)\right) = \Delta^J.$$

  - Might not hold if Δ^J is replaced by a box in $\mathbb{R}^J$.

LP Relaxation Tightness and Disjunctive Programming: $\lambda \in \bigcup_{i\in I} Q(S_i)$, $Q(S_i) = \left\{\lambda \in [0,1]^J \;:\; \lambda_j \le 0\, \forall\, j \notin S_i\right\}$

1. Traditional Linear Size Formulations:

$$\lambda_j \le \sum\nolimits_{\{i:\, j\in S_i\}} x_i, \quad \forall j \in J$$
$$\sum\nolimits_{i\in I} x_i = 1, \quad x_i \in \{0,1\} \quad \forall i \in I$$

   - Simplification of standard *Lifted* Disjunctive Formulation.
   - Preserves *Convex Hull Property* (Jeroslow 88).

2. Independent Branching: $\bigcup_{i\in I} Q(S_i) = \bigcap_{k=1}^{d} \left(Q(L_k) \cup Q(R_k)\right)$
   - For $\lambda \in Q(L_k) \cup Q(R_k)$:

$$\lambda_j \le x_k \quad \forall j \notin L_k, \quad \lambda_j \le (1 - x_k) \quad \forall j \notin R_k$$

3. Constraint Aggregation:

$$\sum\nolimits_{j\notin L_k} \lambda_j \le |J \setminus L_k|\, x_k, \quad \sum\nolimits_{j\notin R_k} \lambda_j \le |J \setminus R_k|\,(1 - x_k)$$

Summary

- First logarithmic formulations for SOS1-SOS2 constraints and piecewiselinear functions of one variable.
- Independent Branching Scheme:
  - Sufficient condition for logarithmic formulation.
  - First logarithmic formulation for piecewiselinear functions of two variables.
- Logarithmic formulations can provide a significant computational advantage.
- Is independent branching a necessary condition?
  - Cardinality constraints: No independent branching, yet standard formulation is logarithmic.
- Extension to piecewise linear function of n variables:
  - $\surd$ Logarithmic on K (for fixed n).
  - X Not Logarithmic on n.