Extended Formulations for Quadratic Mixed Integer Programming

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Joint work with Iain Dunning, Joey Huchette and Miles Lubin

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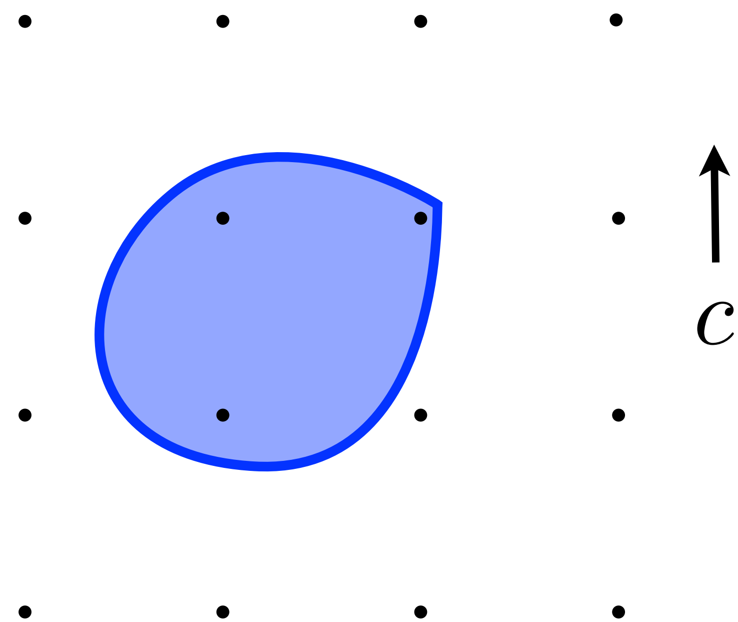
Nonlinear MIP B&B Algorithms

$$\max \quad \sum_{i=1}^{n} c_i x_i$$

$$s.t.$$

$$g_i(x) \leq 0, \; i \in I, \quad x \in \mathbb{Z}^n$$

$$x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$$



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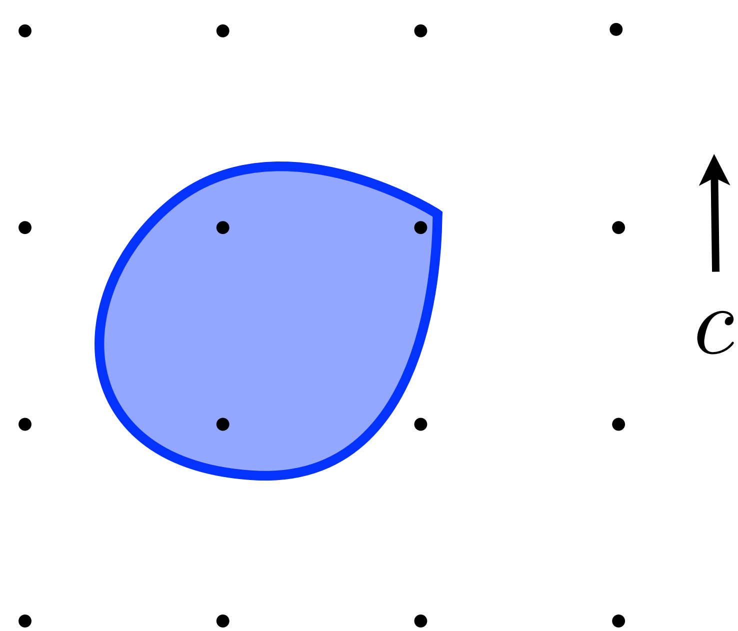

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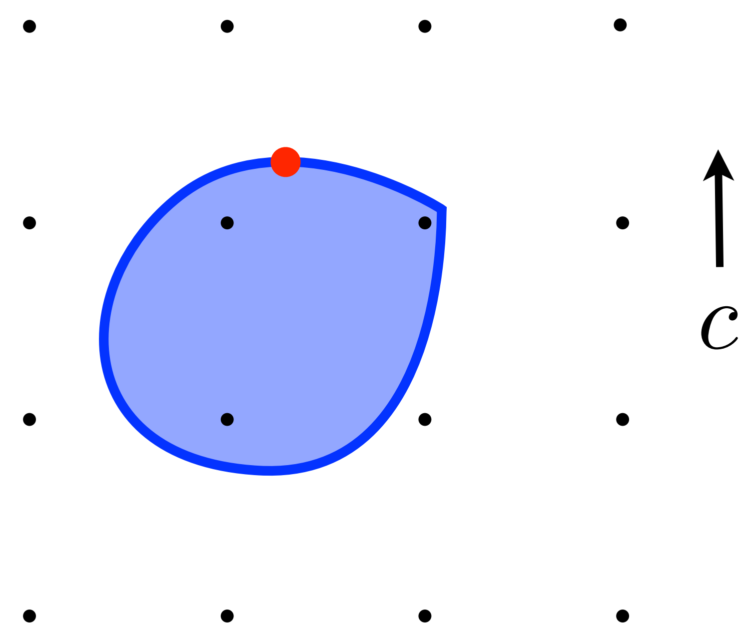

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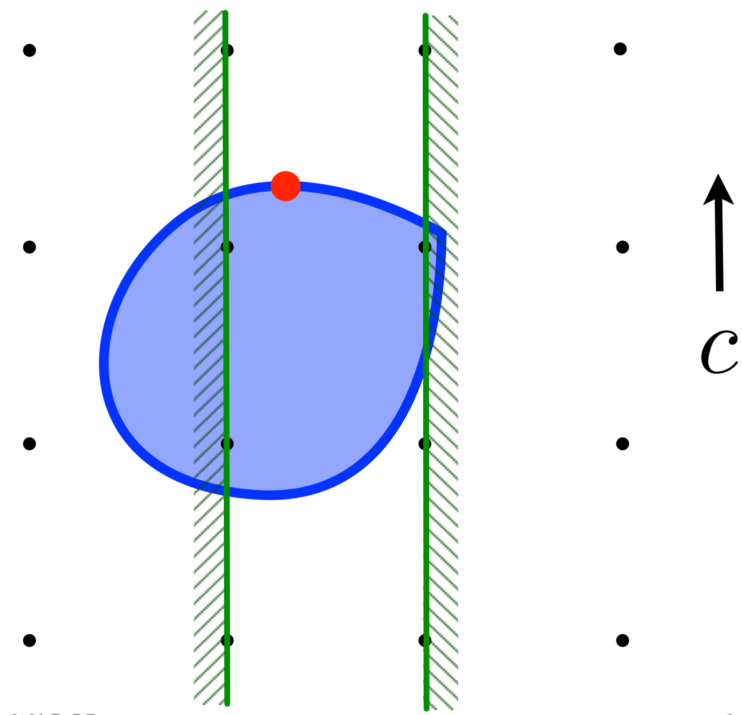

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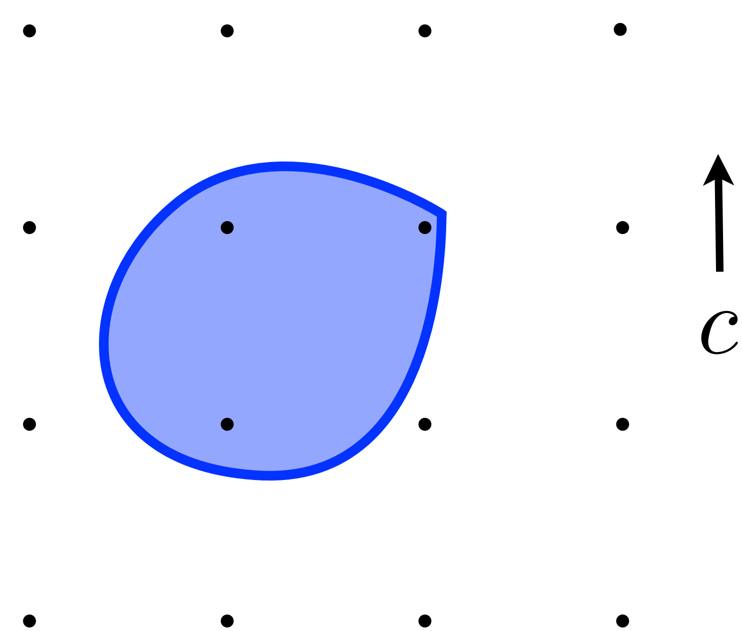

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- NLP (QCP) Based B&B
- (Dynamic) LP Based B&B
  - Few cuts = high speed.
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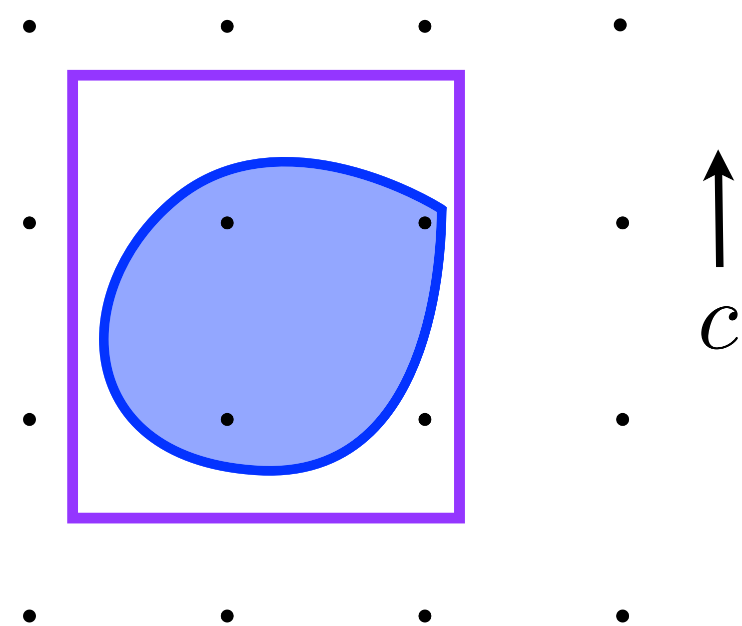

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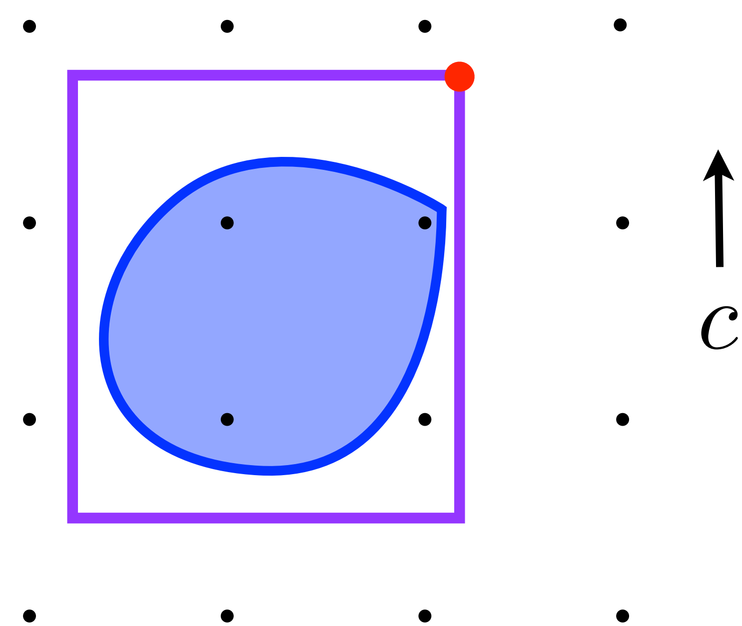

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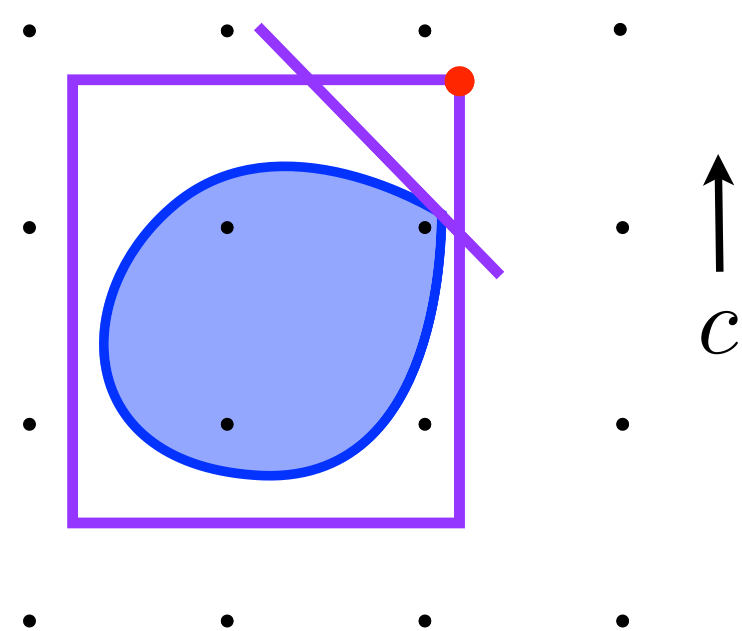

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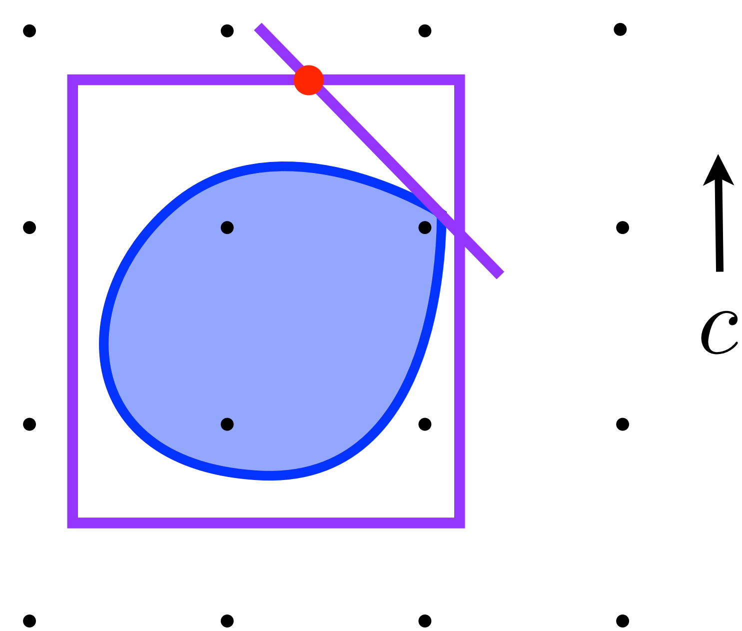

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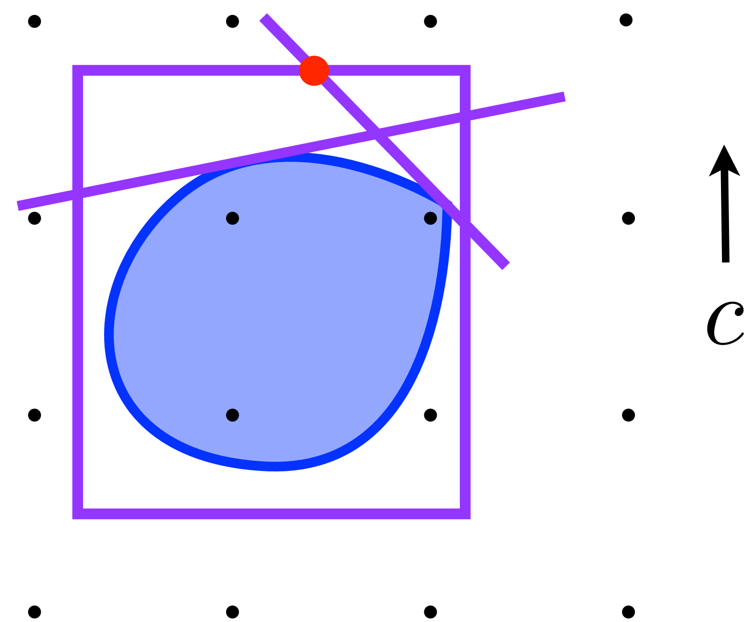

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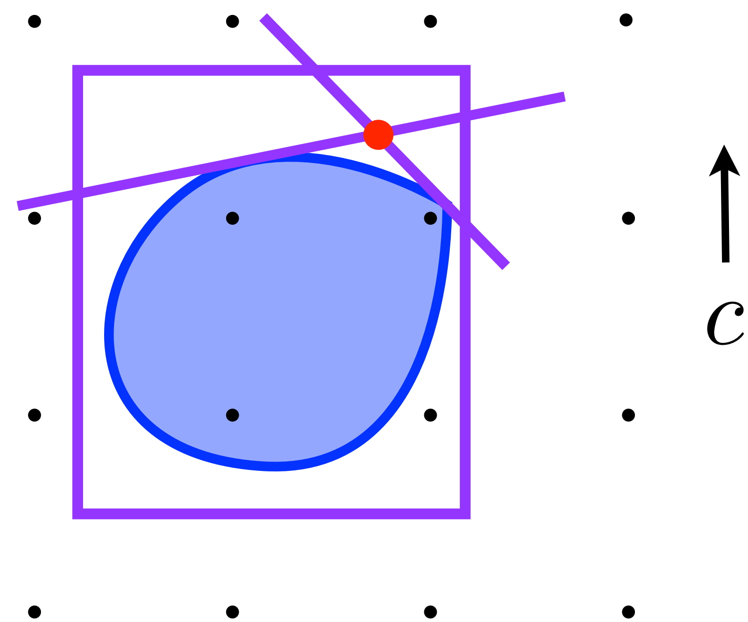

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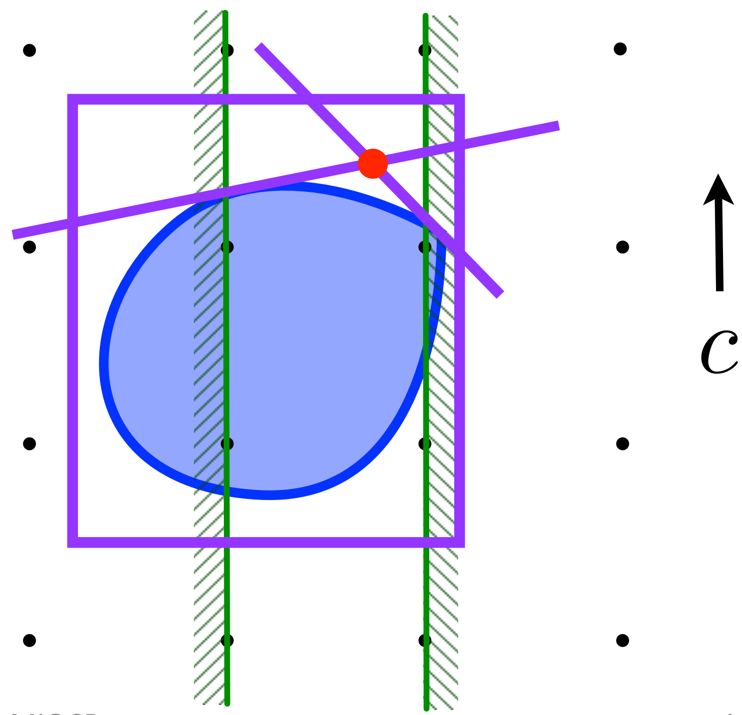

Nonlinear MIP B&B Algorithms

- NLP (QCP) Based B&B
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- Lifted LP B&B
  - Extended or Lifted relaxation.
  - Static relaxation
    - Mimic NLP B&B.
  - Dynamic relaxation
    - Standard LP B&B

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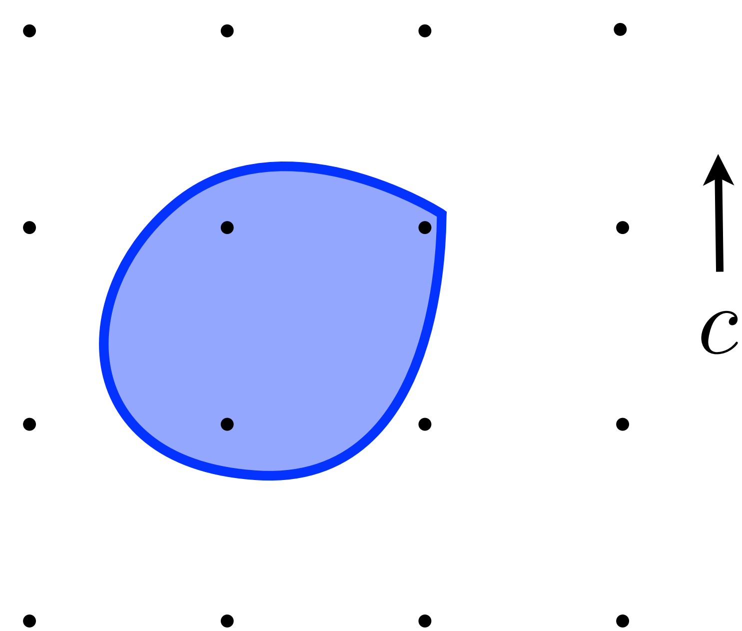

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$$s.t. \quad Ax + Dz \leq b,$$

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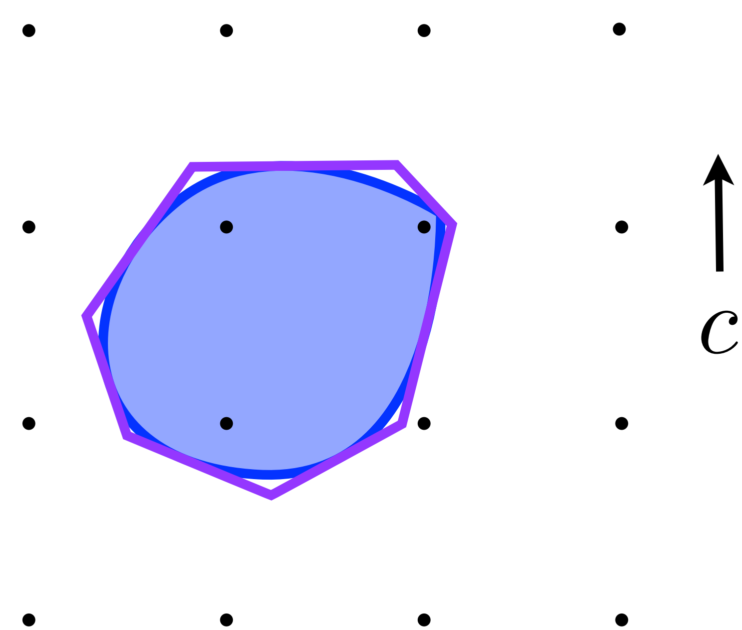
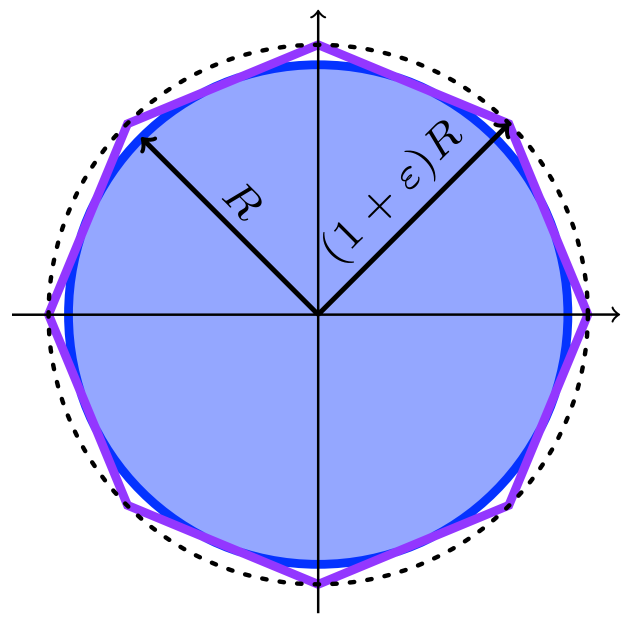


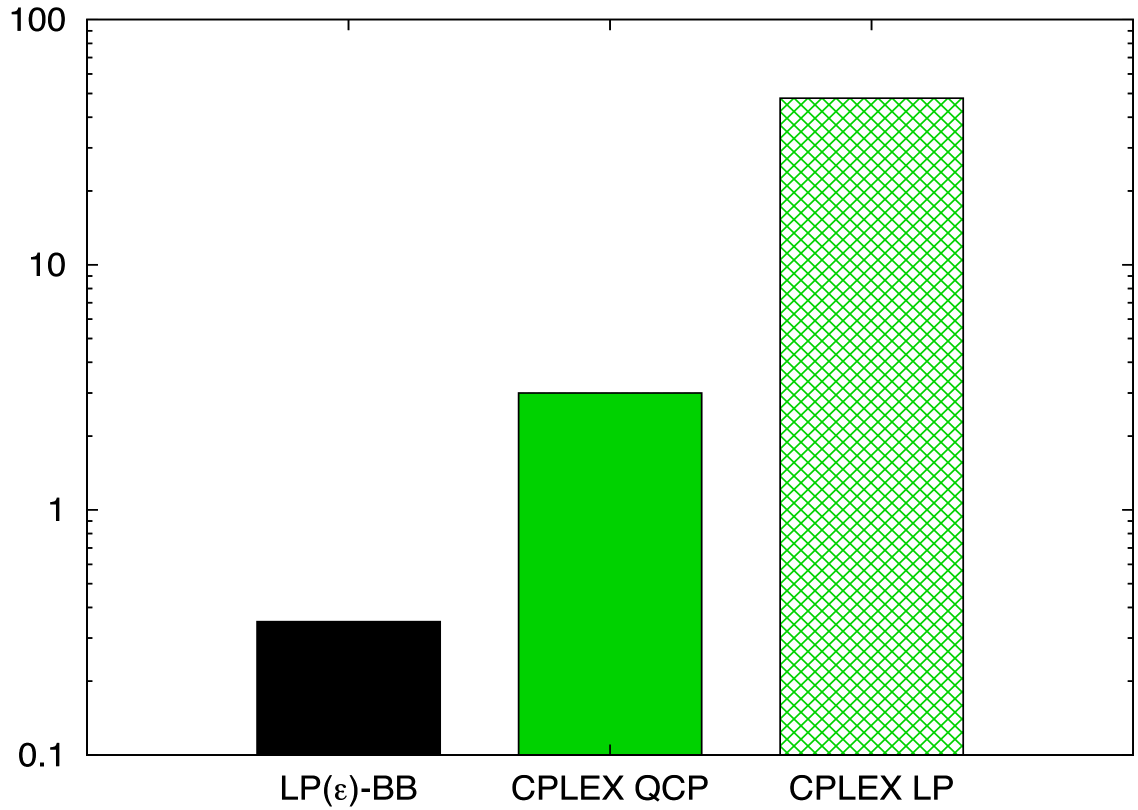
Static Lifted LP for Conic Quadratic MIP

- Approximation of Second Order Cone of dimension n by Ben-Tal and Nemirovski (Glineur).
- $O(n \log(1/\varepsilon))$ variables and constraints for quality ε.
  - Exponential increment in # of constraints through projection.
- Problem:
  - Fixed a-priori quality: no dynamic improvement (e.g. $\varepsilon = 0.01$ best for some portfolio optimization problems)



Correct Quality = Significant Improvement

- Results from V., Ahmed and Nemhauser ’08.



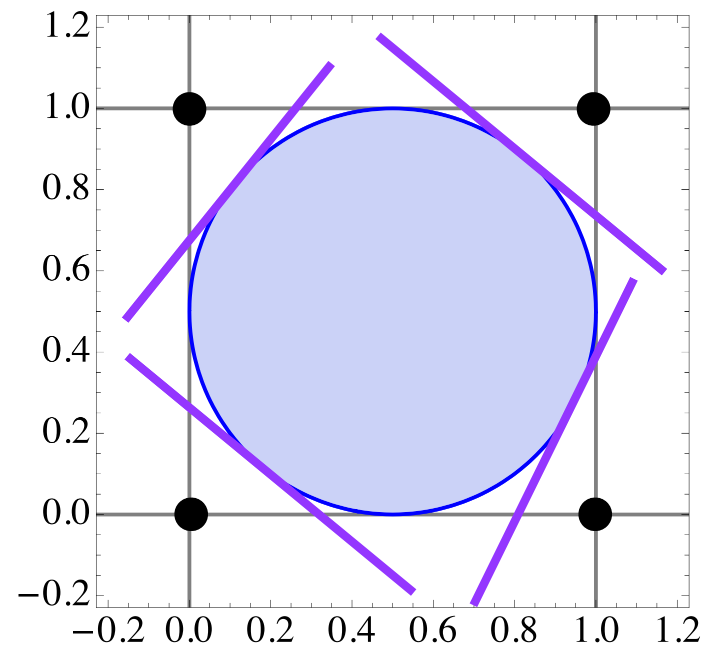
Average Solve Times [s]

Dynamic Lifted LP for Separable Problems

- Motivating example from Hijazi et al. '14

$$F^n := \left\{ x \in \mathbb{R}^n \; : \; \sum_{i=1}^{n} \left(x_i - \frac{1}{2} \right)^2 \leq \frac{n-1}{4} \right\}$$

Showing $F^n \cap \mathbb{Z}^n = \emptyset$ requires 2^n cuts.



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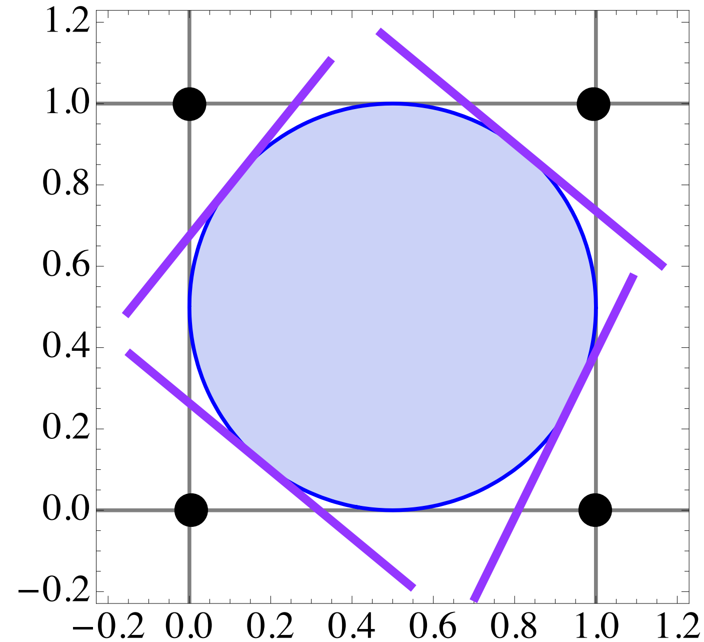
Extended formulation of F^n:

$$\left(x_i - \frac{1}{2} \right)^2 \leq z_i \qquad \forall i \in [n]$$

$$\sum_{i=1}^{n} z_i \leq \frac{n-1}{4}$$



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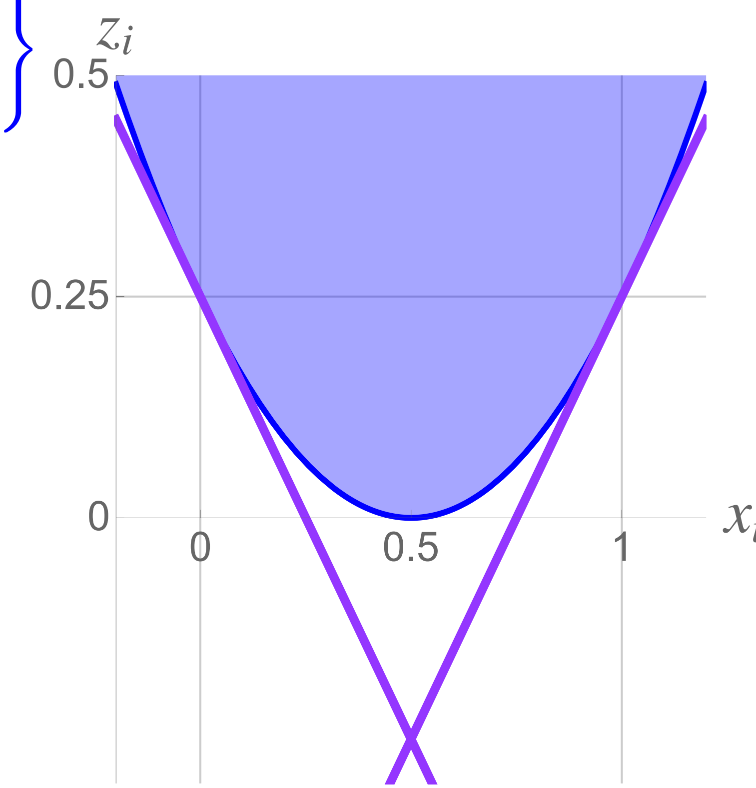

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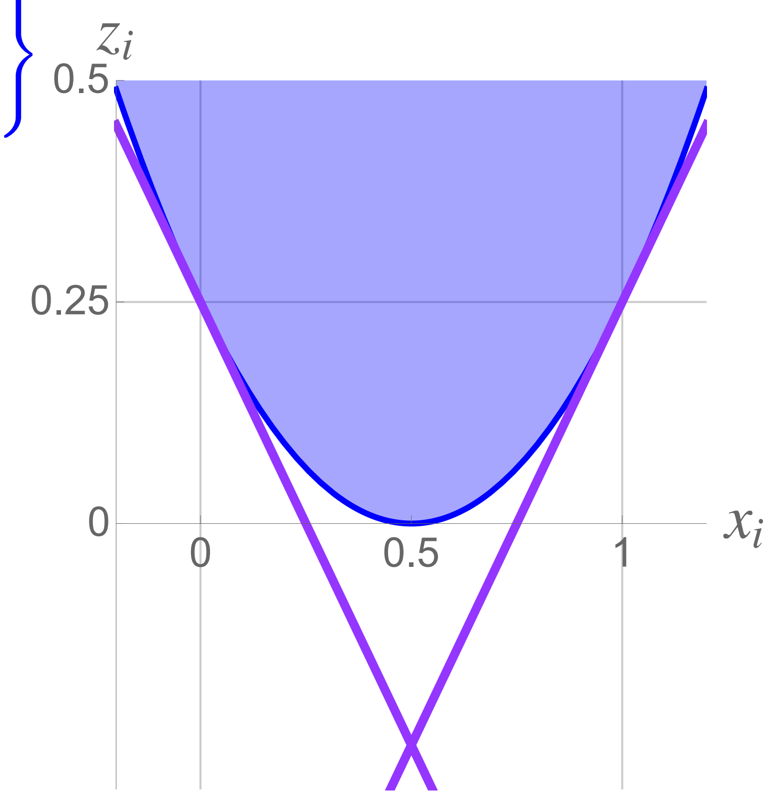

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$B^n \cap \mathbb{Z}^n = \emptyset$ with only $2n$ cuts on extended formulation.

Significant Improvement For Many Problems

- Tawarmalani and Sahinidis '05:

$$f_j : \mathbb{R} \to \mathbb{R} \text{ convex differentiable}$$

Lifted Relaxation of $F := \left\{ (y_0, y) \in \mathbb{R}^{n+1} \ : \ \sum_{j=1}^{n} f_j(y_j) \leq y_0 \right\}$:

$$f_j(\gamma) + f_j'(\gamma)(y_j - \gamma) \leq w_j \quad \forall \gamma \in \Gamma_j, \quad j \in [n]$$

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- Polynomial (degree n) increment in # of constraints through projection

Separable Approach for Conic Quadratic?

- “Separable Sets” include many quadratics:
  - Euclidean Ball

$$B^n := \left\{ y \in \mathbb{R}^n \ : \ \sum\nolimits_{j=1}^n y_j^2 \leq 1 \right\}$$

  - Paraboloids

$$Q^n := \left\{ (y_0, y) \in \mathbb{R}^{n+1} \ : \ \sum\nolimits_{j=1}^n y_j^2 \leq y_0 \right\}$$

- Does not include Lorentz/SOCP cone:

$$L^n := \left\{ (y_0, y) \in \mathbb{R}^{n+1} \ : \ \sqrt{\sum\nolimits_{j=1}^n y_j^2} \leq y_0 \right\}$$

Separable to Conic: Homogenize

- From Euclid to Lorentz

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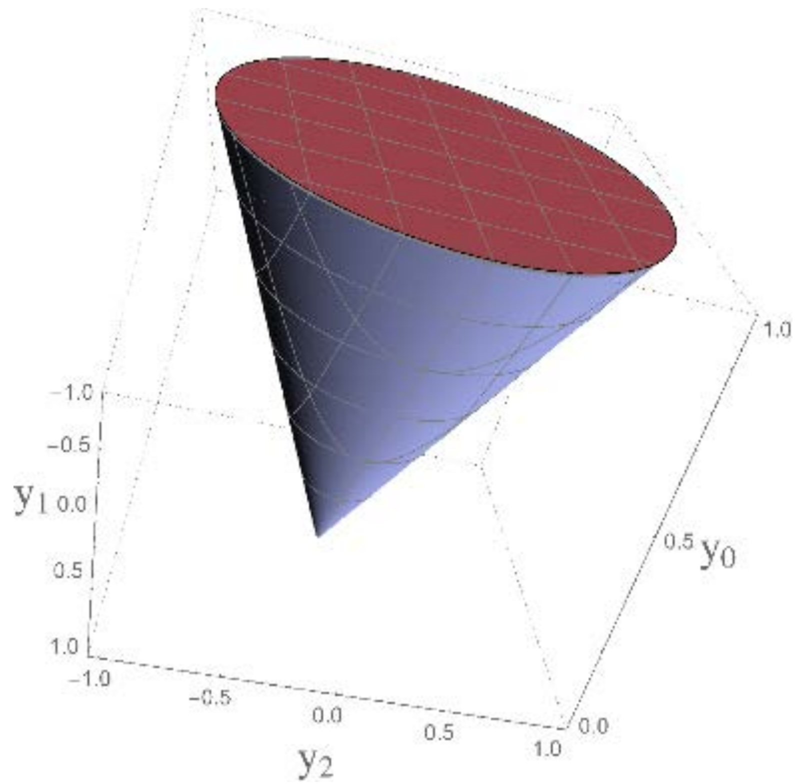
$$L^n = \operatorname{cone}\left(\{1\} \times B^n\right)$$



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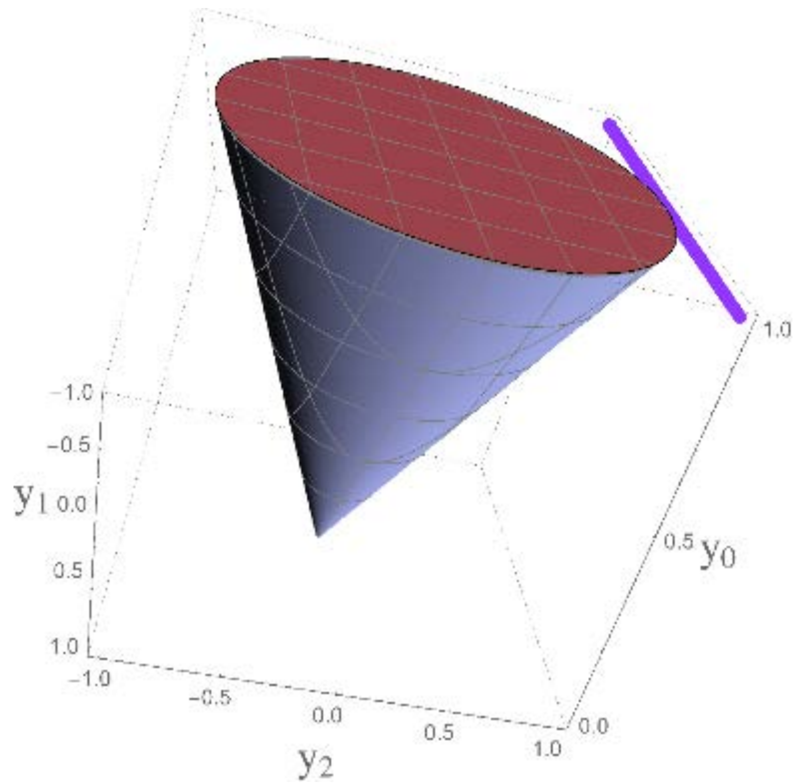
$$B^n \subseteq P$$



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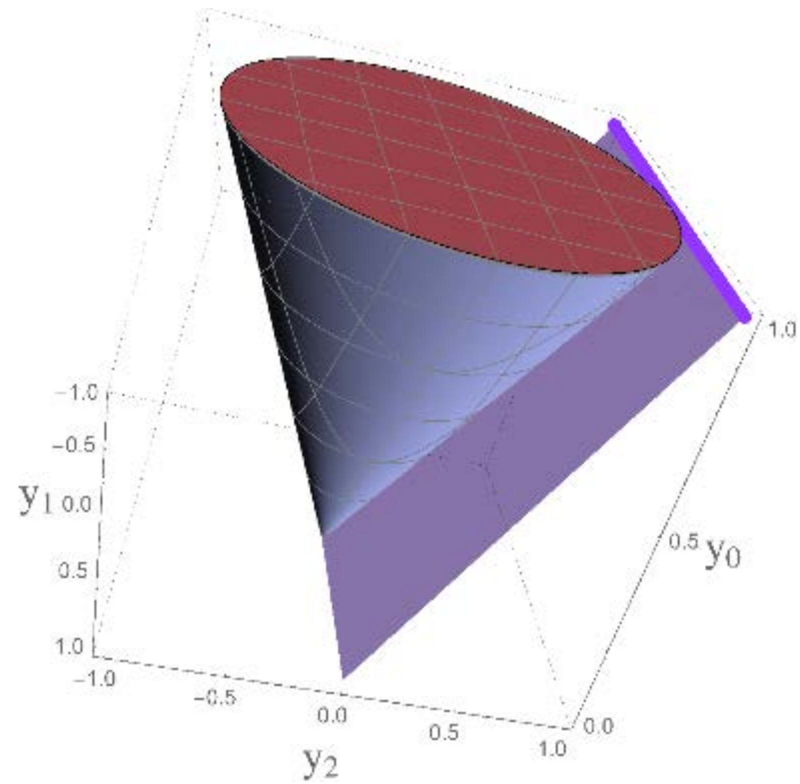

$$B^n \subseteq P \Rightarrow L^n \subseteq \operatorname{cone}\left(\{1\} \times P\right)$$

Lifted Relaxation for Separable Conic Sets

$f_j : \mathbb{R} \to \mathbb{R}$ convex, differentiable and 1-coercive

$$C := \left\{ y \in \mathbb{R}^n \; : \; \sum_{j=1}^{n} f_j(y_j) \leq 1 \right\}$$

Lifted Relaxation of $\operatorname{cone}\left(\{1\} \times C\right)$:

$$\left(f\left(\gamma\right) - \gamma f'\left(\gamma\right)\right) y_0 + f'\left(\gamma\right) y \leq w_j \quad \forall \gamma \in \Gamma_j,\, j \in [n]$$

$$\sum_{j=1}^{n} w_j \leq y_0$$

$$0 \leq y_0$$

Lifted Reformulation for Conic Quadratic Sets

- Lifted Reformulation of $L^n := \left\{(y_0, y) \in \mathbb{R}^{n+1} \,:\, \|y\|_2 \leq y_0\right\}$:

$$y_i^2 \leq z_i \cdot y_0 \qquad \forall i \in [n]$$

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Rotated SOCP cone

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Rotated SOCP cone

- Lifted polyhedral relaxation automatic from standard polyhedral approximation of (rotated) SOCP cone:
  - Dynamic Lifted LP-based algorithm:
    1. Replace every SOCP cone with lifted reformulation
    2. Solve with standard LP-based algorithm

Computational Experiments 1: Solvers

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- NLP-based Branch-and-Bound:
  - **CPLEXCP**
  - **GurobiCP**

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  - **GurobiSepLP**: GurobiLP on lifted reformulation

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- CPLEX v 12.6 and Gurobi v 5.6.3

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- CPLEX v 12.6 and Gurobi v 5.6.3
- Time limit of 3,600 s on i7-3770 3.40GHz

Computational Experiments 1: Instances

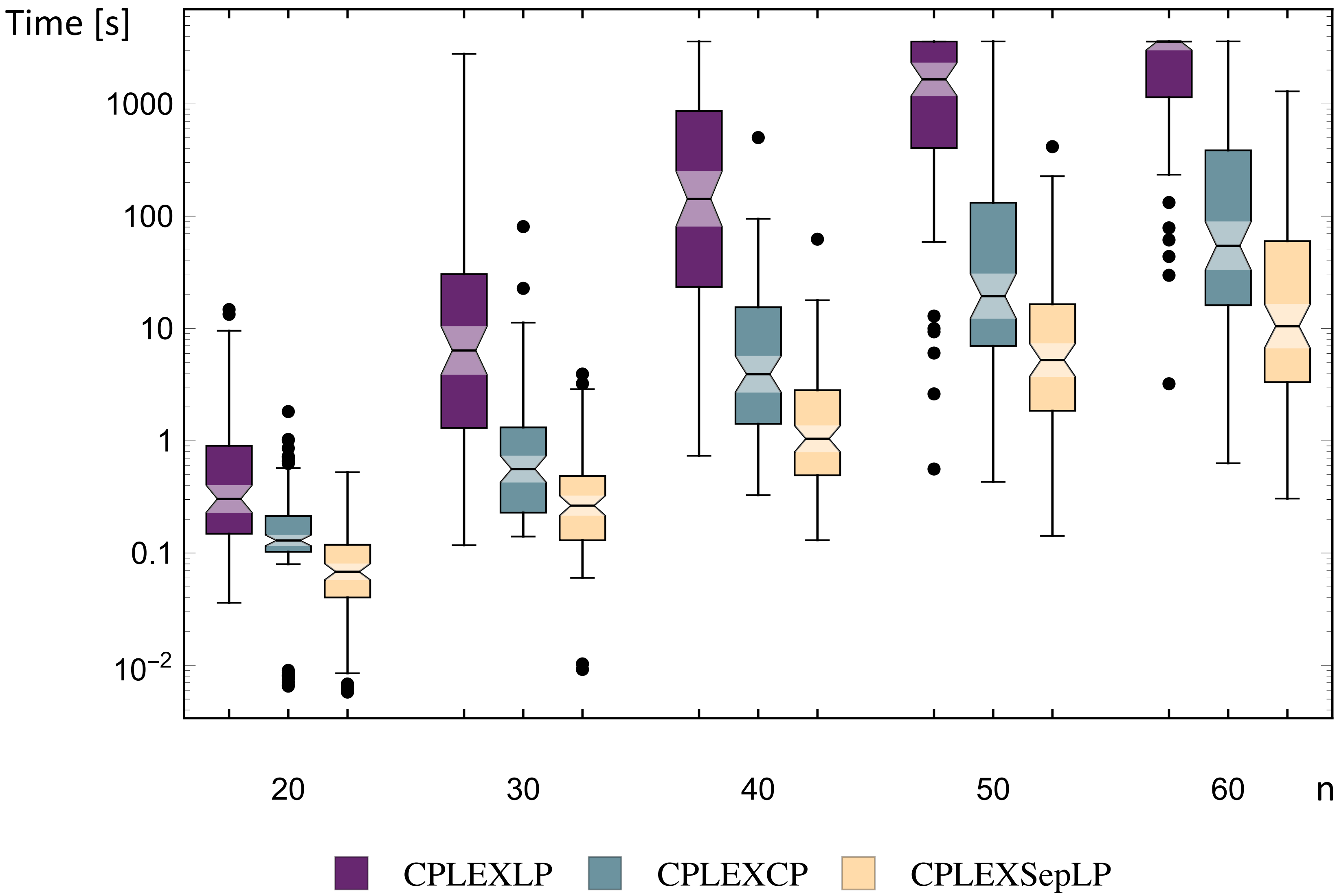
- Portfolio optimization problems:

Classical:

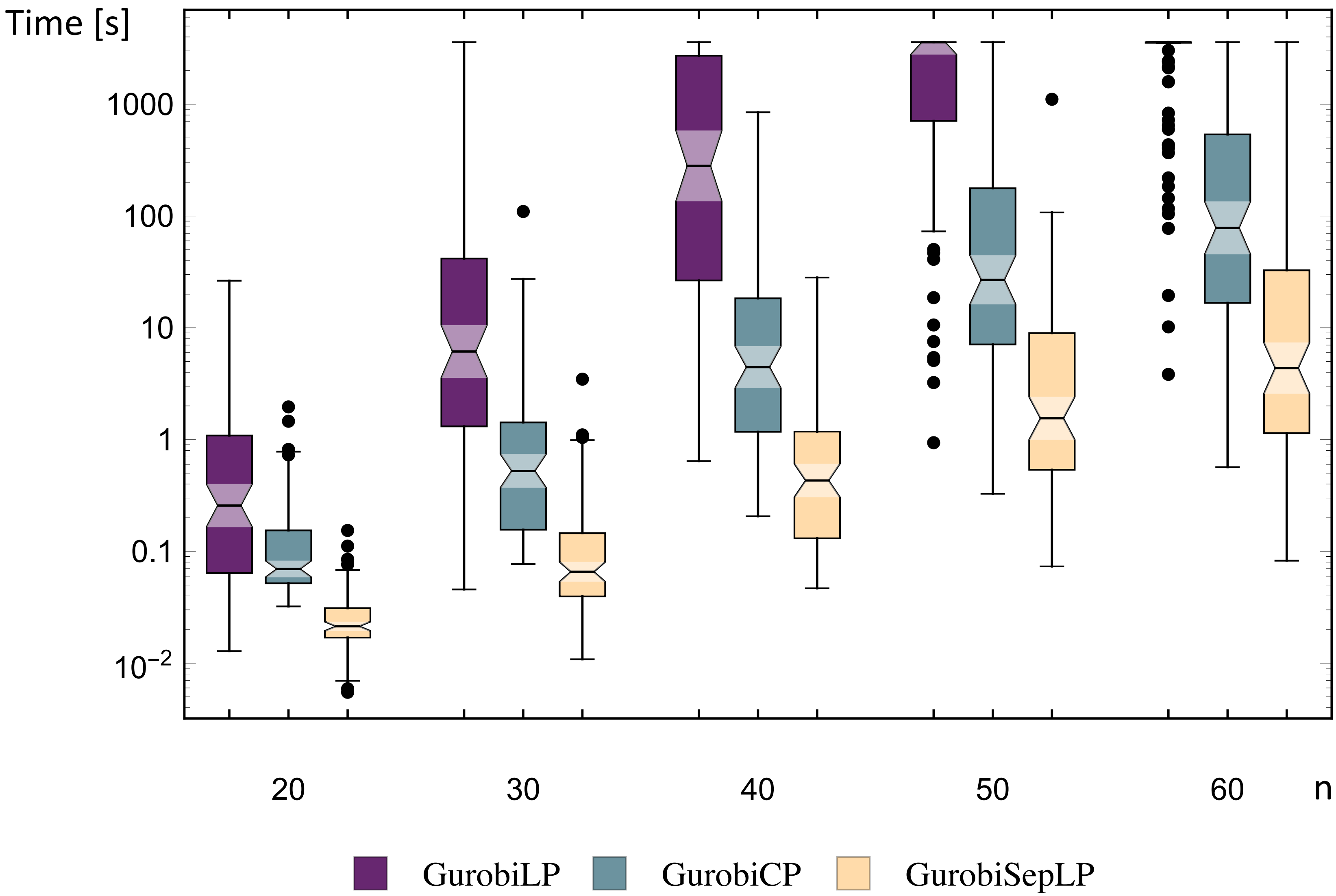
$$
\begin{aligned}
\max \quad & \bar{a}x \\
s.t. \quad & \\
& \left\| Q^{1/2} x \right\|_2 \leq \sigma \\
& \sum_{j=1}^{n} x_j = 1, \quad x \in \mathbb{R}^n_+ \\
& x_j \leq z_j \quad \forall j \in [n] \\
& \sum_{j=1}^{n} z_j \leq K, \quad z \in \{0,1\}^n
\end{aligned}
$$

- $\bar{a}$ expected returns.
- $Q^{1/2}$ square root of covariance matrix.
- K maximum number of assets.
- σ maximum risk.

Results for CPLEX: 100 instances per n



Results for Gurobi: 100 instances per n



Computational Experiments 2: More Solvers

- Static Lifted LP-based Branch-and-Bound:
  - **LiftedLP**: from V., Ahmed and Nemhauser ’08

Computational Experiments 2: More Solvers

- Static Lifted LP-based Branch-and-Bound:
  - **LiftedLP**: from V., Ahmed and Nemhauser ’08
- Static/Dynamic Lifted LP-based Branch-and-Bound:
  - **CPLEXSepLazy** / **GurobiSepLazy** : LiftedLP + Refinement through Separable extended formulation

Computational Experiments 2 : More Instances

- Portfolio optimization problems:

Classical:

$$
\begin{aligned}
\max \quad & \bar{a}x \\
s.t. \quad & \\
& \left\| Q^{1/2}x \right\|_2 \leq \sigma \\
& \sum_{j=1}^{n} x_j = 1, \quad x \in \mathbb{R}^n_+ \\
& x_j \leq z_j \quad \forall j \in [n] \\
& \sum_{j=1}^{n} z_j \leq K, \quad z \in \{0,1\}^n
\end{aligned}
$$

- $\bar{a}$ expected returns.
- $Q^{1/2}$ square root of covariance matrix.
- K maximum number of assets.
- σ maximum risk.

Computational Experiments 2 : More Instances

- Portfolio optimization problems:

Shortfall:

$$\max \quad \bar{a}x$$

$$s.t.$$

$$\Phi^{-1}(\eta_i)\left\|Q^{1/2}y\right\|_2 \leq \bar{a}y - W_i^{low} \qquad i \in \{1,2\}$$

$$\sum_{j=1}^{n} x_j = 1, \quad x \in \mathbb{R}_+^n$$

$$x_j \leq z_j \quad \forall j \in [n]$$

$$\sum_{j=1}^{n} z_j \leq K, \quad z \in \{0,1\}^n$$

- $\bar{a}$ expected returns.
- $Q^{1/2}$ square root of covariance matrix.
- K maximum number of assets.
- $\approx \mathbb{P}\left(\bar{a}x \geq W_i^{low}\right) \geq \eta_i$

Computational Experiments 2 : More Instances

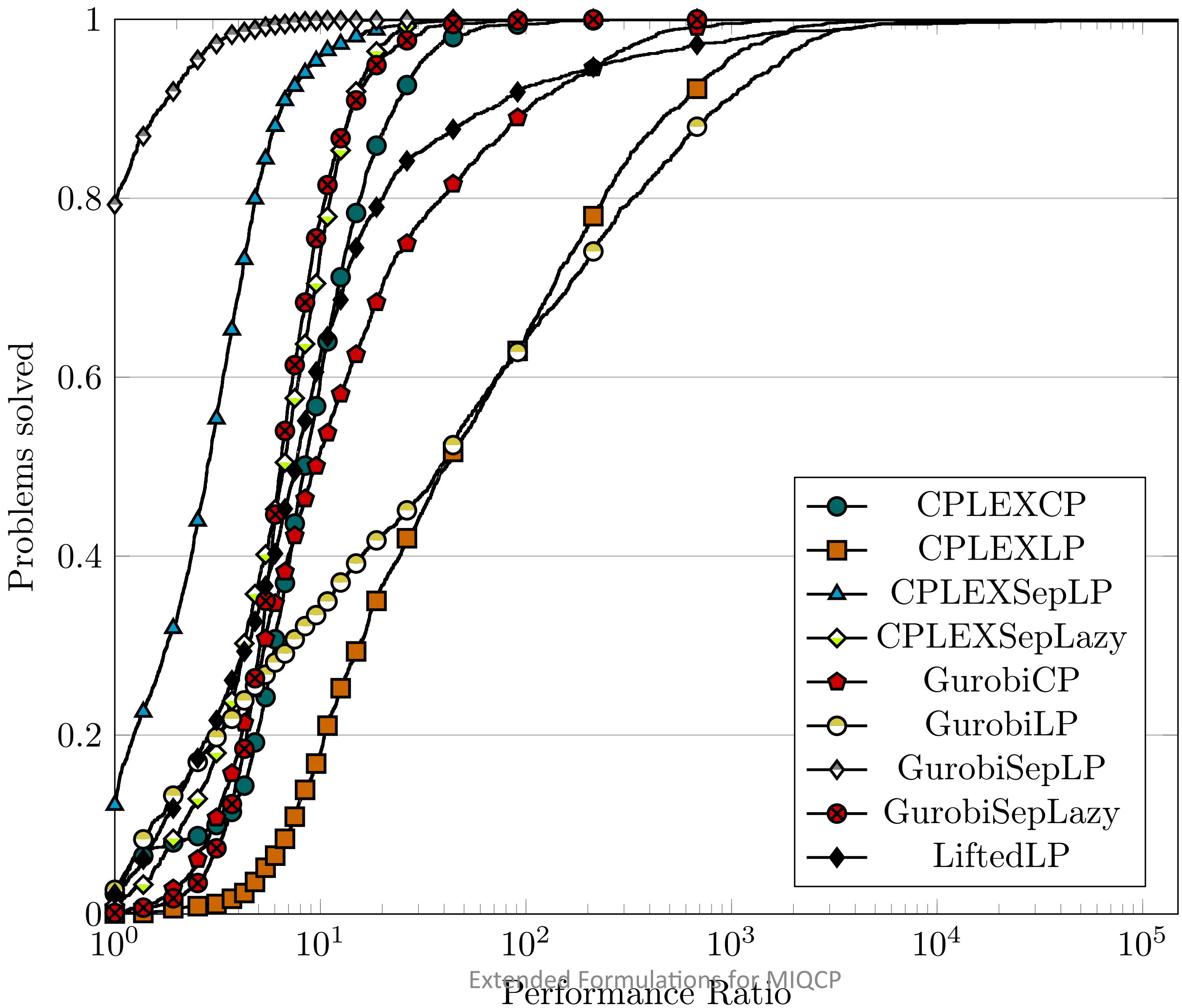
- Portfolio optimization problems:

Robust:

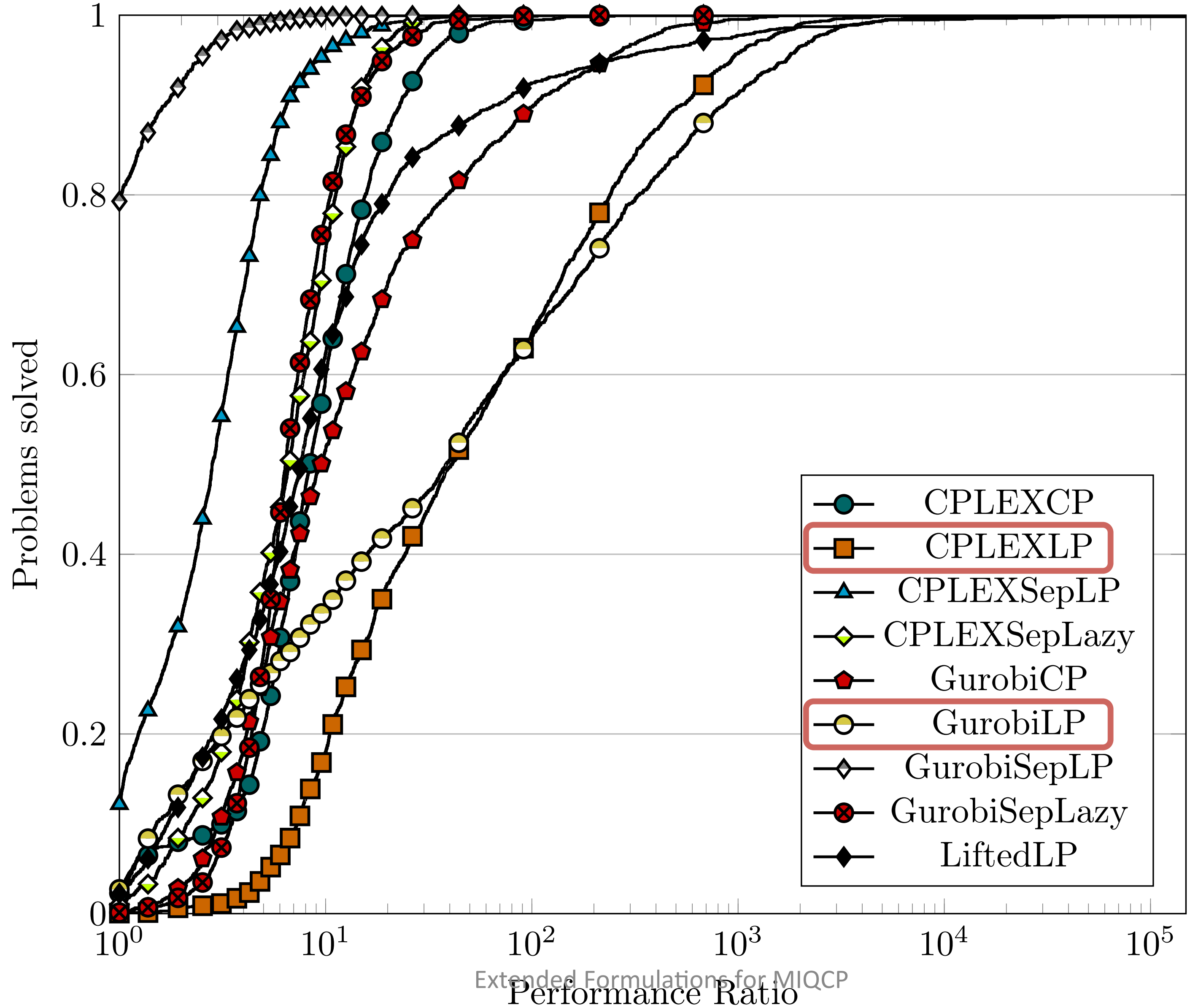
$$\begin{aligned} \max \quad & \bar{a}x - \alpha \left\| R^{1/2} y \right\|_2 \\ s.t. \quad & \\ & \left\| Q^{1/2} x \right\|_2 \leq \sigma \\ & \sum_{j=1}^{n} x_j = 1, \quad x \in \mathbb{R}_+^n \\ & x_j \leq z_j \quad \forall j \in [n] \\ & \sum_{j=1}^{n} z_j \leq K, \quad z \in \{0,1\}^n \end{aligned}$$

- $\bar{a}$ expected returns.
- $Q^{1/2}$ square root of covariance matrix.
- K maximum number of assets.
- σ maximum risk.
- Robust objective.

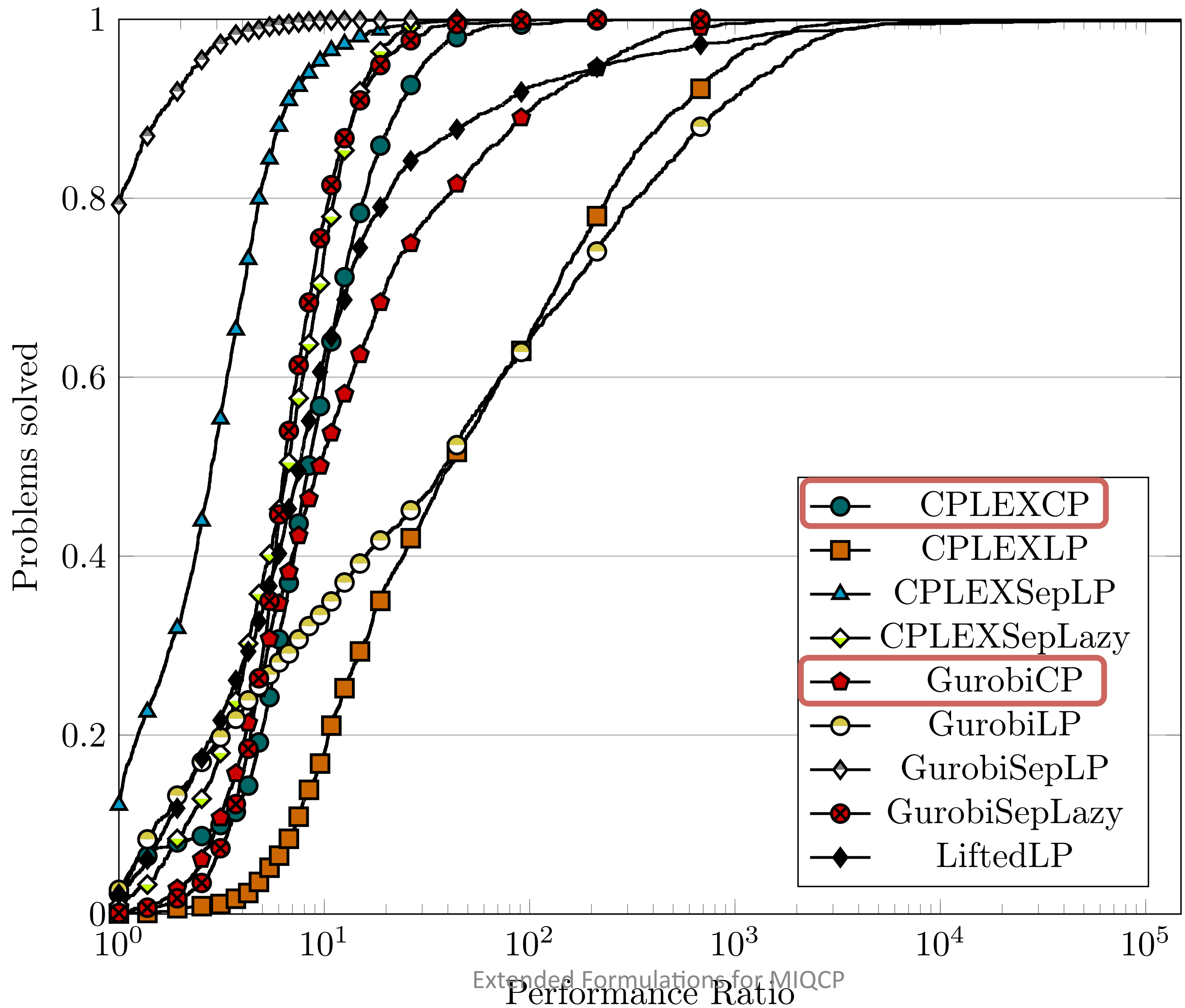
Performance Profile for n “=“ 20-60, 100 and 200



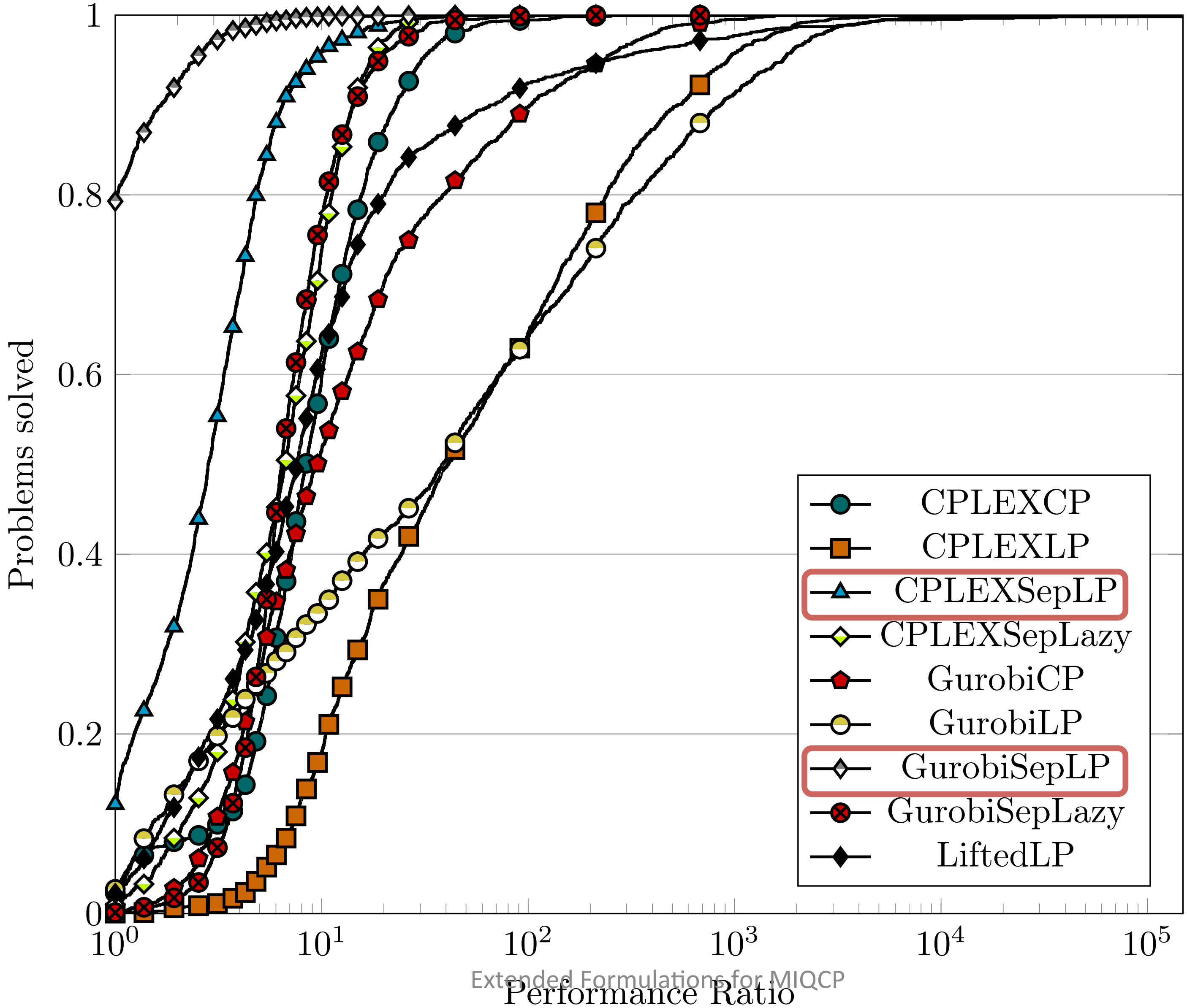
Performance Profile for n “=“ 20-60, 100 and 200



Performance Profile for n “=“ 20-60, 100 and 200



Performance Profile for n "=" 20-60, 100 and 200



Performance Profile for n “=“ 20-60, 100 and 200