Incremental Formulations for SOS1 Variables

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Outline

- Introduction
- Encodings
- General Incremental Formulation
- Incremental Formulation and Branching
- Computational Results
- Summary

Logarithmic Formulation for SOS1

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$\sum_{i=1}^{n} b^i \lambda_i = y$$

$$\lambda \in \mathbb{R}^n_+$$

$$y \in \{0,1\}^m$$

$$\{b^i\}_{i=1}^{n} = \{0,1\}^{\log_2 n}$$

- Li and Lu 2009, Adams and Henry 2011, V. and Nemhauser 2008.
- Sommer, TIMS 1972.
- Log = Binary Encoding
- Other choices of $\{b^i\}_{i=1}^{n}$ lead to standard and incremental formulations

General Logarithmic Formulation

$\{P^i\}_{i=1}^k$ polytopes

$$x \in \bigcup_{i=1}^k P^i \Leftrightarrow \begin{aligned} \sum_{i=1}^k \sum_{v \in \text{ext}(P^i)} v\lambda_v^i &= x \\ \sum_{i=1}^k \sum_{v \in \text{ext}(P^i)} \lambda_v^i &= 1 \\ \sum_{i=1}^k \sum_{v \in \text{ext}(P^i)} b^i \lambda_v^i &= y \\ y \in \{0,1\}^{\lceil \log_2(k) \rceil}, \ \lambda_v^i &\geq 0 \end{aligned}$$

- V., Ahmed and Nemhauser 2010; V. 2012.

General Logarithmic Formulation

$\{P^i\}_{i=1}^k$ polytopes

$$x \in \bigcup_{i=1}^k P^i \Leftrightarrow$$

$$\sum_{i=1}^k \sum_{v \in \operatorname{ext}(P^i)} v\lambda_v^i = x$$

$$\sum_{i=1}^k \sum_{v \in \operatorname{ext}(P^i)} \lambda_v^i = 1$$

$$\sum_{i=1}^k \sum_{v \in \operatorname{ext}(P^i)} b^i\lambda_v^i = y$$

$$y \in \{0,1\}^{\lceil \log_2(k) \rceil}, \ \lambda_v^i \geq 0$$

Also for general polyhedron with common recession cones.

- V., Ahmed and Nemhauser 2010; V. 2012.

Unary and Binary Encodings

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \lambda = y \quad \Leftrightarrow \lambda_i = y_i$$

Unary

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \lambda = y$$

Binary

Incremental Encoding

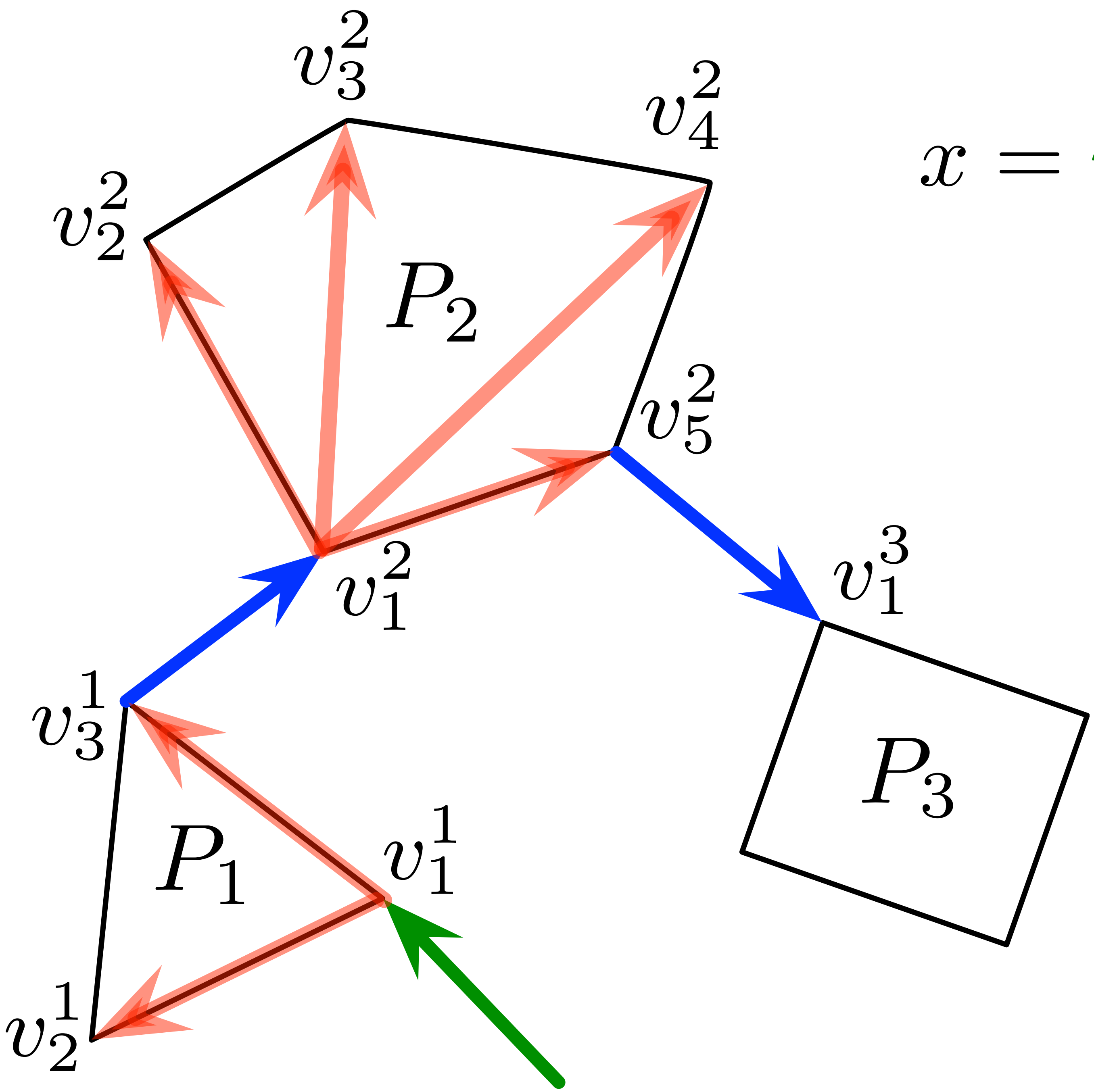
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \lambda = y, \quad \lambda \in \mathbb{R}^8, \; y \in \{0,1\}^7$$

$$\sum_{i=1}^{8} \lambda_i = 1,$$

$$y_1 \geq y_2 \geq \ldots \geq y_7$$

- Linear transformation of λ-formulation gives generalization of incremental δ-formulation of Lee and Wilson 1999.

Incremental "Delta" Formulation



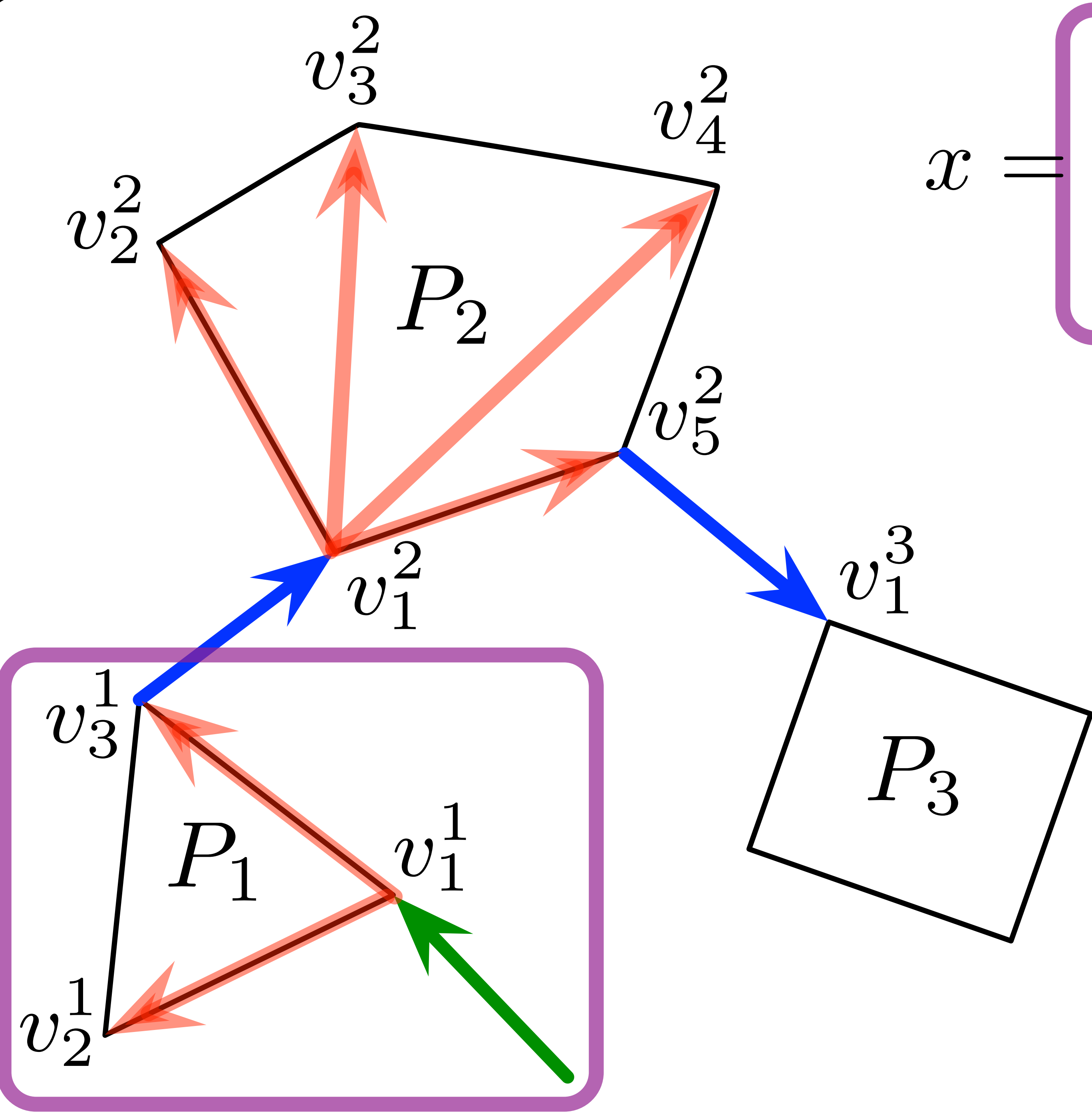
$$x = v_1^1 + \sum_{i=1}^{k} \sum_{j=2}^{r_i} \left(v_j^i - v_1^i\right) \delta_j^i$$

$$+ \sum_{i=2}^{k} \left(v_1^i - v_{r_i}^{i-1}\right) y_i$$

$$\delta_j^i \in [0, 1], \; y_i \in \{0, 1\}$$

- Yıldız and V. 2012

Incremental “Delta” Formulation



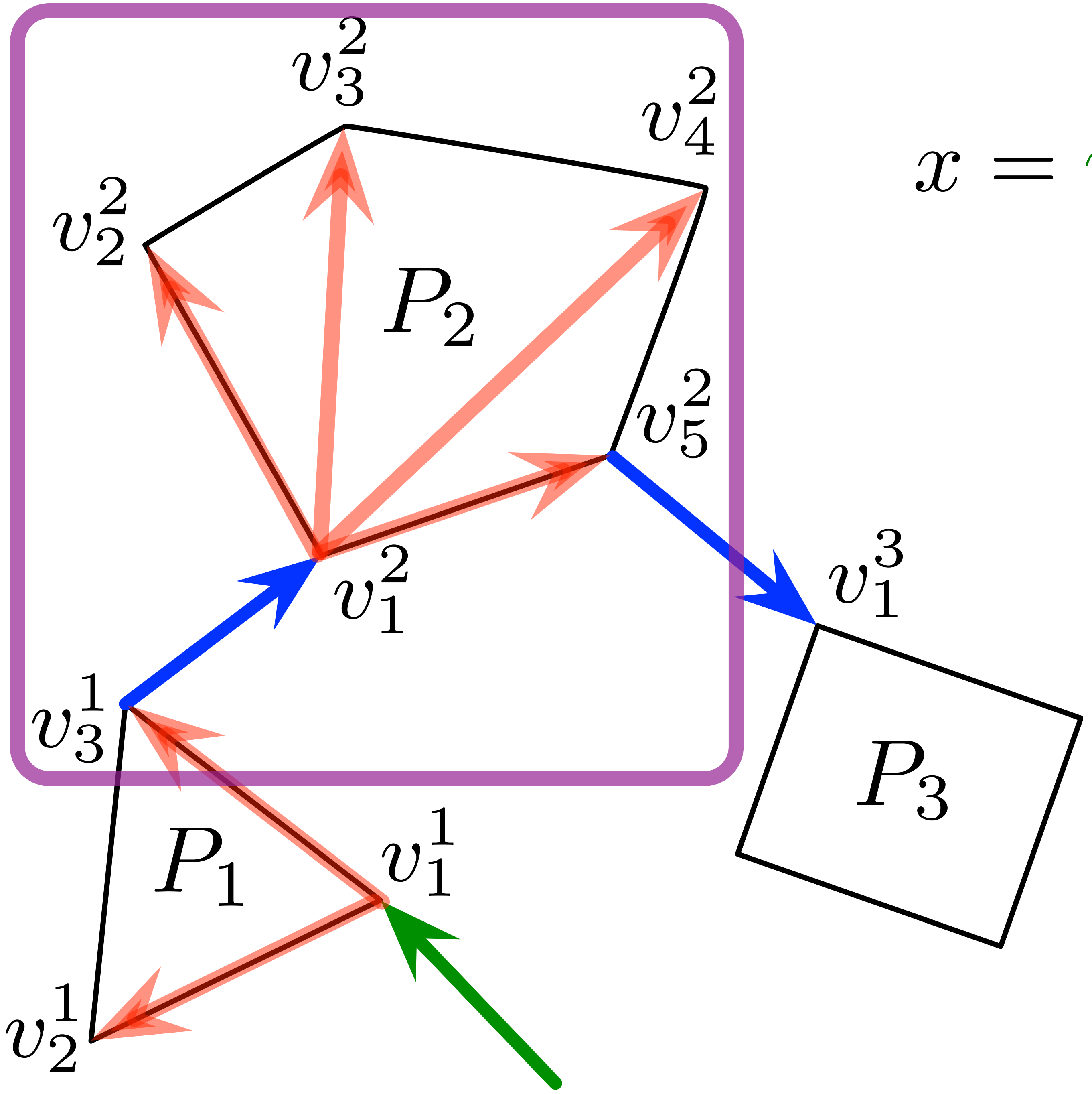
$$x = v_1^1 + \sum_{i=1}^{k} \sum_{j=2}^{r_i} \left(v_j^i - v_1^i\right) \delta_j^i$$

$$+ \sum_{i=2}^{k} \left(v_1^i - v_{r_i}^{i-1}\right) y_i$$

$$\sum_{j=2}^{r_i} \delta_j^i \leq 1, \quad \delta_j^i \geq 0$$

- Yıldız and V. 2012

Incremental “Delta” Formulation

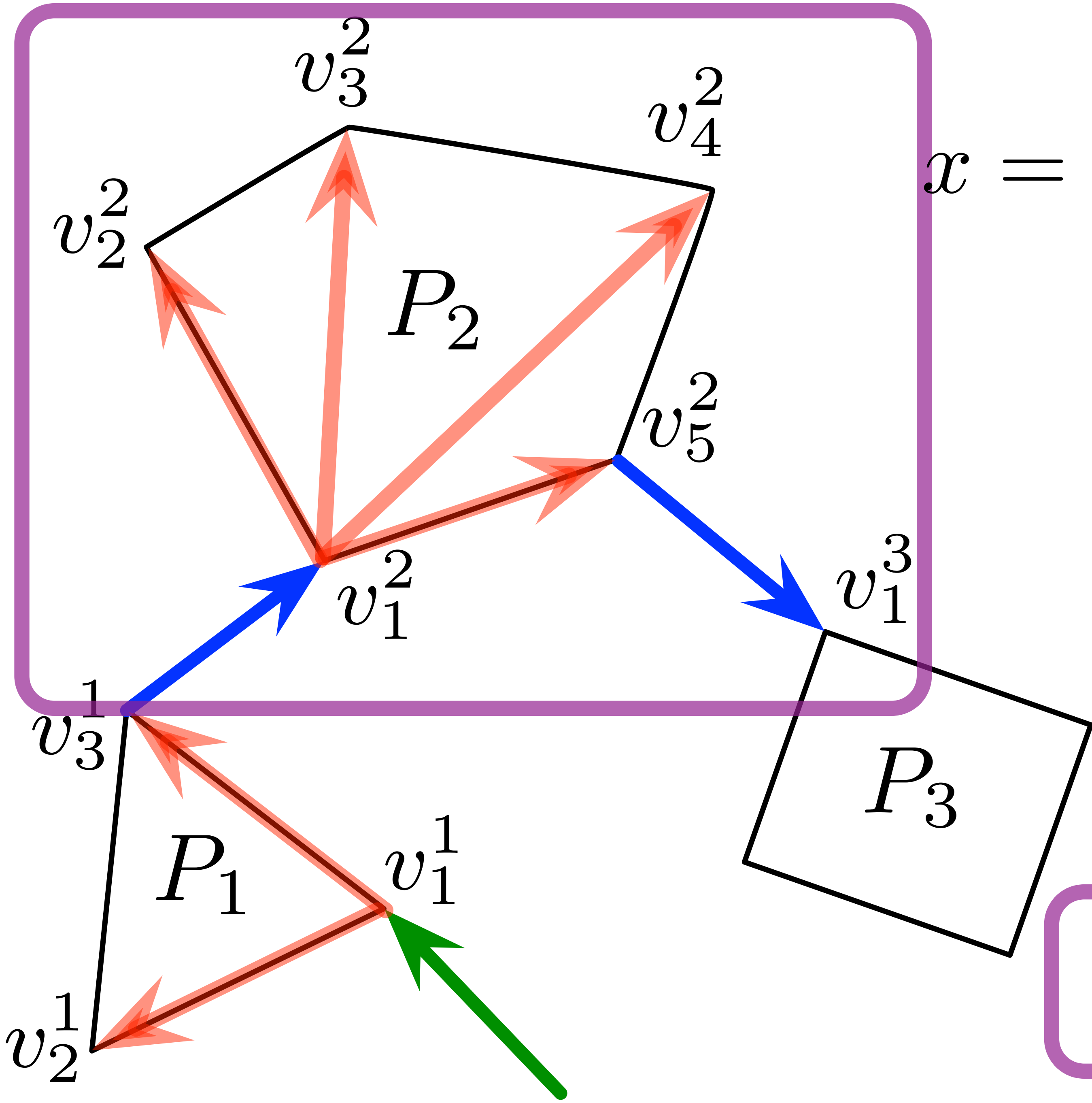


$$x = v_1^1 + \sum_{i=1}^{k} \sum_{j=2}^{r_i} \left(v_j^i - v_1^i\right) \delta_j^i + \sum_{i=2}^{k} \left(v_1^i - v_{r_i}^{i-1}\right) y_i$$

$$\sum_{j=2}^{r_i} \delta_j^i \leq y_i$$

- Yıldız and V. 2012

Incremental “Delta” Formulation



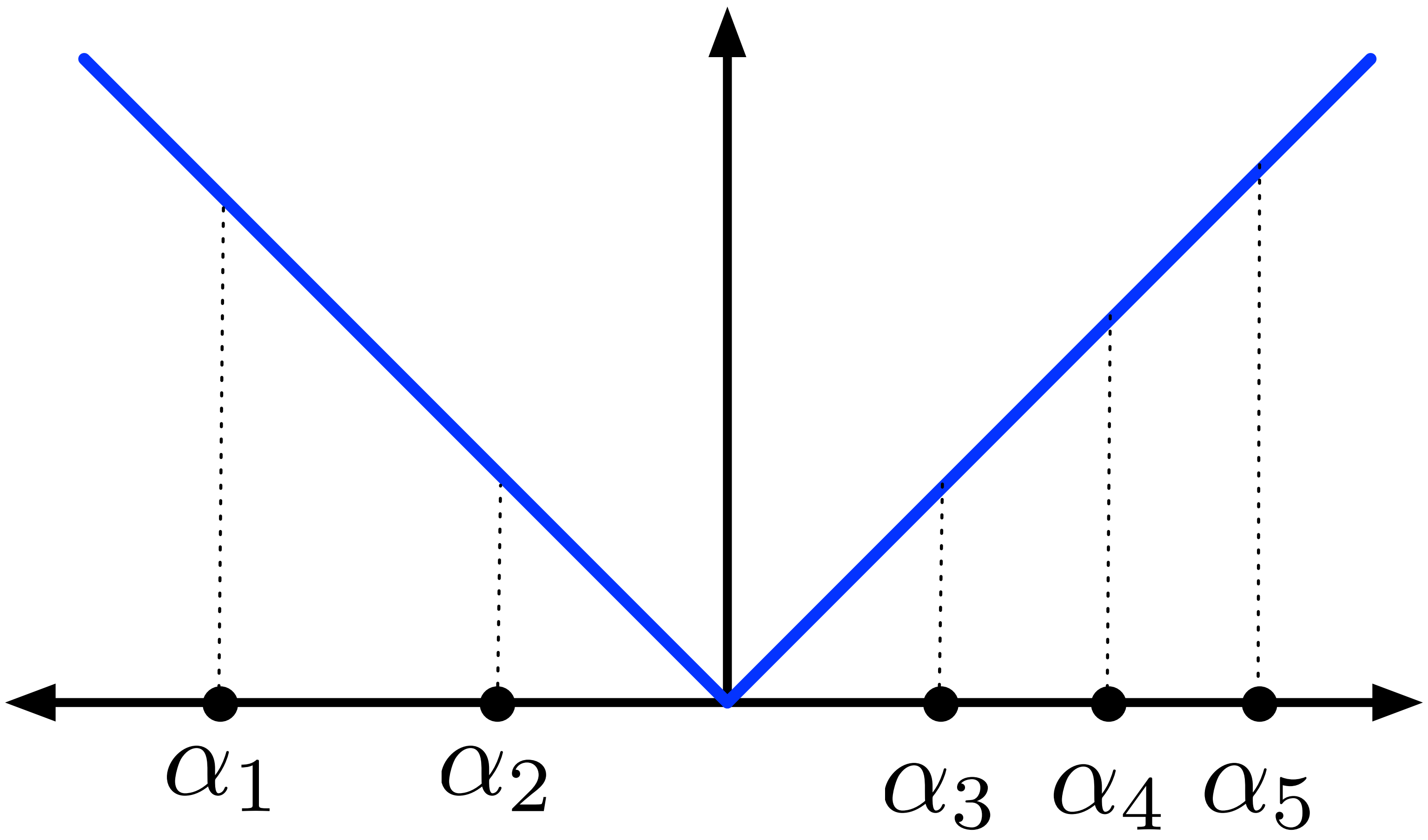
$$x = v_1^1 + \sum_{i=1}^{k} \sum_{j=2}^{r_i} \left(v_j^i - v_1^i\right) \delta_j^i + \sum_{i=2}^{k} \left(v_1^i - v_{r_i}^{i-1}\right) y_i$$

$$y_{i+1} \le \delta_{r_i}^i, \quad y_{i+1} \le y_i$$

- Yıldız and V. 2012

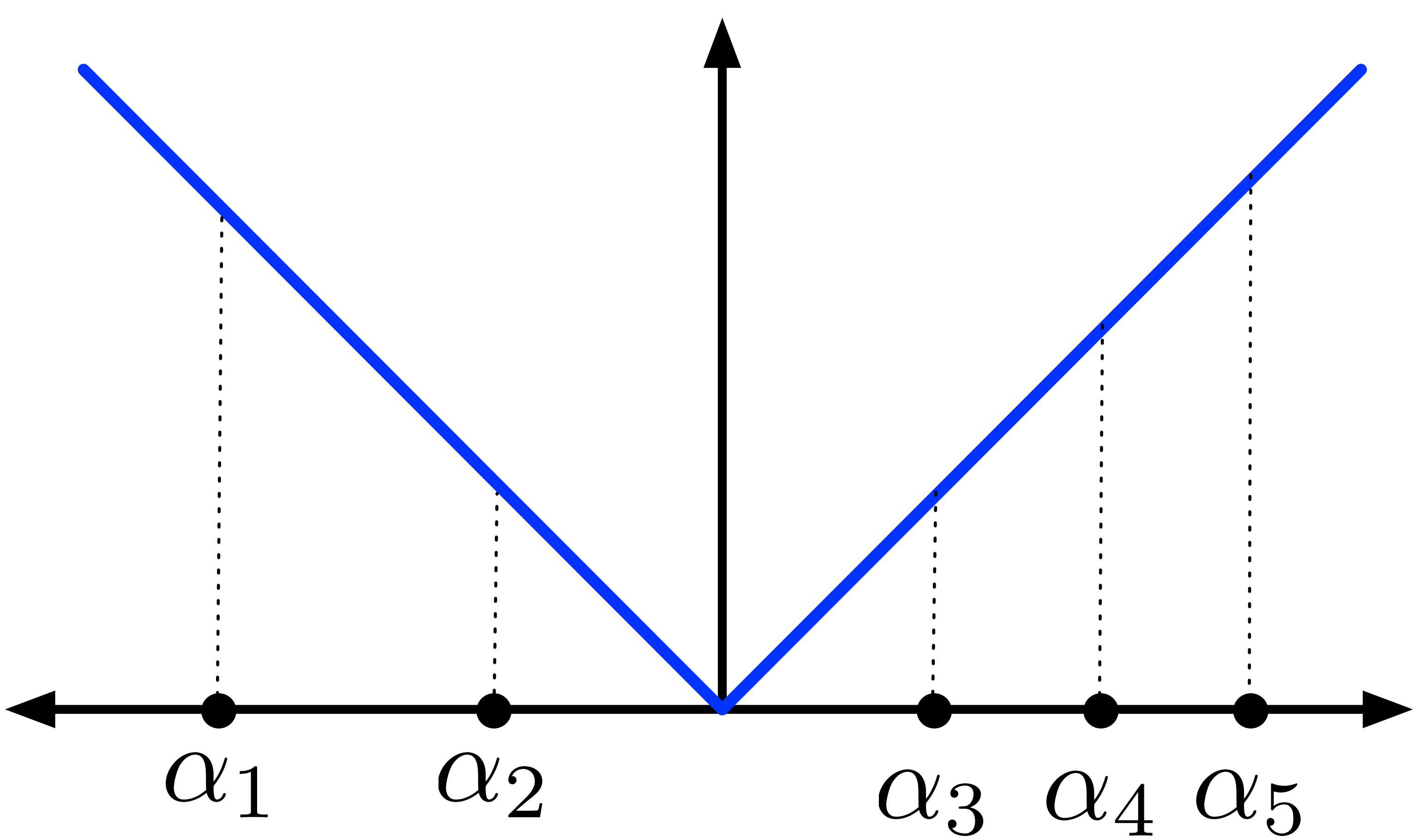
Example: # of B & B Nodes

$$\min \quad |x|$$
$$s.t.$$
$$x \in \{\alpha_i\}_{i=1}^n$$



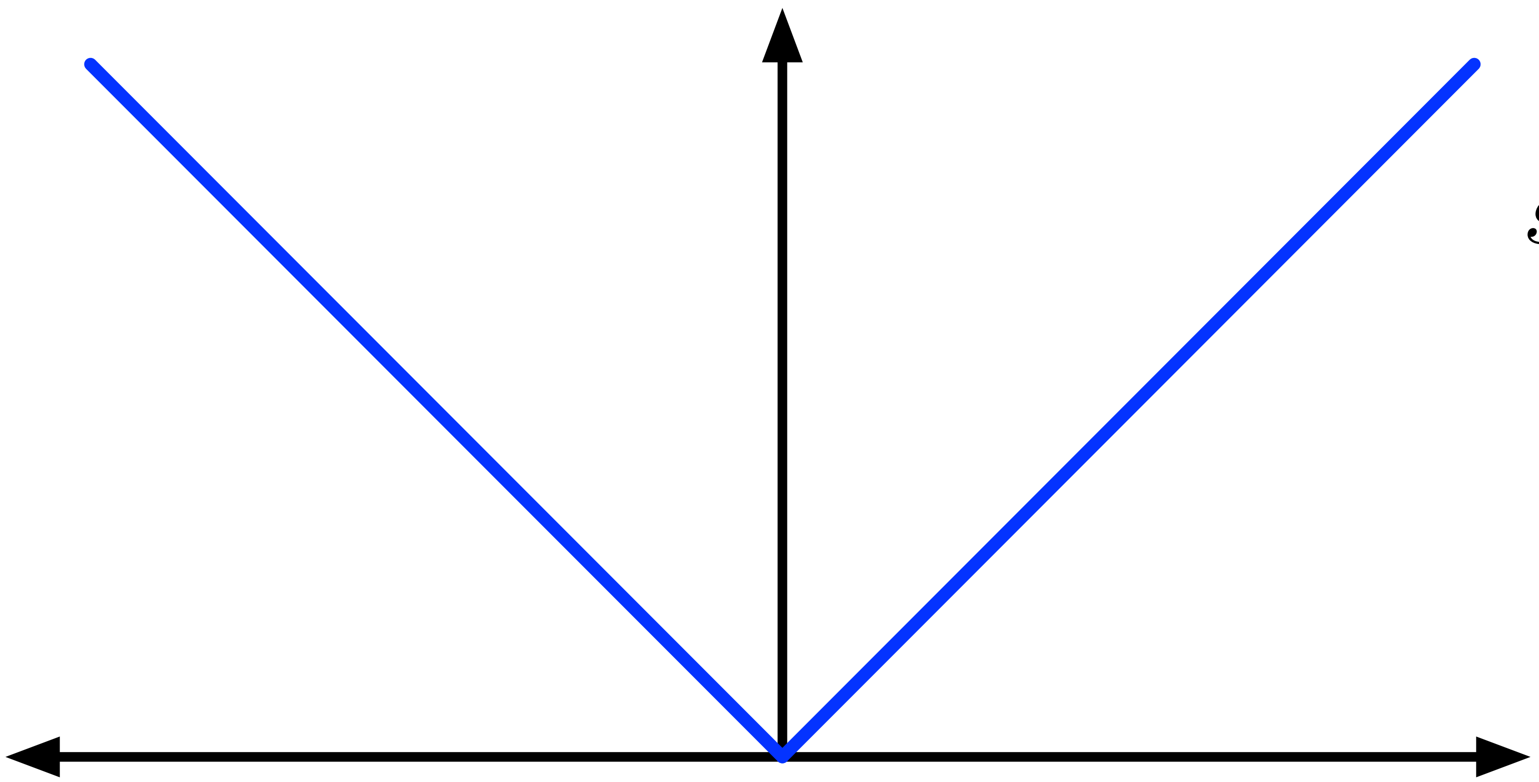
Example: # of B & B Nodes

$$\begin{aligned} \min \quad & |x| \\ s.t. \quad & \\ & x \in \{\alpha_i\}_{i=1}^n \end{aligned}$$



$$\begin{aligned} \min \quad & t \\ s.t. \quad & \\ & \sum_{i=1}^n \lambda_i \alpha_i \leq t \\ & -\sum_{i=1}^n \lambda_i \alpha_i \leq t \\ & \sum_{i=1}^n \lambda_i = 1 \\ & \sum_{i=1}^n b^i \lambda_i = y \\ & \lambda \in \mathbb{R}_+^n \\ & y \in \{0,1\}^m \end{aligned}$$

Example: Unary Encoding



$$
\begin{aligned}
\min \quad & t \\
s.t. \quad & \\
& \sum_{i=1}^{n} \lambda_i \alpha_i \leq t \\
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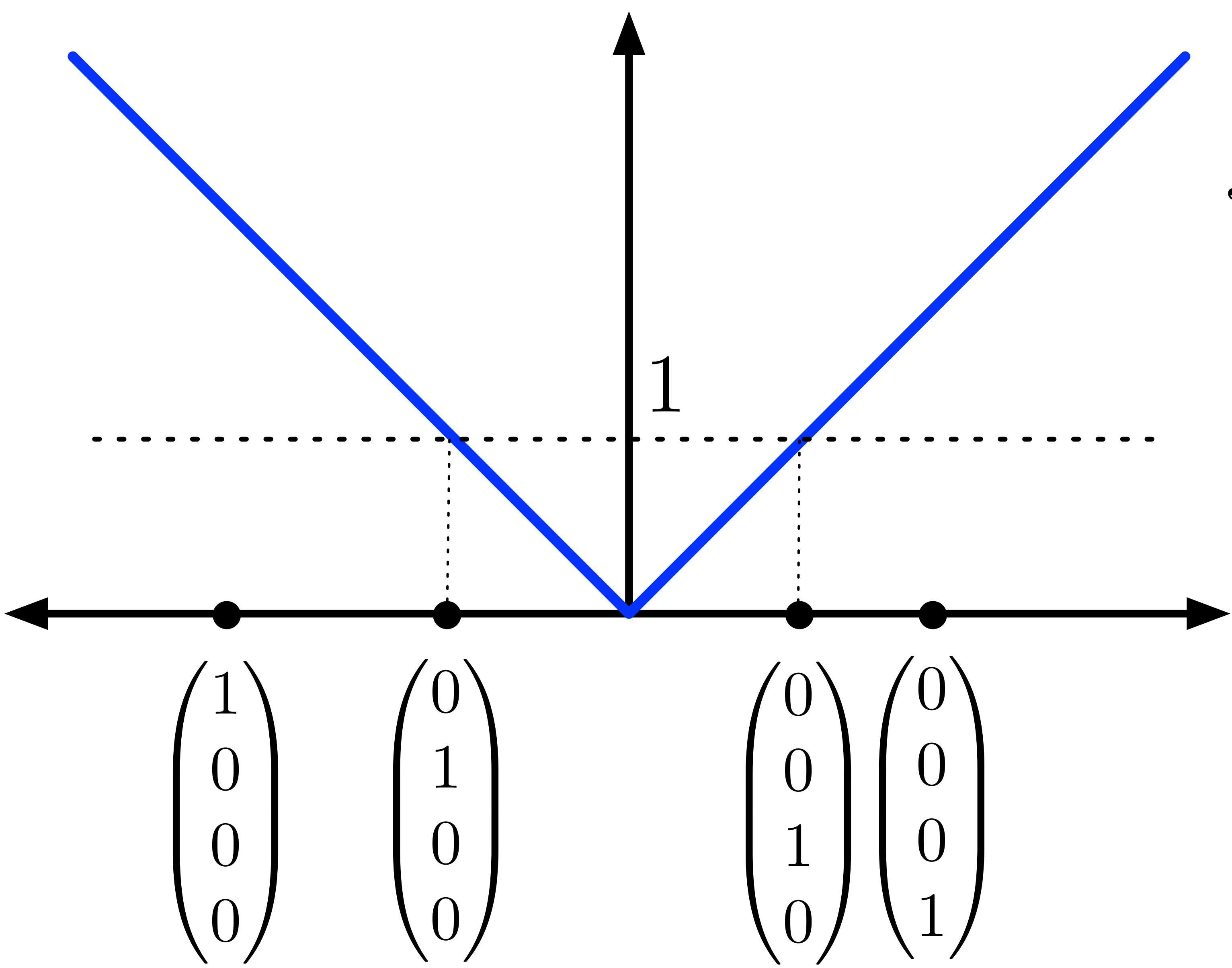
& \sum_{i=1}^{n} b^i \lambda_i = y \\
& \lambda \in \mathbb{R}^n_+ \\
& y \in \{0,1\}^m
\end{aligned}
$$

Example: Unary Encoding



$$\begin{aligned}
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s.t. \quad & \\
& \sum_{i=1}^{n} \lambda_i \alpha_i \leq t \\
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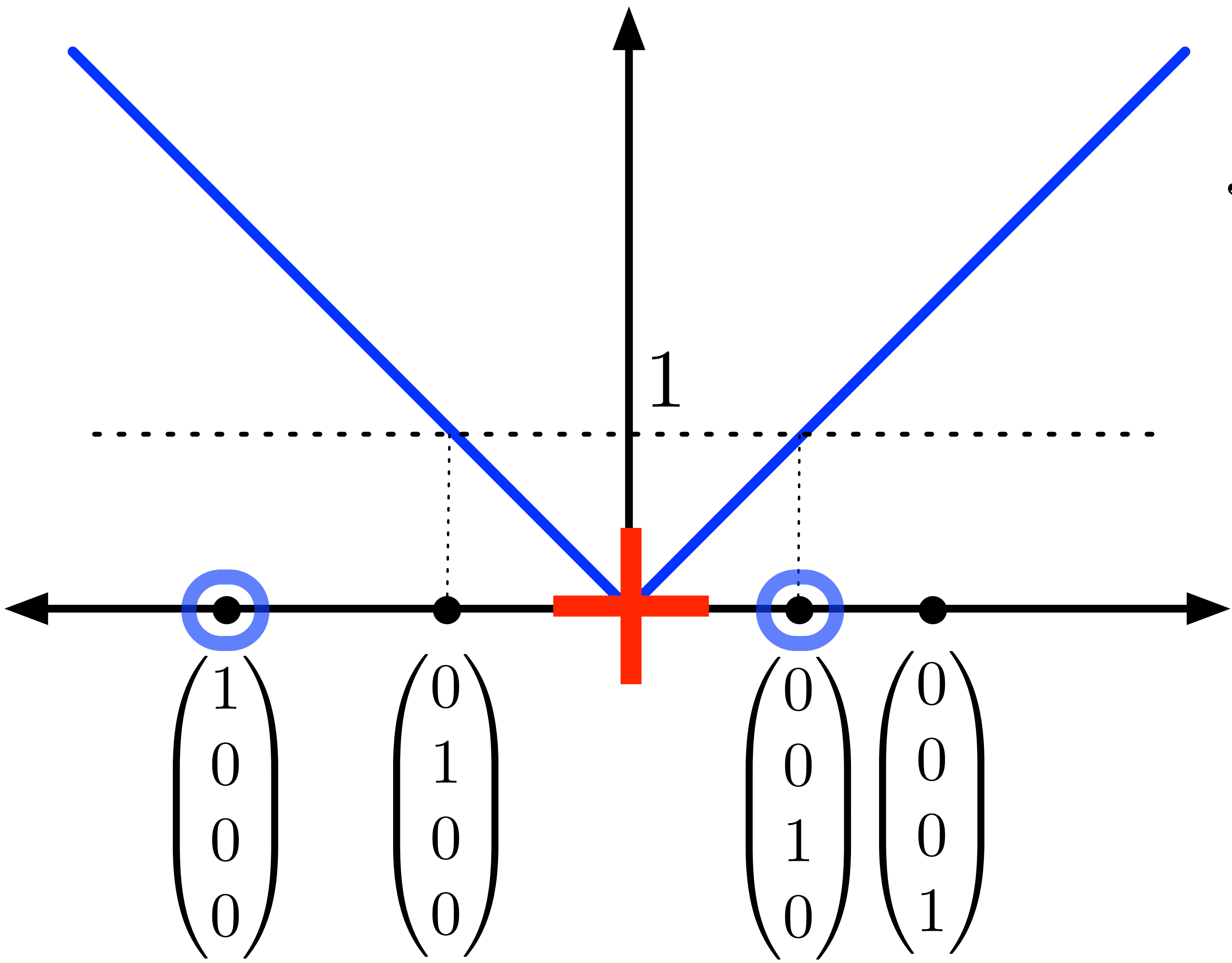
$t^* = 1, \; t_{LP} = 0$

Example: Unary Encoding



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\end{aligned}
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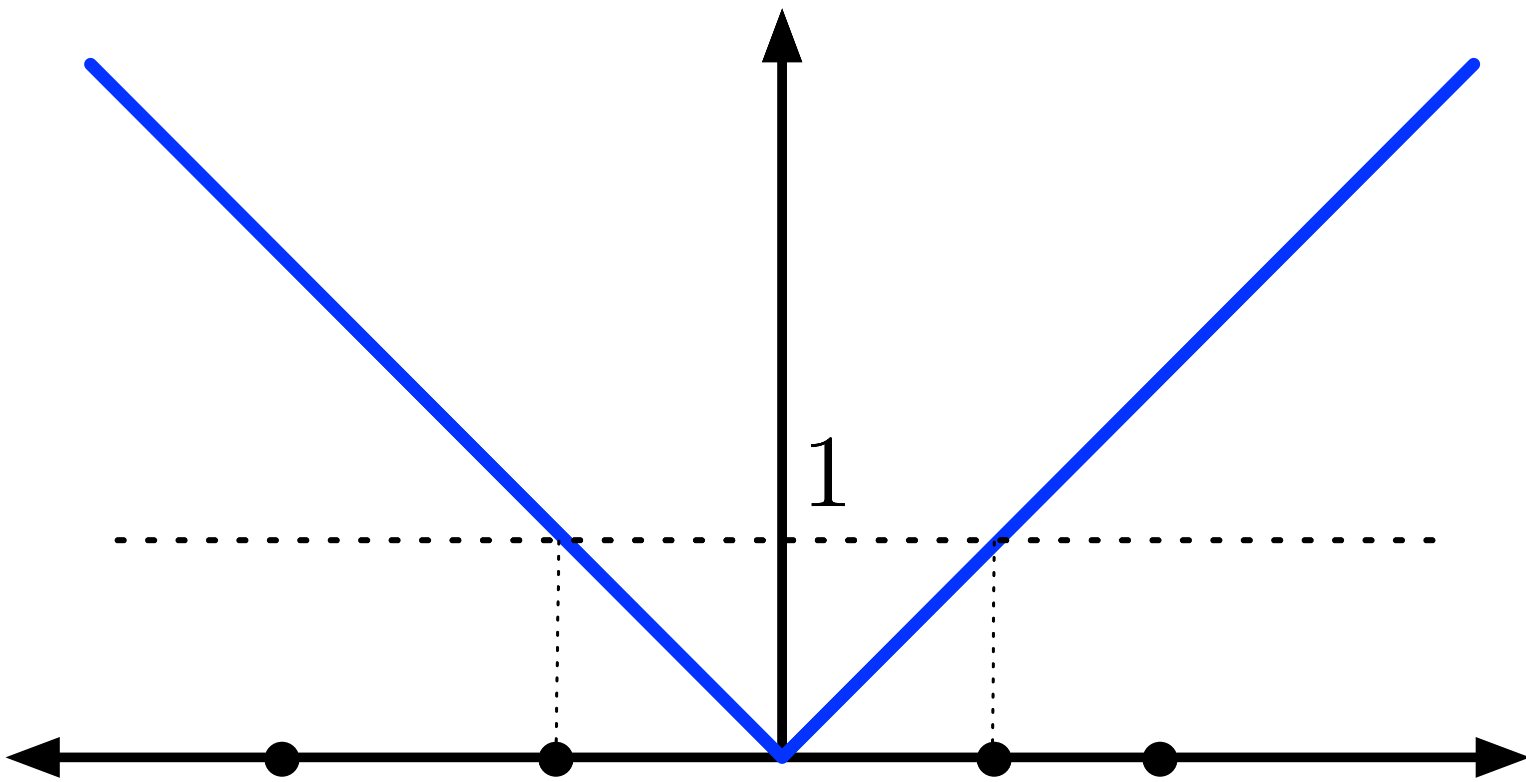

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}$$

$$\begin{aligned}
\min \quad & t \\
s.t. \quad & \sum_{i=1}^{n} \lambda_i \alpha_i \leq t \\
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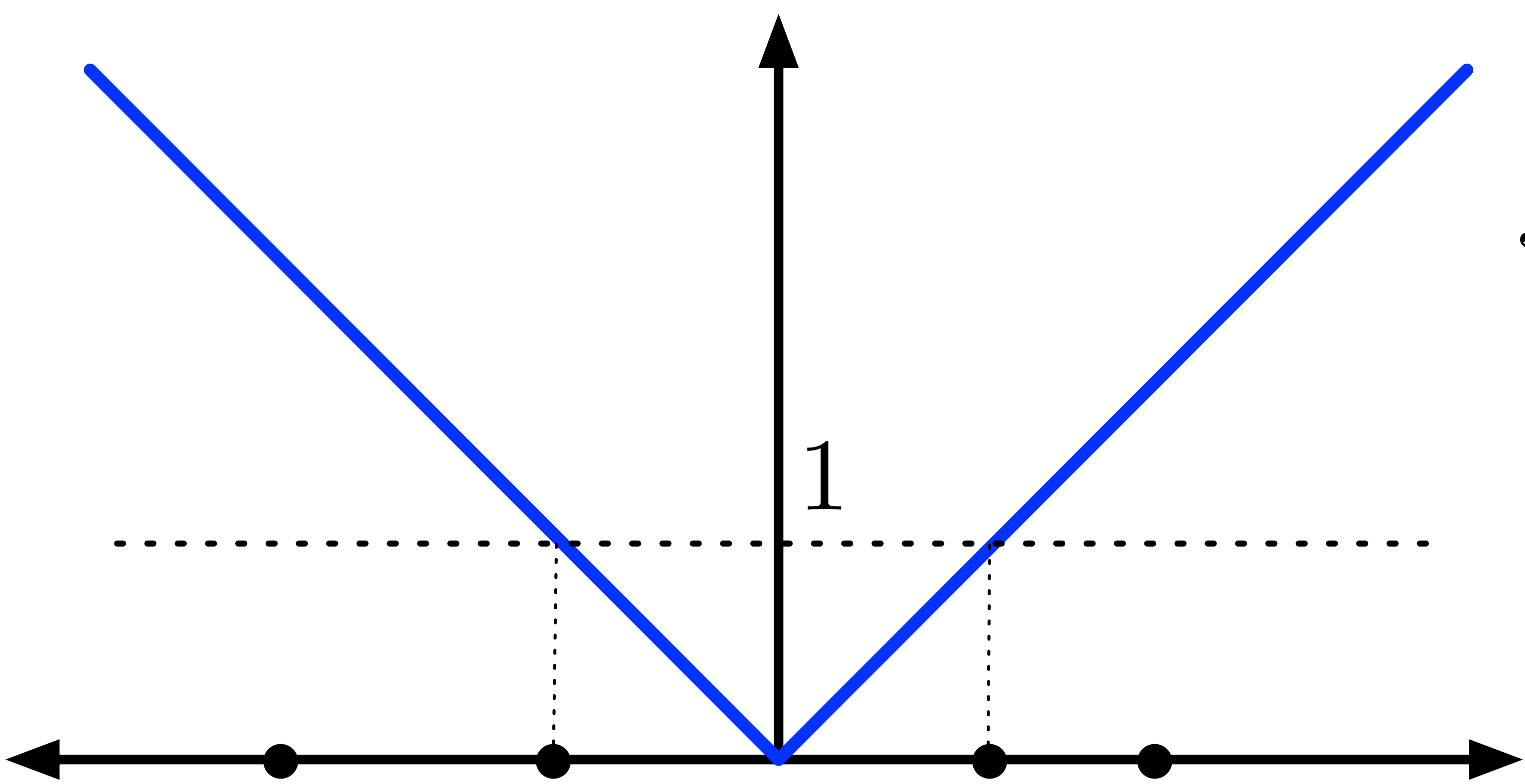

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}$$

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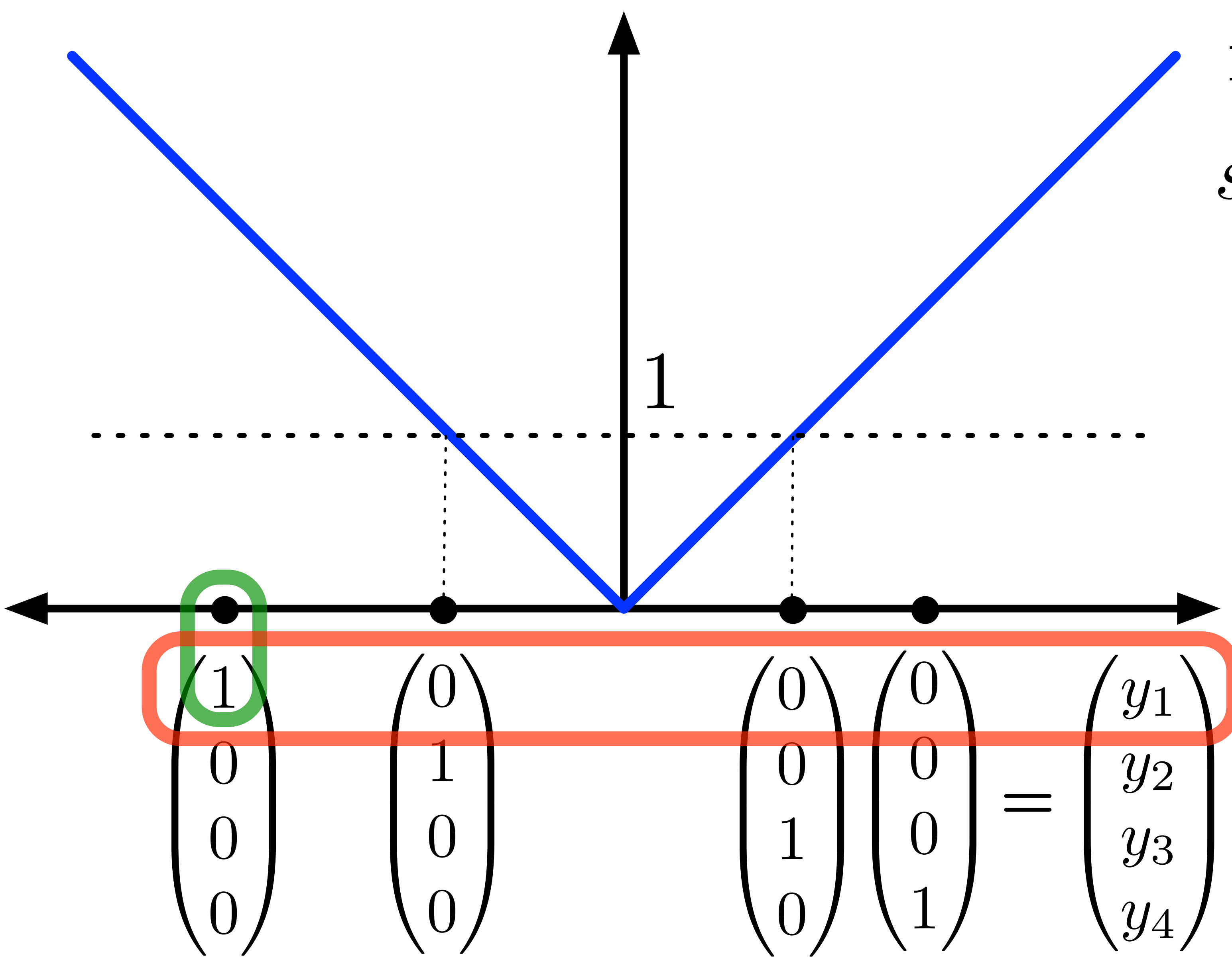
$t^* = 1,\ t_{LP} = 0 \qquad y_1 =$

Example: Unary Encoding



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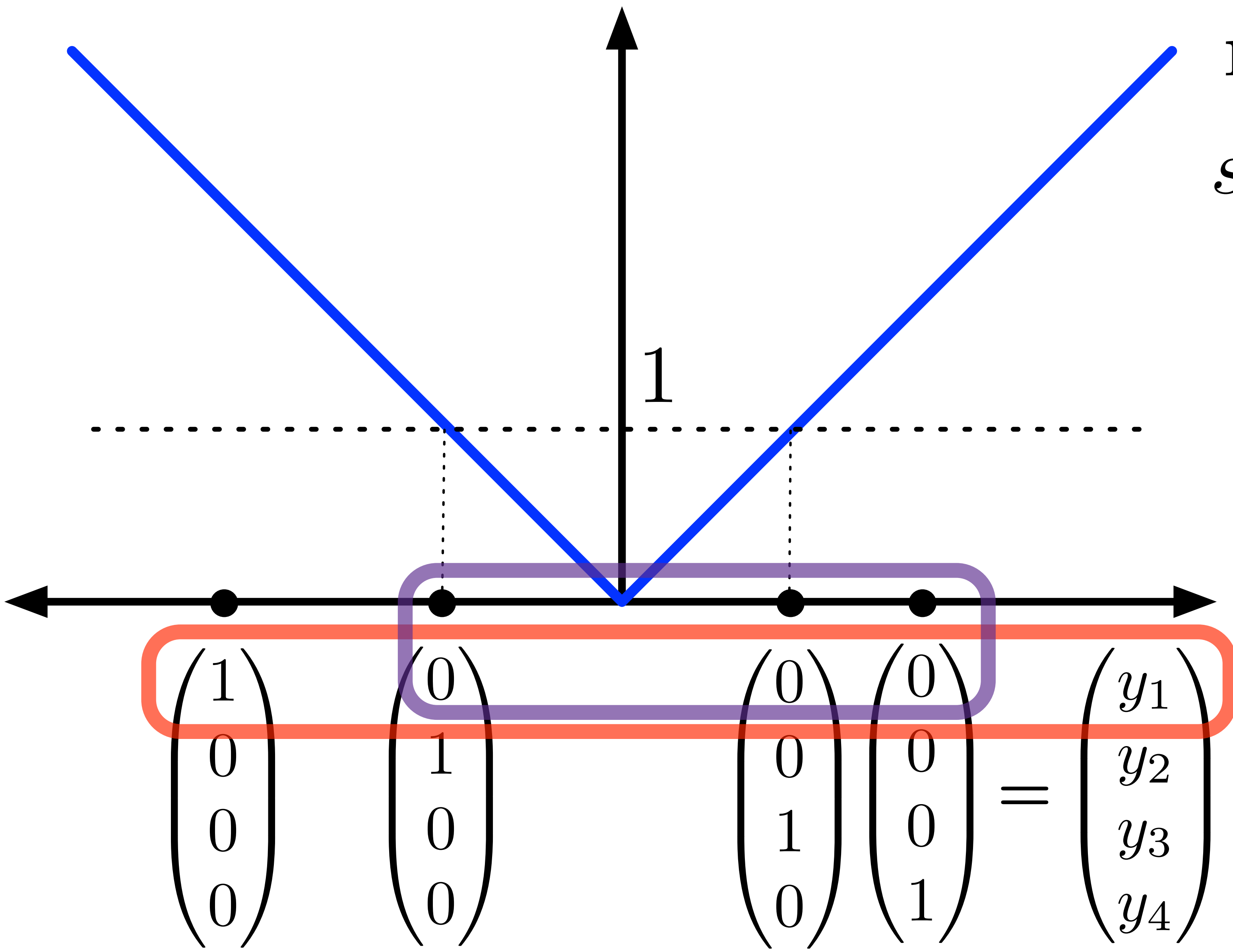
$t^* = 1,\ t_{LP} = 0 \qquad y_1 = 1$

Example: Unary Encoding



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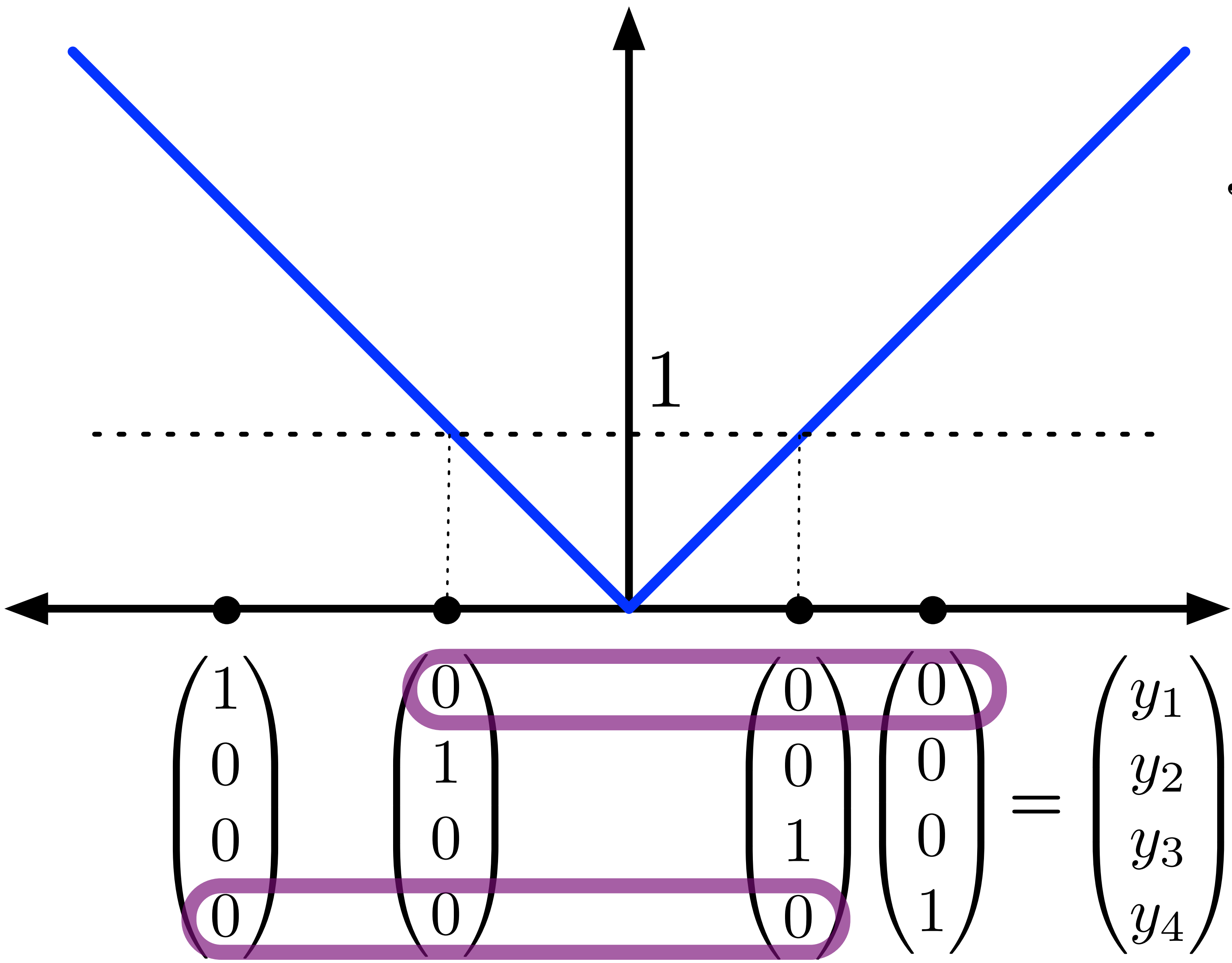
$t^* = 1,\ t_{LP} = 0 \qquad y_1 = 0$

Example: Unary Encoding



$$\min \quad t$$

$$s.t.$$

$$\sum_{i=1}^{n} \lambda_i \alpha_i \leq t$$

$$-\sum_{i=1}^{n} \lambda_i \alpha_i \leq t$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

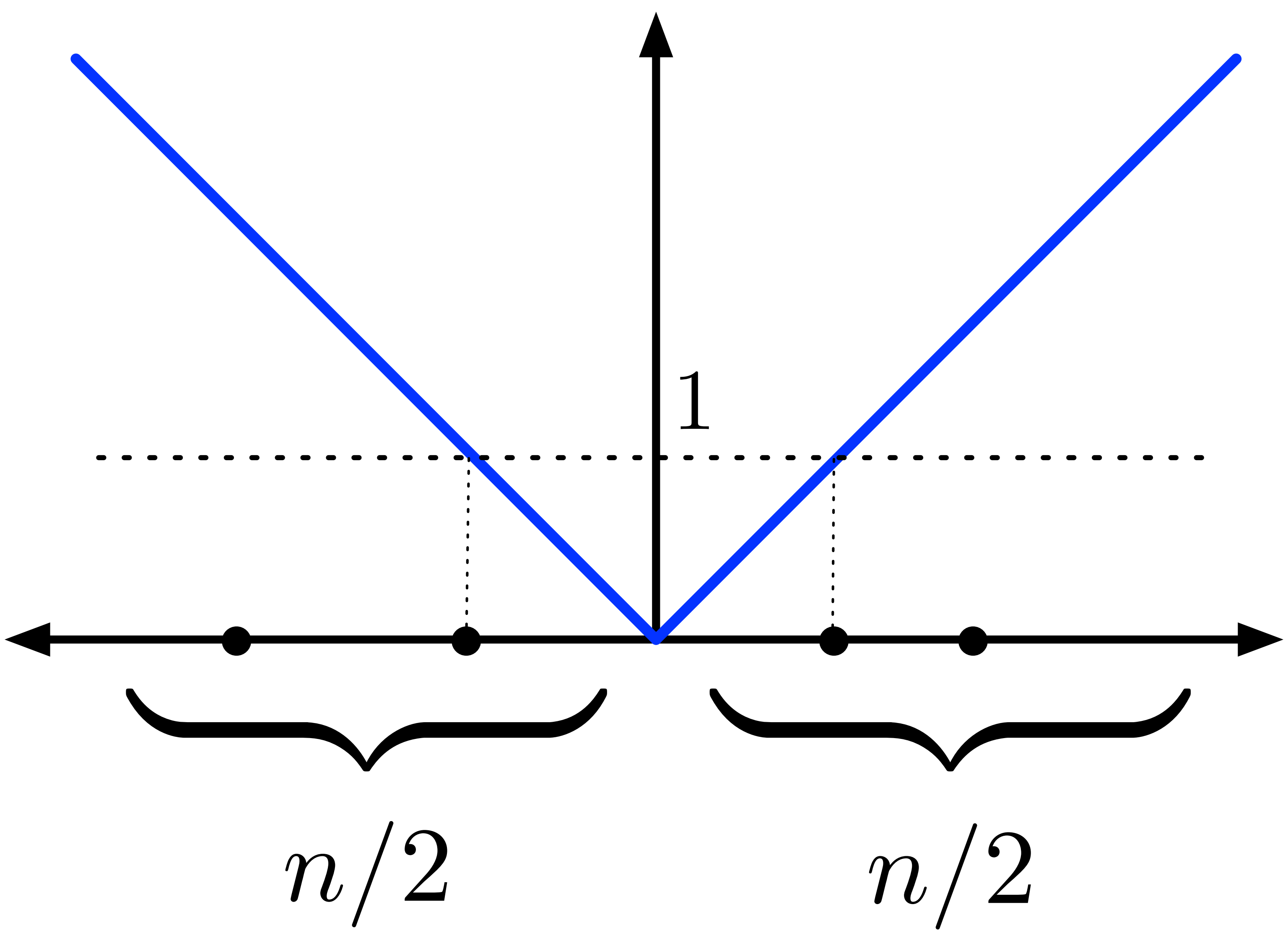
$$\sum_{i=1}^{n} b^i \lambda_i = y$$

$$\lambda \in \mathbb{R}^n_+$$

$$y \in \{0,1\}^m$$

$t^* = 1,\ t_{LP} = 0$ $\quad y_1 = y_4 = 0$

Example: Unary Encoding



$t_{LP} = 0$ unless:

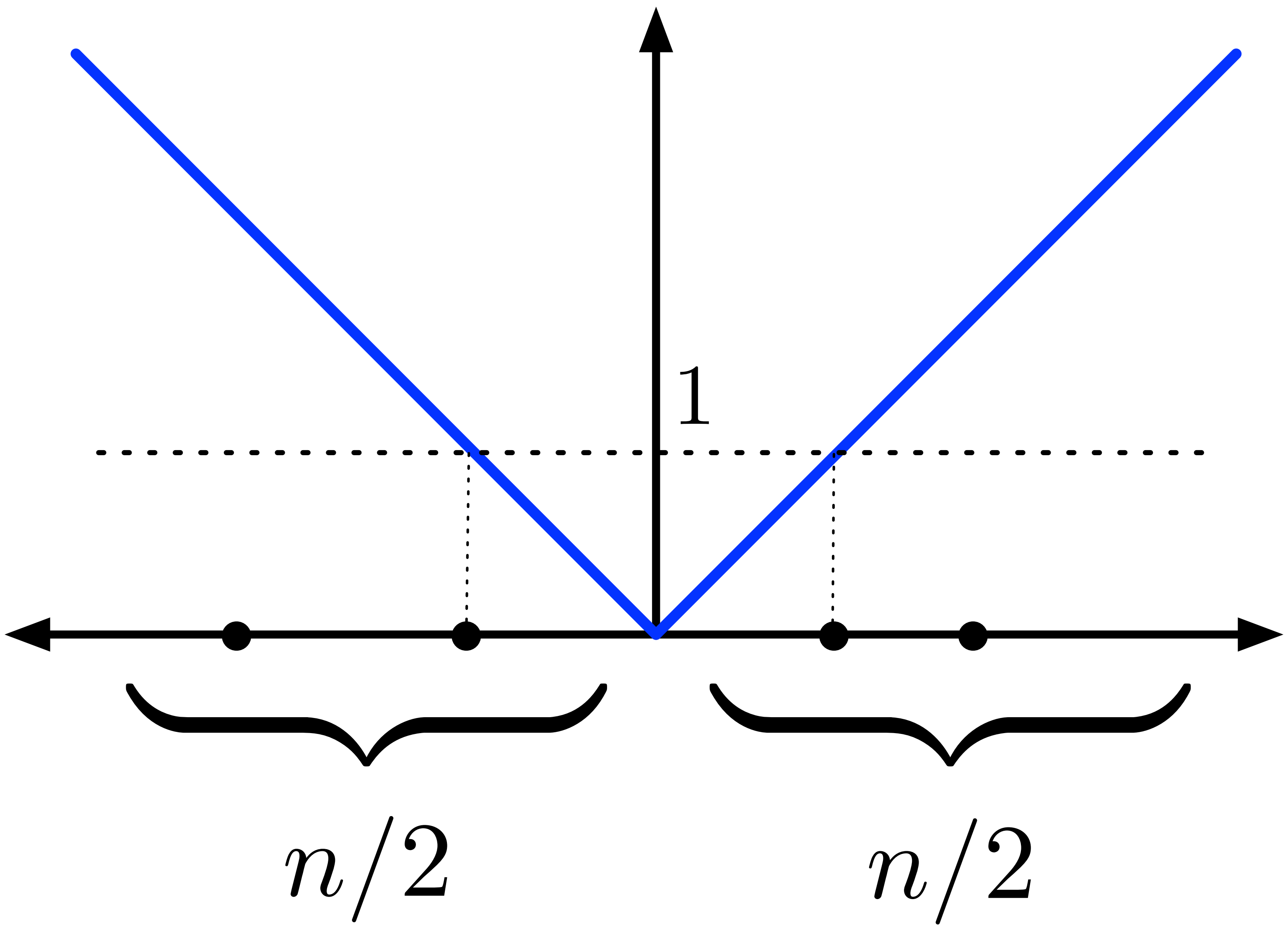
$$y_i = 0 \quad \forall i \leq n/2$$

or

$$y_i = 0 \quad \forall i \geq n/2$$

$t^* = 1,\ t_{LP} = 0$

Example: Unary Encoding



$t_{LP} = 0$ unless:

$$y_i = 0 \quad \forall i \leq n/2$$

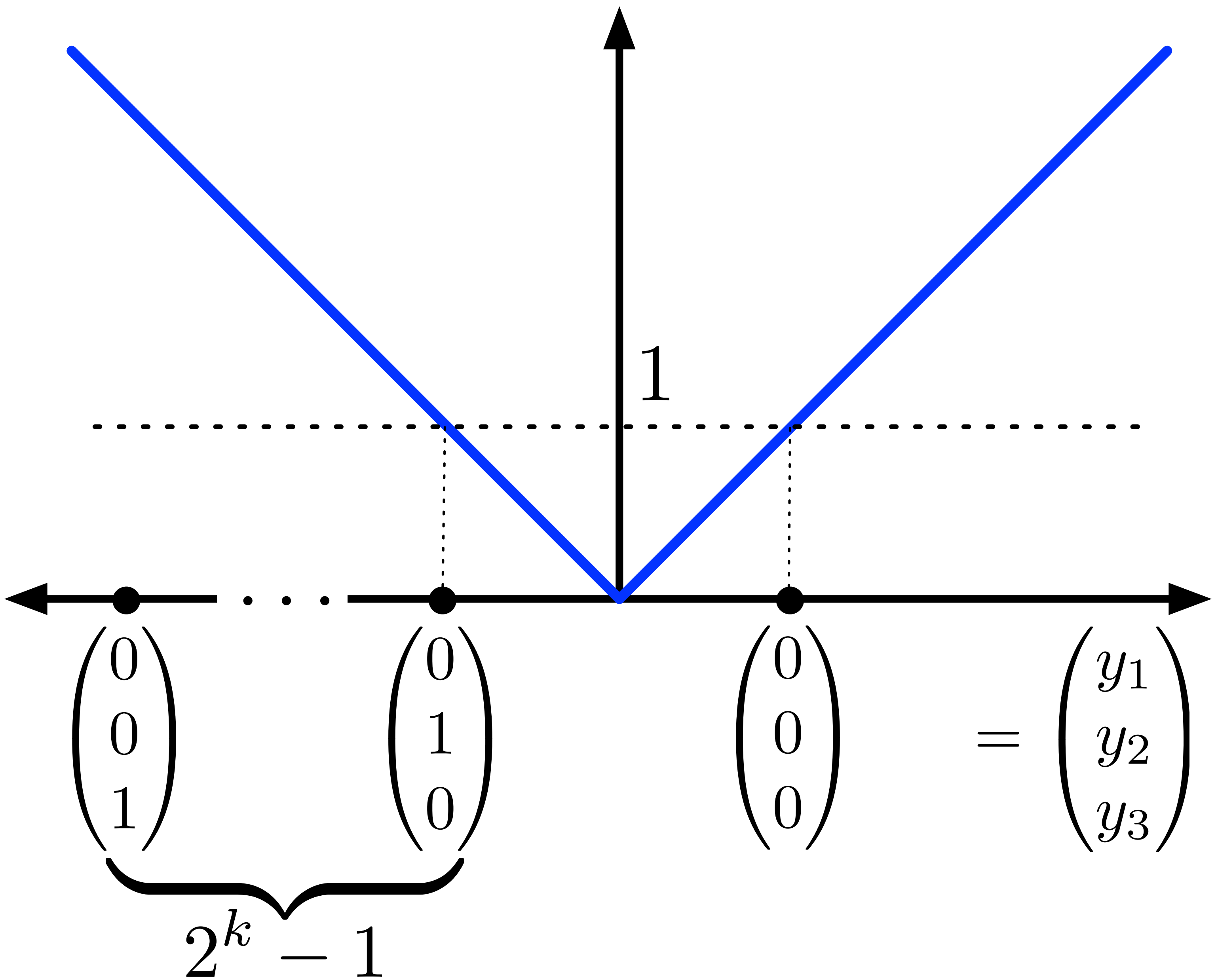
or

$$y_i = 0 \quad \forall i \geq n/2$$

Need $n/2$ branches to solve.

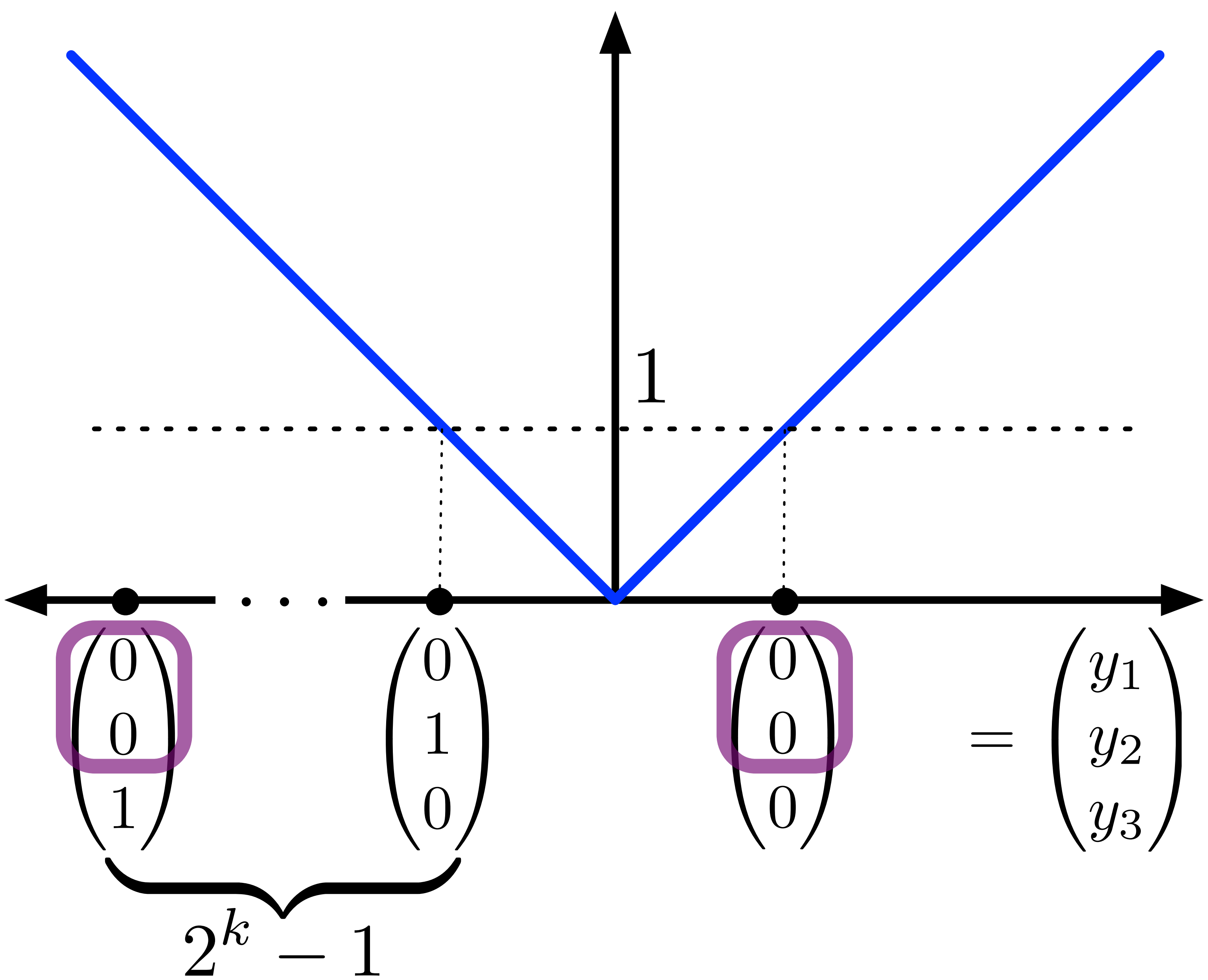
$t^* = 1,\ t_{LP} = 0$

Example: Binary Encoding



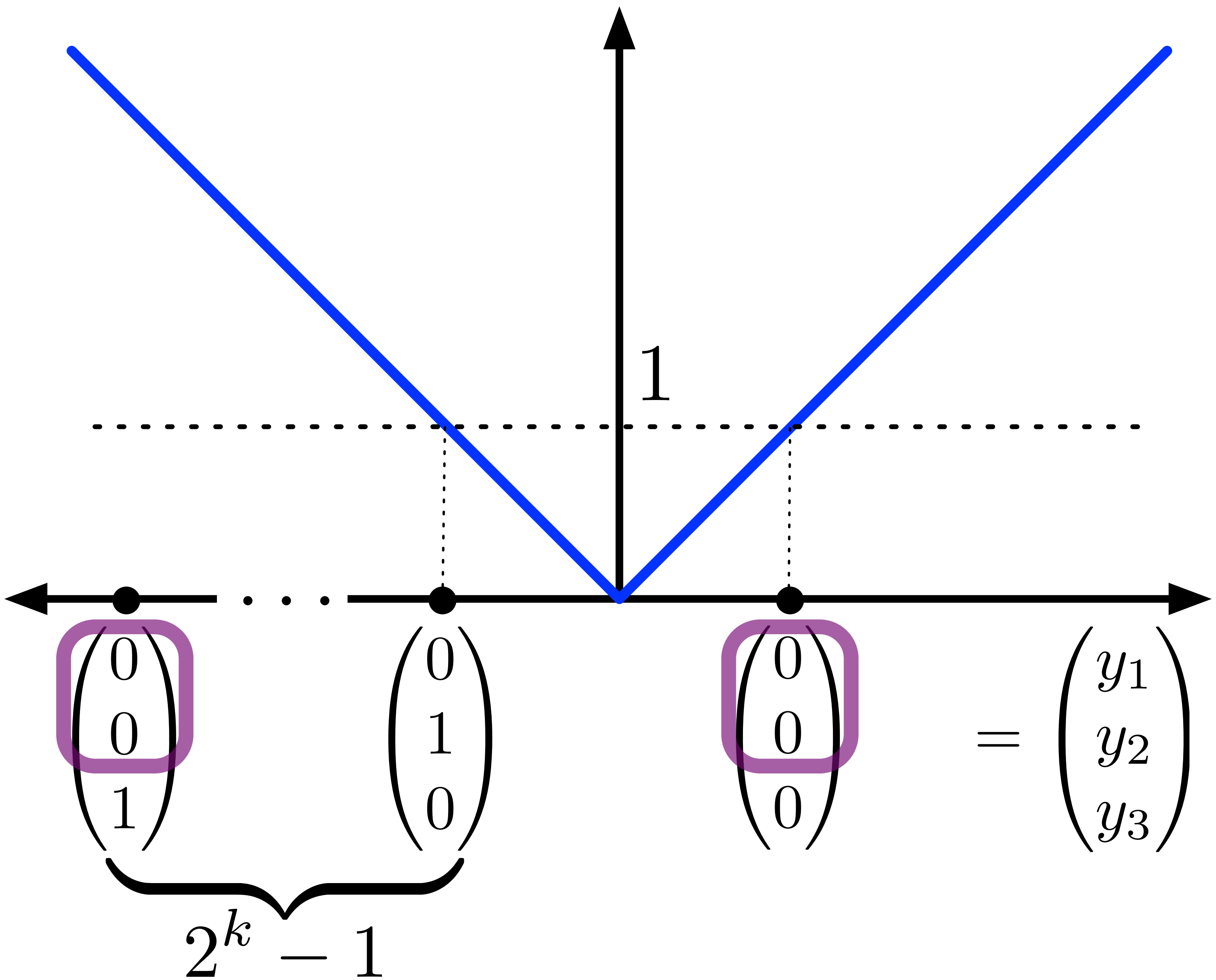
$$t^* = 1,\ t_{LP} = 0$$

Example: Binary Encoding



$t^* = 1,\ t_{LP} = 0 \quad y_1 = y_2 = 0$

Example: Binary Encoding

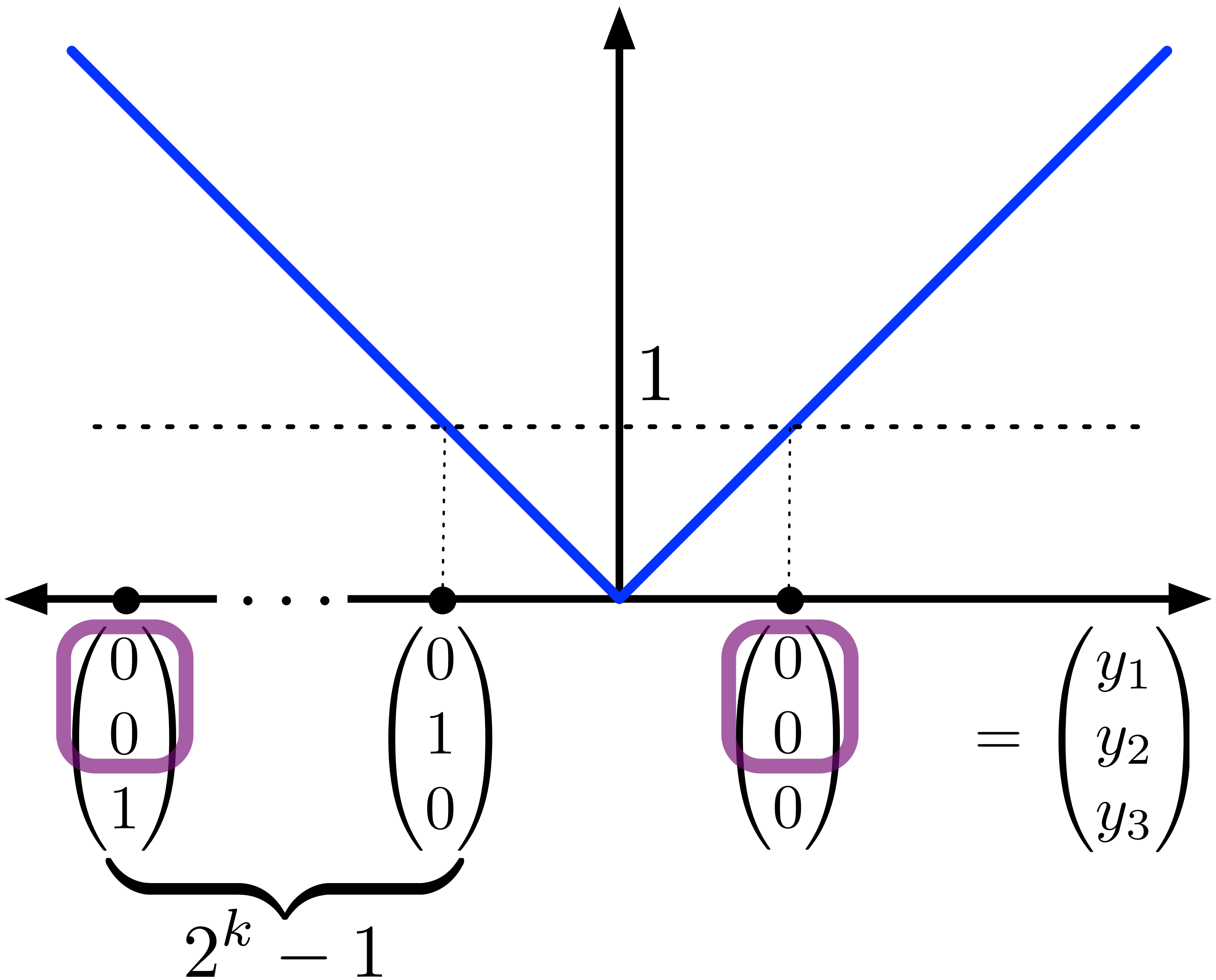


$t_{LP} = 0$ unless:

$$y_i = 0 \quad \forall i$$

$t^* = 1,\ t_{LP} = 0 \quad y_1 = y_2 = 0$

Example: Binary Encoding



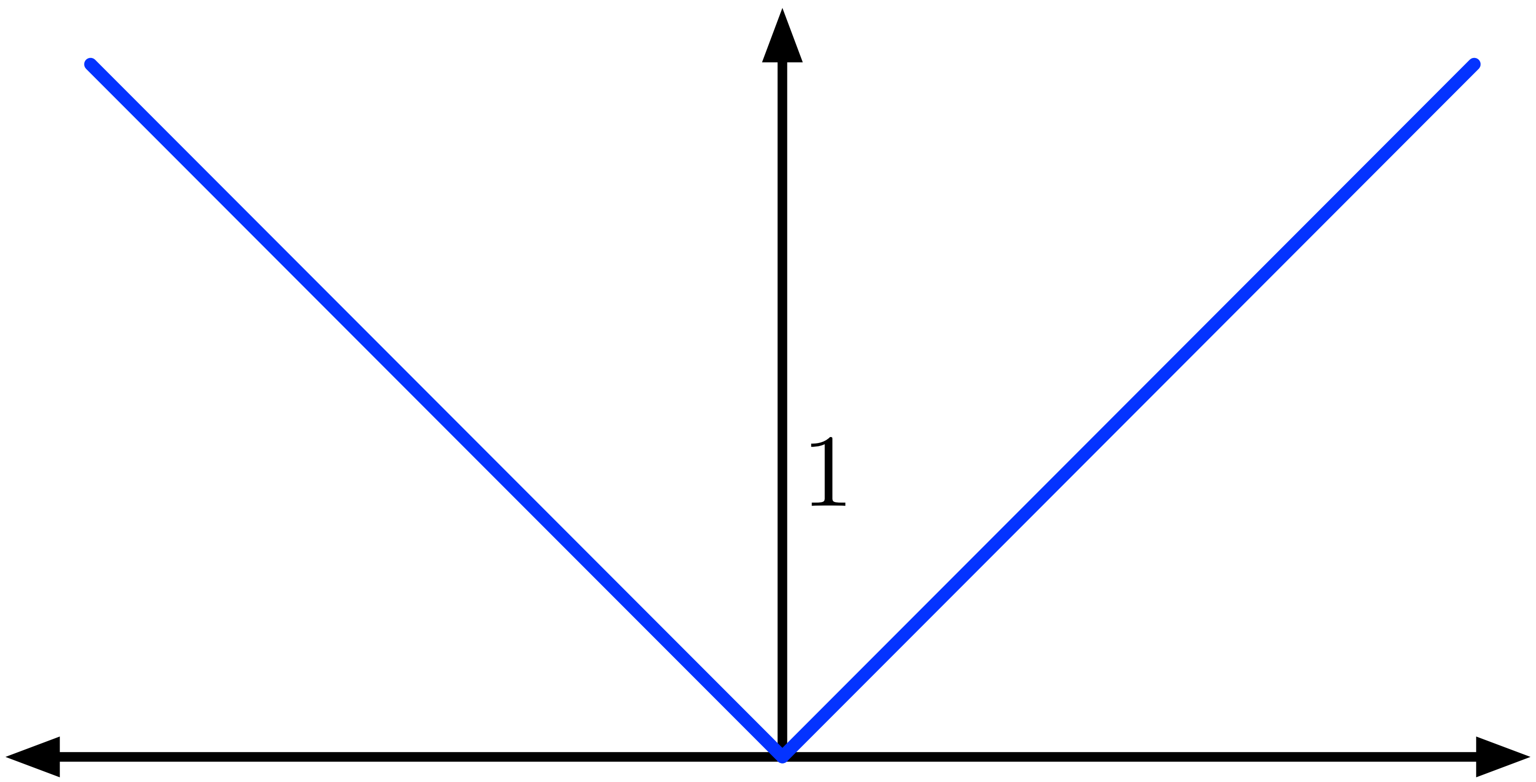
$t_{LP} = 0$ unless:

$y_i = 0 \quad \forall i$

$t^* = 1,\ t_{LP} = 0 \quad y_1 = y_2 = 0$

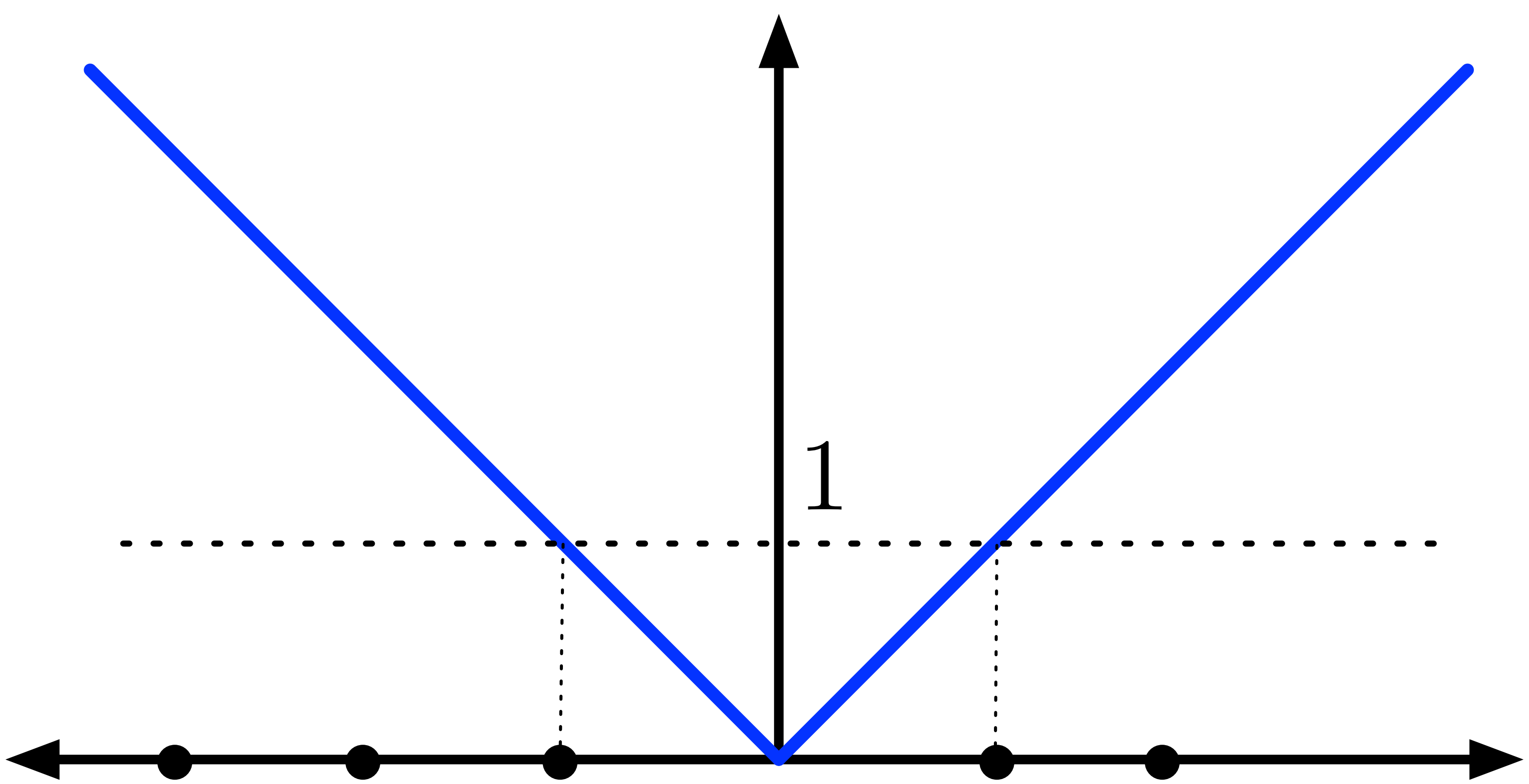
Need all $k = \log_2 n$ branches to solve.

Example: Incremental Encoding



$t^* = 1,\, t_{LP} = 0$

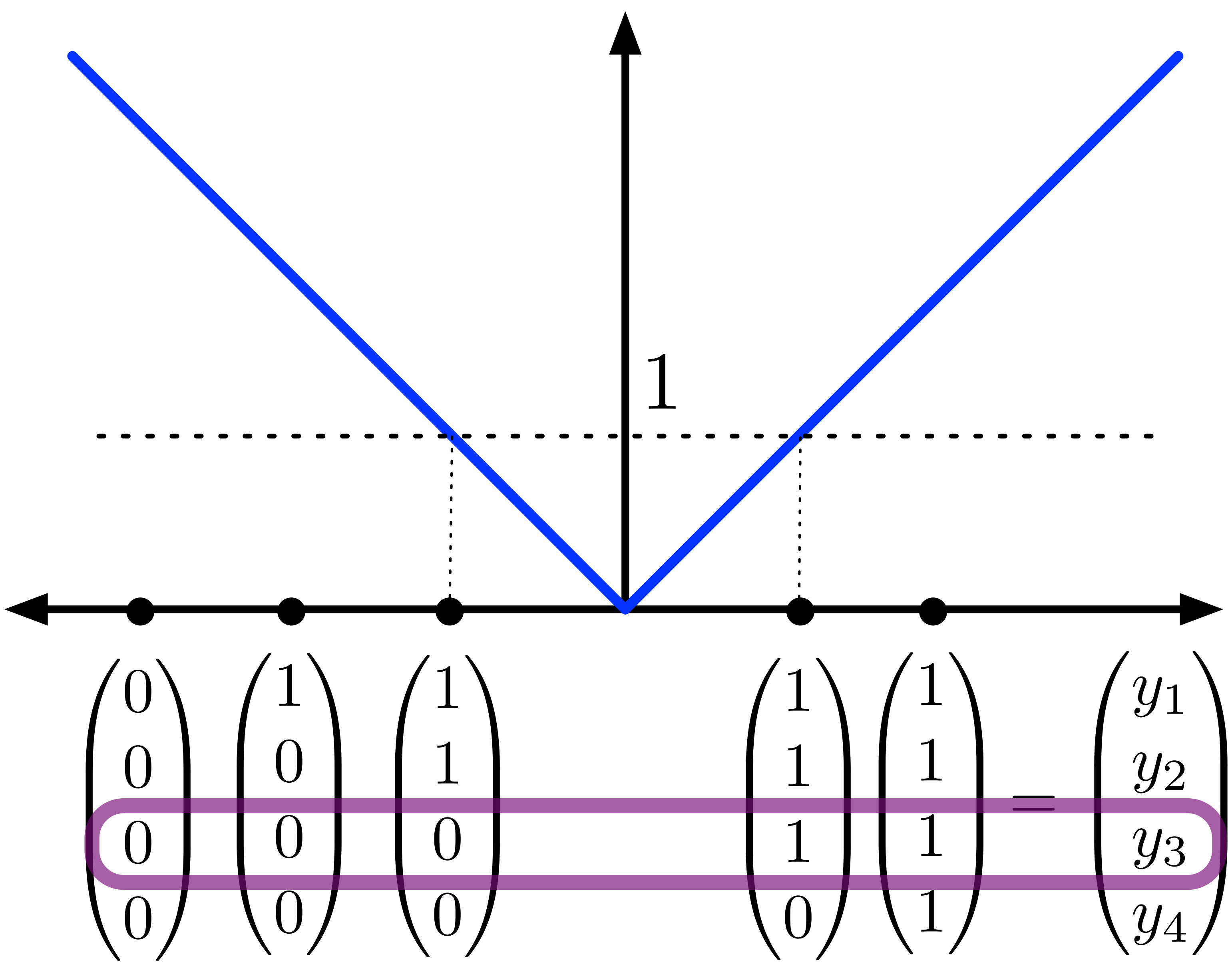
Example: Incremental Encoding



$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}1\\1\\0\\0\end{pmatrix} \quad \begin{pmatrix}1\\1\\1\\0\end{pmatrix} \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}$$

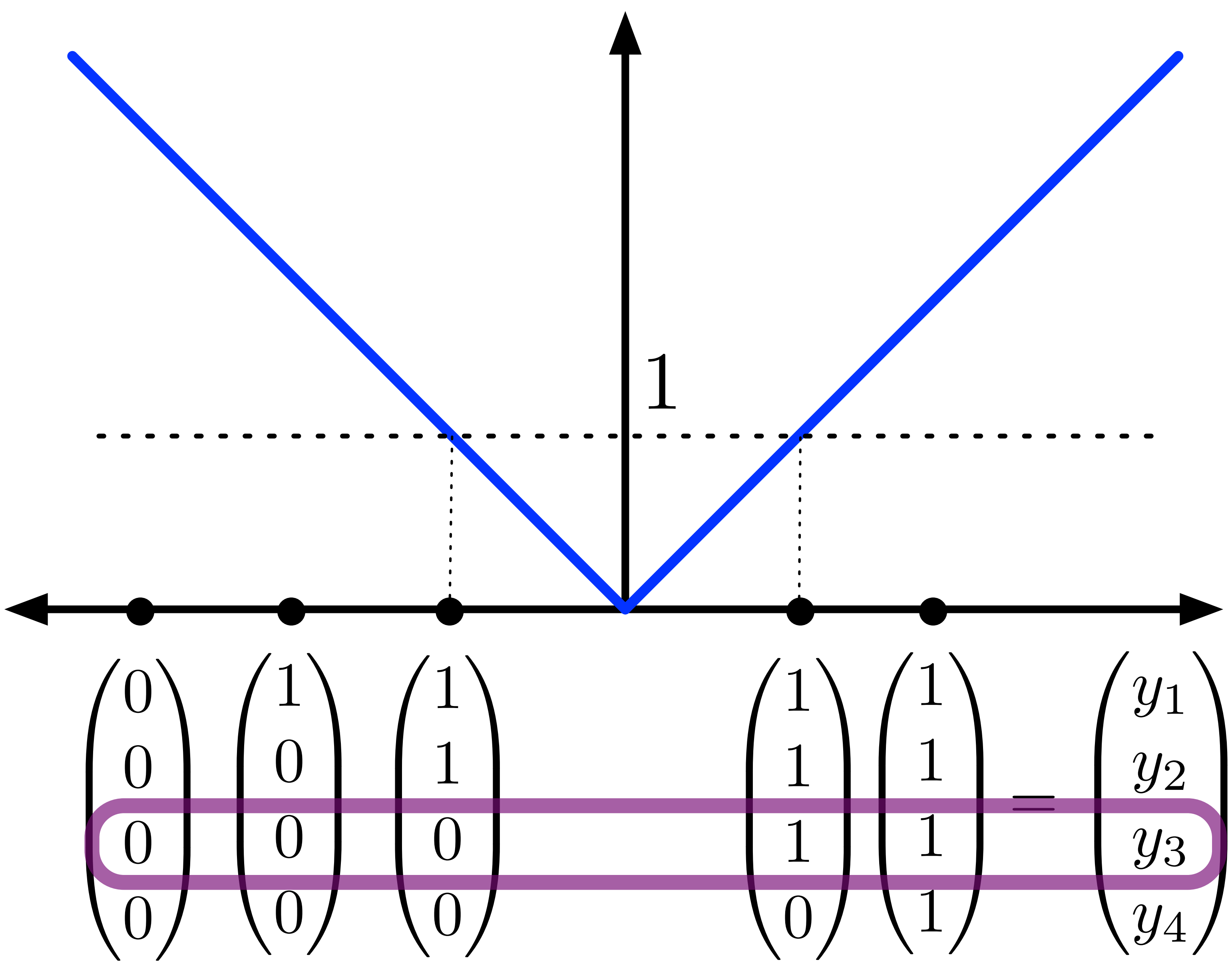
$t^* = 1,\ t_{LP} = 0$

Example: Incremental Encoding



$t^* = 1, \, t_{LP} = 0 \quad y_3 = 1 \vee y_3 = 0$

Example: Incremental Encoding



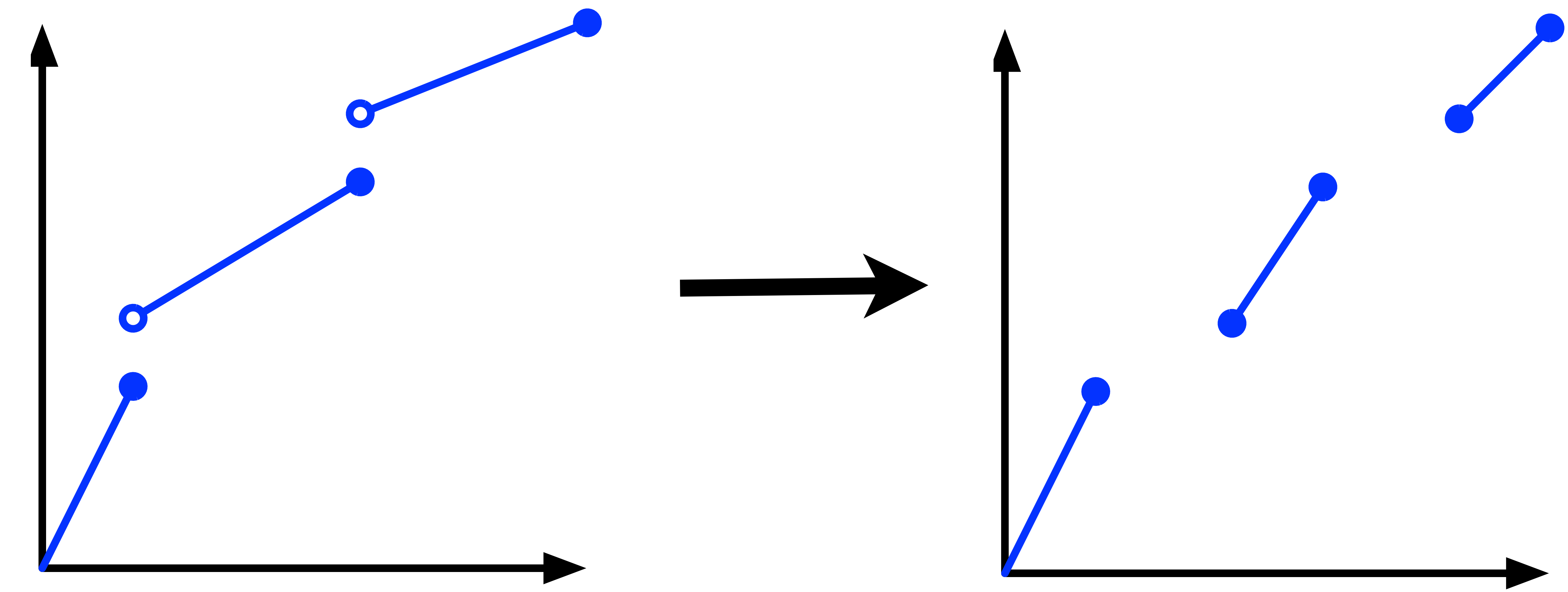
$t_{LP} = 1$ if:

$$y_{i^*} = 0 \vee y_{i^*} = 1$$

Only need 1 branch!

$t^* = 1,\ t_{LP} = 0$ $\quad y_3 = 1 \vee y_3 = 0$

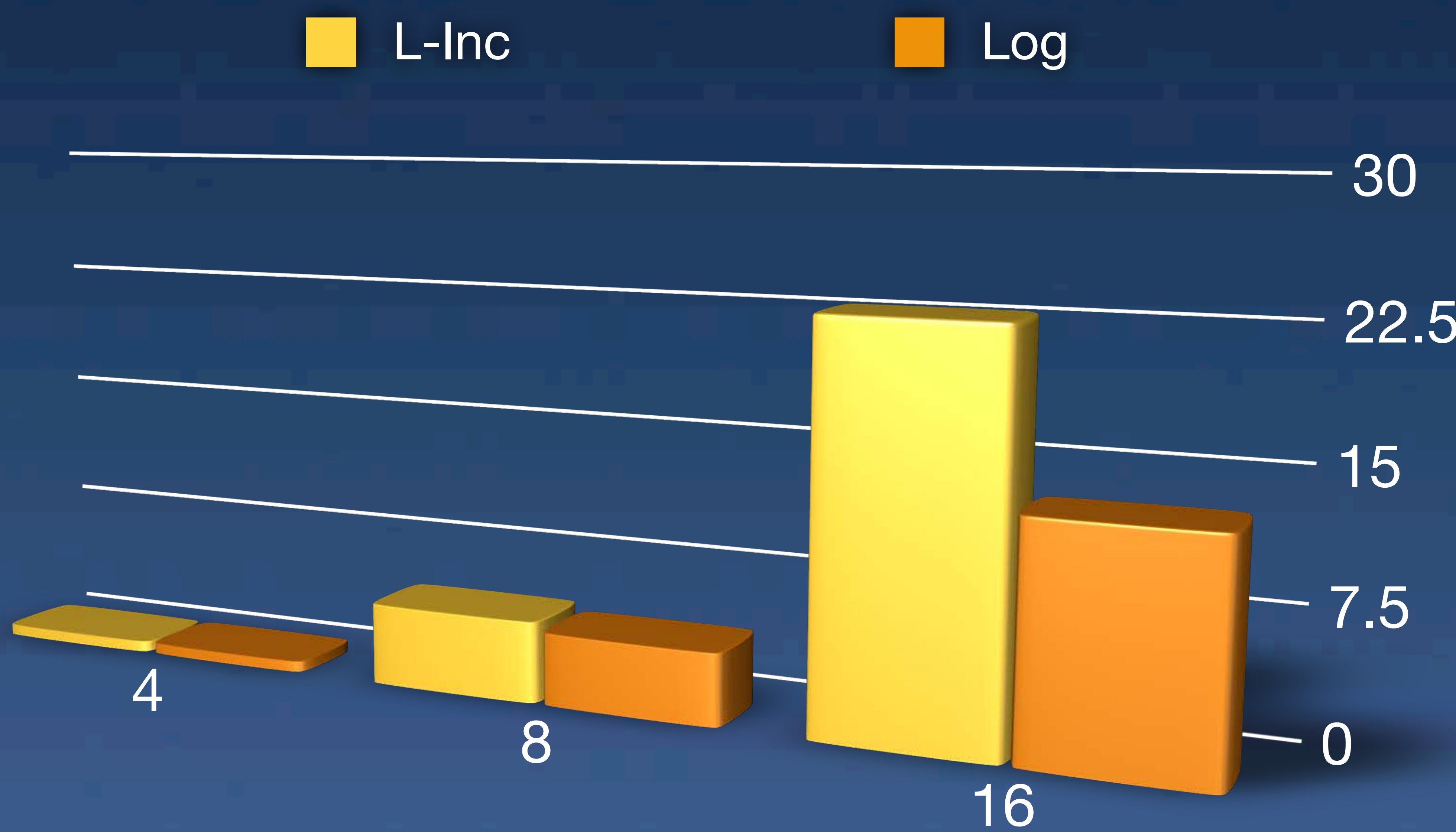
Revisiting Transportation Instances



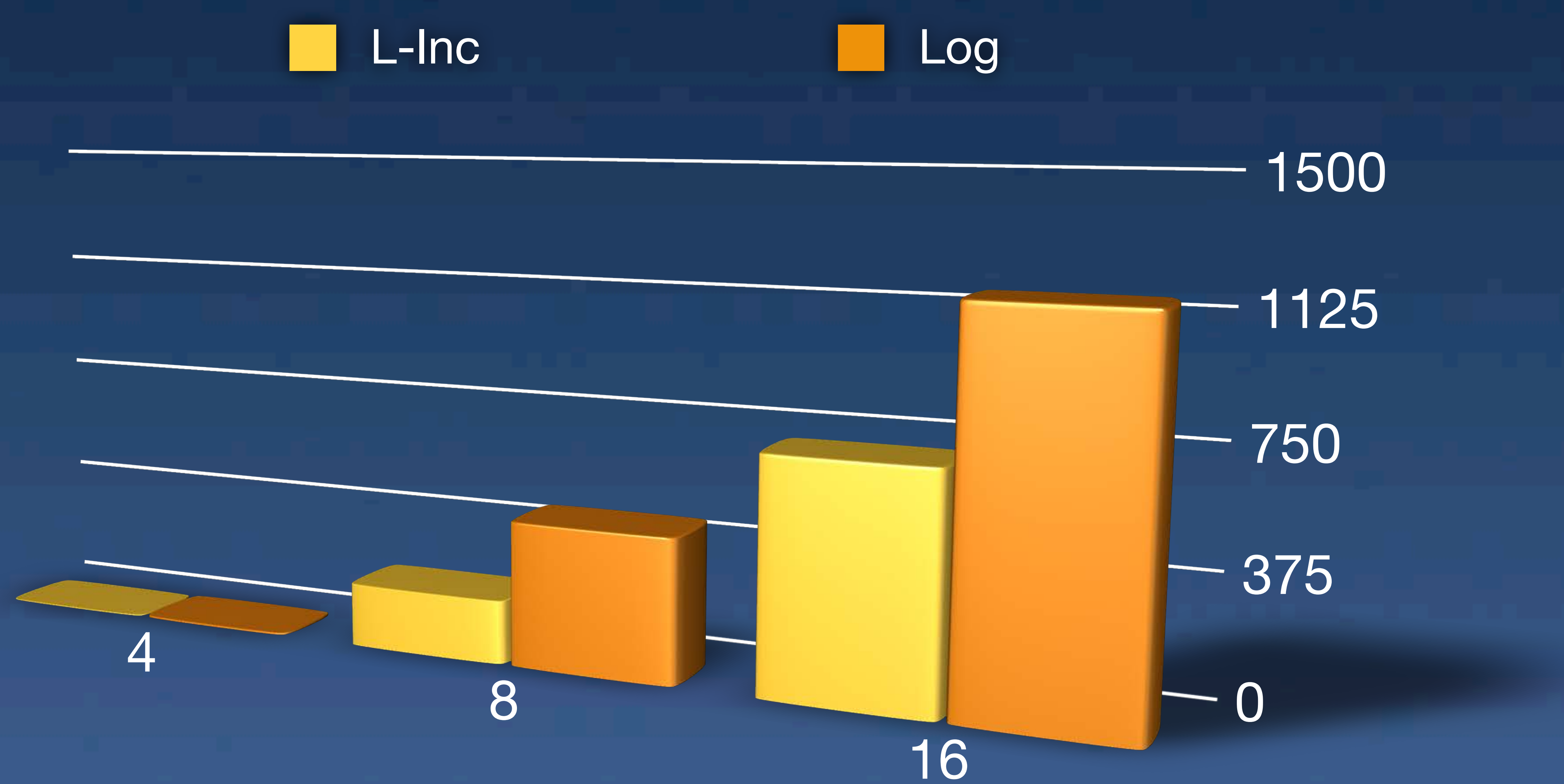
Discontinuous
Piecewise Linear

Disc. PWL
+
“Semicontinuous”

Piecewise Linear



Piecewise Linear + Semi Continuous



Summary

- General "encoding" formulation
  - General incremental formulation
- Incremental formulation can be better than logarithmic formulation!
- Paper ready soon, meanwhile:
  - Survey: V., "Mixed Integer Linear Programming Formulation Techniques", Web and Opt-Online.