Advanced Mixed Integer Programming Formulations for Non-Convex Optimization Problems in Statistical Learning

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(Custom) Product Recommendations via CBCA

Feature	SX530	RX100
Zoom	50x	3.6x
Prize	$249.99	$399.99
Weight	15.68 ounces	7.5 ounces
Prefer	☑	☐

Feature	TG-4	G9
Waterproof	Yes	No
Prize	$249.99	$399.99
Weight	7.36 lb	7.5 lb
Prefer	☐	☑

Feature	TG-4	Galaxy 2
Waterproof	Yes	No
Prize	$249.99	$399.99
Viewfinder	Electronic	Optical
Prefer	☑	☐

We recommend:

Towards Optimal Product Recommendation

- Find enough information about preferences to recommend

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We recommend:

- How do I pick the **next (1st) question** to obtain the largest reduction of uncertainty or “variance” on preferences

Choice-based Conjoint Analysis

Feature	Chewbacca	BB-8
Wookiee	Yes	No
Droid	No	Yes
Blaster	Yes	No
I would buy toy	☑	☐
Product Profile	x^1	x^2

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = x^2$$

MNL Preference Model

- Utilities for 2 products, n features (e.g. n = 12)

$$U_1 = \beta \cdot x^1 + \epsilon_1 = \sum_{i=1}^{n} \beta_i x_i^1 + \epsilon_1$$

$$U_2 = \beta \cdot x^2 + \epsilon_2 = \sum_{i=1}^{n} \beta_i x_i^2 + \epsilon_2$$

part-worths

product profile

noise (gumbel)

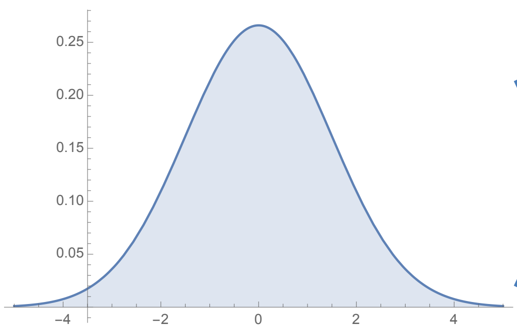
- Utility maximizing customer: $x^1 \succeq x^2 \Leftrightarrow U_1 \text{“}\geq\text{”} U_2$

- Noise can result in response error:

$$L\left(\beta \,|x^1 \succeq x^2\right) = \mathbb{P}\left(x^1 \succeq x^2 \mid \beta\right) = \frac{e^{\beta \cdot x^1}}{e^{\beta \cdot x^1} + e^{\beta \cdot x^2}}$$

Next Question To Reduce "Variance": Bayesian

Prior Distribution of β



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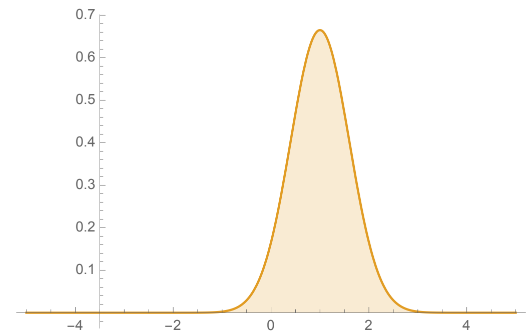
Bayesian Update

Posterior Distribution



MCMC

Feature	TG-4	Galaxy 2
Waterproof	Yes	No
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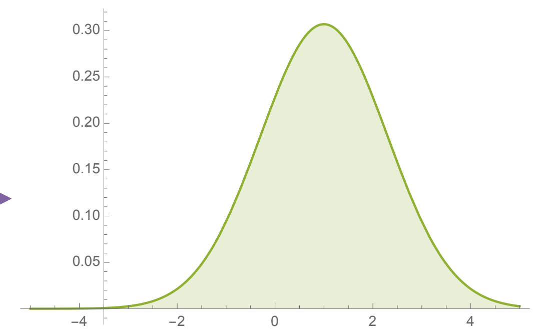
Bayesian Update

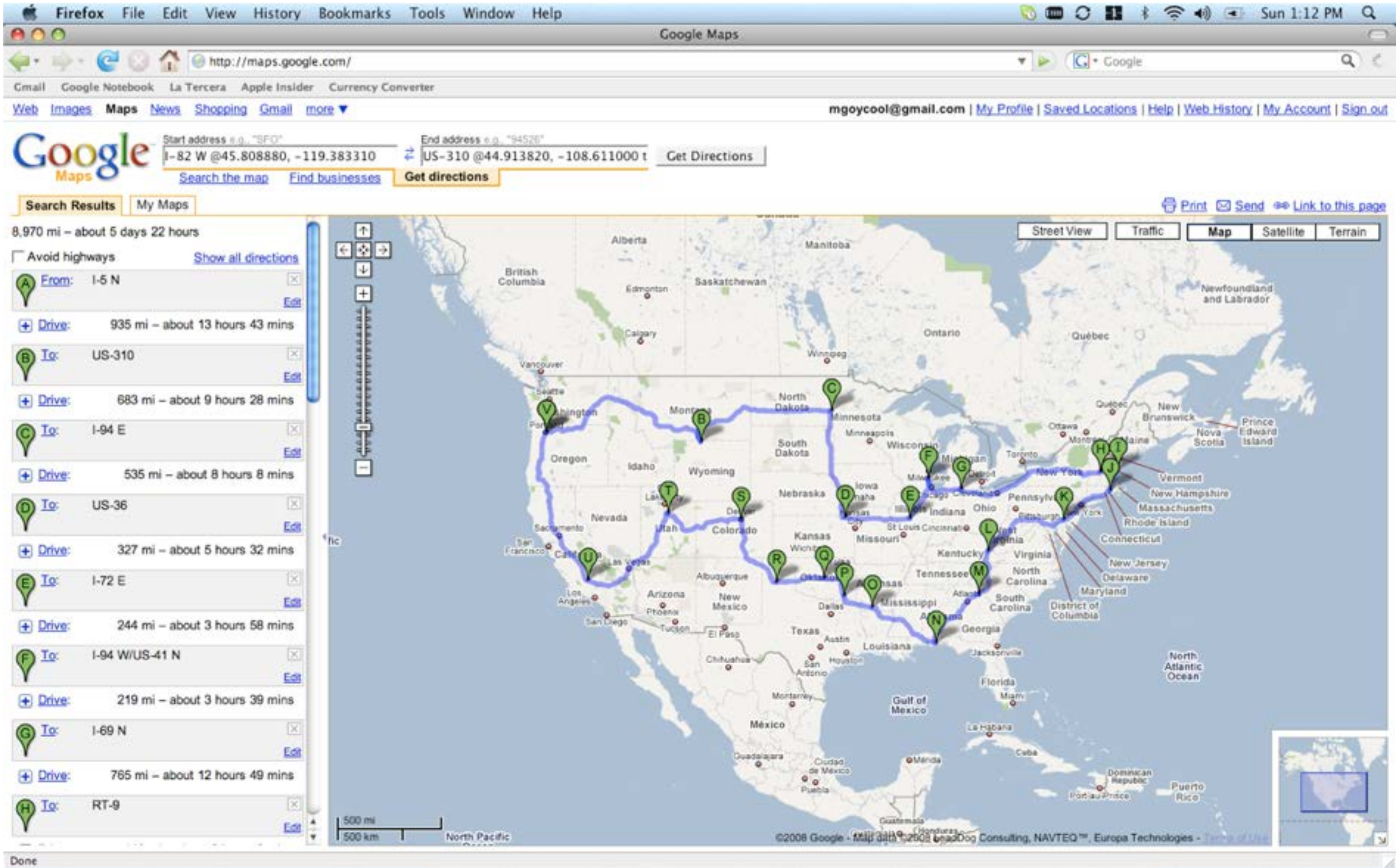
Posterior Distribution



- Black-box objective: Question Selection = Enumeration ☹
- Question selection by Mixed Integer Programming (MIP)

Avoiding Enumeration with MIP

Traveling Salesman Problem (TSP): Visit Cities Fast



How about 49 cities?

- Number of tours $= 48!/2 \approx 10^{60}$
- Fastest supercomputer $\approx 10^{17}$flops
- Assuming one floating point operation per tour:
  $> 10^{35}$ years $\approx 10^{25}$ times the age of the universe!
- How long does it take on an iphone?
  - Less than a second!
  - 4 iterations of **cutting plane** method!
  - Dantzig, Fulkerson and Johnson 1954 did it by hand!
  - For more info see tutorial in ConcordeTSP app
  - **Cutting planes** are the key for effectively solving (even NP-hard) MIP problems in practice.

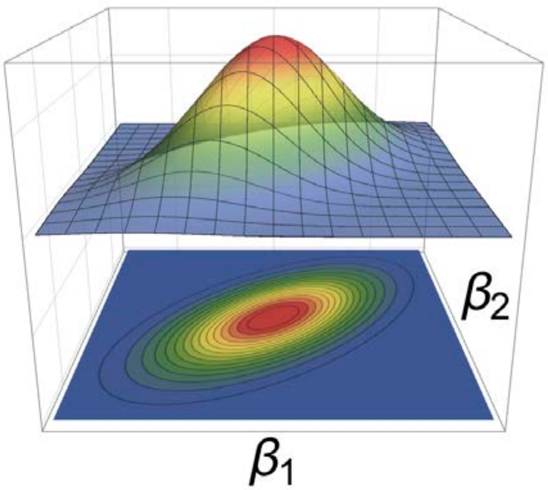
50+ Years of MIP = Significant Solver Speedups

- Algorithmic Improvements (Machine Independent):
  - CPLEX v1.2 (1991) – v11 (2007): 29,000x speedup
  - Gurobi v1 (2009) – v6.5 (2015): 48.7x speedup
  - Commercial, but free for academic use
- (Reasonably) effective free / open source solvers:
  - GLPK, CBC and SCIP (free only for non-commercial)
- Easy to use, fast and versatile modeling languages
  - Julia based JuMP modelling language
- Linear MIP solvers very mature and effective:
  - Convex nonlinear MIP getting there (quadratic nearly there)

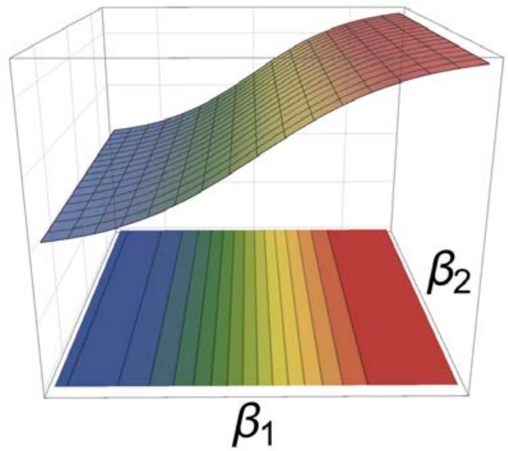
Question Selection with MIP

Bayesian Update and Geometric Updates

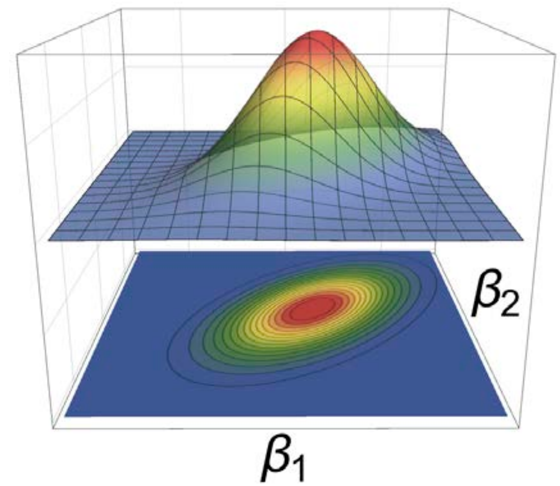
Prior distribution



Answer likelihood



Posterior distribution



$$\beta \sim N(\mu, \Sigma)$$

$$x^1 \succeq x^2$$

$$f\left(\beta \mid x^1 \succeq x^2\right)$$

$$\phi(\beta; \mu, \Sigma)$$

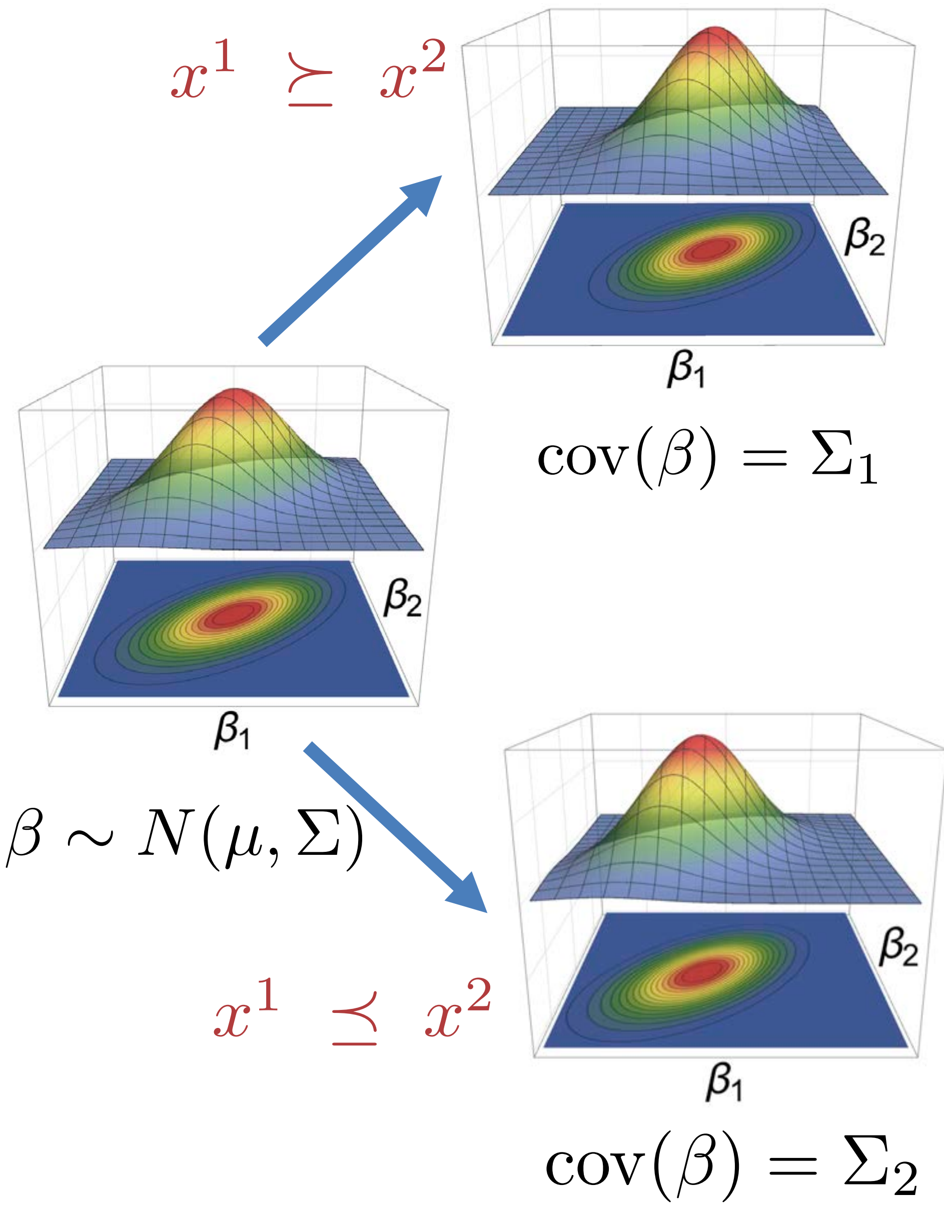
$$L\left(\beta \mid x^1 \succeq x^2\right)$$

Multidimensional Integration?

$$f\left(\beta \mid x^1 \succeq x^2\right) = \frac{\boxed{\phi(\beta\,;\,\mu, \Sigma)\, L\left(\beta \mid x^1 \succeq x^2\right)}}{\int_{\mathbb{R}} \phi(\beta\,;\,\mu, \Sigma)\, L\left(\beta \mid x^1 \succeq x^2\right) d\beta}$$

non-convex on $x^1, x^2 \in \{0,1\}^n$

D-Efficiency and Posterior Covariance Matrix



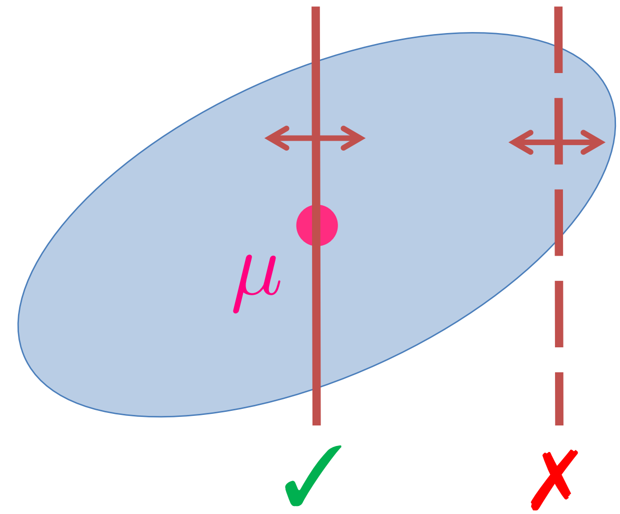
- "Variance" = D-Efficiency:
- $f\left(x^1, x^2\right) := \mathbb{E}_{\beta, x^1 \preceq/\succeq x^2}\left(\det(\Sigma_i)^{1/p}\right)$
- Non-convex function
- Even evaluating expected D-Efficiency for a question requires multidimensional integration

Standard Question Selection Criteria

$$(\beta - \mu)' \cdot \Sigma^{-1} \cdot (\beta - \mu) \leq r$$

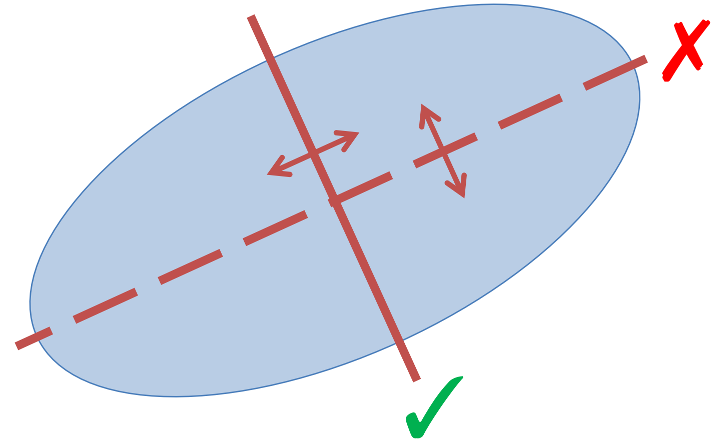
- Choice balance:
  - Minimize distance to center

$$\mu \cdot (x^1 - x^2)$$



- Postchoice symmetry:
  - Maximize variance of question

$$(x^1 - x^2)' \cdot \sum \cdot (x^1 - x^2)$$

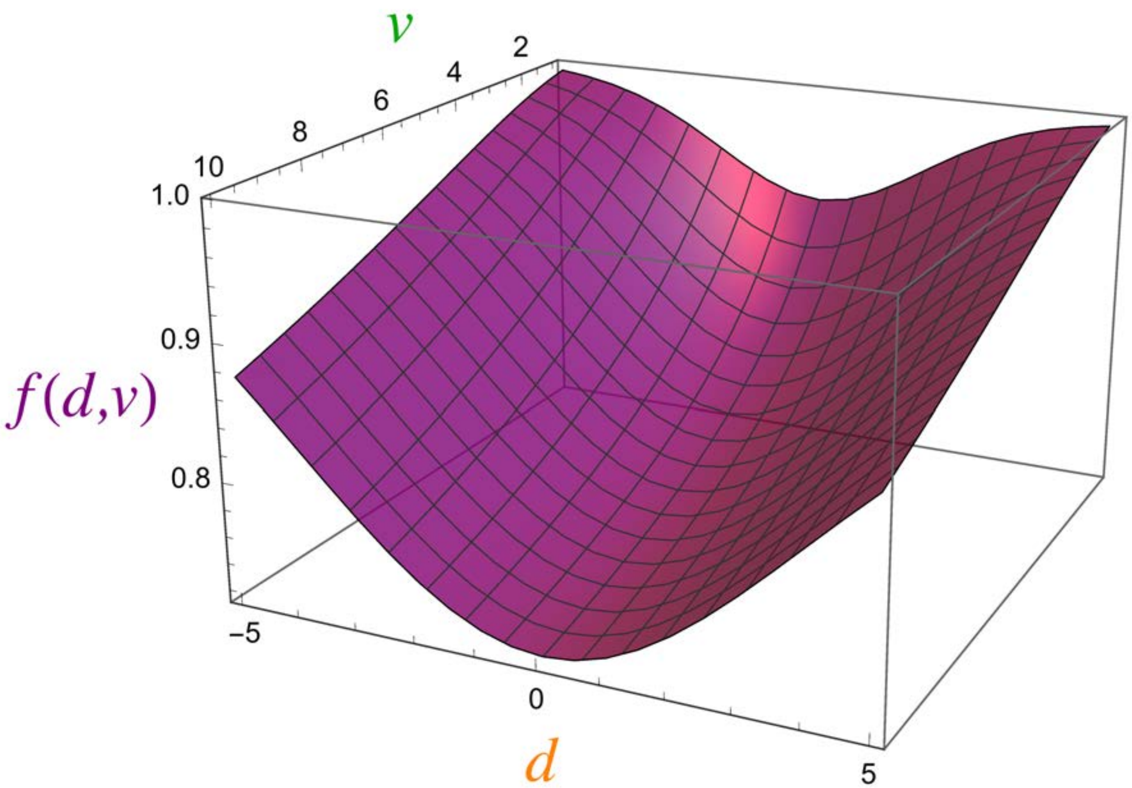


D-efficiency: Balance Question Trade-off

- D-efficiency = Non-convex function $f(d, v)$ of

  distance: $d := \mu \cdot (x^1 - x^2)$

  variance: $v := (x^1 - x^2)' \cdot \sum \cdot (x^1 - x^2)$



Can evaluate $f(d, v)$ with 1-dim integral 🙂

Optimization Model

$$\min \quad f(d, v) \quad ✗$$

$$s.t.$$

$$\mu \cdot (x^1 - x^2) = d \quad ✓$$

$$(x^1 - x^2)' \cdot \Sigma \cdot (x^1 - x^2) = v \quad ✗$$

$$A^1 x^1 + A^2 x^2 \leq b \quad ✓$$

$$x^1 \neq x^2 \quad ✗$$

$$x^1, x^2 \in \{0, 1\}^n$$

Technique 1: Binary Quadratic $x^1, x^2 \in \{0,1\}^n$

$$(x^1 - x^2)' \cdot \Sigma \cdot (x^1 - x^2) = v$$

$$X^l_{i,j} = x^l_i \cdot x^l_j \quad (l \in \{1,2\}, \quad i,j \in \{1,\ldots,n\}):$$

$$X^l_{i,j} \leq x^l_i, \quad X^l_{i,j} \leq x^l_j, \quad X^l_{i,j} \geq x^l_i + x^l_j - 1, \quad X^l_{i,j} \geq 0$$

$$W_{i,j} = x^1_i \cdot x^2_j :$$

$$W_{i,j} \leq x^1_i, \quad W_{i,j} \leq x^2_j, \quad W_{i,j} \geq x^1_i + x^2_j - 1, \quad W_{i,j} \geq 0$$

$$\sum_{i,j=1}^{n} \left(X^1_{i,j} + X^2_{i,j} - W_{i,j} - W_{j,i}\right) \Sigma_{i,j} = v$$

Technique 1: Binary Quadratic $x^1, x^2 \in \{0,1\}^n$

$$x^1 \neq x^2 \quad \Leftrightarrow \quad \|x^1 - x^2\|_2^2 \geq 1$$

$$X^l_{i,j} = x^l_i \cdot x^l_j \quad (l \in \{1,2\}, \quad i,j \in \{1,\ldots,n\}):$$

$$X^l_{i,j} \leq x^l_i, \quad X^l_{i,j} \leq x^l_j, \quad X^l_{i,j} \geq x^l_i + x^l_j - 1, \quad X^l_{i,j} \geq 0$$

$$W_{i,j} = x^1_i \cdot x^2_j :$$

$$W_{i,j} \leq x^1_i, \quad W_{i,j} \leq x^2_j, \quad W_{i,j} \geq x^1_i + x^2_j - 1, \quad W_{i,j} \geq 0$$

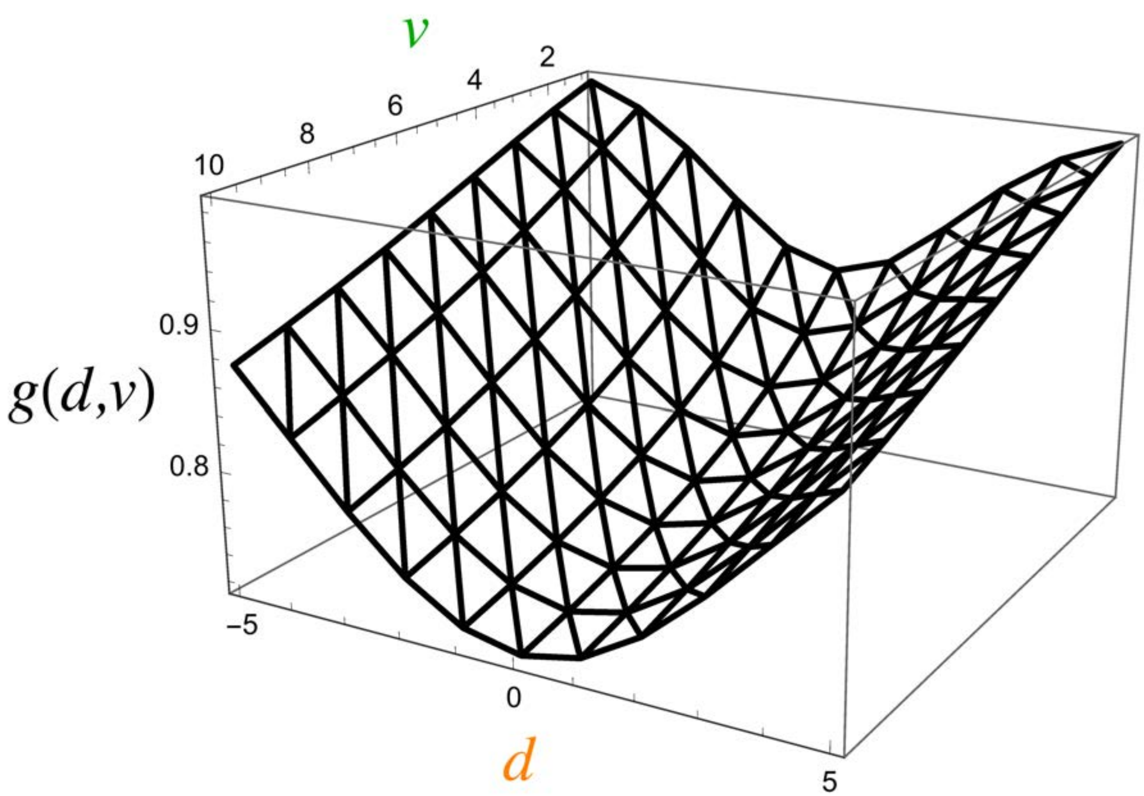
$$\sum_{i,j=1}^{n} \left(X^1_{i,j} + X^2_{i,j} - W_{i,j} - W_{j,i}\right) \geq 1$$

Technique 2: Piecewise Linear Functions

- D-efficiency = Non-convex function $f(d, v)$ of

  distance: $d := \mu \cdot (x^1 - x^2)$

  variance: $v := (x^1 - x^2)' \cdot \sum \cdot (x^1 - x^2)$



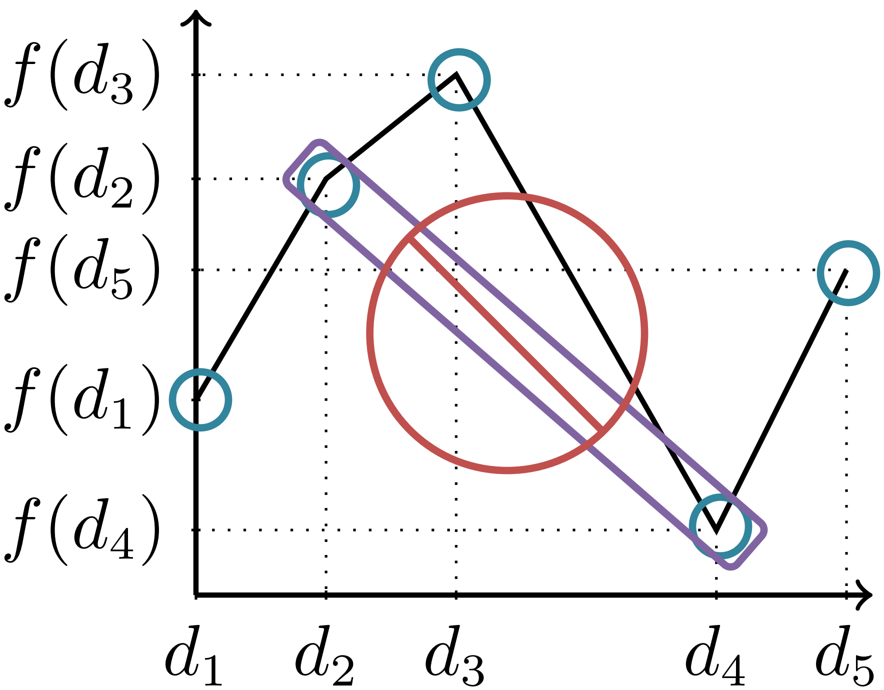
Can evaluate $f(d, v)$ with 1-dim integral 🙂

Piecewise Linear Interpolation

MIP formulation

Simple Formulation for Univariate Functions

$$z = f(x)$$



$$\text{Size} = O(\#\text{ of segments})$$

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^{5} \lambda_j, \quad \lambda_j \geq 0$$

$$y \in \{0,1\}^4, \quad \sum_{i=1}^{4} y_i = 1$$

$$0 \leq \lambda_1 \leq y_1$$

$$0 \leq \lambda_2 \leq y_1 + y_2$$

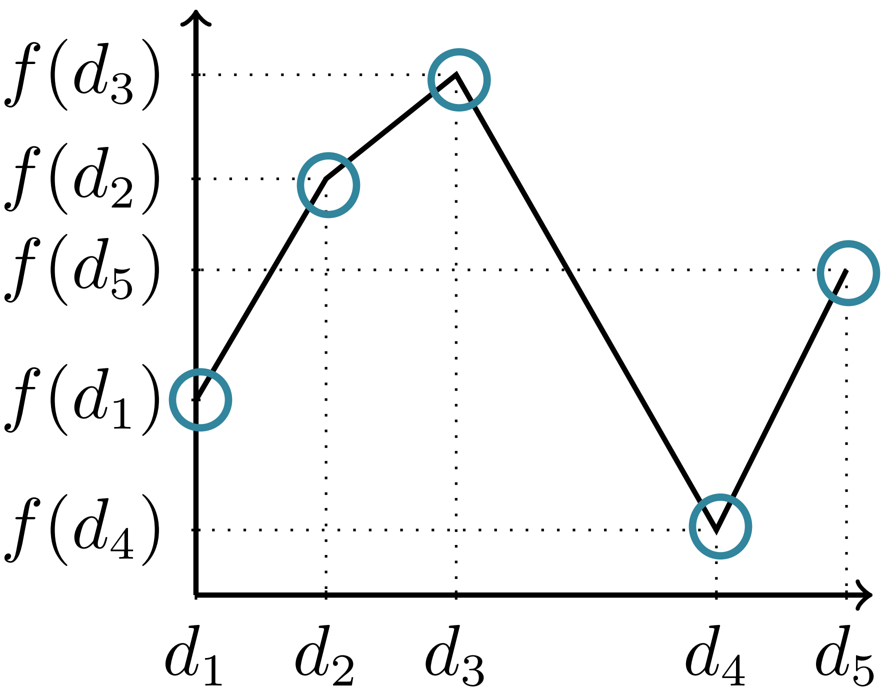
$$0 \leq \lambda_3 \leq y_2 + y_3$$

$$0 \leq \lambda_4 \leq y_3 + y_4$$

$$0 \leq \lambda_5 \leq y_4$$

Advanced Formulation for Univariate Functions

$$z = f(x)$$



$$\text{Size} = O\left(\log_2 \# \text{ of segments}\right)$$

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^{5} \lambda_j, \quad \lambda_j \geq 0$$

$$y \in \{0,1\}^2$$

$$0 \leq \lambda_1 + \lambda_5 \leq 1 - y_1$$

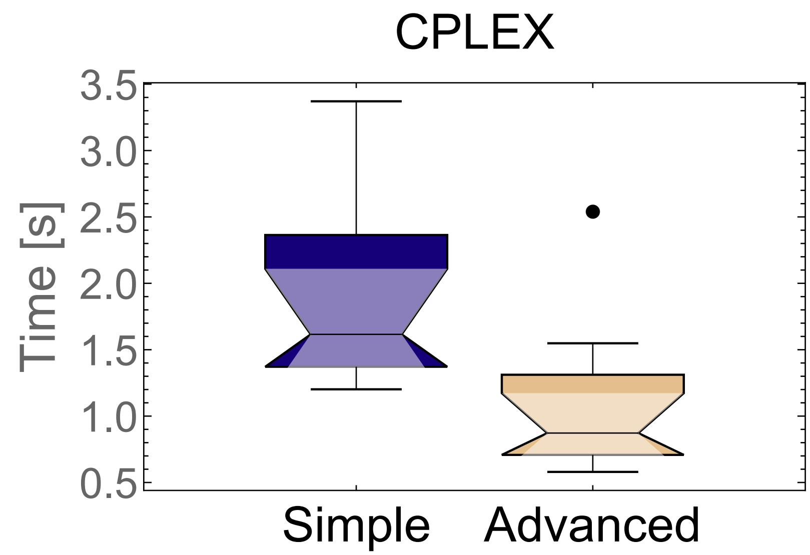
$$0 \leq \lambda_3 \leq y_1$$

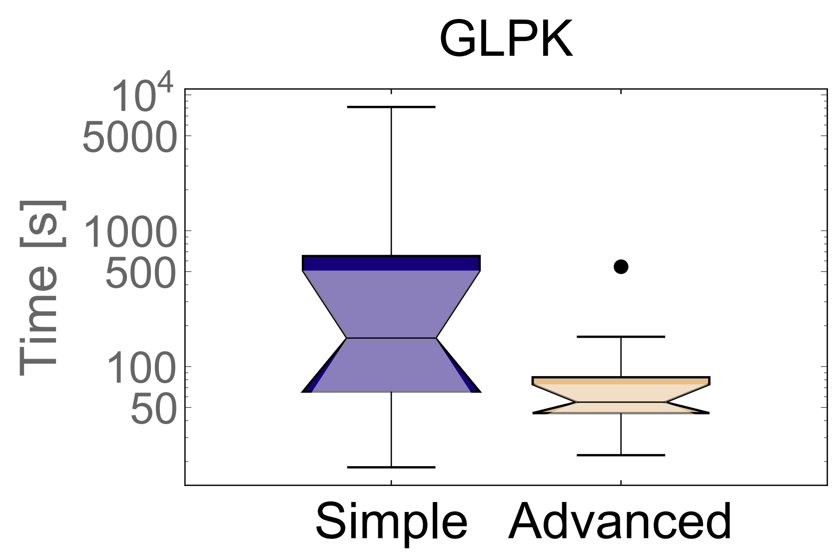
$$0 \leq \lambda_4 + \lambda_5 \leq 1 - y_2$$

$$0 \leq \lambda_1 + \lambda_2 \leq y_2$$

Computational Performance

- Advanced formulations provide an computational advantage
- Advantage is significantly more important for free solvers
- State of the art commercial solvers can be significantly better that free solvers
- Still, free is free!





Summary and Main Messages

- Always choose Chewbacca!

- MIP can solve very challenging problems in practice
- Commercial solvers best, but free solvers reasonable
  - Easily accessible and integrated into complex systems through the JuMP modeling language github.com/JuliaOpt/JuMP.jl
- Formulations = speed-ups and are (relatively) easy to learn
  - Mixed integer linear programming formulation techniques. J. P. Vielma. SIAM Review 57, 2015. pp. 3-57.
- CBC application: http://ssrn.com/abstract=2798984