Extended Formulations for Quadratic Mixed Integer Programming

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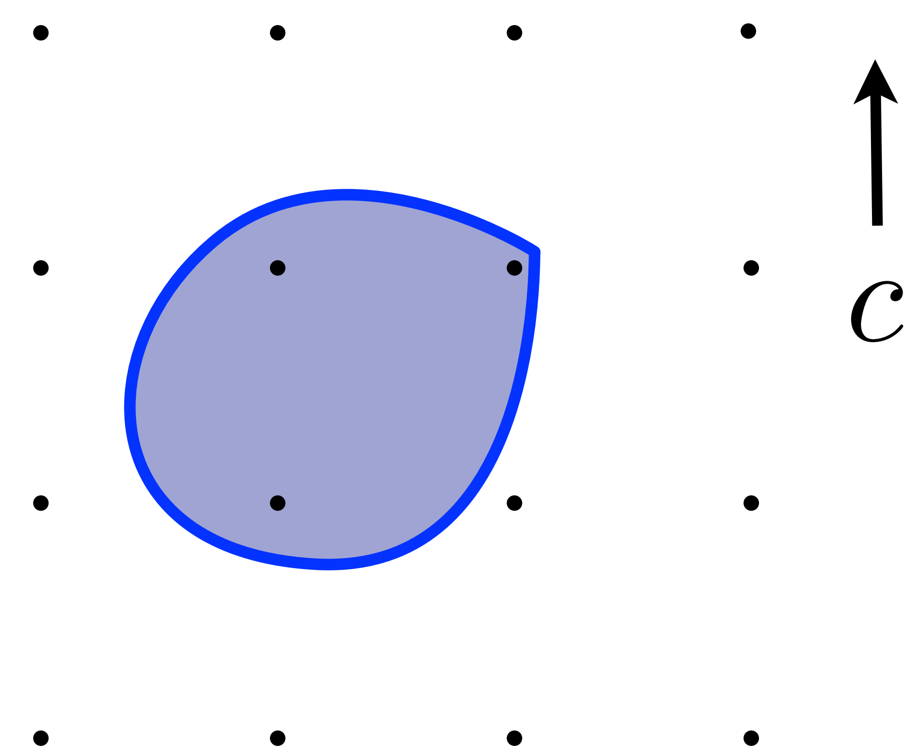
Nonlinear MIP B&B Algorithms

$$\max \quad \sum_{i=1}^{n} c_i x_i$$

$$s.t.$$

$$g_i(x) \leq 0,\; i \in I,$$

$$x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$$



Nonlinear MIP B&B Algorithms

- NLP (QCP) Based B&B

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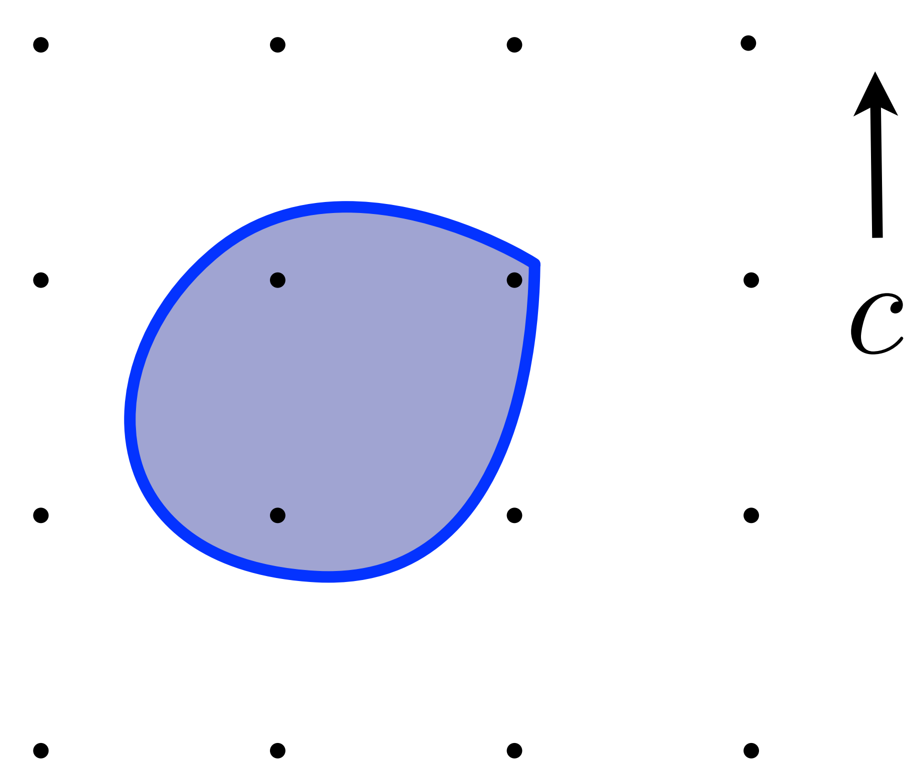

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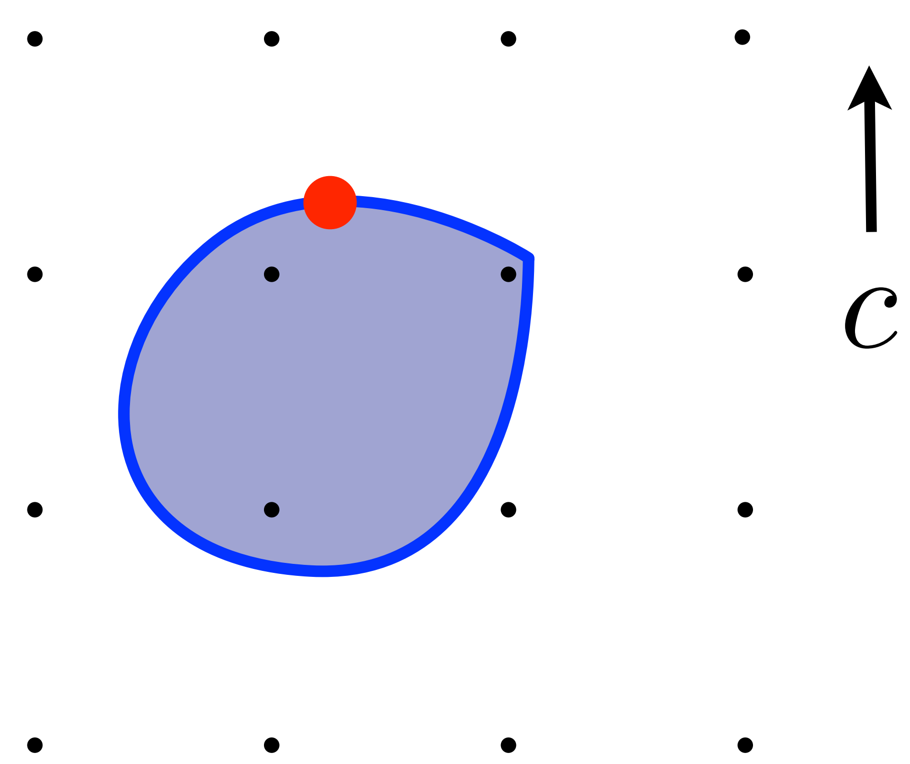

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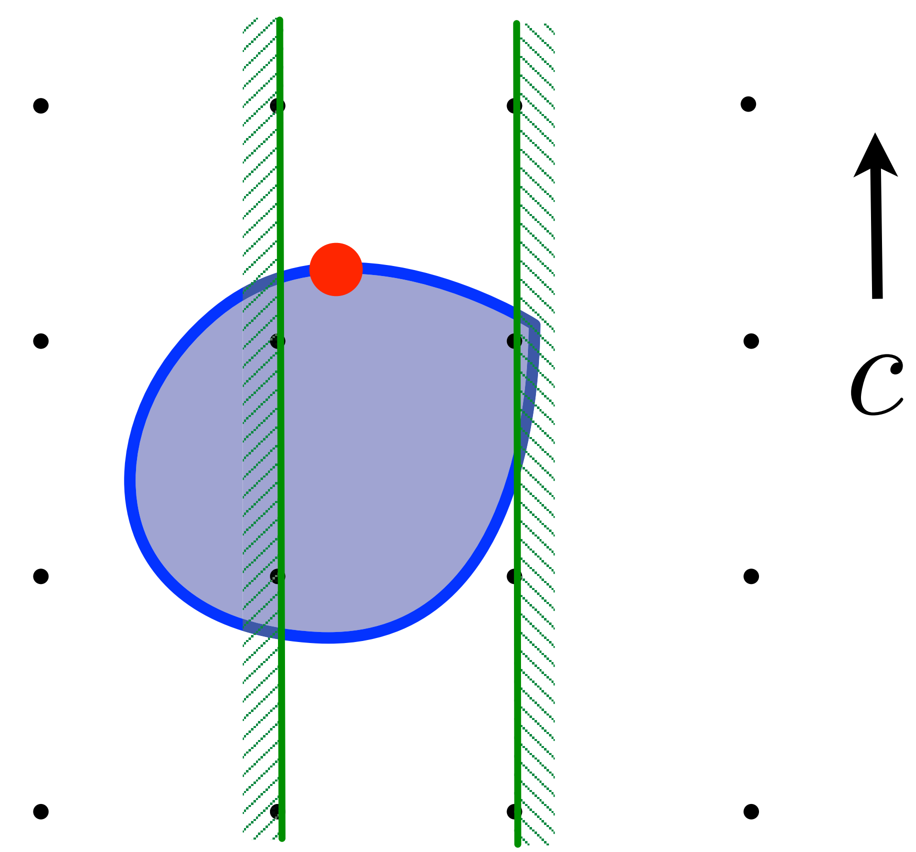

MINLP B&B Algorithms

- NLP (QCP) Based B&B
- (Dynamic) LP Based B&B
  - Few cuts = high speed.
  - Possible slow convergence.

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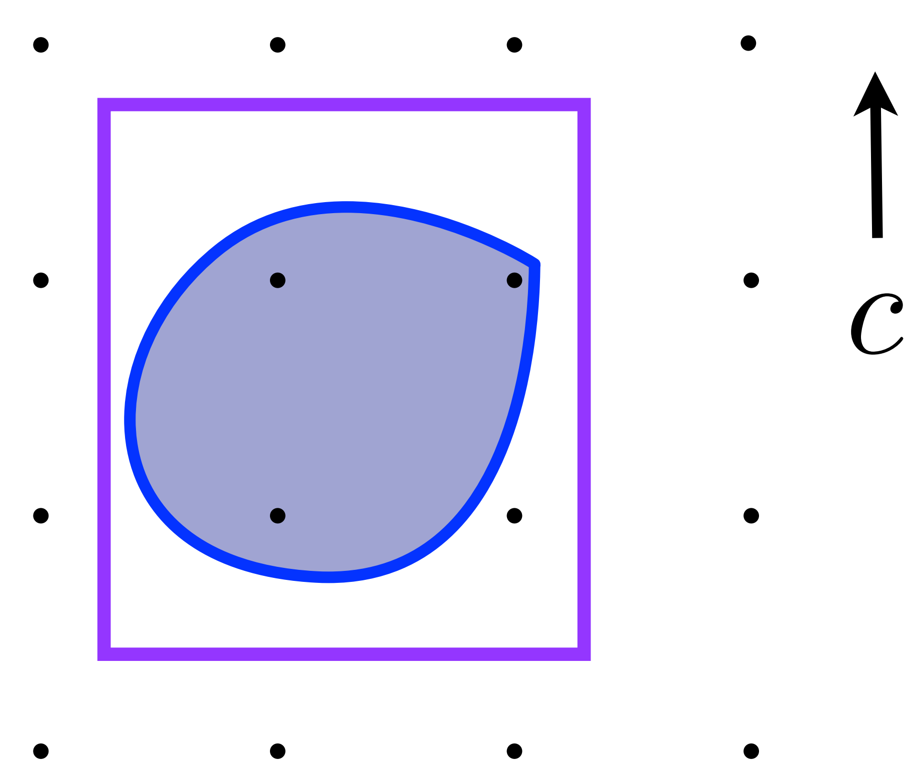

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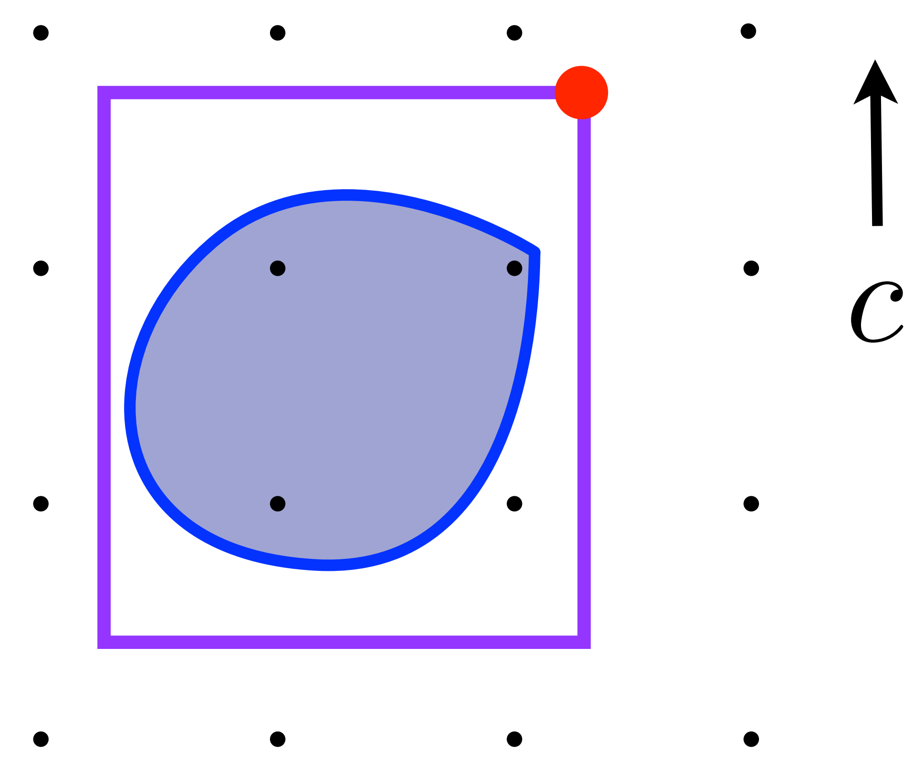

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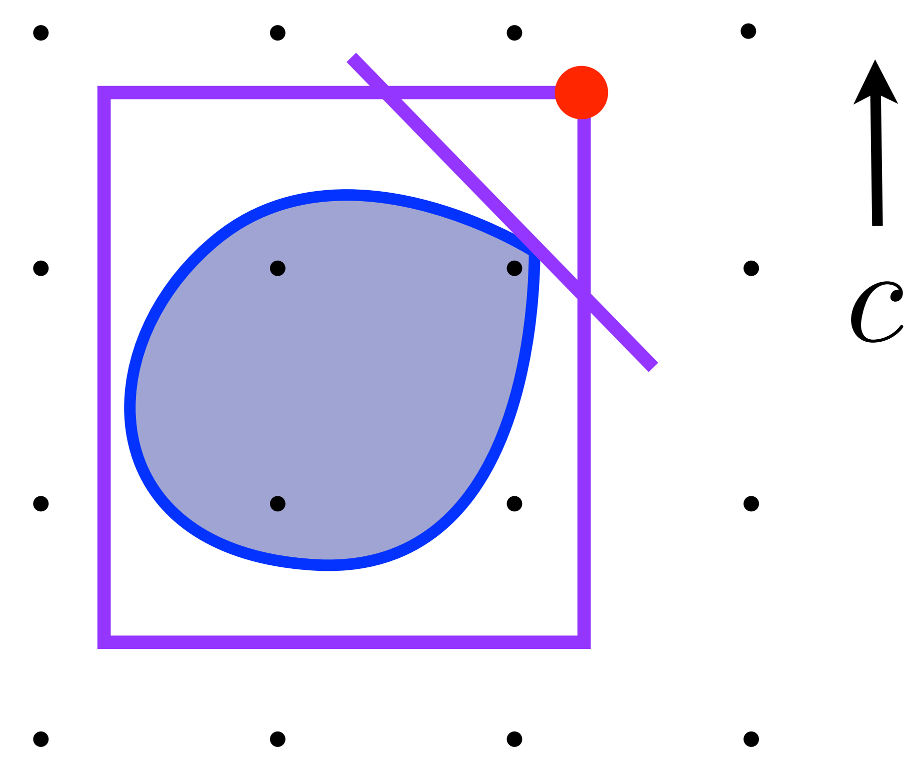

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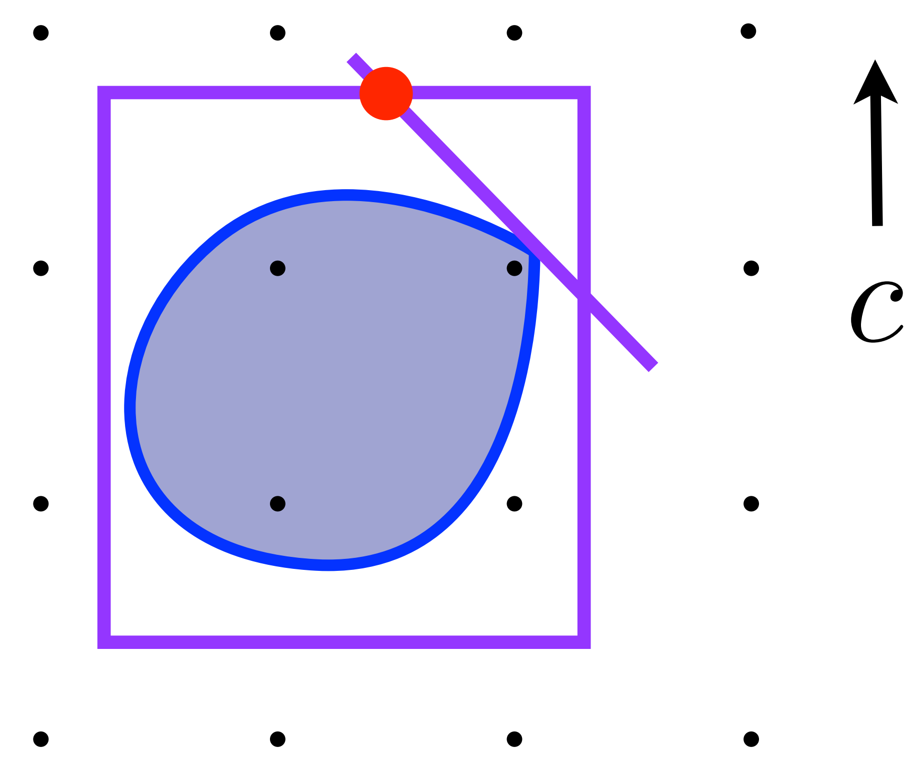

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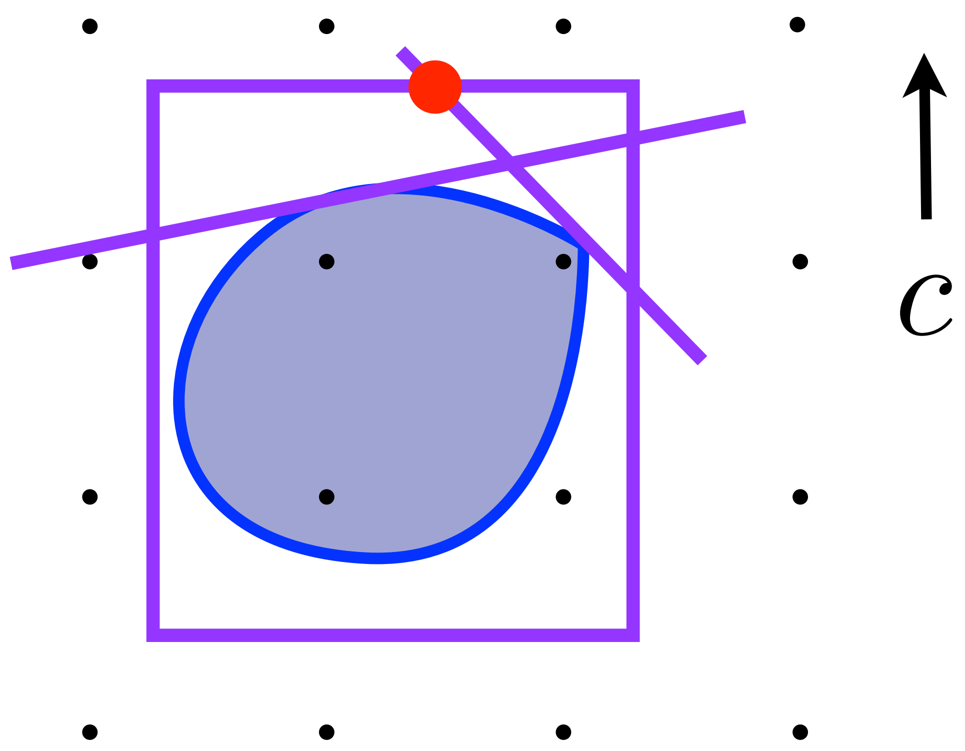

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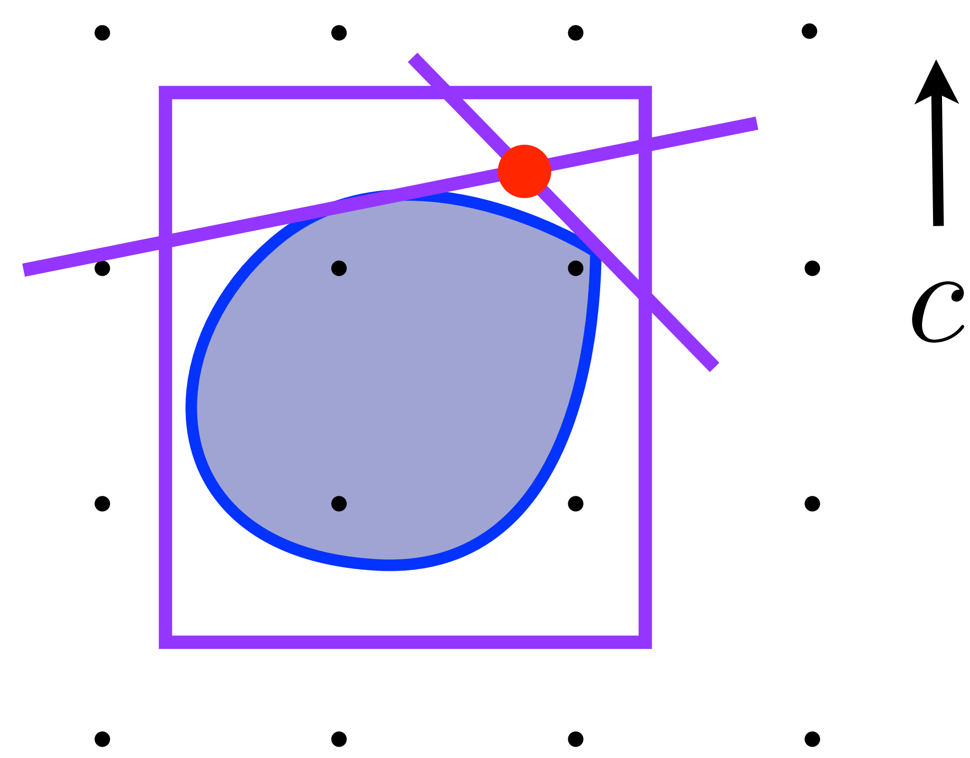

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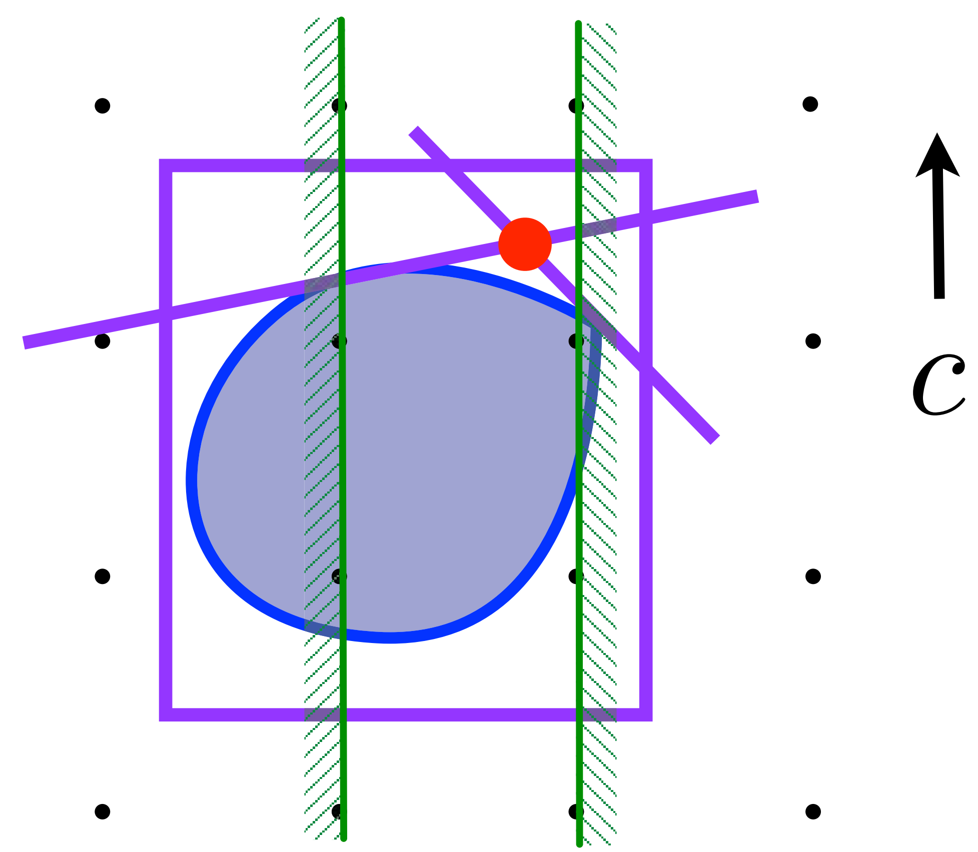

MINLP B&B Algorithms

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  - Fixed extended relaxation.
  - Mimic NLP B&B.

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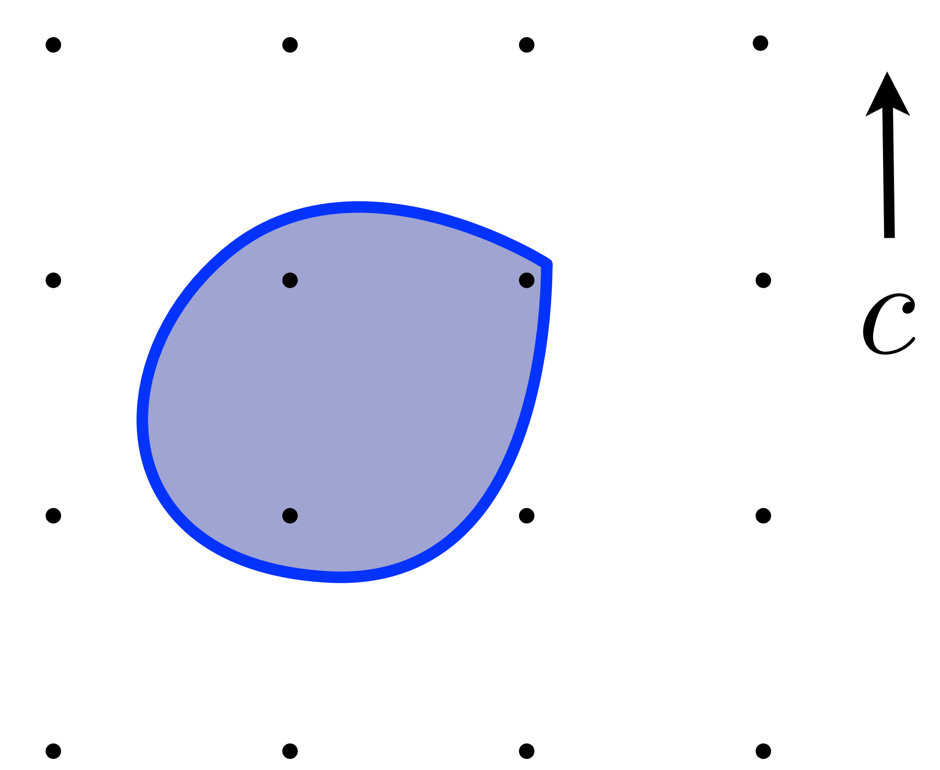

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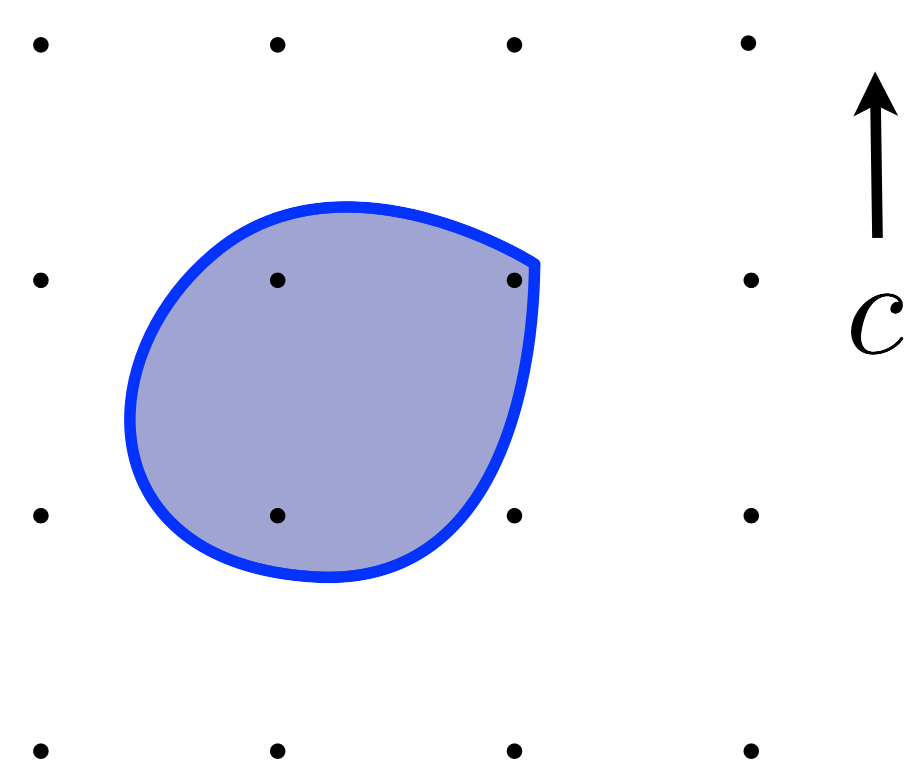

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$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} c_i x_i \\
s.t. \quad & Ax + Dz \leq b, \\
& \cancel{g_i(x) \leq 0,\ i \in I,} \\
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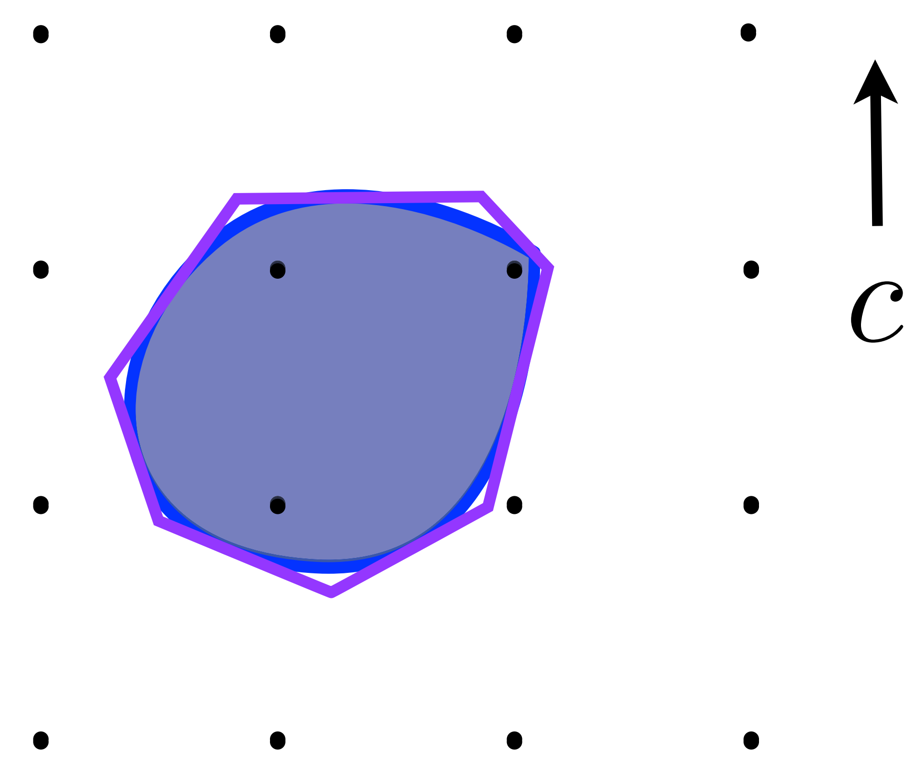
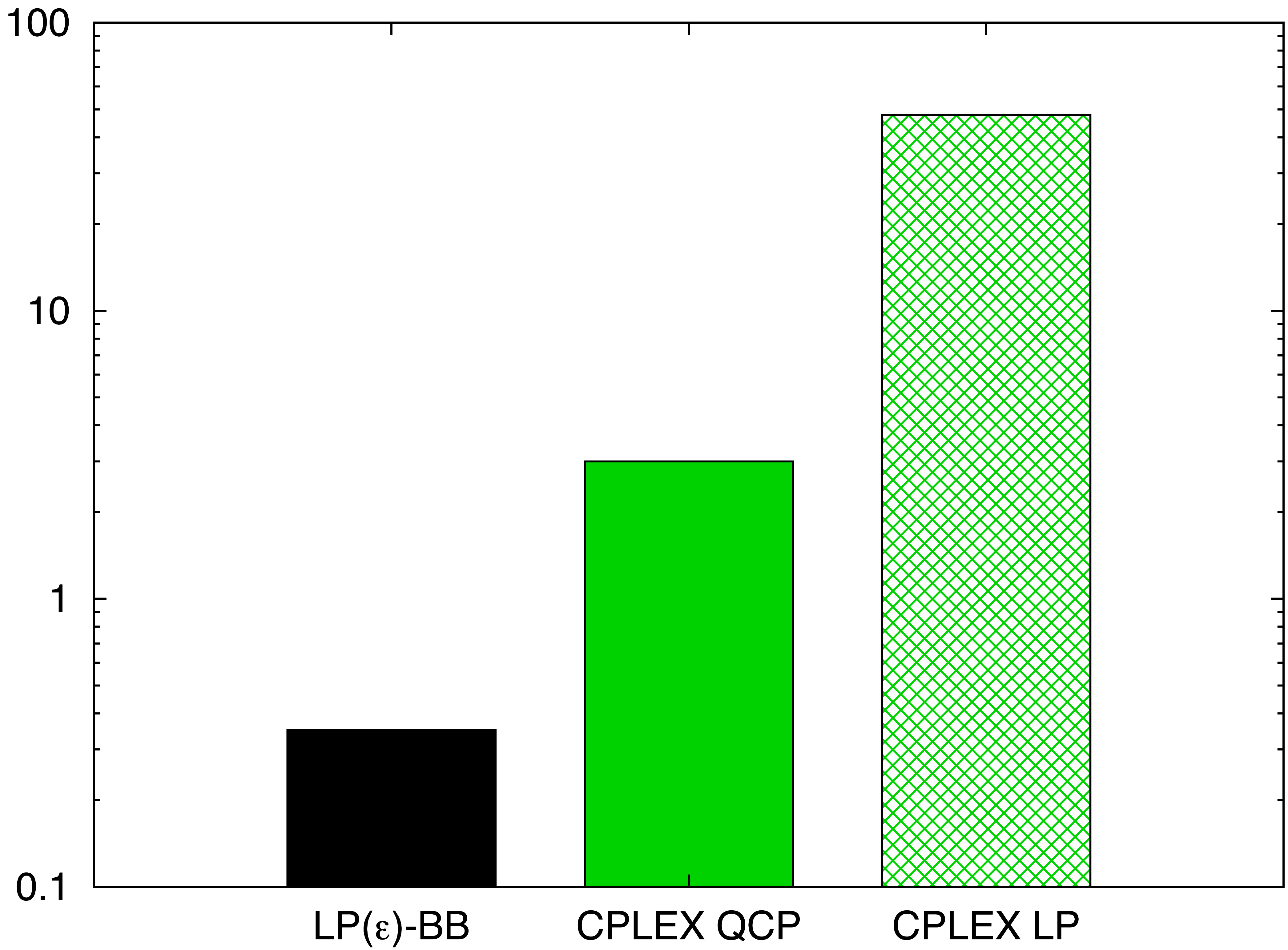


Problem 1: Classical

$$
\begin{aligned}
\max_{x,y} \quad & \bar{a}y \\
s.t. \quad & \\
& ||Q^{1/2}y||_2 \leq \sigma \\
& \sum_{j=1}^{n} y_j = 1 \\
& y_j \leq x_j \qquad \forall j \in \{1, \ldots, n\} \\
& \sum_{j=1}^{n} x_j \leq 10 \\
& x \in \{0,1\}^n \\
& y \in \mathbb{R}^n_+
\end{aligned}
$$

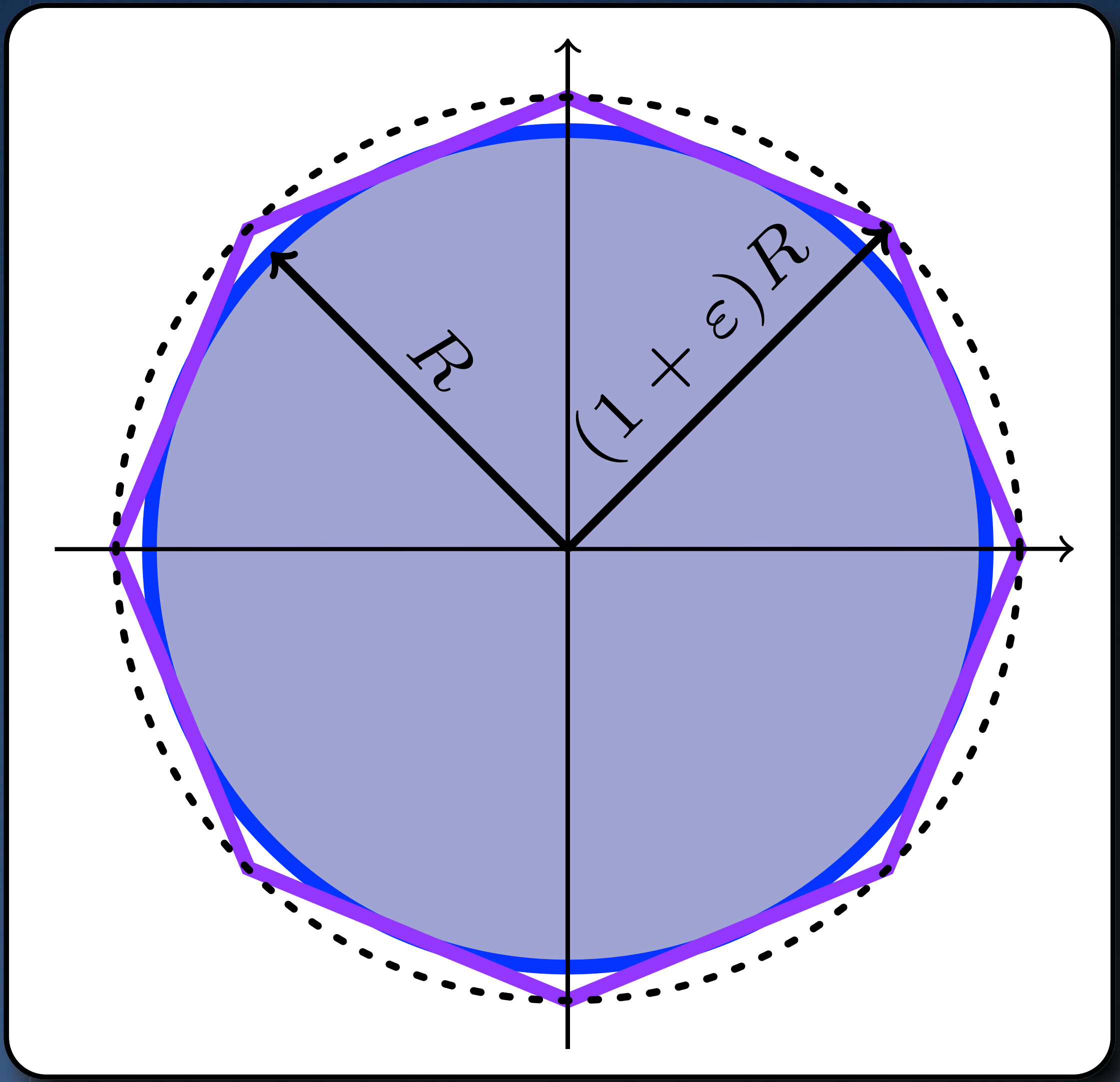
- y fraction of the portfolio invested in each of n assets.
- $\bar{a}$ expected returns of assets.
- $Q^{1/2}$ positive semidefinite square root of the covariance matrix Q of returns.
- Hold at most 10 assets.

Avg. of Solve Times [s] for $n \in \{20, 30\}$ (CPLEX v11)



Extended Formulation for Lifted LP

- Approximation of Second Order Cone by Ben-Tal and Nemirovski (Glineur).
- $O(d \log(1/\varepsilon))$ variables and constraints for quality ε.
- Problem:
  - Fixed a-priori quality: no dynamic improvement.
  - e.g. $\varepsilon = 0.01$ for portfolio had to be calibrated.



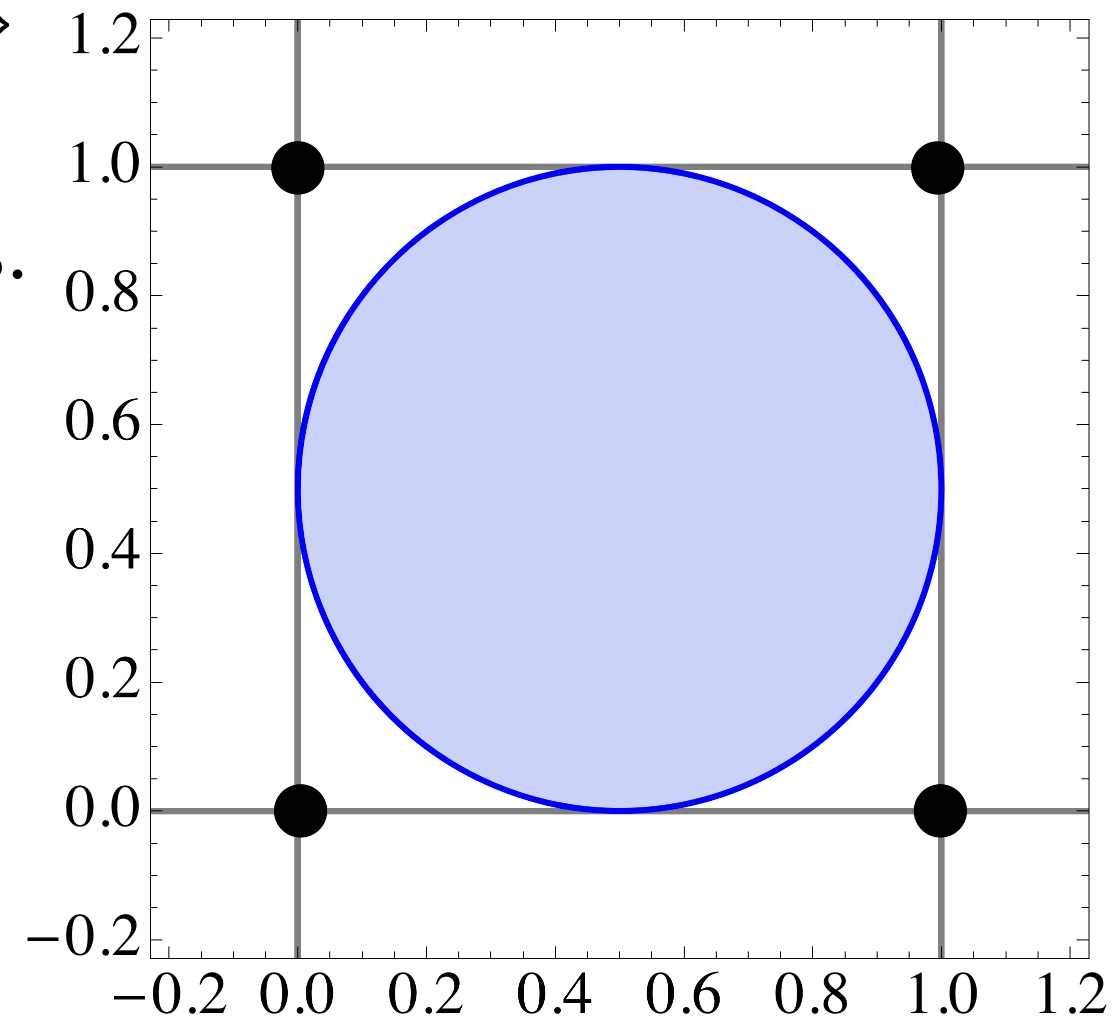
Dynamic Lifted Approximations

Towards a Dynamic Lifted LP

- Separable approach by Tawarmalani and Sahinidis ’05 and Hijazi et al. ‘14

$$B^n := \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} \left(x_i - \frac{1}{2} \right)^2 \leq \frac{n-1}{4} \right\}$$

Showing $B^n \cap \mathbb{Z}^n = \emptyset$ requires 2^n cuts.



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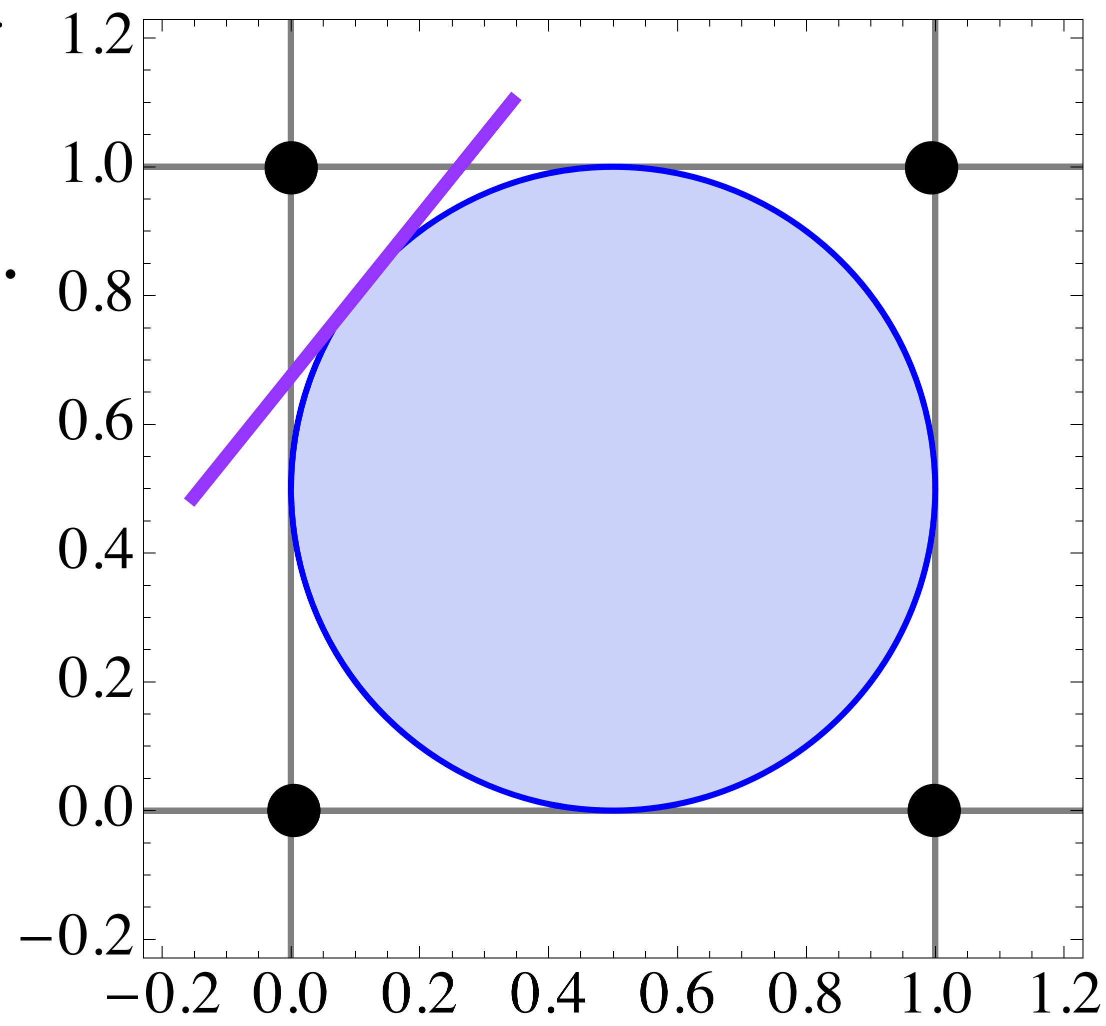

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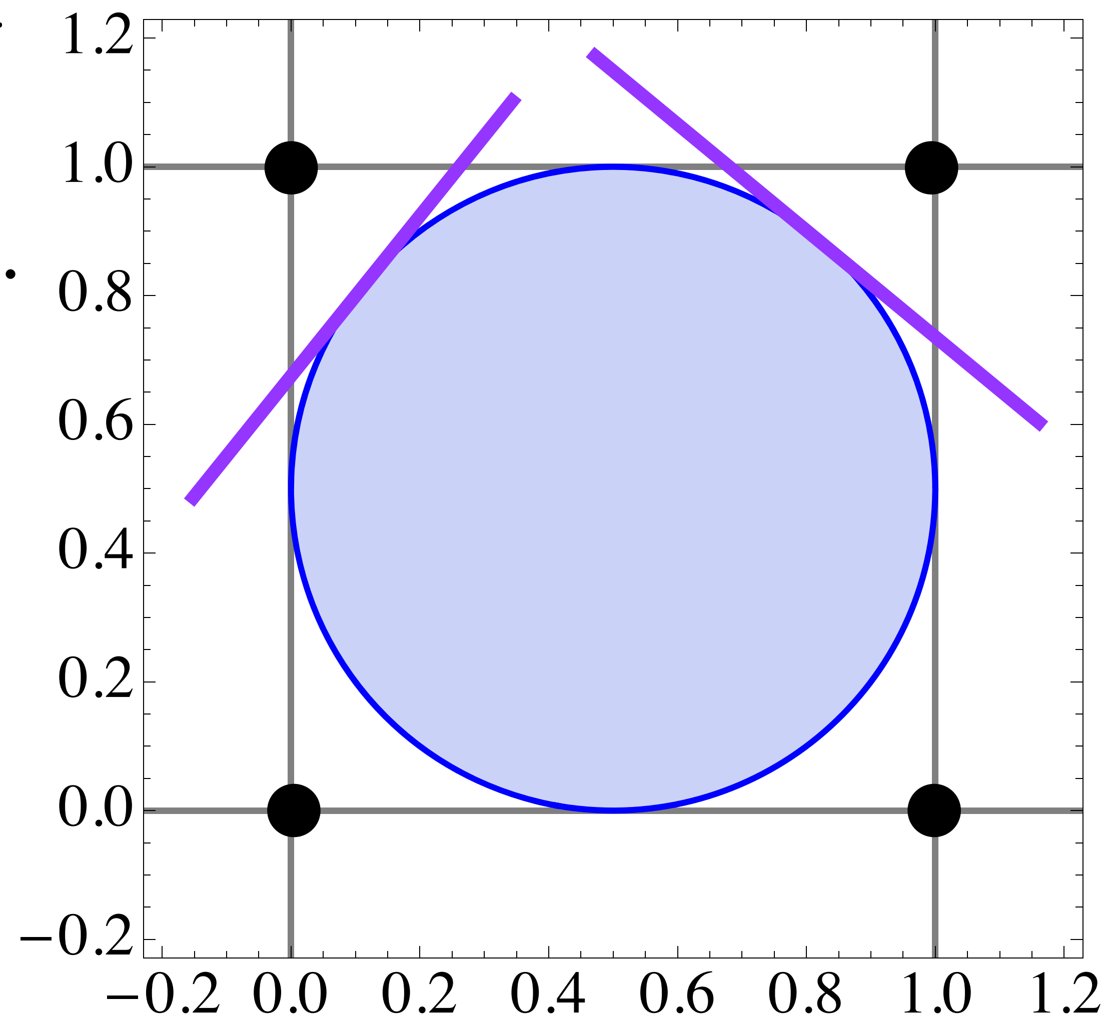

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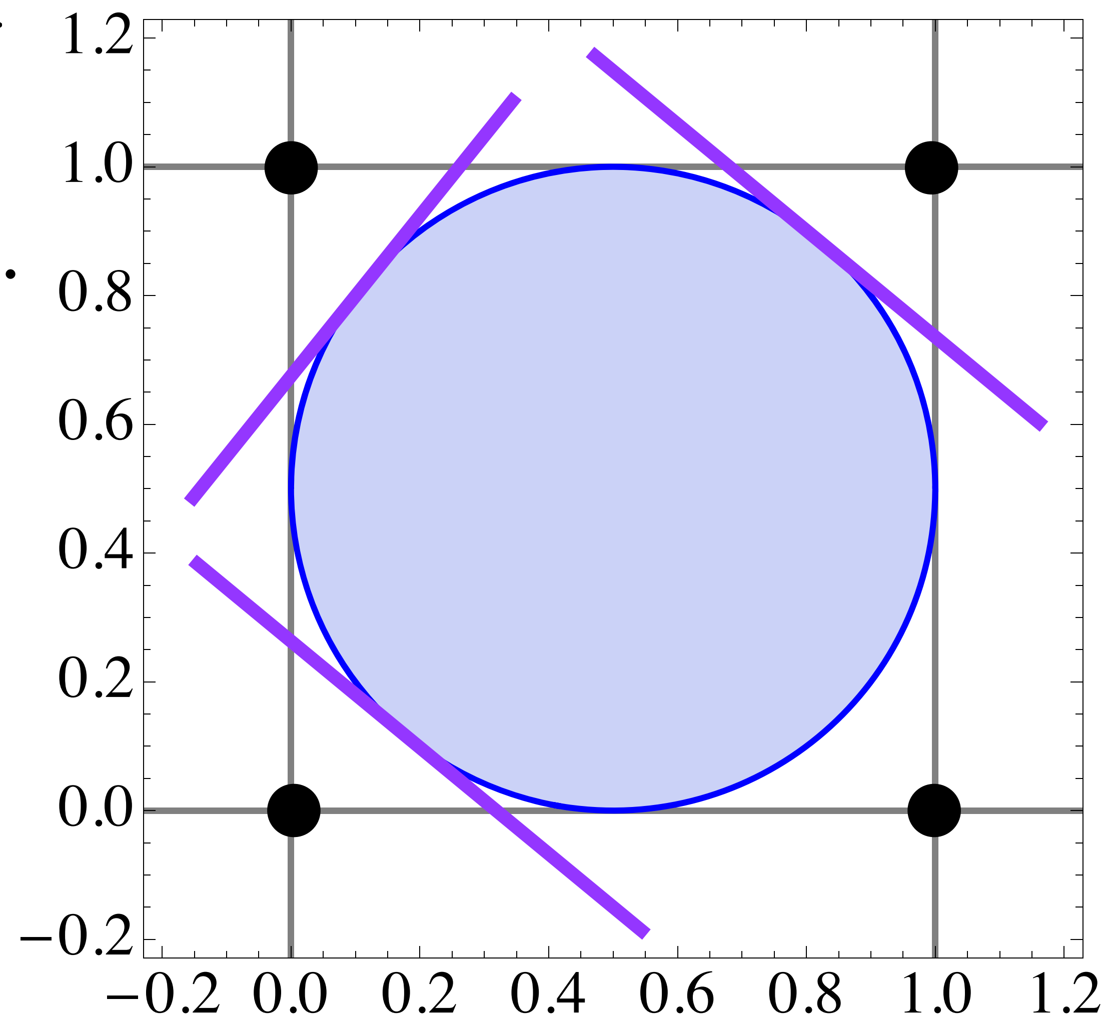

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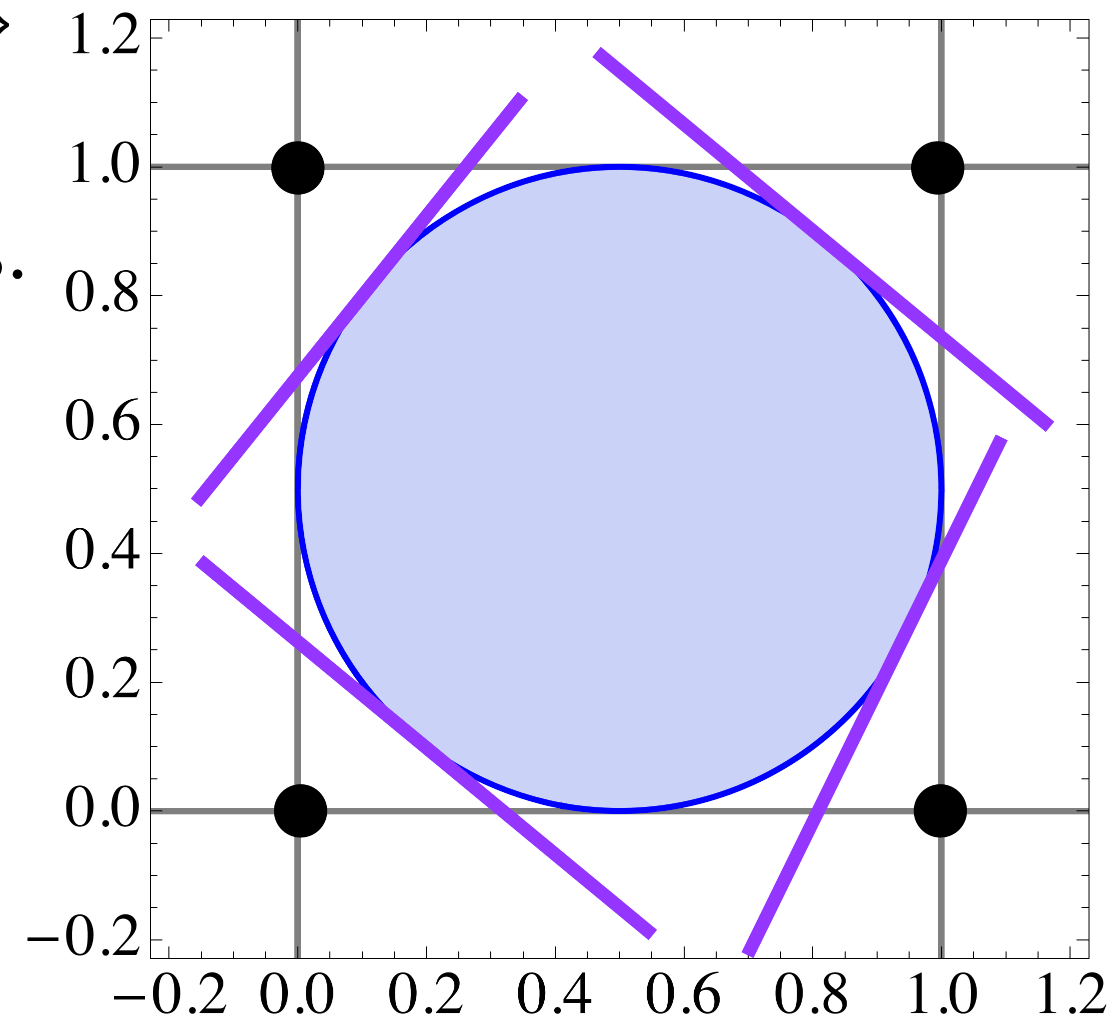

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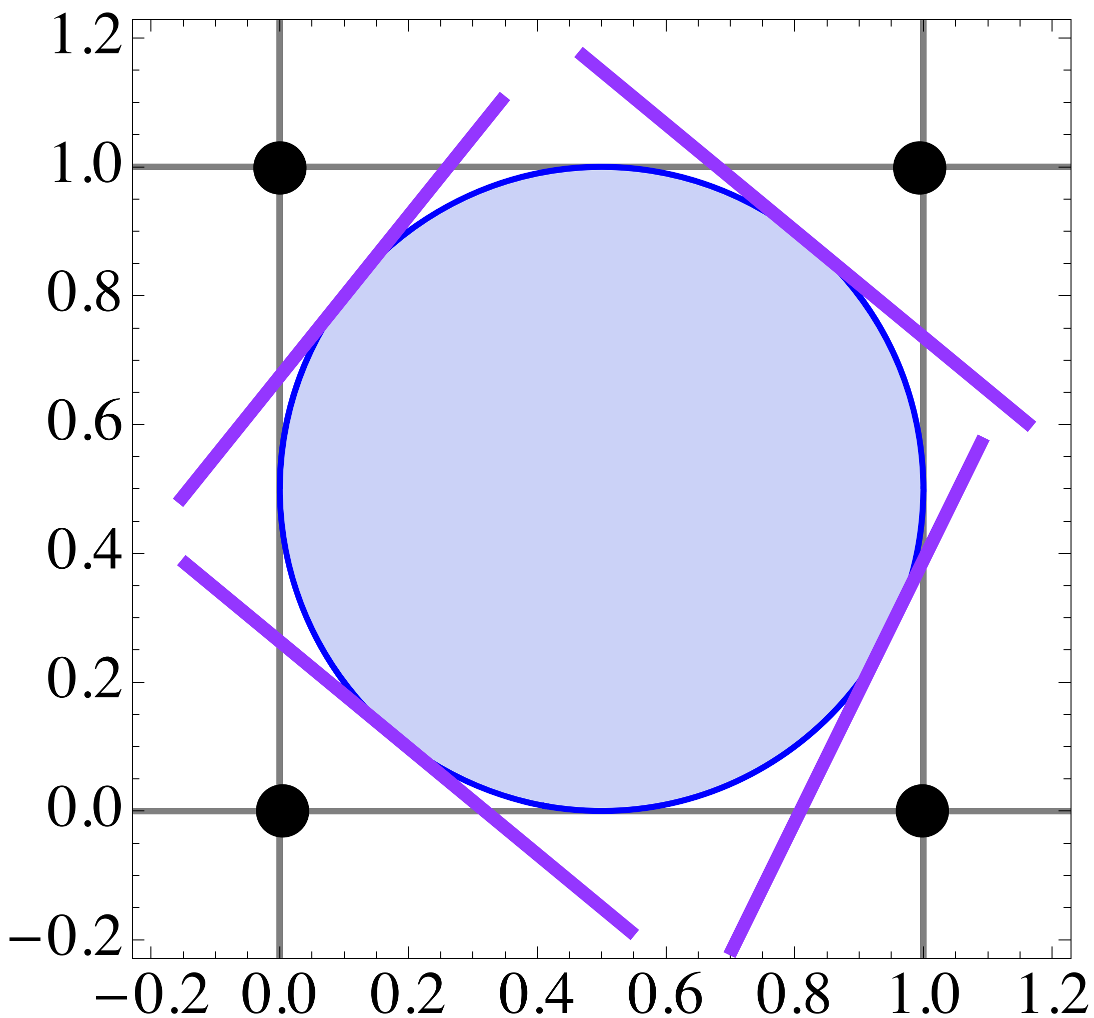
$$B^n := \left\{ x \in \mathbb{R}^n \;:\; \sum_{i=1}^{n} \left(x_i - \frac{1}{2} \right)^2 \le \frac{n-1}{4} \right\}$$

Extended formulation of B^n:

$$\left(x_i - \frac{1}{2} \right)^2 \le z_i \qquad \forall i \in [n]$$

$$\sum_{i=1}^{n} z_i \le \frac{n-1}{4}$$



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Extended formulation of B^n:

$$\left(x_i - \frac{1}{2} \right)^2 \leq z_i \qquad \forall i \in [n]$$

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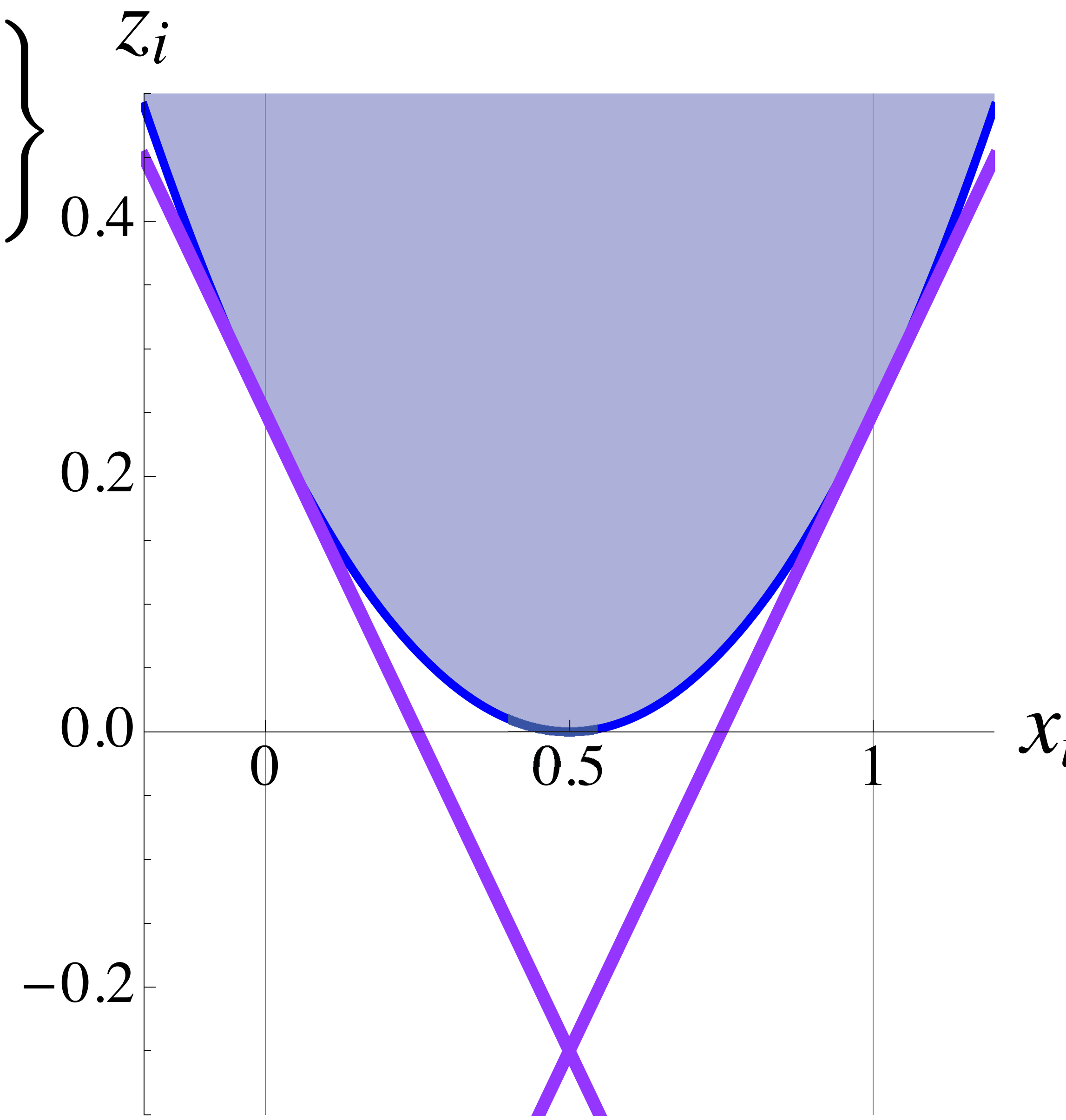
$B^n \cap \mathbb{Z}^n = \emptyset$ with only $2n$ cuts on extended formulation.



Towards a Dynamic Lifted LP

- Separable approach by Tawarmalani and Sahinidis '05 and Hijazi et al. '14

Separable approach works for any set of the form:

$$C = \left\{ x \in \mathbb{R} \;:\; \sum_{i=1}^{n} f_i(x_i) \leq 1 \right\}$$

or

$$C = \left\{ (x, t) \in \mathbb{R} \times \mathbb{R} \;:\; \sum_{i=1}^{n} f_i(x_i) \leq t \right\}$$

for convex $f_i : \mathbb{R} \to \mathbb{R}$

Problem 1: Classical

$$
\begin{aligned}
\max_{x,y} \quad & \bar{a}y \\
s.t. \quad & \\
& ||Q^{1/2}y||_2 \leq \sigma \\
& \sum_{j=1}^{n} y_j = 1 \\
& y_j \leq x_j \qquad \forall j \in \{1, \ldots, n\} \\
& \sum_{j=1}^{n} x_j \leq K \\
& x \in \{0, 1\}^n \\
& y \in \mathbb{R}^n_+
\end{aligned}
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- y fraction of the portfolio invested in each of n assets.
- $\bar{a}$ expected returns of assets.
- $Q^{1/2}$ positive semidefinite square root of the covariance matrix Q of returns.
- K maximum number of assets to hold.

Problem 2 : Shortfall

$$\max_{x,y} \quad \bar{a}y$$

$$s.t.$$

$$||Q^{1/2}y||_2 \leq \sigma$$

$$\sum_{j=1}^{n} y_j = 1$$

$$y_j \leq x_j \qquad \forall j \in \{1, \ldots, n\}$$

$$\sum_{j=1}^{n} x_j \leq K$$

$$x \in \{0,1\}^n$$

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Problem 2 : Shortfall

$$\max_{x,y} \quad \bar{a}y$$

$$s.t.$$

$$||Q^{1/2}y||_2 \leq \frac{\bar{a}y - W_i^{low}}{\Phi^{-1}(\eta_i)} \qquad i \in \{1,2\}$$

$$\sum_{j=1}^{n} y_j = 1$$

$$y_j \leq x_j \qquad \forall j \in \{1,\ldots,n\}$$

$$\sum_{j=1}^{n} x_j \leq K$$

$$x \in \{0,1\}^n$$

$$y \in \mathbb{R}_+^n$$

- y fraction of the portfolio invested in each of n assets.
- $\bar{a}$ expected returns of assets.
- $Q^{1/2}$ positive semidefinite square root of the covariance matrix Q of returns.
- K maximum number of assets to hold.
- Approximation of $\mathrm{Prob}(\bar{a}y \geq W_i^{low}) \geq \eta_i$

Extended Formulation for SOCP

$$L^n = \left\{(x,t) \in \mathbb{R} \times \mathbb{R} \,:\, \|x\| \leq t\right\}$$

Extended formulation of $L^n =$ homogenization of B^n formulation:

$$x_i^2 \leq z_i \cdot t \qquad\qquad \forall i \in [n]$$

$$\sum_{i=1}^{n} z_i \leq t$$

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Extended formulation of L^n = homogenization of B^n formulation:

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Rotated SOCP cone

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- Dynamic Lifted LP B&B
  - LP B&B on extended form.

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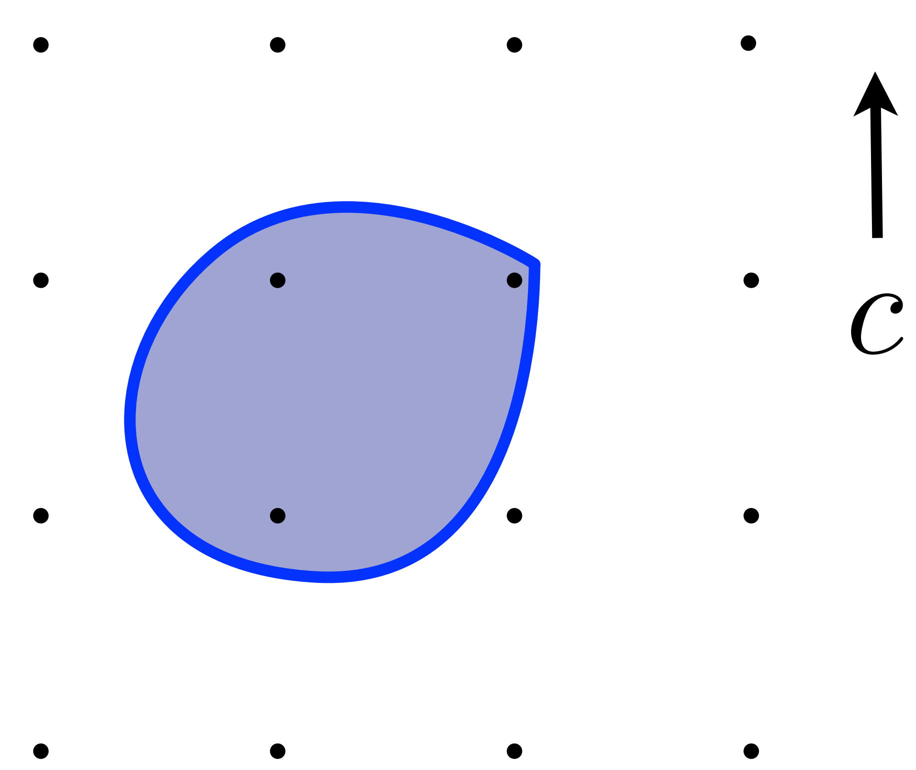

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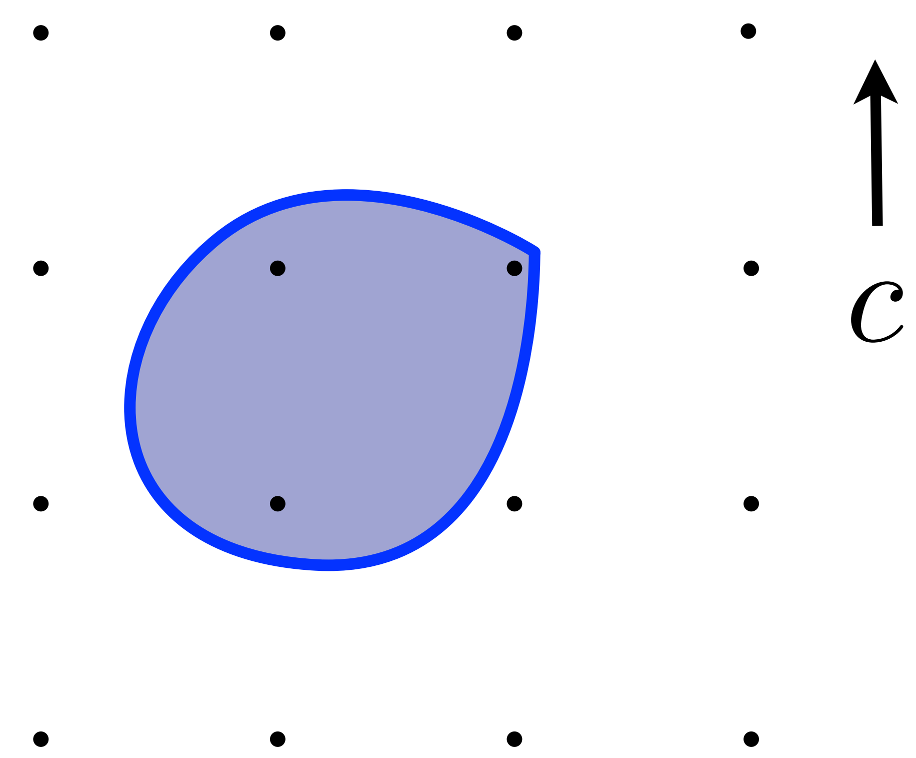

Computational Results

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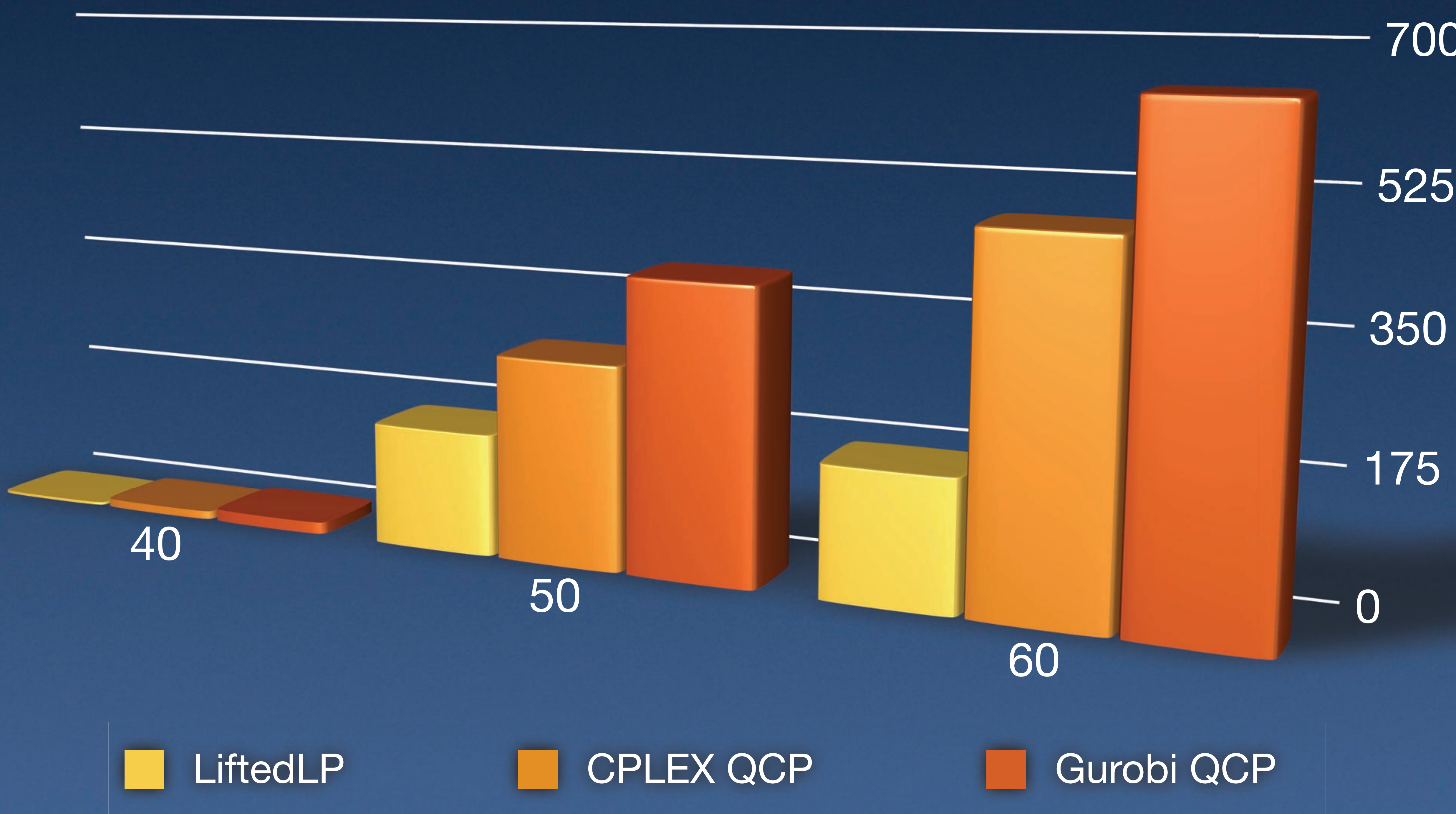
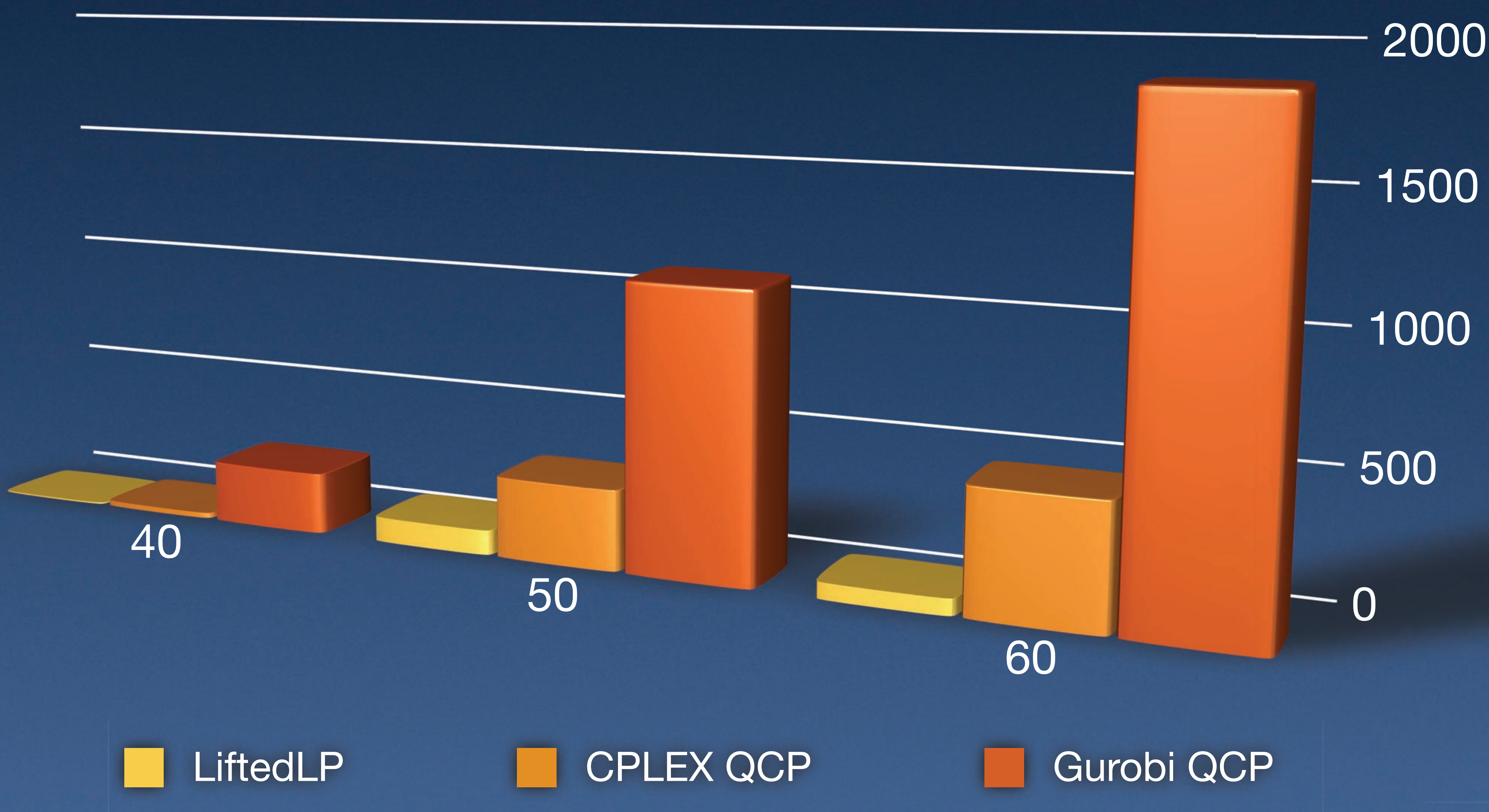
- Averages over 20 instances:
  - Classical and Shortfall. 40, 50 and 60 stocks.
- Solvers:
  - CPLEX/Gurobi QCP-BB on original formulation .
  - Lifted LP: Implemented in JuMP using CPLEX's branch, incumbent and heuristic callback.
  - CPLEX/Gurobi LP-BB on extended "separable" reformulation.

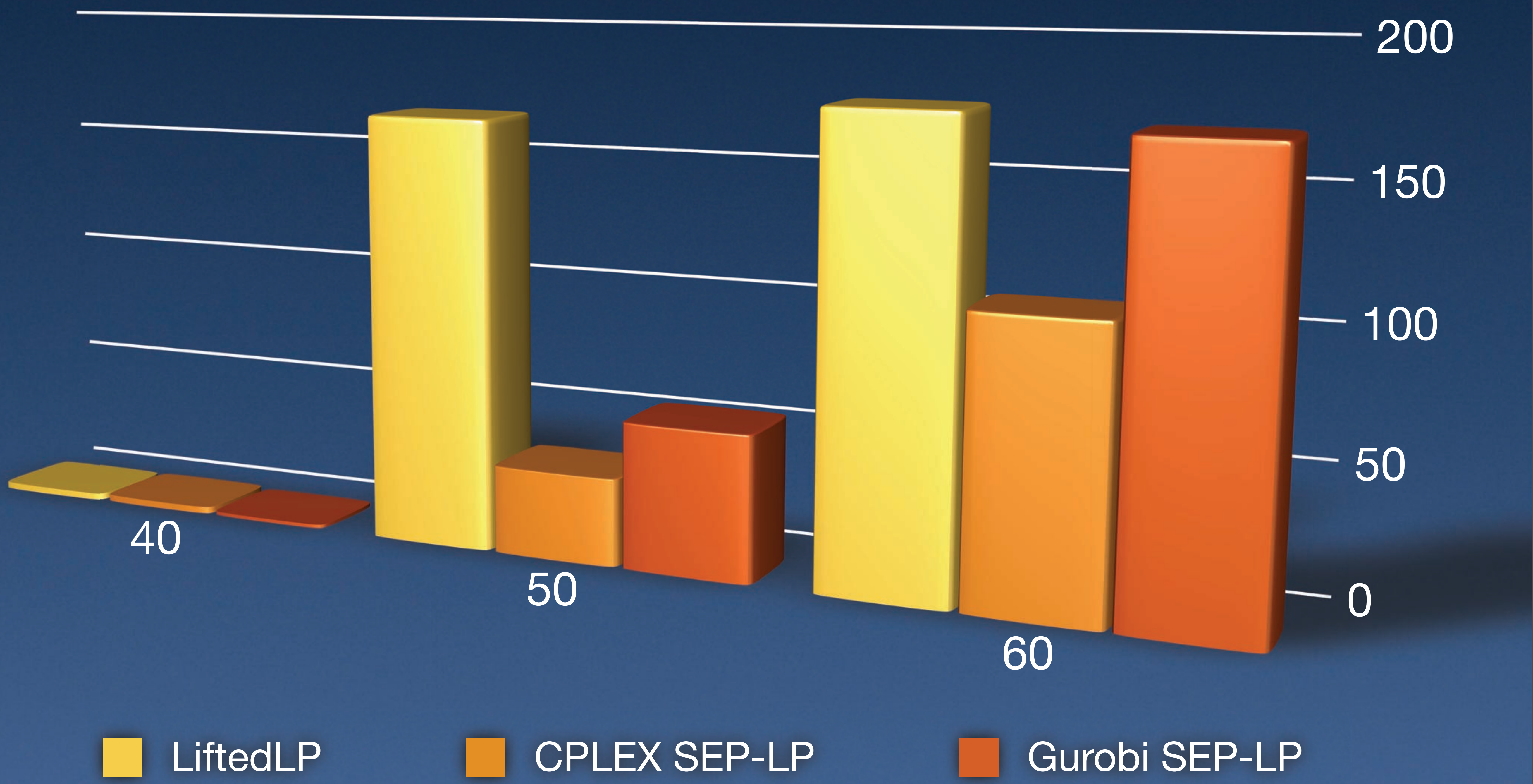
LiftedLP v/s QCP: Classical



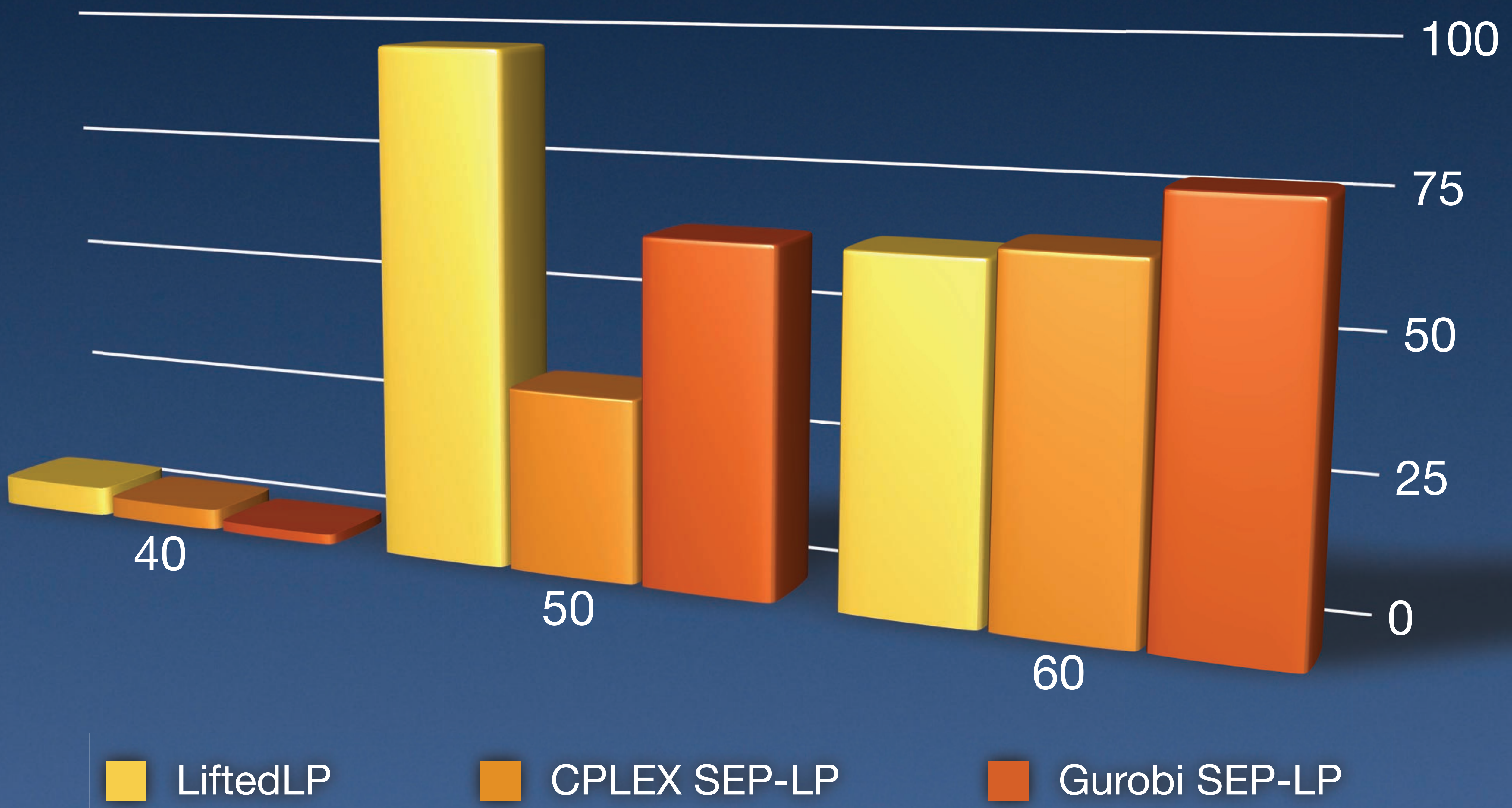
LiftedLP v/s QCP: Shortfall



LiftedLP v/s Dynamic : Classical



LiftedLP v/s Dynamic : Shortfall



Summary

- Lifted LP: 200 lines of JuMP code in a weekend.
  - Developed by ORC students Iain Dunning, Joey Huchette and Miles Lubin
  - https://github.com/JuliaOpt/JuMP.jl
  - Poster at MIP 2014. OSU, July 21st
  - Talk at INFORMS. San Francisco, November
- Dynamic Lifted LP:
  - Comparable performance with simple reformulation.