Advanced Mixed Integer Programming (MIP) Formulation Techniques

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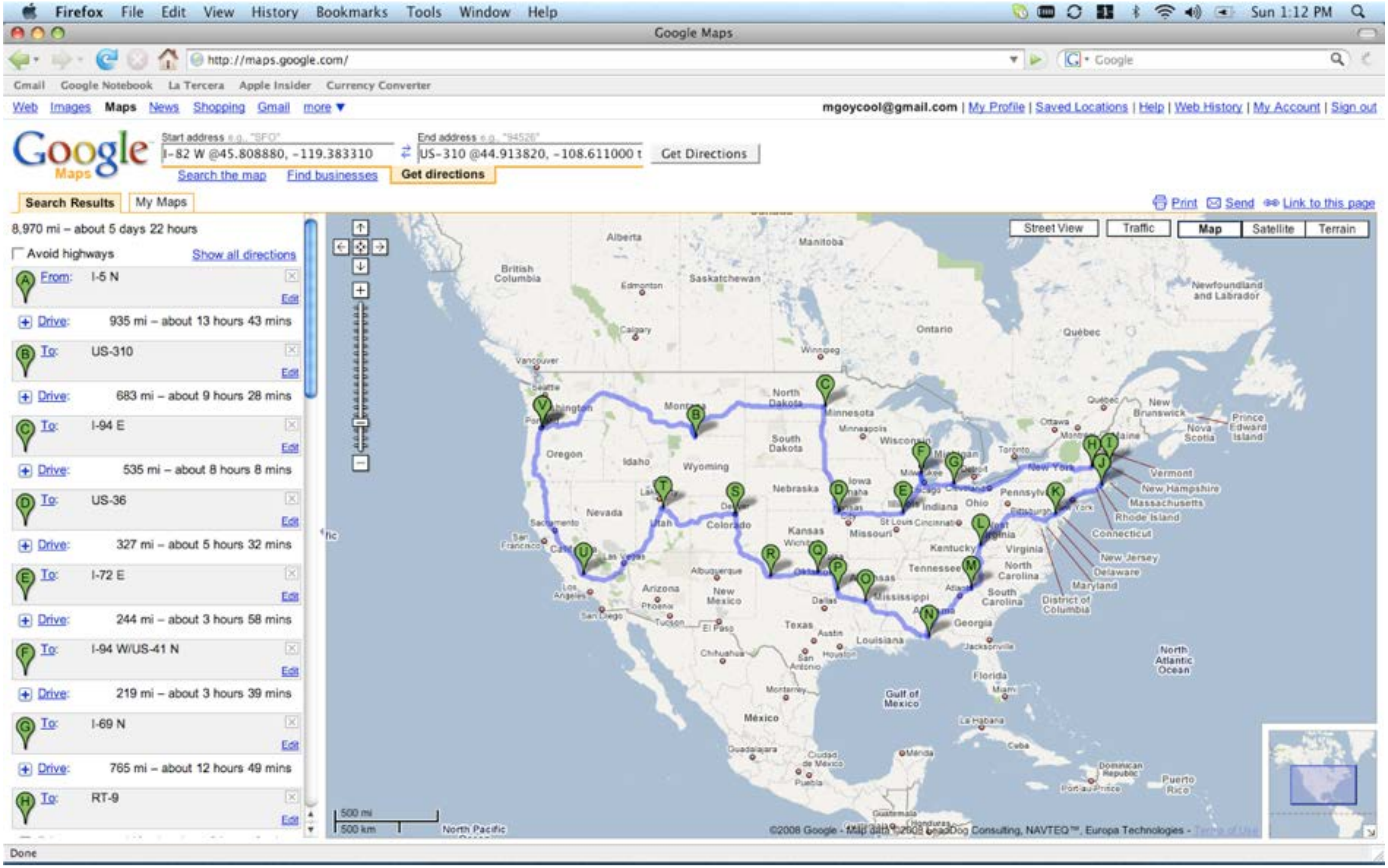
Los Alamos, New Mexico, August, 2016.

Traveling Salesman Problem (TSP): Visit Cities Fast



Start address e.g., "SFO": I-82 W @45.808880, -119.383310

End address e.g., "94526": US-310 @44.913820, -108.611000 t

8,970 mi – about 5 days 22 hours

☐ Avoid highways

Show all directions

A From: I-5 N

Drive: 935 mi – about 13 hours 43 mins

B To: US-310

Drive: 683 mi – about 9 hours 28 mins

C To: I-94 E

Drive: 535 mi – about 8 hours 8 mins

D To: US-36

Drive: 327 mi – about 5 hours 32 mins

E To: I-72 E

Drive: 244 mi – about 3 hours 58 mins

F To: I-94 W/US-41 N

Drive: 219 mi – about 3 hours 39 mins

G To: I-69 N

Drive: 765 mi – about 12 hours 49 mins

H To: RT-9

MIP = Avoid Enumeration

- Number of tours for 49 cities $= 48!/2 \approx 10^{60}$
- Fastest supercomputer $\approx 10^{17}\text{flops}$
- Assuming one floating point operation per tour: $> 10^{35}$ years $\approx 10^{25}$ times the age of the universe!
- How long does it take on an iphone?
  - Less than a second!
  - 4 iterations of **cutting plane** method!
  - Dantzig, Fulkerson and Johnson 1954 did it by hand!
  - For more info see tutorial in ConcordeTSP app
  - **Cutting planes** are the key for effectively solving (even NP-hard) MIP problems in practice.

50+ Years of MIP = Significant Solver Speedups

- Algorithmic Improvements (Machine Independent):
  - CPLEX v1.2 (1991) – v11 (2007): 29,000x speedup
  - Gurobi v1 (2009) – v6.5 (2015): 48.7x speedup
  - Commercial, but free for academic use
- (Reasonably) effective free / open source solvers:
  - GLPK, CBC and SCIP (free only for non-commercial)
- Easy to use, fast and versatile modeling languages
  - Julia based JuMP modelling language
- Linear MIP solvers very mature and effective:
  - Convex nonlinear MIP getting there (quadratic nearly there)

What do YOU need to do to use MIP?

1. Use JuMP
2. Construct a MIP formulation of your problem

- This talk:
  - From non-convex constraints to linear MIP formulations
  - One illustrative example
- Beyond linear MIP:
  - Convex nonlinear MIP
  - See Miles talk on Thursday

Example: Experimental Design in Marketing

Think “Simulation-Based” Optimization

(Custom) Product Recommendations via CBCA

Feature	SX530	RX100
Zoom	50x	3.6x
Prize	$249.99	$399.99
Weight	15.68 ounces	7.5 ounces
Prefer	☑	☐

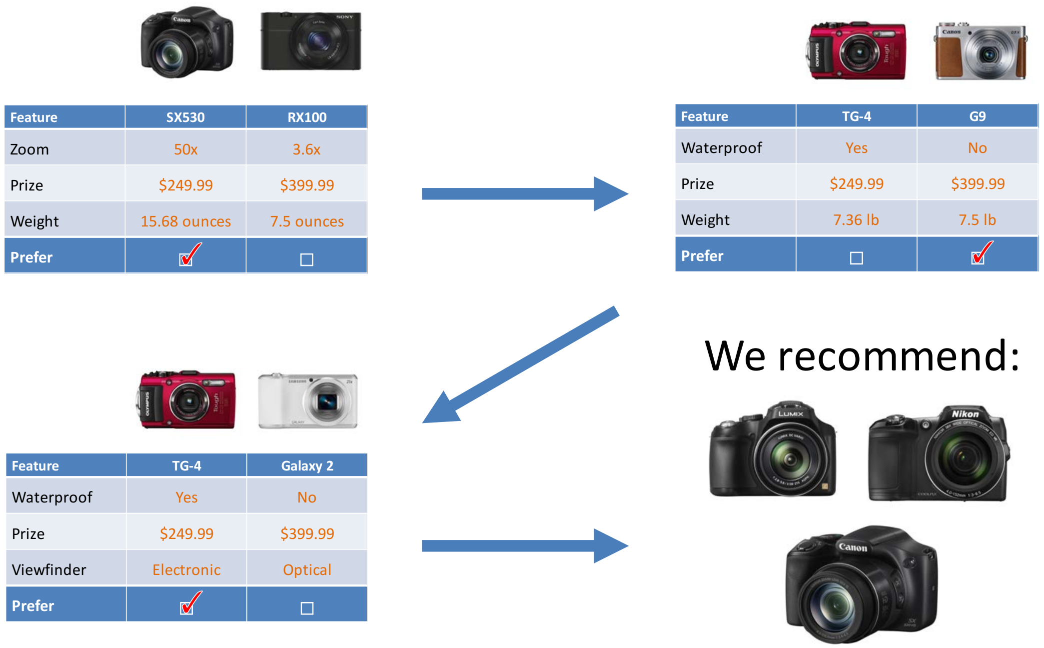
Feature	TG-4	G9
Waterproof	Yes	No
Prize	$249.99	$399.99
Weight	7.36 lb	7.5 lb
Prefer	☐	☑

Feature	TG-4	Galaxy 2
Waterproof	Yes	No
Prize	$249.99	$399.99
Viewfinder	Electronic	Optical
Prefer	☑	☐

We recommend:

Towards Optimal Product Recommendation

- Find enough information about preferences to recommend



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We recommend:

- How do I pick the **next (1^{st}) question** to obtain the largest reduction of uncertainty or “variance” on preferences

Choice-based Conjoint Analysis

Feature	Chewbacca	BB-8
Wookiee	Yes	No
Droid	No	Yes
Blaster	Yes	No
I would buy toy	☑	☐
Product Profile	x^1	x^2

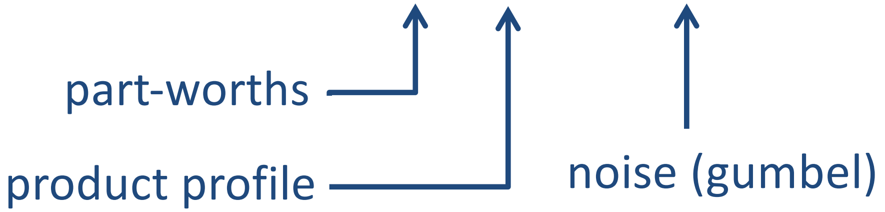
$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = x^2$$

MNL Preference Model

- Utilities for 2 products, n features (e.g. n = 12)

$$U_1 = \beta \cdot x^1 + \epsilon_1 = \sum\nolimits_{i=1}^{n} \beta_i x_i^1 + \epsilon_1$$

$$U_2 = \beta \cdot x^2 + \epsilon_2 = \sum\nolimits_{i=1}^{n} \beta_i x_i^2 + \epsilon_2$$



part-worths

product profile

noise (gumbel)

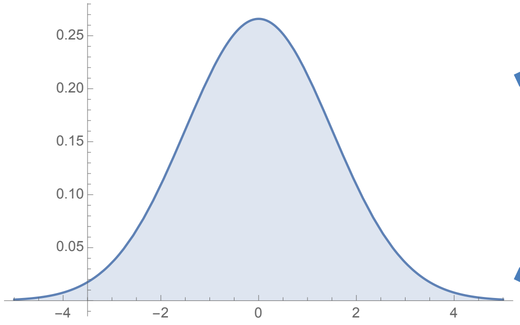
- Utility maximizing customer: $x^1 \succeq x^2 \Leftrightarrow U_1 \text{``}\geq\text{''} U_2$

- Noise can result in response error:

$$L\left(\beta \,|x^1 \;\succeq\; x^2\,\right) = \mathbb{P}\left(x^1 \;\succeq\; x^2 \mid \beta\right) = \frac{e^{\beta \cdot x^1}}{e^{\beta \cdot x^1} + e^{\beta \cdot x^2}}$$

Next Question To Reduce “Variance”: Bayesian

Prior Distribution of β



Feature	SX530	RX100
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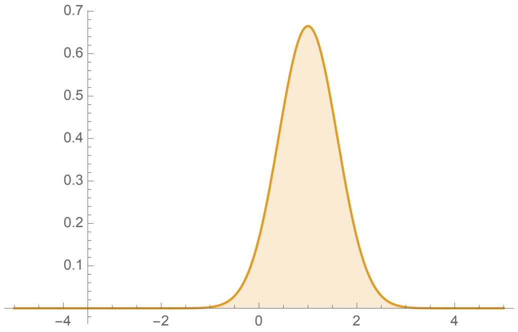
Bayesian Update

Posterior Distribution

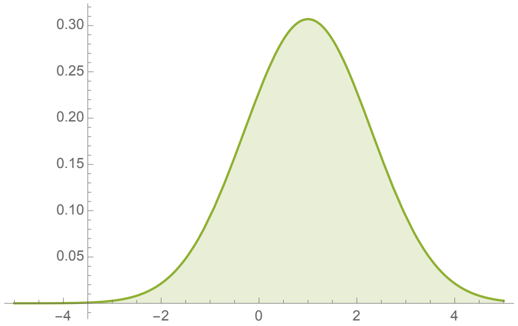


MCMC

Feature	TG-4	Galaxy 2
Waterproof	Yes	No
Prize	$249.99	$399.99
Viewfinder	Electronic	Optical
Prefer	☑	☐

Bayesian Update

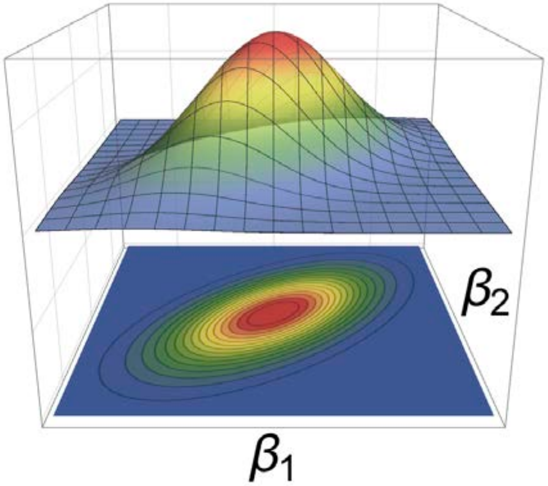
Posterior Distribution



- Black-box objective: Question Selection = Enumeration ☹
- Question selection by Mixed Integer Programming (MIP)

Bayesian Update and Geometric Updates

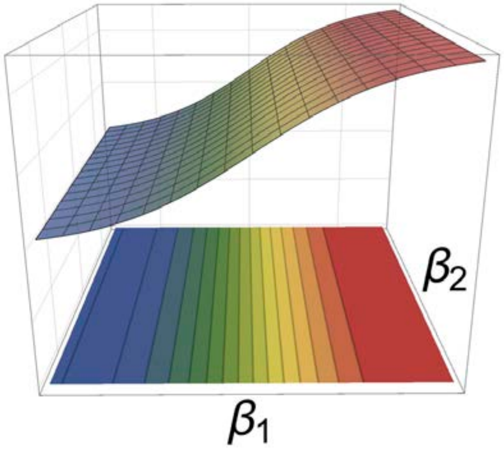
Prior distribution



$$\beta \sim N(\mu, \Sigma)$$

$$\phi(\beta; \mu, \Sigma)$$

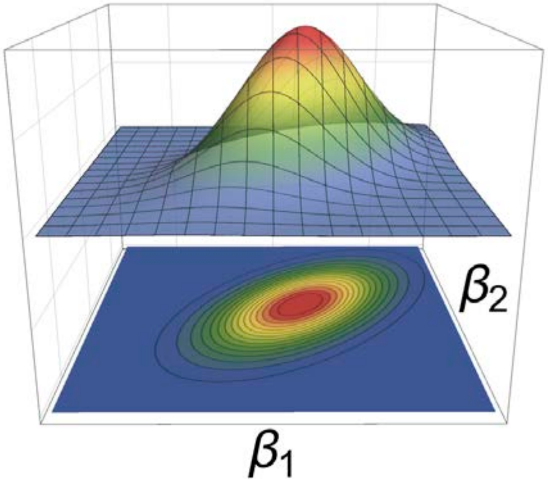
Answer likelihood



$$x^1 \succeq x^2$$

$$L\left(\beta \mid x^1 \succeq x^2\right)$$

Posterior distribution



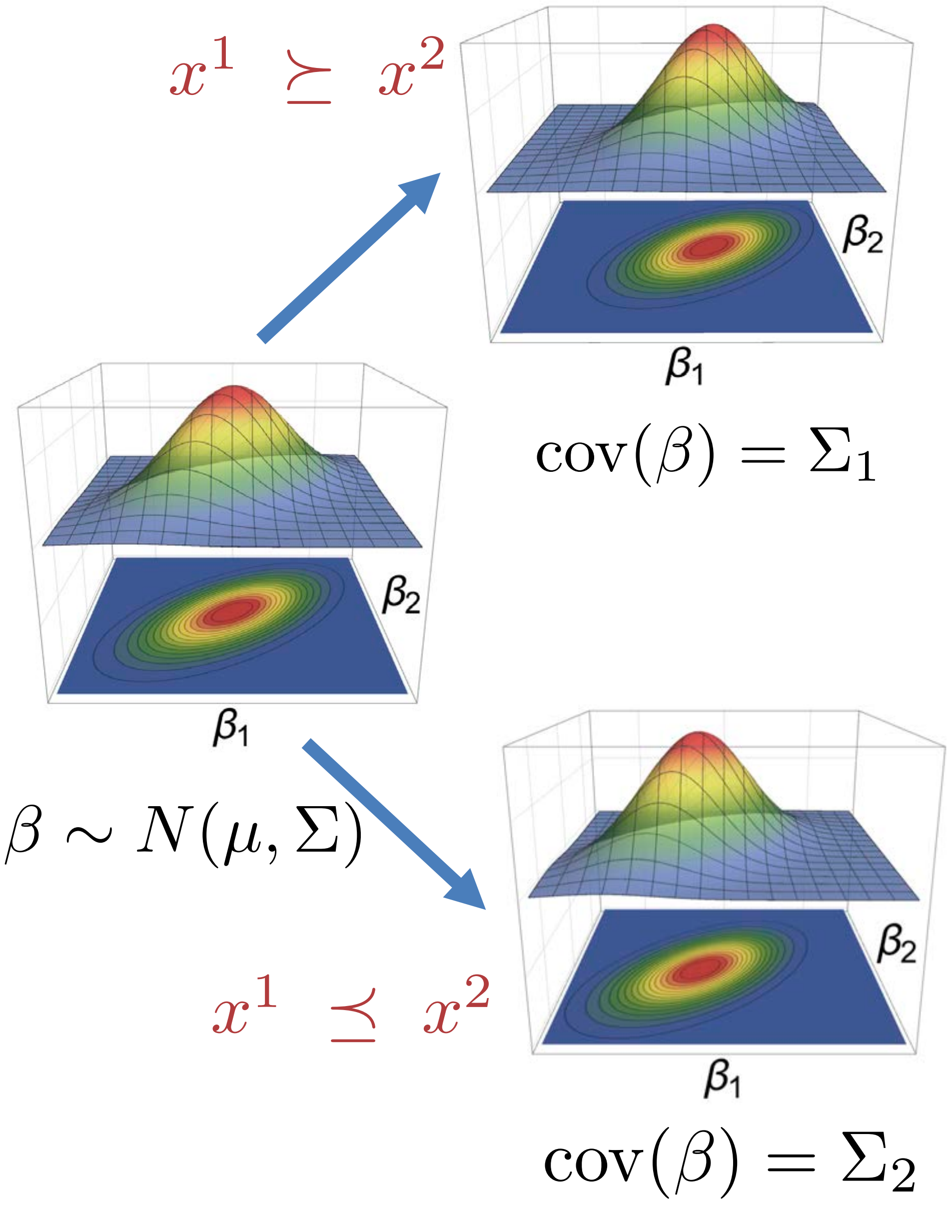
$$f\left(\beta \mid x^1 \succeq x^2\right)$$

Multidimensional Integration?

$$f\left(\beta \mid x^1 \succeq x^2\right) = \frac{\boxed{\phi\left(\beta\,;\ \mu, \Sigma\right) L\left(\beta \mid x^1 \succeq x^2\right)}}{\int_{\mathbb{R}} \phi\left(\beta\,;\ \mu, \Sigma\right) L\left(\beta \mid x^1 \succeq x^2\right) d\beta}$$

non-convex on $x^1, x^2 \in \{0, 1\}^n$

D-Efficiency and Posterior Covariance Matrix



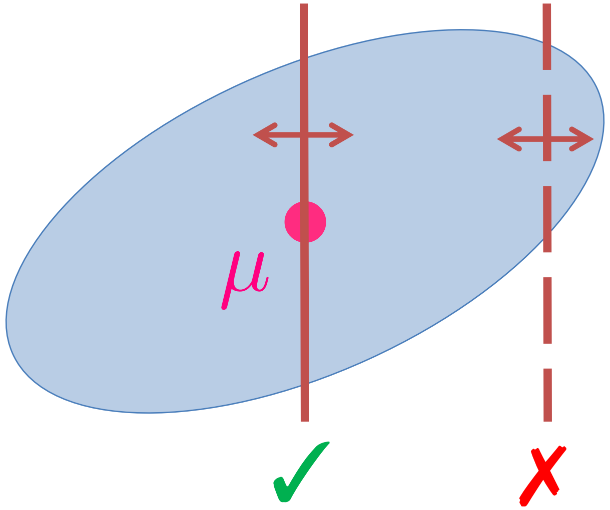
- “Variance” = D-Efficiency:
- $f\left(x^1, x^2\right) := \mathbb{E}_{\beta, x^1 \preceq/\succeq x^2}\left(\det(\Sigma_i)^{1/p}\right)$
- Non-convex function
- Even evaluating expected D-Efficiency for a question requires multidimensional integration

Standard Question Selection Criteria

$$(\beta - \mu)' \cdot \Sigma^{-1} \cdot (\beta - \mu) \leq r$$

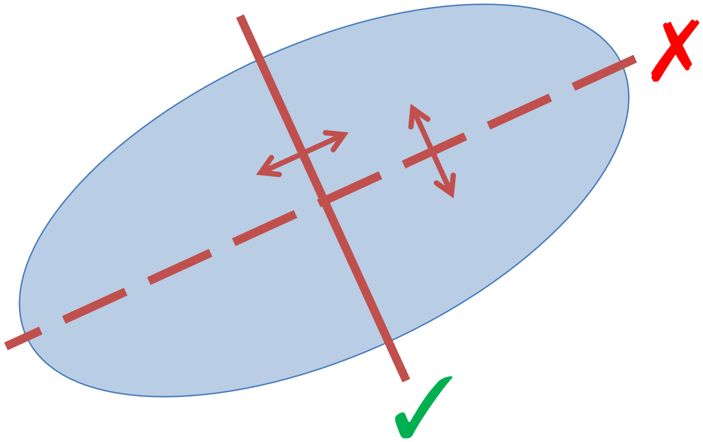
- Choice balance:
  - Minimize distance to center

$$\mu \cdot \left(x^1 - x^2\right)$$



- Postchoice symmetry:
  - Maximize variance of question

$$\left(x^1 - x^2\right)' \cdot \sum \cdot \left(x^1 - x^2\right)$$

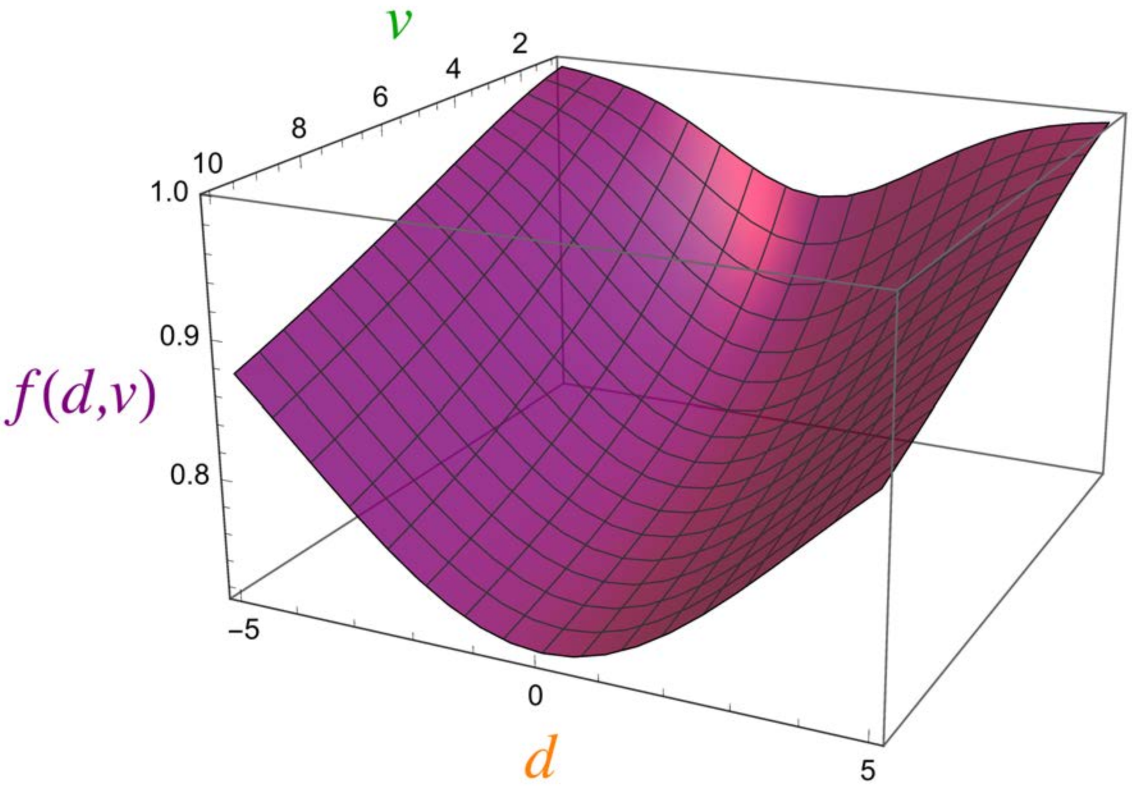


D-efficiency: Balance Question Trade-off

- D-efficiency = Non-convex function $f(d, v)$ of

  distance: $d := \mu \cdot (x^1 - x^2)$

  variance: $v := (x^1 - x^2)' \cdot \sum \cdot (x^1 - x^2)$



Can evaluate $f(d, v)$ with 1-dim integral 🙂

Optimization Model

$$\min \qquad f(d, v)$$ ✗

$$s.t.$$

$$\mu \cdot \left(x^1 - x^2\right) = d$$ ✓

$$\left(x^1 - x^2\right)' \cdot \sum \cdot \left(x^1 - x^2\right) = v$$ ✗

$$A^1 x^1 + A^2 x^2 \leq b$$ ✓

$$x^1 \neq x^2$$ ✗

$$x^1, x^2 \in \{0, 1\}^n$$

Technique 1: Binary Quadratic $x^1, x^2 \in \{0,1\}^n$

$$(x^1 - x^2)' \cdot \Sigma \cdot (x^1 - x^2) = v$$

$$X^l_{i,j} = x^l_i \cdot x^l_j \quad (l \in \{1,2\}, \quad i,j \in \{1,\ldots,n\}) :$$

$$X^l_{i,j} \leq x^l_i, \quad X^l_{i,j} \leq x^l_j, \quad X^l_{i,j} \geq x^l_i + x^l_j - 1, \quad X^l_{i,j} \geq 0$$

$$W_{i,j} = x^1_i \cdot x^2_j :$$

$$W_{i,j} \leq x^1_i, \quad W_{i,j} \leq x^2_j, \quad W_{i,j} \geq x^1_i + x^2_j - 1, \quad W_{i,j} \geq 0$$

$$\sum_{i,j=1}^{n} \left(X^1_{i,j} + X^2_{i,j} - W_{i,j} - W_{j,i}\right) \Sigma_{i,j} = v$$

Technique 1: Binary Quadratic $x^1, x^2 \in \{0,1\}^n$

$$x^1 \neq x^2 \quad \Leftrightarrow \quad \|x^1 - x^2\|_2^2 \geq 1$$

$$X^l_{i,j} = x^l_i \cdot x^l_j \quad (l \in \{1,2\}, \quad i,j \in \{1,\ldots,n\}):$$

$$X^l_{i,j} \leq x^l_i, \quad X^l_{i,j} \leq x^l_j, \quad X^l_{i,j} \geq x^l_i + x^l_j - 1, \quad X^l_{i,j} \geq 0$$

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$$W_{i,j} \leq x^1_i, \quad W_{i,j} \leq x^2_j, \quad W_{i,j} \geq x^1_i + x^2_j - 1, \quad W_{i,j} \geq 0$$

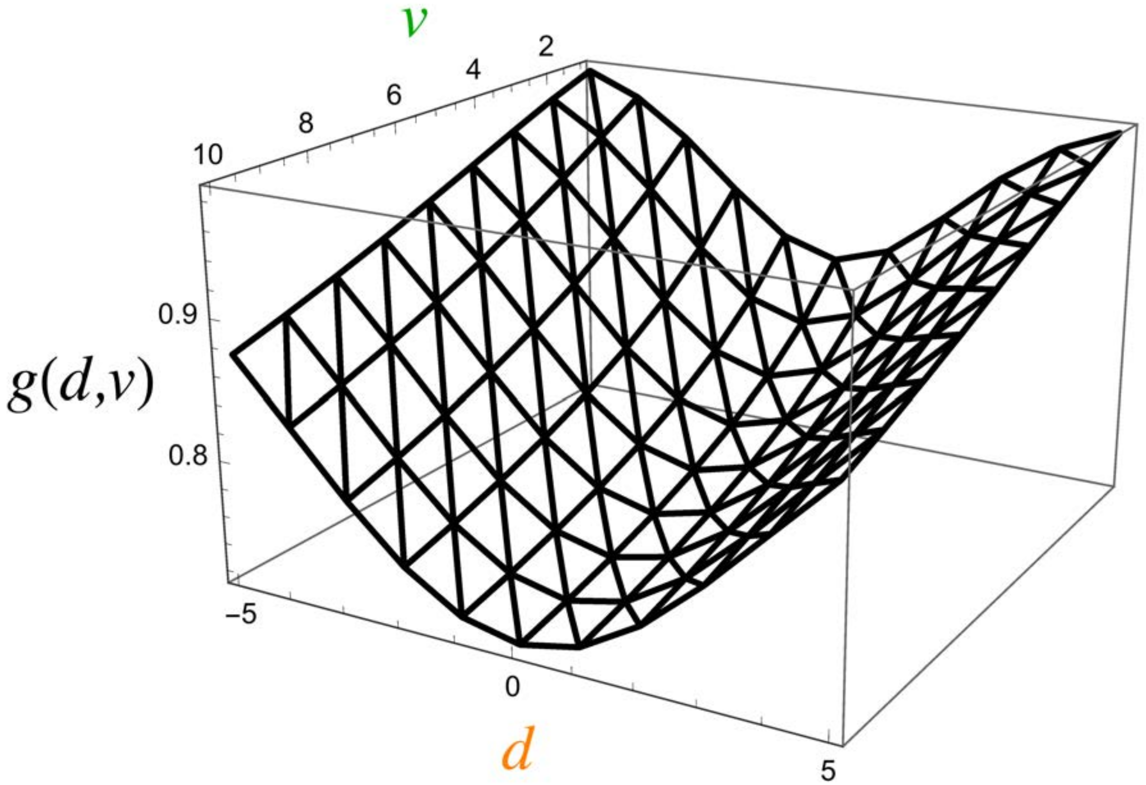
$$\sum_{i,j=1}^{n} \left(X^1_{i,j} + X^2_{i,j} - W_{i,j} - W_{j,i}\right) \geq 1$$

Technique 2: Piecewise Linear Functions

- D-efficiency = Non-convex function $f(d,v)$ of

distance: $d := \mu \cdot (x^1 - x^2)$

variance: $v := (x^1 - x^2)' \cdot \sum \cdot (x^1 - x^2)$



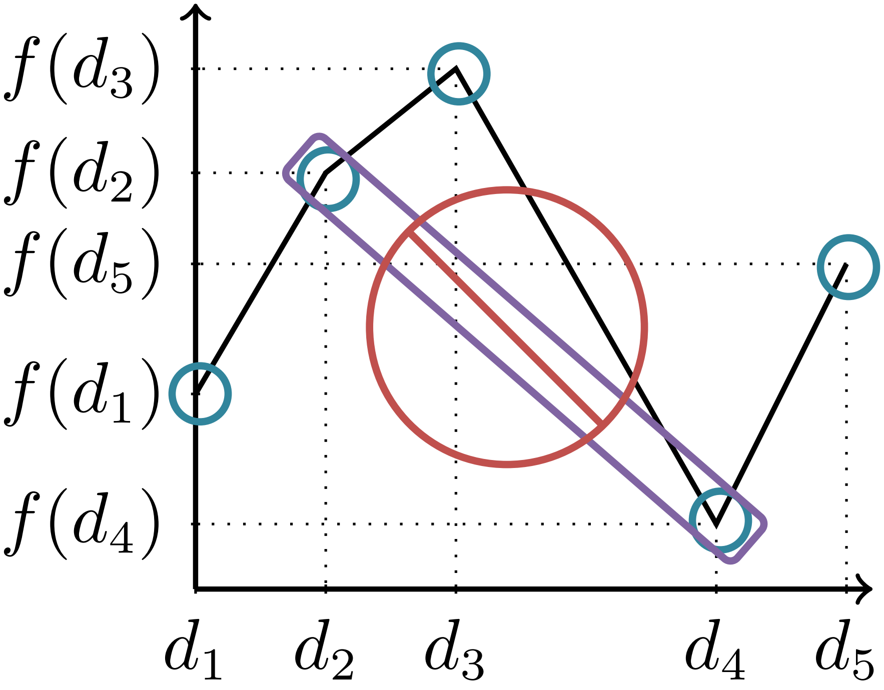
Can evaluate $f(d,v)$ with 1-dim integral 🙂

Piecewise Linear Interpolation

MIP formulation

Simple Formulation for Univariate Functions

$$z = f(x)$$



$$\text{Size} = O(\#\text{ of segments})$$

Non-Ideal: Fractional Extreme Points

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^{5} \lambda_j, \quad \lambda_j \geq 0$$

$$y \in \{0,1\}^4, \quad \sum_{i=1}^{4} y_i = 1$$

$$0 \leq \lambda_1 \leq y_1$$

$$0 \leq \lambda_2 \leq y_1 + y_2$$

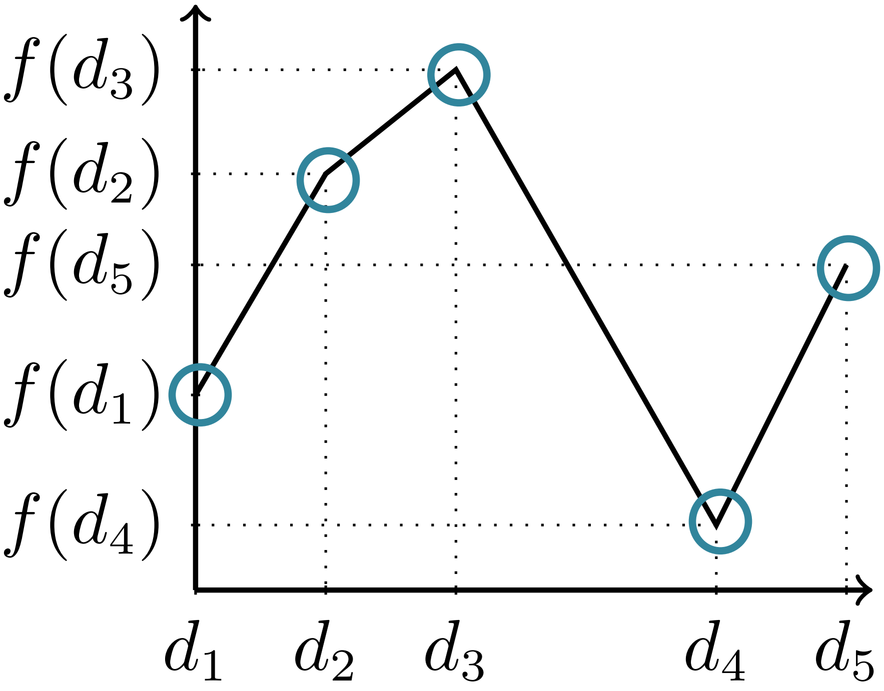
$$0 \leq \lambda_3 \leq y_2 + y_3$$

$$0 \leq \lambda_4 \leq y_3 + y_4$$

$$0 \leq \lambda_5 \leq y_4$$

Advanced Formulation for Univariate Functions

$$z = f(x)$$



$$\text{Size} = O\left(\log_2 \# \text{ of segments}\right)$$

Ideal: Integral Extreme Points

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^{5} \lambda_j , \quad \lambda_j \geq 0$$

$$y \in \{0,1\}^2$$

$$0 \leq \lambda_1 + \lambda_5 \leq 1 - y_1$$

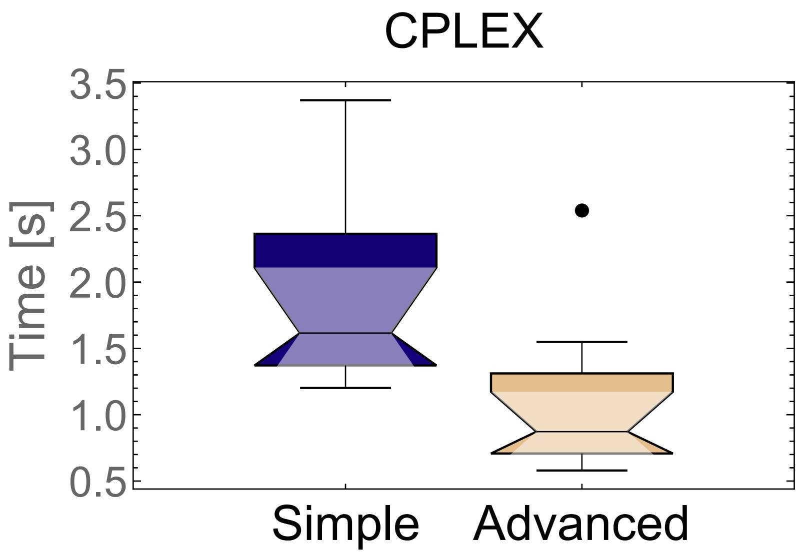
$$0 \leq \lambda_3 \leq y_1$$

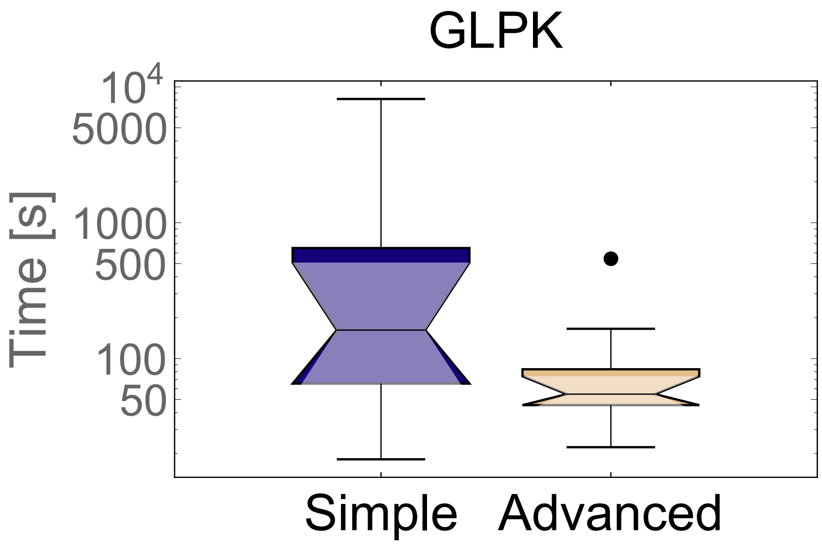
$$0 \leq \lambda_4 + \lambda_5 \leq 1 - y_2$$

$$0 \leq \lambda_1 + \lambda_2 \leq y_2$$

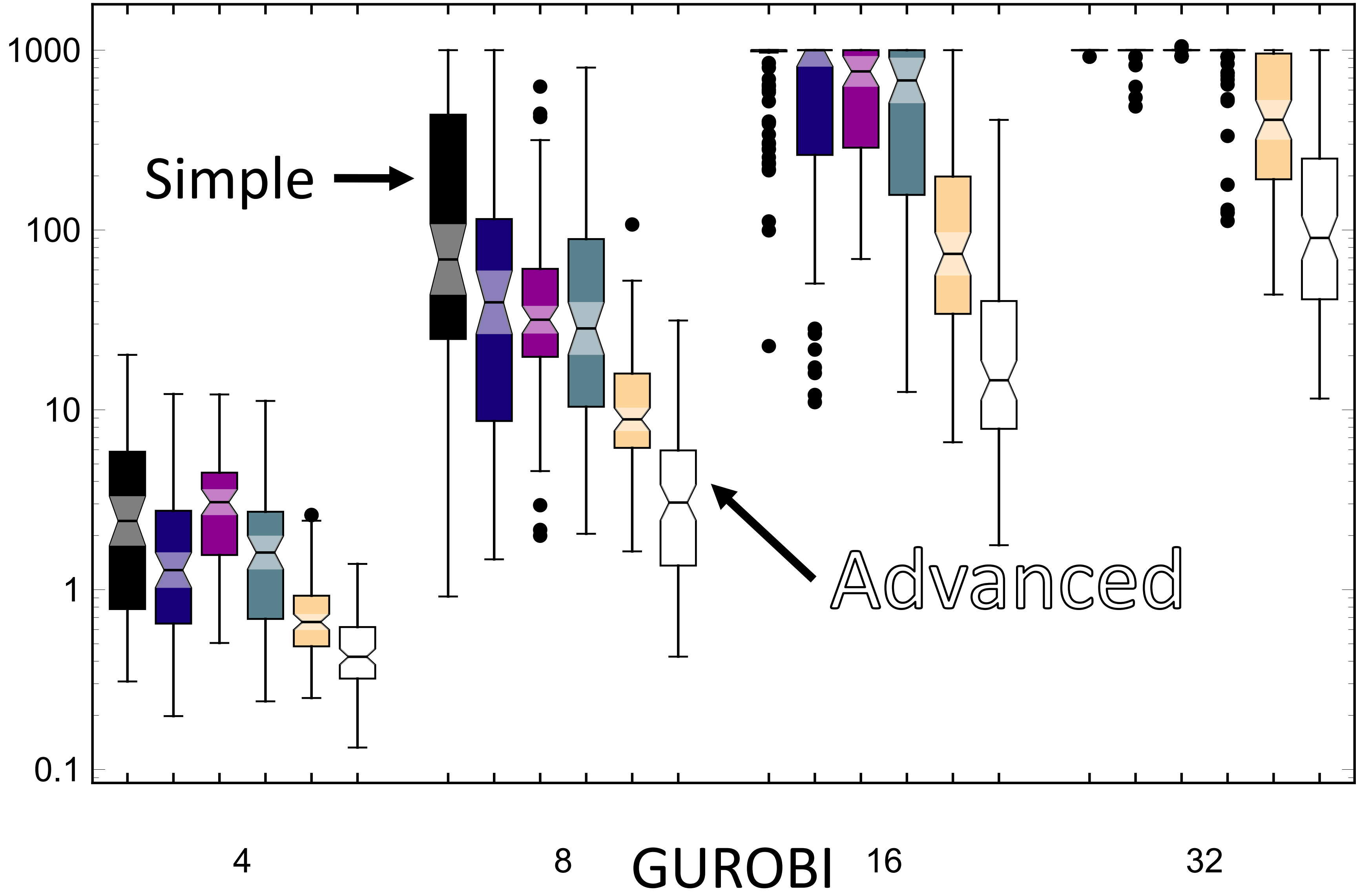
Computational Performance

- Advanced formulations provide an computational advantage
- Advantage is significantly more important for free solvers
- State of the art commercial solvers can be significantly better that free solvers
- Still, free is free!





Formulation Improvements can be Significant



Constructing Advanced Formulations

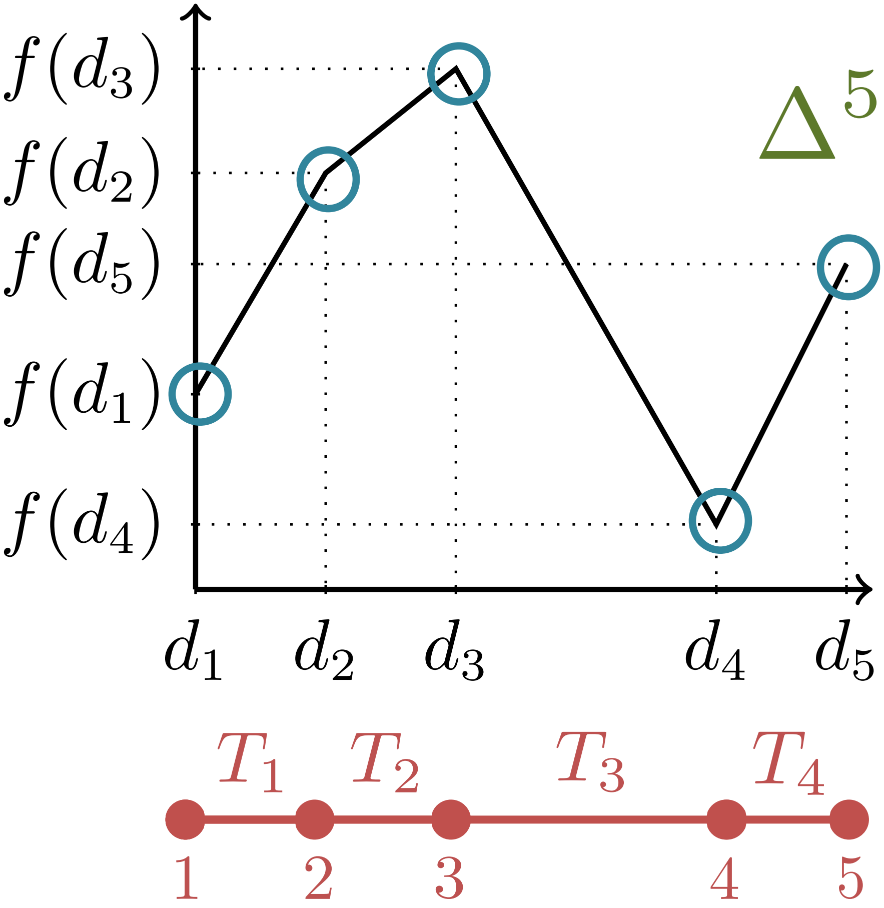
Abstracting Univariate Functions

$$P_i := \{\lambda \in \Delta^5 : \lambda_j = 0 \quad \forall j \notin T_i\}$$

$$z = f(x)$$

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$T_i := \{i, i+1\} \quad i \in \{1, \ldots, 4\}$$

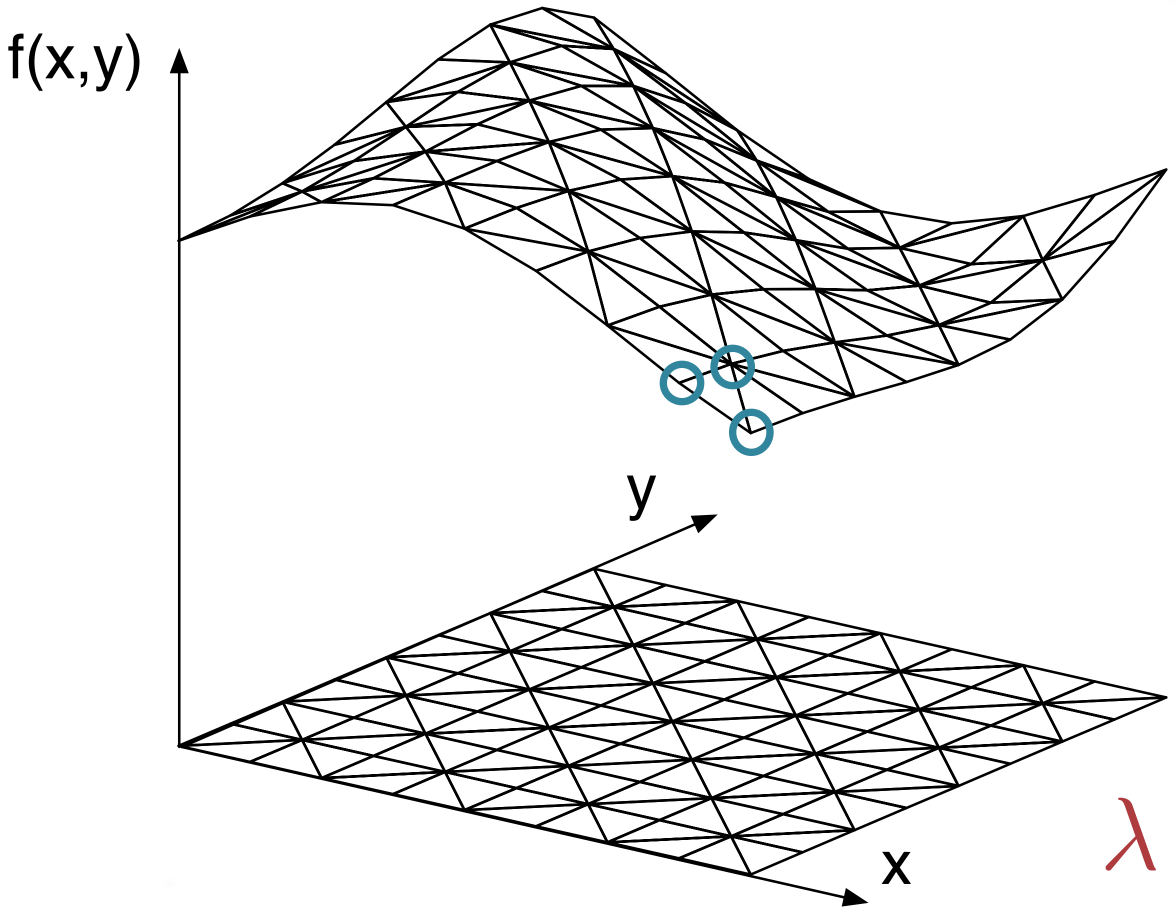


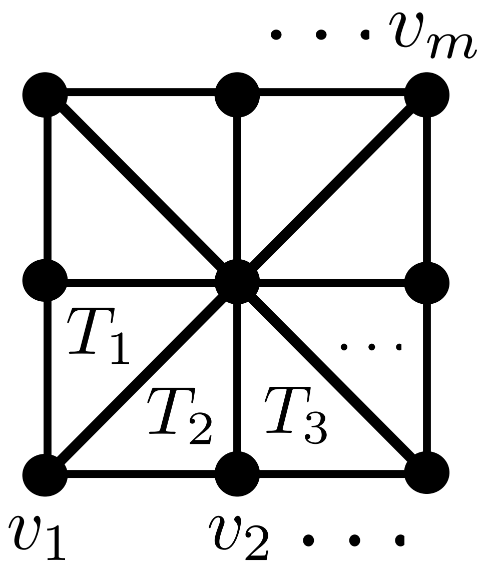
$$\Delta^5 = \boxed{1 = \sum_{j=1}^{5} \lambda_j , \quad \lambda_j \geq 0}$$

$$\lambda \in \bigcup_{i=1}^{4} P_i \subseteq \Delta^5$$

Abstraction Works for Multivariate Functions

$$P_i := \{\lambda \in \Delta^m \ : \ \lambda_j = 0 \quad \forall v_j \notin T_i\}$$

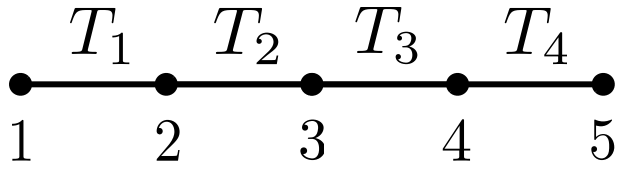


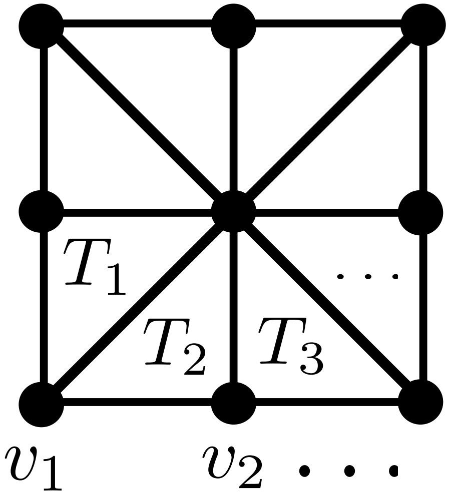


$$\lambda \in \bigcup_{i=1}^{n} P_i \subseteq \Delta^m$$

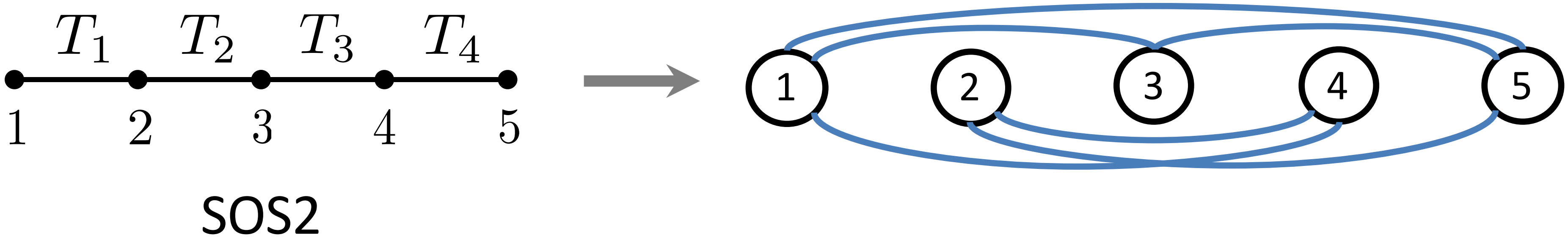
Complete Abstraction

- $\Delta^V := \left\{ \lambda \in \mathbb{R}^V_+ : \sum_{v \in V} \lambda_v = 1 \right\},$
- $P_i = \left\{ \lambda \in \Delta^V : \lambda_v = 0 \quad \forall v \notin T_i \right\}$
- $\lambda \in \bigcup_{i=1}^{n} P_i$
- $T_i =$ cliques of a graph

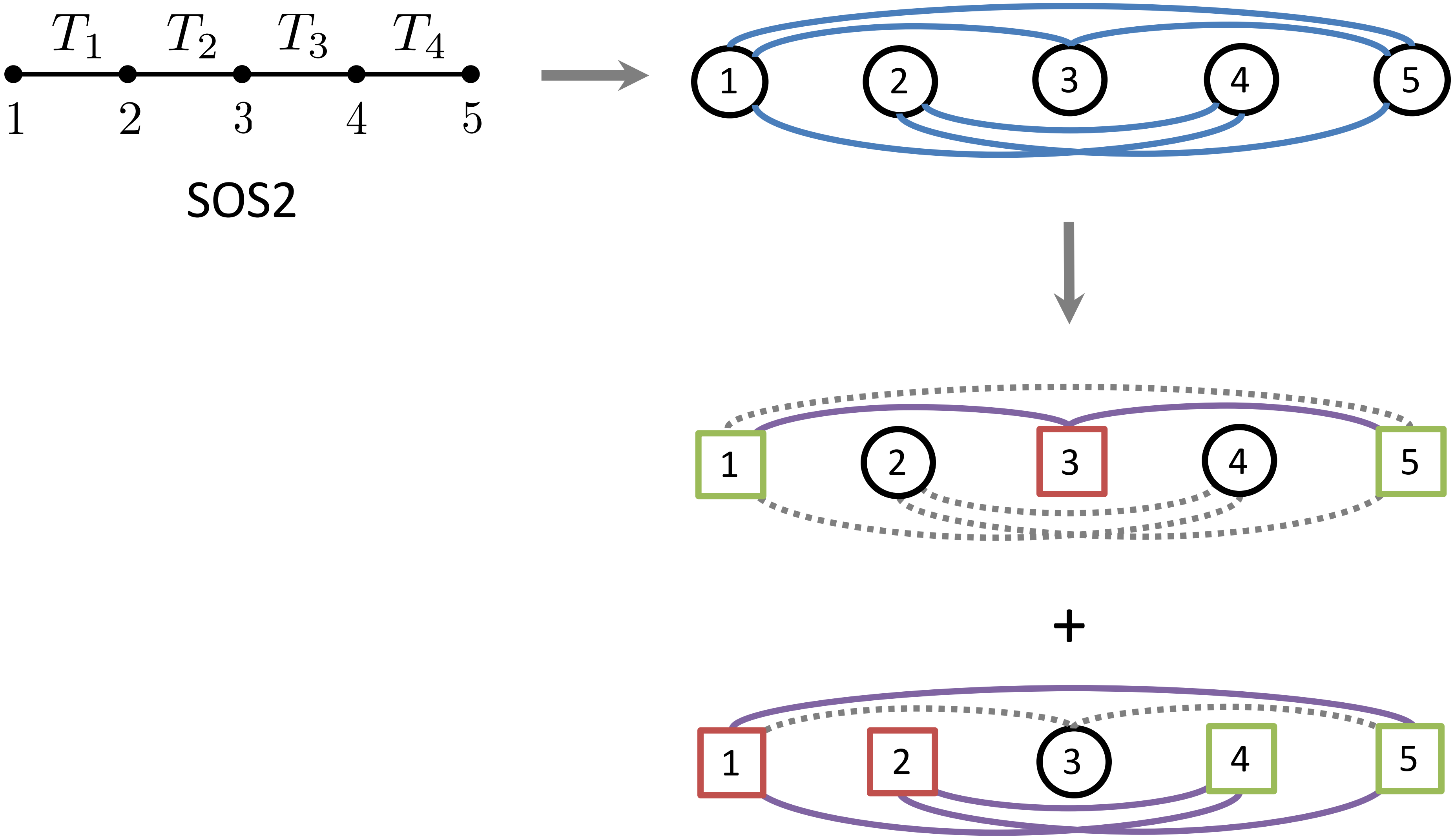




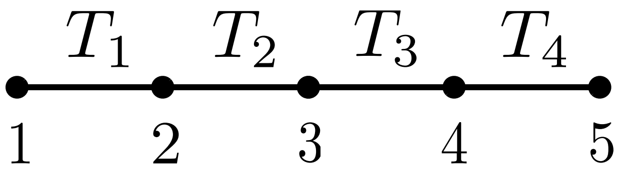
From Cliques to (Complement) Conflict Graph



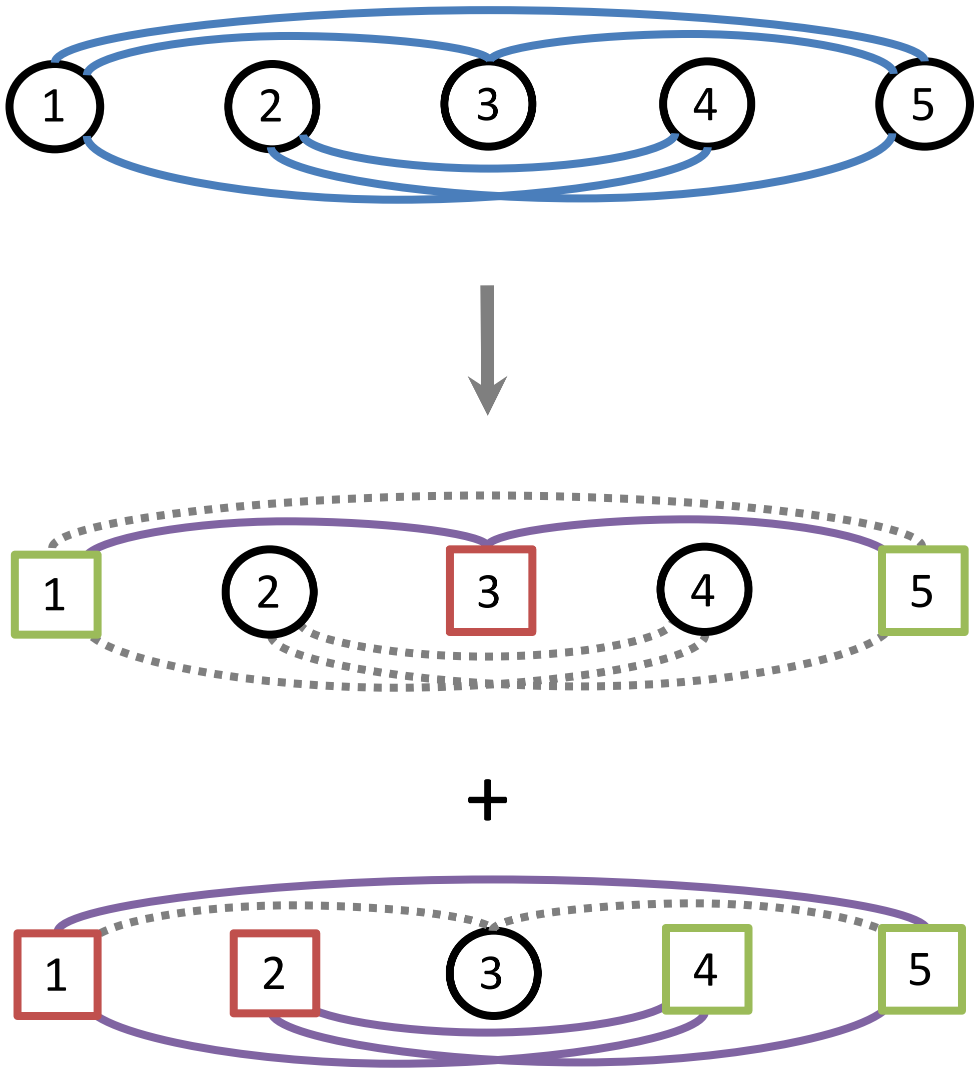
From Conflict Graph to Bi-clique Cover



From Bi-clique Cover to Formulation



SOS2



$$\begin{aligned} 0 &\leq \lambda_1 + \lambda_5 \leq 1 - y_1 \\ 0 &\leq \lambda_3 \qquad\quad\;\; \leq y_1 \\ 0 &\leq \lambda_4 + \lambda_5 \leq 1 - y_2 \\ 0 &\leq \lambda_1 + \lambda_2 \leq y_2 \end{aligned}$$

Ideal Formulation from Bi-clique Cover

- Conflict Graph $G = (V, E)$

$$E = \{(u, v) \ : \ u, v \in V, \ u \neq v, \quad \nexists i \text{ s.t. } \ u, v \in T_i\}$$

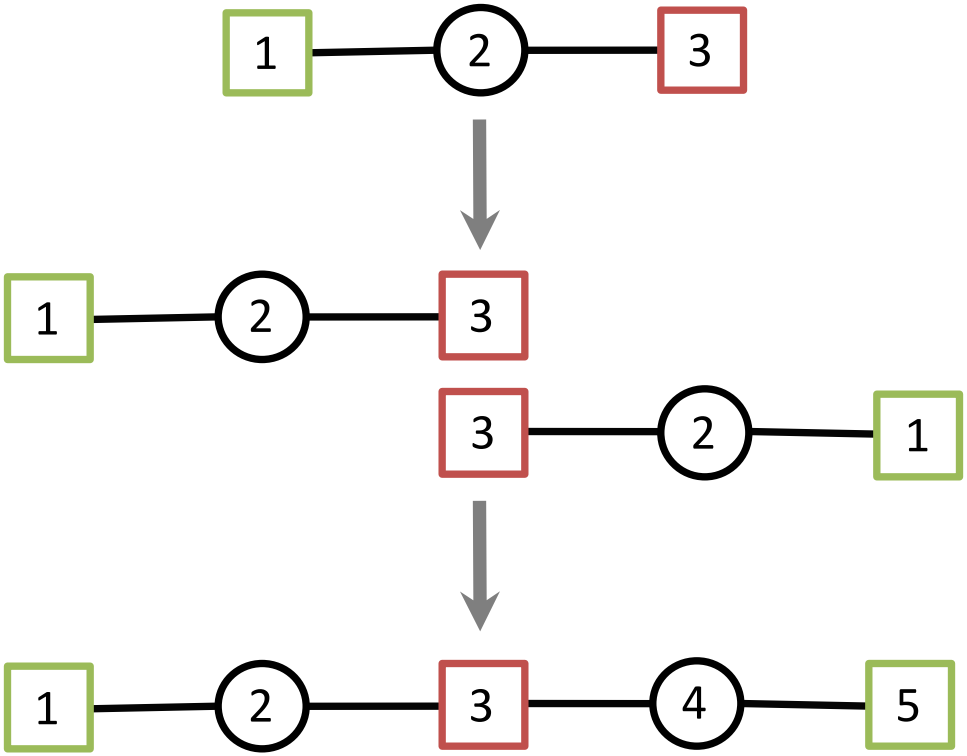
- Bi-clique cover $\left\{\left(A^j, B^j\right)\right\}_{j=1}^{t}, \quad A^j, B^j \subseteq V$

$$\forall \{u, v\} \in E \quad \exists j \text{ s.t. } \ u \in A^j \ \wedge v \in B^j$$

- Formulation

$$\sum_{v \in A^j} \lambda_v \leq 1 - y_j \quad \forall j \in [t]$$
$$\sum_{v \in B^j} \lambda_v \leq y_j \qquad \forall j \in [t]$$
$$y \in \{0, 1\}^t$$

Recursive Construction of Cover for SOS2, Step 1



Base case $n=2^1$:

Step 1 recursion :

Reflect Graph / Cover

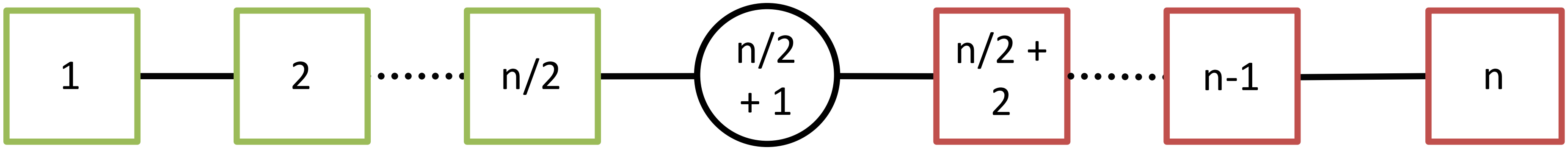
Stick Graph / Cover

Repeat for all bi-cliques from 2^{k-1} to cover all edges within first and last half of conflict graph

Recursive Construction of Cover for SOS2, Step 2

Only edges missing are those between
first and last half of conflict graph

Step 2 : Add one more bi-clique

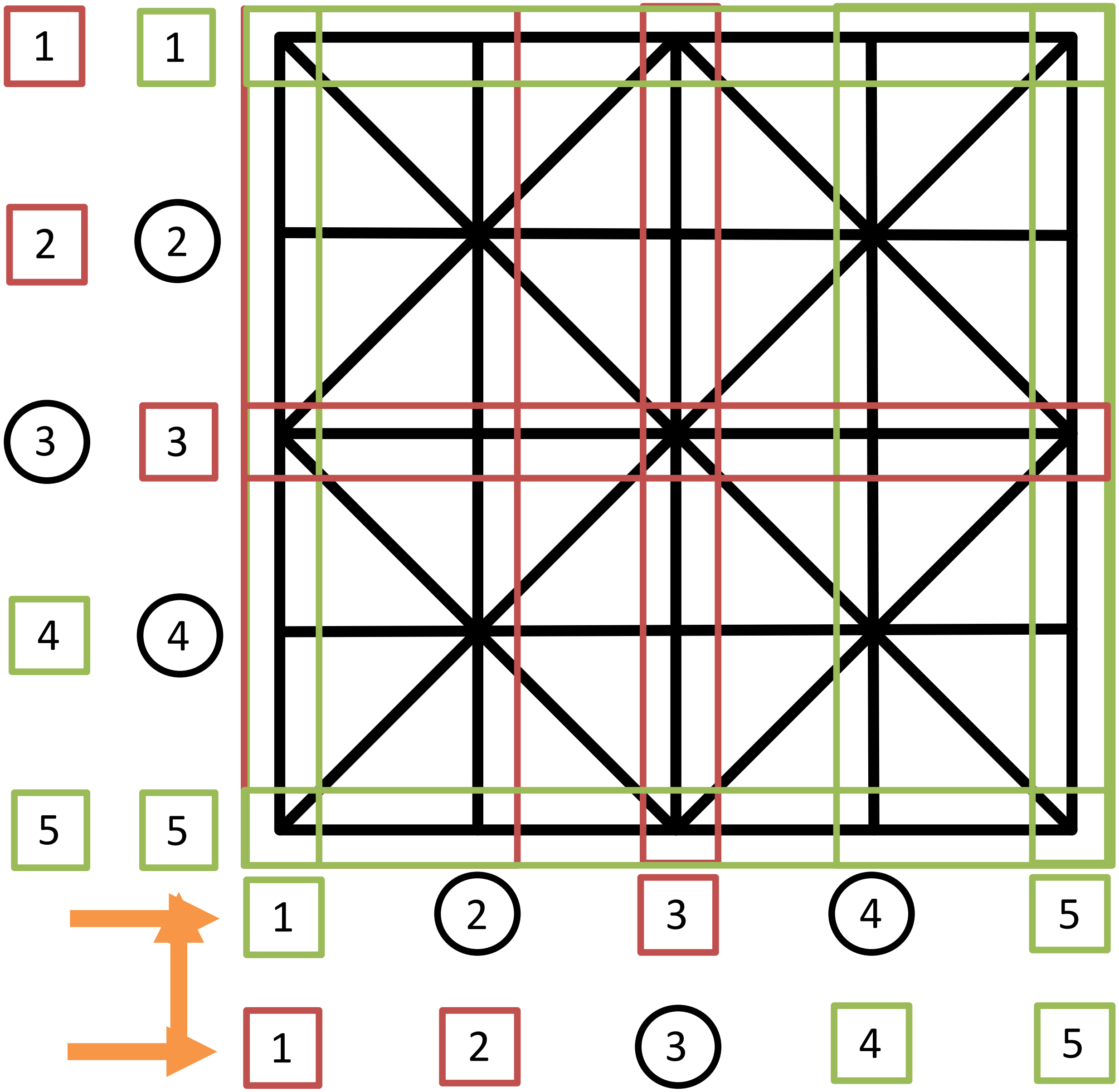


$$\text{Cover has } \log_2 n \text{ bi-cliques.}$$

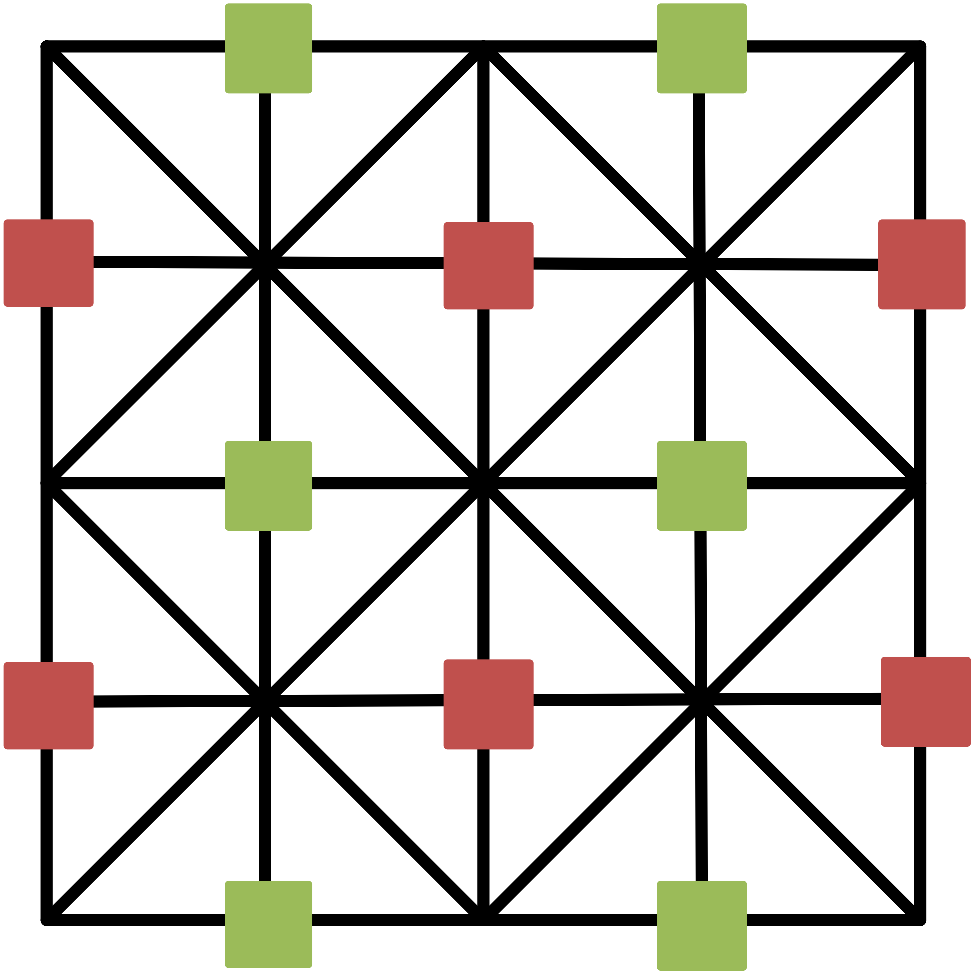
For non-power of two just delete extra nodes.

Grid Triangulations: Step 1 = SOS2 for Inter-Box

Covers all arcs between boxes



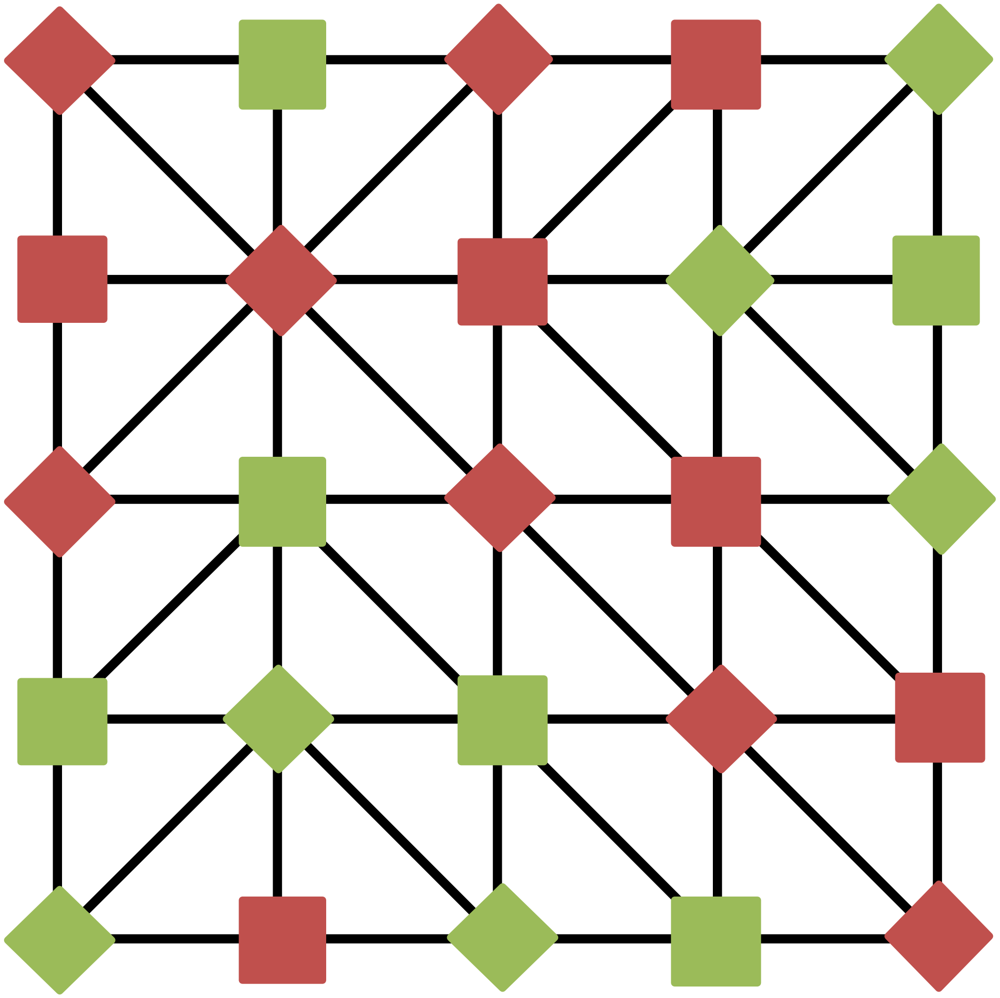
Grid Triangulations: Step 2 = Ad-hoc Intra-Box



Covers all arcs within boxes

Sometimes 1 additional cover

Grid Triangulations: Step 2 = Ad-hoc Intra-Box



Sometimes **2** additional covers

Sometimes more, but always less than **9**

Simple rules to get (near) optimal in Fall ‘16

Summary and Main Messages

- Always choose Chewbacca!

- MIP can solve very challenging problems in practice
- Commercial solvers best, but free solvers reasonable
  - Both easily accessible and integrated into complex systems through the JuMP
- Advanced formulations yield important speed-ups and are (relatively) easy to learn

More Information

- JuMP:
  - Ask Miles and https://github.com/JuliaOpt/JuMP.jl
- MIP Formulations:
  - Mixed integer linear programming formulation techniques. V. SIAM Review 57, 2015. pp. 3-57.
- Advanced Formulation:
  - Small independent branching formulations for unions of V-polyhedra. Joey Huchette and V. 2016. arXiv:1607.04803
- Marketing Application:
  - Ellipsoidal methods for adaptive choice-based conjoint analysis. Denis Saure and V. 2016. http://ssrn.com/abstract=2798984