Modeling Disjunctive Constraints with a Logarithmic Number of Binary Variables and Constraints

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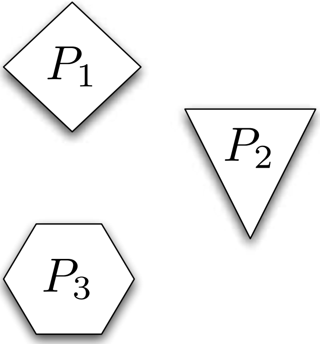
ISyE DOS Optimization Seminar, 2008 – Atlanta

Outline

Disjunctive Constraint: Union of Polyhedra



- For a finite index set I

$$z \in \bigcup_{i \in I} P_i \subset \mathbb{R}^n.$$

- $P_i = \{z \in \mathbb{R}^n \ : \ A^i z \leq b^i\}$.
- Assume P_i's are polytopes for simplicity.
- Balas (79), Blair (76), Jeroslow (77), Sherali and Shetty (80),. . .

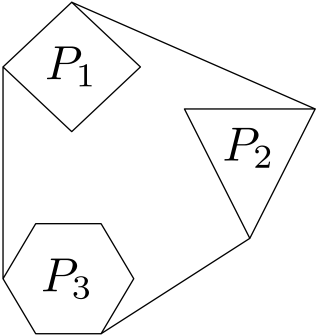
Modeling a Disjunctive Constraint as an MIP

- For finite index set I, $z \in \bigcup_{i\in I} \left\{z \in \mathbb{R}^n \ : \ A^i z \leq b^i\right\}$ can be modeled as the following standard MIP

$$
\begin{aligned}
z &= \sum_{i\in I} z^i, \\
A^i z^i &\leq x_i b^i \qquad \forall i \in I, \\
\sum_{i\in I} x_i &= 1, \\
x_i &\in \{0,1\} \qquad \forall i \in I, \\
z^i &\in \mathbb{R}^n \qquad \forall i \in I.
\end{aligned}
$$

- Balas (79), Jeroslow and Lowe (84), . . .
- Number of binary variables and constraints are linear in $|I|$.

The Standard MIP is Tight



- Projection of LP relaxation into original z variables is

$$\operatorname{conv}\left(\bigcup_{i\in I} P_i\right).$$

- Having multiple copies of continuous variables is usually necessary for a tight formulation.
- Reducing the number of continuous variables has been studied by Balas (88), Blair (90), Jeroslow (88).
- Reducing the number of binary variables has received little attention Ibaraki (76).

Reducing the Number of Binary Variables

For $I = [0, u] \cap \mathbb{Z}$

$$x \in [0, u] \cap \mathbb{Z} = \bigcup_{i \in I} \{i\}$$

the traditional model can be simplified to

$$z = \sum_{i \in I} i\, x_i, \quad \sum_{i \in I} x_i = 1, \quad x_i \in \{0, 1\} \quad \forall i \in I.$$

But we can reduce the number of binaries from $|I| = u + 1$ to

$$z = \sum_{i=0}^{\lfloor \log_2 u \rfloor} 2^i\, x_i, \quad z \leq u, \quad x_i \in \{0, 1\} \quad \forall i \in \{0, \ldots, \lfloor \log_2 u \rfloor\}.$$

Special Type of Disjunctive Constraints: Only some subsets of variables can be non-zero at the same time

- SOS1: $\lambda \in [0,1]^n$ such that at most one λ_j is non-zero.
- SOS2: $(\lambda_j)_{j=0}^n \in [0,1]^{n+1}$ such that at most two λ_j's are non-zero. Two non-zero λ_j's must be adjacent.
  - $\surd$ $(0,1,\frac{1}{2},0,0)$
  - X $(0,1,0,\frac{1}{2},0)$
- In general, for finite set J and finite family $\{S_i\}_{i\in I} \subset J$

$$\lambda \in \bigcup_{i\in I} Q(S_i) \subset \mathbb{R}_+^J$$

where $Q(S_i) = \left\{\lambda \in \mathbb{R}_+^J \,:\, \lambda_j \le 0 \,\forall j \notin S_i\right\}$.
- For SOS1: $J = I = \{1,\ldots,n\}$ and $S_i = \{i\}$ for all $i \in I$.
- For SOS2: $J = \{0,\ldots,n\}$, $I = \{1,\ldots,n\}$ and $S_i = \{i-1,i\}$ for all $i \in I$.

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First MIP Model

- For "simplicity" we restrict to the simplex $\Delta^J := \{\lambda \in \mathbb{R}_+^J \ : \ \sum_{j\in J} \lambda_j \leq 1\}$ and consider

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- Standard MIP simplifies to:

$$\lambda \in \Delta^J$$
$$\lambda_j \leq \sum_{\{i : j \in S_i\}} x_i \quad \forall j \in J$$
$$\sum_{i\in I} x_i = 1$$
$$x_i \in \{0,1\} \qquad \forall i \in I$$

- $|I|$ binaries and $|J|$ extra constraints.

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- Standard MIP simplifies to:

$I(j) = \{i \in I : j \in S_i\}$

$$\lambda \in \Delta^J$$
$$\lambda_j \leq \sum_{\{i : j \in S_i\}} x_i \quad \forall j \in J$$
$$\sum_{i \in I} x_i = 1$$
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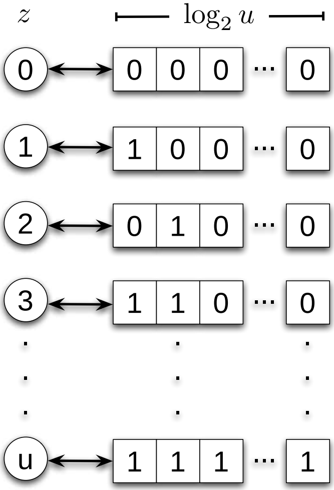
Using Binary Expansion for $z \in [0, u] \cap \mathbb{Z}$ for $u = 2^k - 1$



- One-to-One correspondence between integers in $[0, u]$ and vectors in $\{0,1\}^{\log_2 u}$.
- One-to-One correspondence between elements of I and vectors in $\{0,1\}^{\log_2 |I|}$.
- In general, we need an injective function:

$$B : I \to \{0,1\}^{\lceil \log_2 |I| \rceil}$$

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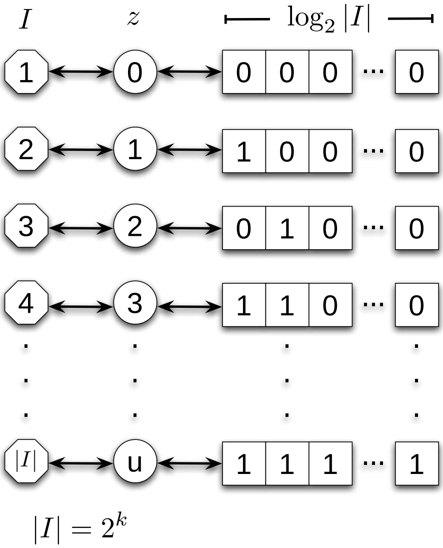

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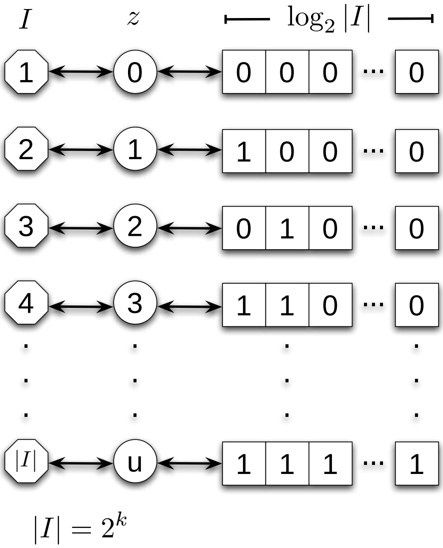

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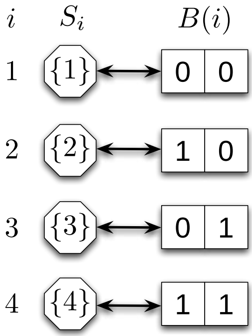
Model for SOS1 over $\lambda \in \Delta^J \subset \mathbb{R}^4_+$, $(I = J = \{1,\ldots,4\})$

- For injective function $B : I \to \{0,1\}^{\lceil \log_2 |I| \rceil}$:



$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \le 1, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$$

- In general: $\lceil \log_2 |I| \rceil$ binaries and $|I|$ extra constraints.

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i	S_i	$B(i)$	
1	$\{1\}$	0	0
2	$\{2\}$	1	0
3	$\{3\}$	0	1
4	$\{4\}$	1	1
		x_1	x_2

$x_1, x_2 \in \{0,1\}$

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$$\lambda_2 + \lambda_3 + \lambda_4 \leq x_1 + x_2$$

$$x_1, x_2 \in \{0,1\}$$

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Reducing the Number of Constraints for SOS1 over $\lambda \in \Delta^J \subset \mathbb{R}^4_+$, $(I = J = \{1, \ldots, 4\})$

- For injective function $B : I \to \{0,1\}^{\lceil \log_2 |I| \rceil}$:

i	S_i		$B(i)$: x_1	$B(i)$: x_2
1	$\{1\}$	$\longleftrightarrow$	0	0
2	$\{2\}$	$\longleftrightarrow$	1	0
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$x_1 \quad x_2 \qquad \in \{0,1\}$

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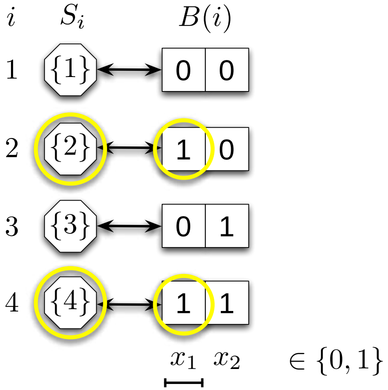

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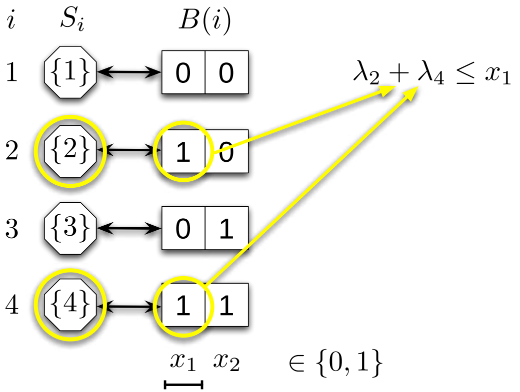

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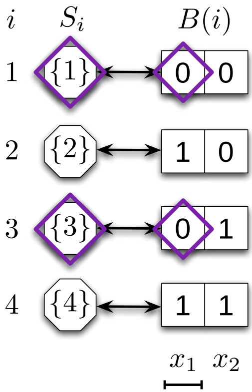

$$\lambda_2 + \lambda_4 \le x_1$$

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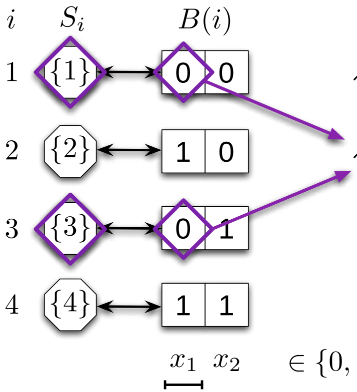

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$\lambda_2 + \lambda_4 \leq x_1$

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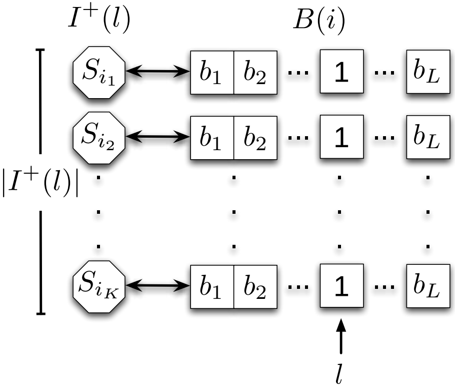
Model with a Logarithmic Number of Binary Variables and Extra Constraints

- $I^+(l) := \{i \in I \,:\, B(i)_l = 1\}$.



$$\lambda_j \geq 0 \quad \forall j \in J$$

$$\sum_{j \in J} \lambda_j \leq 1$$

$$x_l \in \{0,1\} \quad \forall l \in \{1,\ldots,L\}$$

$$L = \lceil \log_2 |I| \rceil$$

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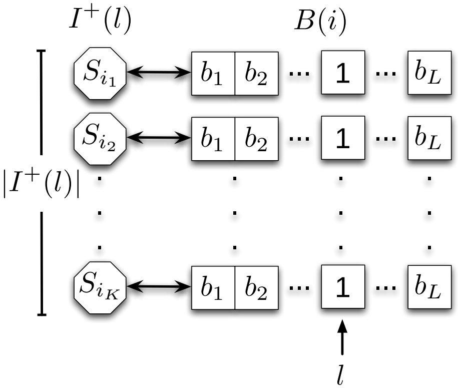

If $I(j) \subset I^+(l)$, $x_l = 0 \Rightarrow \lambda_j = 0$

$$\lambda_j \geq 0 \quad \forall j \in J$$

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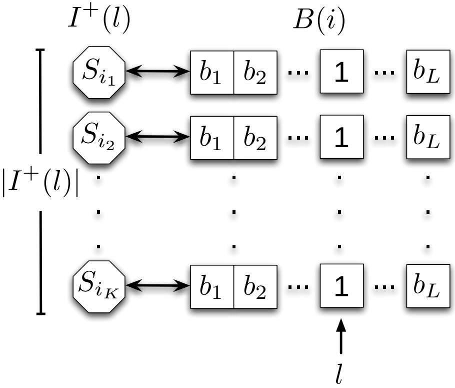

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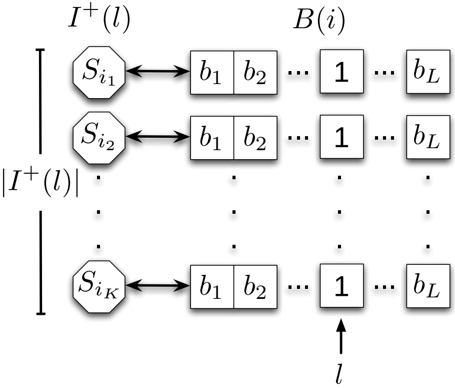

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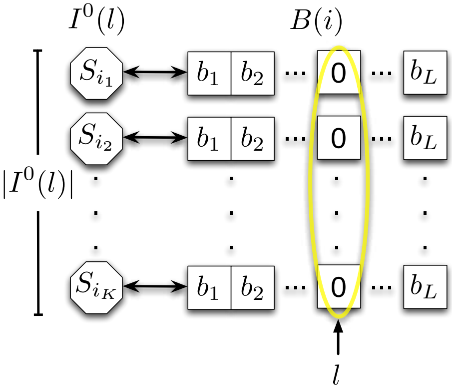

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Model with a Logarithmic Number of Binary Variables and Extra Constraints

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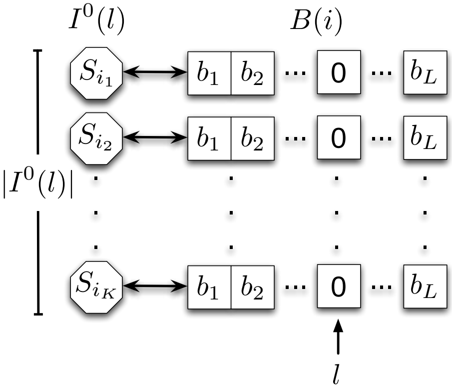

$$J^0(l) := \{j \in J \,:\, I(j) \subset I^0(l)\}$$

$$\lambda_j \geq 0 \quad \forall j \in J$$

$$\sum_{j \in J} \lambda_j \leq 1$$

$$\sum_{j \in J^+(l)} \lambda_j \leq x_l$$

$$\sum_{j \in J^0(l)} \lambda_j \leq (1 - x_l)$$

$$x_l \in \{0,1\} \quad \forall l \in \{1,\ldots,L\}$$

$$L = \lceil \log_2 |I| \rceil$$

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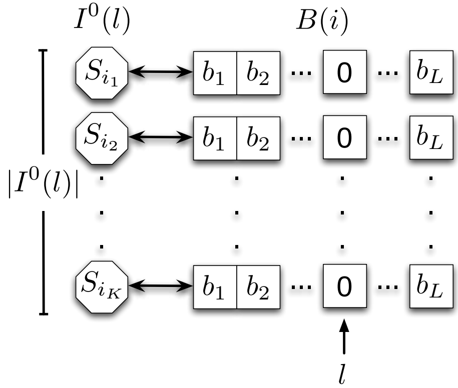

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$$x_l \in \{0,1\} \quad \forall l \in \{1, \ldots, L\}$$

$\lceil \log_2 |I| \rceil$ binary variables and $2\lceil \log_2 |I| \rceil$ extra constraints.

Logarithmic Model for SOS2 over $(\lambda_j)_{j=0}^4 \in \Delta^J \subset \mathbb{R}^5_+$

- $J = \{0, \ldots, 4\}$, $I = \{1, \ldots, 4\}$.

i	S_i		$B(i)$	
1	$\{0,1\}$	$\longleftrightarrow$	0 0	$\lambda_4 \leq x_1$
2	$\{1,2\}$	$\longleftrightarrow$	1 0	$\lambda_0 \leq (1 - x_1)$
3	$\{2,3\}$	$\longleftrightarrow$	0 1	$\lambda_0 + \lambda_1 \leq (1 - x_2)$
4	$\{3,4\}$	$\longleftrightarrow$	1 1	$\lambda_3 + \lambda_4 \leq x_2$
			$x_1 \; x_2 \quad \in \{0,1\}$	

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

-

Logarithmic Model for SOS2 over $(\lambda_j)_{j=0}^4 \in \Delta^J \subset \mathbb{R}^5_+$

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$$x_1 \quad x_2 \qquad \in \{0,1\}$$

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \le 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$$

- λ_2 does not show in any constraint!

Logarithmic Model for SOS2 over $(\lambda_j)_{j=0}^4 \in \Delta^J \subset \mathbb{R}^5_+$

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$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

- First Option: Add $\lambda_2 \leq x_1 + x_2$, $\quad \lambda_2 \leq 2 - x_1 - x_2$.

Logarithmic Model for SOS2 over $(\lambda_j)_{j=0}^4 \in \Delta^J \subset \mathbb{R}^5_+$

- $J = \{0, \ldots, 4\}$, $I = \{1, \ldots, 4\}$.

i	S_i		$B(i)$: x_1	x_2	
1	$\{0,1\}$	$\longleftrightarrow$	0	0	$\lambda_4 \leq x_1$
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$x_1 \; x_2 \in \{0,1\}$

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

- Second Option: Modify $B(i)$.

Logarithmic Model for SOS2 over $(\lambda_j)_{j=0}^4 \in \Delta^J \subset \mathbb{R}_+^5$

- $J = \{0,\ldots,4\}$, $I = \{1,\ldots,4\}$.

i	S_i	$B(i)$: x_1	$B(i)$: x_2
1	$\{0,1\}$	0	0
2	$\{1,2\}$	1	0
3	$\{2,3\}$	1	1
4	$\{3,4\}$	0	1

$x_1 \quad x_2 \quad \in \{0,1\}$

$$\lambda_4 \leq x_1$$

$$\lambda_0 \leq (1 - x_1)$$

$$\lambda_0 + \lambda_1 \leq (1 - x_2)$$

$$\lambda_3 + \lambda_4 \leq x_2$$

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

- Second Option: Modify $B(i)$.

Logarithmic Model for SOS2 over $(\lambda_j)_{j=0}^4 \in \Delta^J \subset \mathbb{R}^5_+$

- $J = \{0,\ldots,4\}$, $I = \{1,\ldots,4\}$.

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1	$\{0,1\}$	$\longleftrightarrow$	0 0	$\lambda_2 \leq x_1$
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$$x_1 \quad x_2 \quad \in \{0,1\}$$

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Logarithmic Model for SOS2 over $(\lambda_j)_{j=0}^{4} \in \Delta^J \subset \mathbb{R}^5_+$

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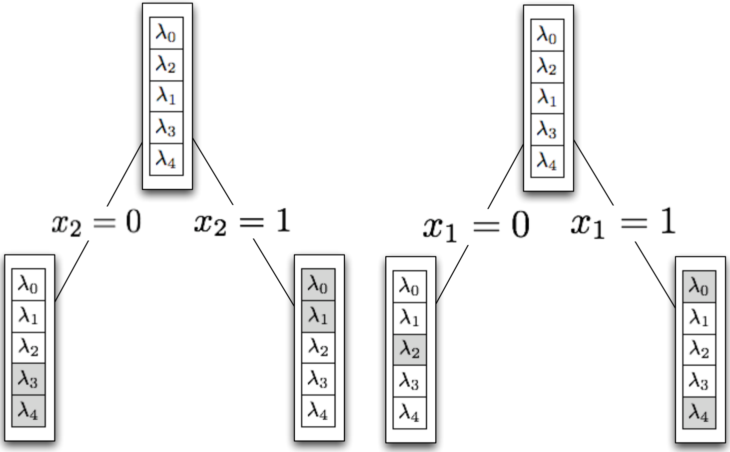
$$x_1 \quad x_2 \qquad \in \{0, 1\}$$

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

- Condition: $B(i)$ and $B(i+1)$ only differ in one component.

Logarithmic Model and Independent Branching



$$\lambda_0 + \lambda_1 \leq (1 - x_2)$$

$$\lambda_3 + \lambda_4 \leq x_2$$

$$\lambda_2 \leq x_1$$

$$\lambda_0 + \lambda_4 \leq (1 - x_1)$$

Independent Branching Scheme for $\lambda \in \bigcup_{i \in I} Q(S_i)$

$$\bigcup_{i \in I} Q(S_i) = \bigcap_{k=1}^{d} \left(Q(L_k) \cup Q(R_k) \right)$$

For $\{L_k, R_k\}_{k=1}^{d}$ with $L_k, R_k \subset J$.

$d :=$ "depth"

Independent Branching Scheme for $\lambda \in \bigcup_{i\in I} Q(S_i)$

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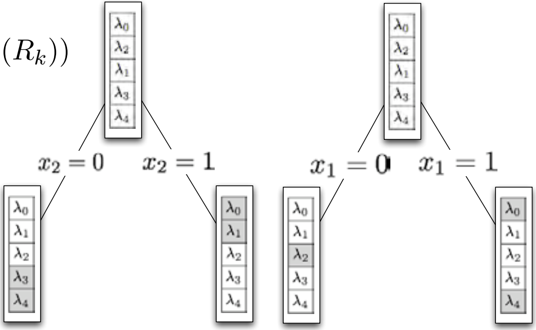


$$Q(\{0,1\}) \cup Q(\{1,2\}) \cup Q(\{2,3\}) \cup Q(\{3,4\}) =$$
$$\Big(Q(\{0,1,2\}) \cup Q(\{2,3,4\})\Big) \cap \Big(Q(\{0,1,3,4\}) \cup Q(\{1,2,3\})\Big)$$

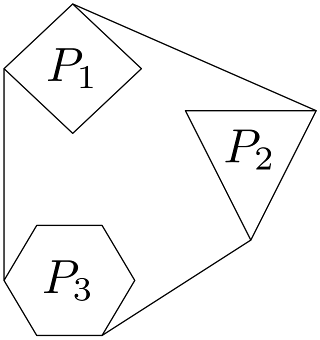
Formulation from Independent Branching Scheme

- For an independent branching $\{L_k, R_k\}_{k=1}^{d}$ of $\lambda \in \bigcup_{i \in I} Q(S_i)$:

$$
\begin{aligned}
\lambda_j &\geq 0 && \forall j \in J \\
\sum_{j \in J} \lambda_j &\leq 1 && \\
\sum_{j \notin L_k} \lambda_j &\leq x_k && \forall k \in \{1, \ldots, d\} \\
\sum_{j \notin R_k} \lambda_j &\leq (1 - x_k) && \forall k \in \{1, \ldots, d\} \\
x_k &\in \{0, 1\} && \forall k \in \{1, \ldots, d\}
\end{aligned}
$$

- d binary variables and $2d$ extra constraints.
- Independent branchings for SOS1 and SOS2 have $d = \lceil \log_2 |I| \rceil$.

Independent Branching Formulation is Tight



- Formulation:

$$\lambda \in \Delta^J$$

$$\sum_{j \notin L_k} \lambda_j \leq x_k, \quad \sum_{j \notin R_k} \lambda_j \leq (1 - x_k),$$

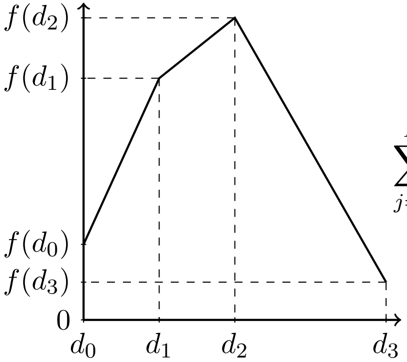
$$x_k \in \{0,1\} \quad \forall k \in \{1, \ldots, d\}$$

- Projection of LP relaxation into λ variables is

$$\operatorname{conv}\left(\bigcup_{i \in I} Q(S_i)\right) = \Delta^J.$$

- Might not hold if Δ^J is replaced by a box in $\mathbb{R}^J$.

SOS2 Model for Continuous Non-convex Piecewiselinear Functions



$K = 3$

$$\sum_{j=0}^{K} d_j \lambda_j = x$$

$$\sum_{j=0}^{K} f(d_j) \lambda_j = f(x)$$

$$\sum_{j=0}^{K} \lambda_j = 1$$

$$\lambda_j \geq 0 \qquad \forall j \in \{0, \ldots, K\}$$

$(\lambda_j)_{j=0}^{K}$ is SOS2

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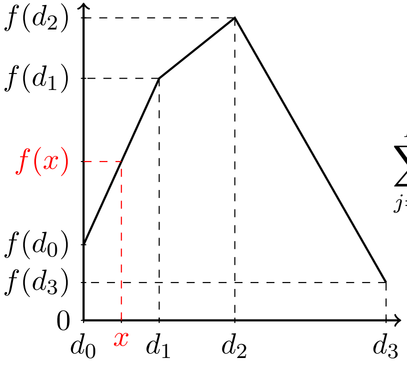

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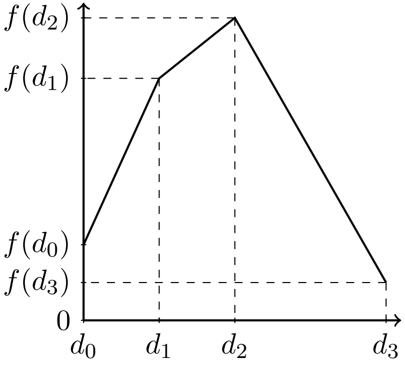


$K = 3$

$$\sum_{j \in J} d_j \lambda_j = x$$

$$\sum_{j \in J} f(d_j) \lambda_j = f(x)$$

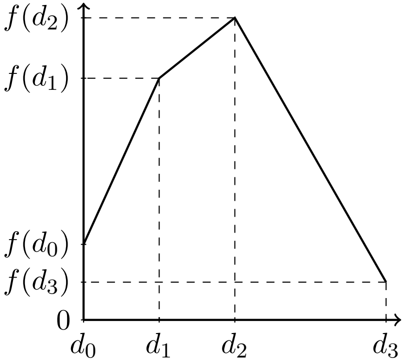
$$\sum_{j \in J} \lambda_j = 1$$

$$\lambda_j \geq 0 \qquad \forall j \in J$$

$$(\lambda_j)_{j=0}^{K} \text{ is SOS2}$$

- $J = \{0, \ldots, K\}$.

SOS2 Model for Continuous Non-convex Piecewiselinear Functions



$K = 3$

$$\sum_{j\in J} d_j \lambda_j = x$$

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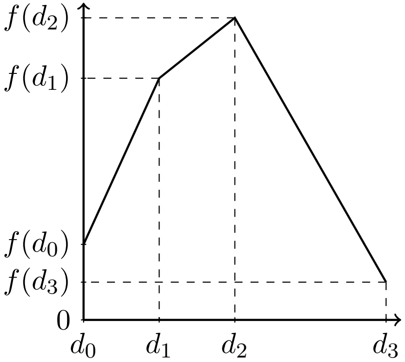


$K = 3$

$$\sum_{j\in J} d_j \lambda_j = x$$

$$\sum_{j\in J} f(d_j)\lambda_j = f(x)$$

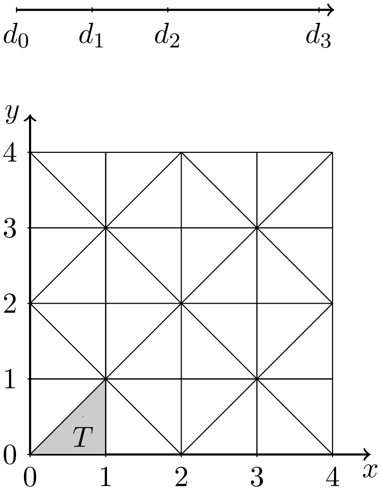
$$\sum_{j\in J} \lambda_j \geq 1$$

$$\lambda \in \Delta^J$$

$$(\lambda_j)_{j=0}^K \text{ is SOS2}$$

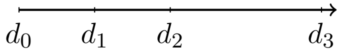
- $J = \{0, \ldots, K\}$.
- Log formulation for SOS2 yields formulation with $\lceil \log_2 K \rceil$ binary variables and extra constraints.

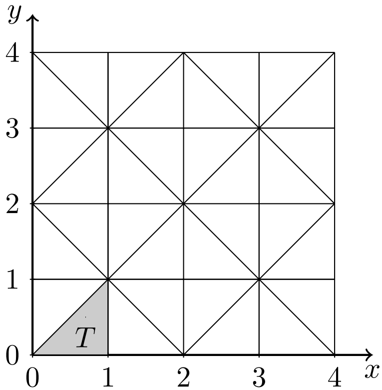
Extension to Non-Separable Piecewiselinear Functions of Two Variables: $f(x,y)$



$K = 4$

Extension to Non-Separable Piecewiselinear Functions of Two Variables: $f(x,y)$





$K = 4$

$$\sum_{j\in J} (j_1, j_2)^T \lambda_j = (x,y)^T$$

$$\sum_{j\in J} f(j_1, j_2)\lambda_j = f(x,y)$$

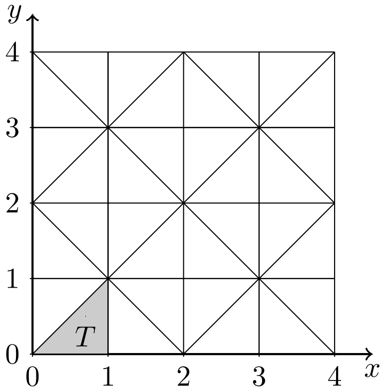
$$\sum_{j\in J} \lambda_j \geq 1$$

$$\lambda \in \Delta^J$$

$$\lambda \in \bigcup_{i\in I} Q(S_i)$$

- $J = \{0,\ldots,K\}^2 = \{\text{vertices}\}$.
- $I = \{\text{triangles}\}$,
  $S_i = \{\text{vertices of triangle } i\}$
  $(S_T = \{(0,0),(1,0),(1,1)\})$.

Independent Branching for Two Variable Functions



$K = 4$

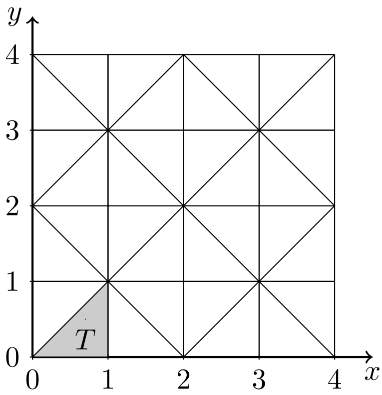
- Select a triangle by forbidding the use of vertices ($J = \{\text{vertices}\}$):

$$\sum_{j \notin L_k} \lambda_j \leq x_k$$

$$\sum_{j \notin R_k} \lambda_j \leq (1 - x_k)$$

$$x_k \in \{0, 1\}$$

Independent Branching for Two Variable Functions



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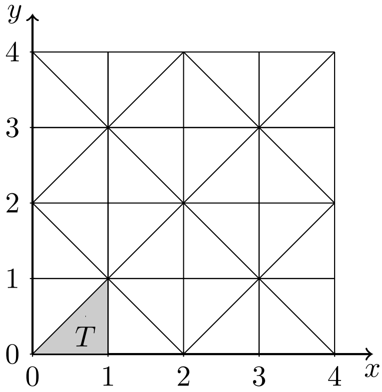


$K = 4$

- Select a triangle by forbidding the use of vertices ($J = \{\text{vertices}\}$):

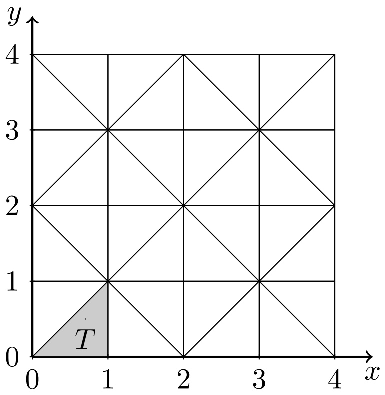
$$\sum_{j \in \overline{L}_k} \lambda_j \leq x_k$$

$$\sum_{j \in \overline{R}_k} \lambda_j \leq (1 - x_k)$$

$$x_k \in \{0, 1\}$$

- $\overline{L}_k = J \setminus L_k$, $\overline{R}_k = J \setminus R_k$.

Independent Branching for Two Variable Functions



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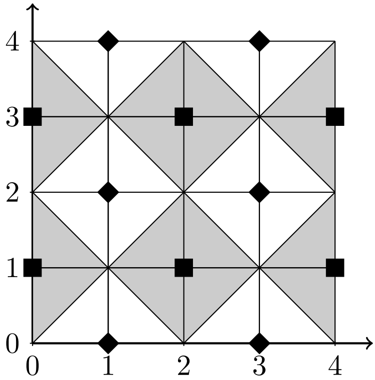
$$x_k \in \{0, 1\}$$

- $\overline{L}_k = J \setminus L_k$, $\overline{R}_k = J \setminus R_k$.
- Two phases:
  1. Square selection: applying SOS2 independent branching to each component.
  2. Triangle selection.

Triangle Selecting Independent Branching

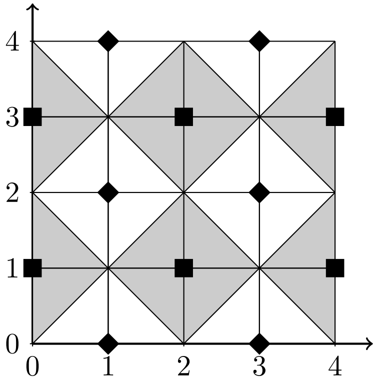


- Forbid white triangles in one branch and grey triangles in the other.

$$\bar{L} = \{(r,s) \in J \ : \ r \text{ even and } s \text{ odd}\}$$
$$= \{\text{square vertices}\}$$
$$\bar{R} = \{(r,s) \in J \ : \ r \text{ odd and } s \text{ even}\}$$
$$= \{\text{diamond vertices}\}$$

- Triangle branching allows only one triangle in each square.
- Depth of independent branching is $\lceil \log_2 \mathcal{T} \rceil$ for $\mathcal{T}$ = total # of triangles.

Triangle Selecting Independent Branching



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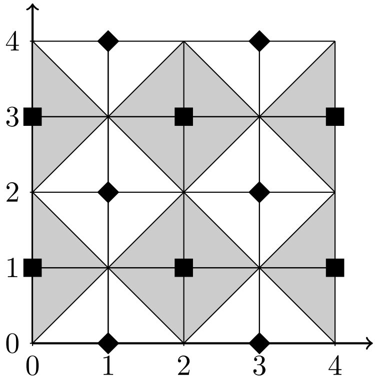

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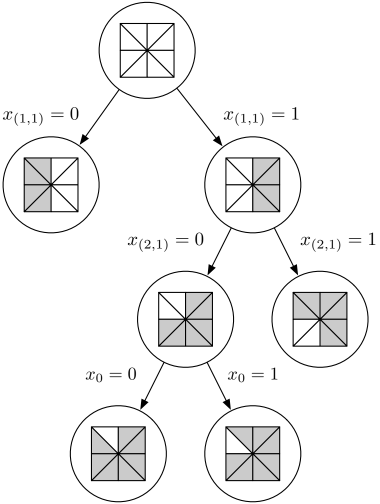
Example for Two Variable Function



$$
\begin{aligned}
\lambda_{(0,0)} + \lambda_{(0,1)} + \lambda_{(0,2)} &\leq x_{(1,1)}, \\
\lambda_{(2,0)} + \lambda_{(2,1)} + \lambda_{(2,2)} &\leq 1 - x_{(1,1)} \\
\lambda_{(0,0)} + \lambda_{(1,0)} + \lambda_{(2,0)} &\leq x_{(2,1)}, \\
\lambda_{(0,2)} + \lambda_{(1,2)} + \lambda_{(2,2)} &\leq 1 - x_{(2,1)} \\
\lambda_{(0,1)} + \lambda_{(2,1)} &\leq x_0, \\
\lambda_{(1,0)} + \lambda_{(1,2)} &\leq 1 - x_0.
\end{aligned}
$$

Computational Experiments (Instances)

- Single Variable:
  - 10×10 transportation problems.
  - Minimize $\sum_{e \in E} f_e(x_e)$. x_e flow in arc e.
  - $f_e(x_e)$ non-decreasing continuous concave piecewiselinear.
  - Number of segments where $f_e(x_e)$ is linear: $K = \{4, 8, 16, 32\}$.
  - 5 base instances. 20 randomly generated objectives for each base instance and each K. Total of 100 instances for each K.
- Two Variables:
  - 5×5 two-commodity transportation problems.
  - Minimize $\sum_{e \in E} f_e(x_e^1, x_e^2)$. x_e^i flow of commodity i in arc e.
  - $f_e(x_e^1, x_e^2)$ interpolation on grid of $g\left(\left\|\left(x_e^1, x_e^2\right)\right\|\right)$. g non-decreasing continuous concave piecewiselinear.
  - Interpolation grid resolution: 4×4, 8×8 and 16×16.
  - 5 base instances. 20 randomly generated objectives for each base instance and gird resolution. Total of 100 instances per grid resolution.

Computational Experiments (Solver and Formulations)

- Solver and Machine Stats:
  - CPLEX 11.
  - Dual 2.4GHz Linux workstation with 2GB of RAM.
  - Time Limit of 10,000 seconds.
- Formulations:

(Log) Logarithmic formulation.

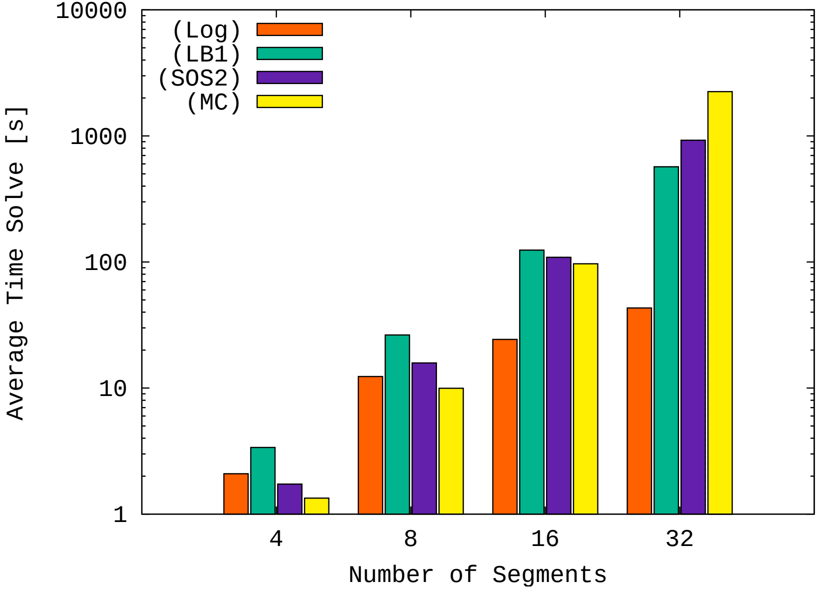
(LB1) Independent branching formulations of linear depth (Fuqua 2007). Only for single variable.

(LB2) Independent branching formulations of linear depth (Martin et. al. 2006).

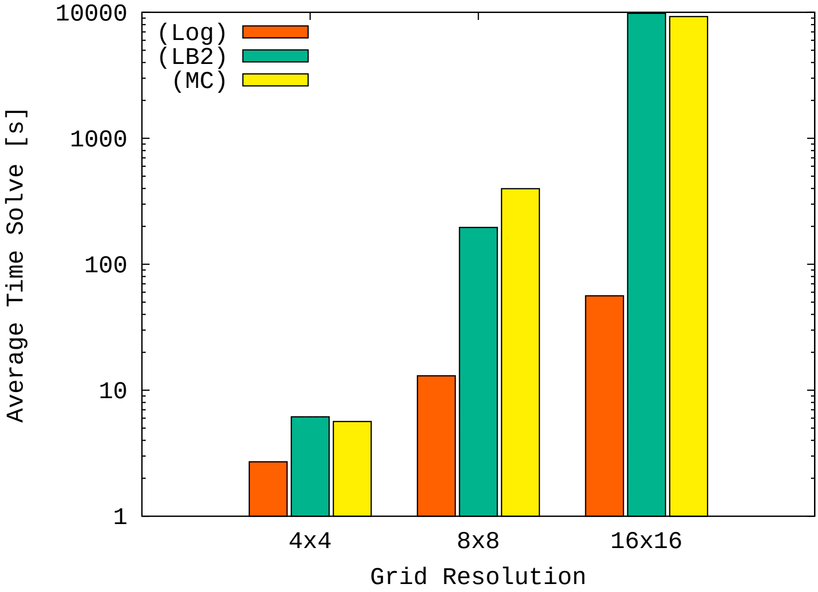
(SOS2) SOS2 based formulation. Only for single variable.

(MC) Multiple choice formulation (Jeroslow and Lowe 1984, Balakrishnan and Graves 1989, Croxton et. al 2003).

Average Solve Times for One Variable Functions



Average Solve Times for One Variable Functions



Summary

- Modeling a class of disjunctive constraints with a logarithmic number of binary variables and constraints:
  - First logarithmic formulations for SOS1-SOS2 constraints and piecewiselinear functions of one variable.
- Independent Branching Scheme:
  - Sufficient condition for logarithmic formulation.
  - First logarithmic formulation for piecewiselinear functions of two variables.
- Logarithmic formulations can provide a significant computational advantage.
  - Independent branching effectively turns CPLEX's variable branching into a specialized branching (e.g. SOS2 branching).

Future Work

- Formulation for piecewiselinear can be extended to functions of n variables in a K^n grid.
  - Only works for specific triangulation.
  - For fixed n, variable K,

$$\text{\# of variables and extra constr} \sim \log_2(\text{\# simplices}),$$

but for fixed K, variable n,

$$\log_2(\text{\# simplices}) = o(\text{\# of variables and extra constr}),$$

- Independent branching is not a necessary condition for logarithmic formulation:
  - Cardinality constraints: limit at most K components of $\lambda \in [0,1]^n$ to be non-zero. $J = \{1, \ldots, n\}$, $|I| = \binom{n}{K}$
  - Doesn't have independent branching, but for $K = n/2$ has formulation of size $O(\log_2(|I|))$:

$$\sum_{j=1}^{n} x_j \leq K; \quad \lambda_j \in [0,1], \quad \lambda_j \leq x_j, \quad x_j \in \{0,1\} \quad \forall j \in J.$$

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