Embedding Formulations and Complexity for Unions of Polyhedra

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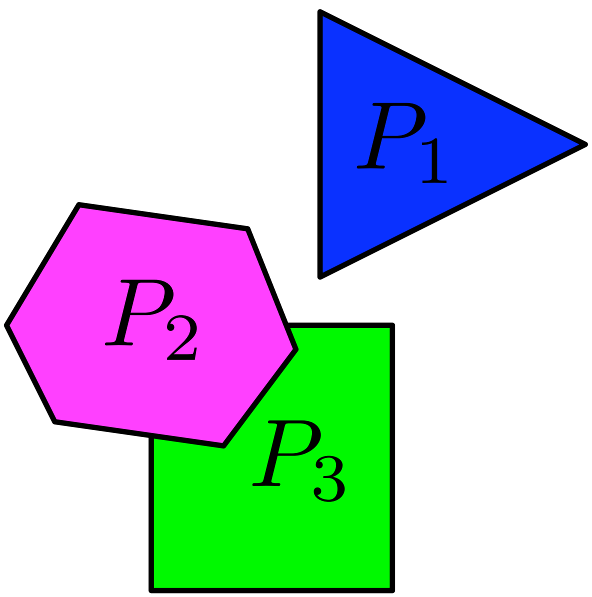
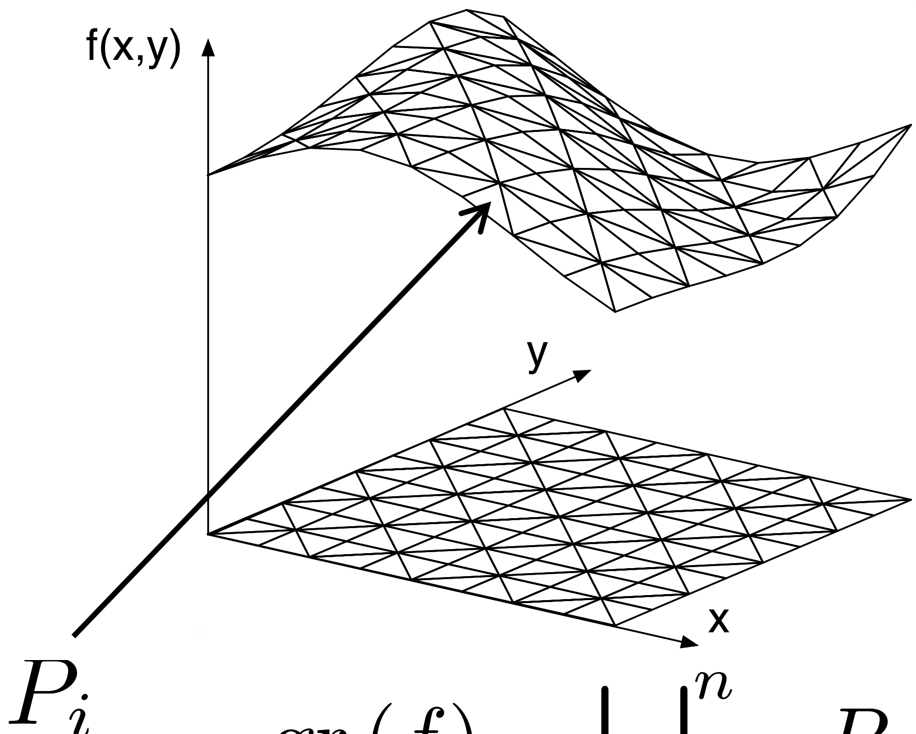
(Linear) Mixed <u>0-1</u> Integer Formulations

- Modeling Finite Alternatives = Unions of Polyhedra

$$x \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^d$$





$$\mathrm{gr}\,(f) = \bigcup_{i=1}^{n} P_i$$

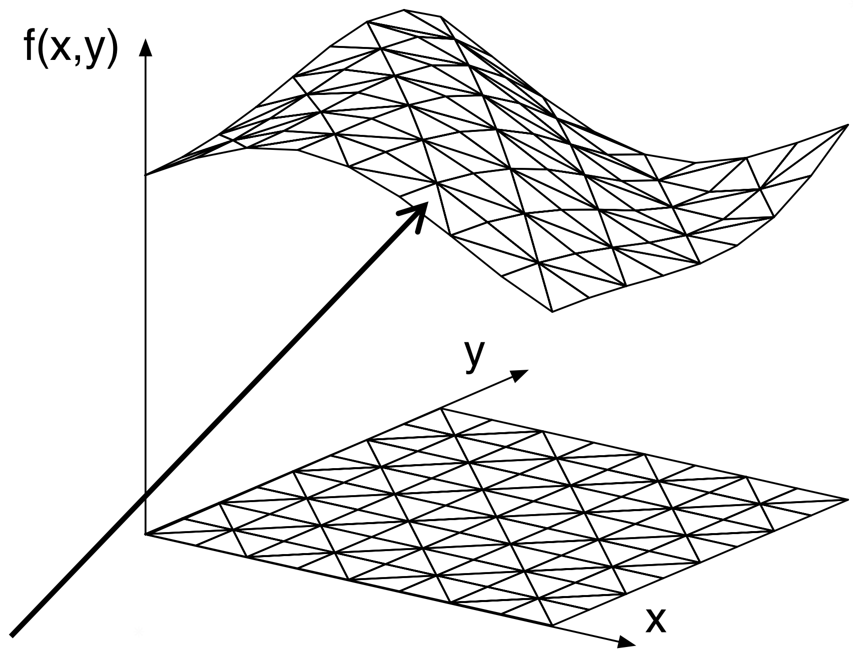
(Linear) Mixed 0-1 Integer Formulations

- Modeling Finite Alternatives = Unions of Polyhedra

$$\min \quad \sum_{j=1}^{m} f_j(x_j, y_j)$$

$$s.t.$$

$$(x, y) \in X$$



$$P_i$$

$$\operatorname{gr}(f) = \bigcup_{i=1}^{n} P_i$$

Outline

- Introduction
  - Classical Formulations v/s Specialized Branching
- Encoding Formulations
  - Role of Binary Variables and Specialized Branching
- Embedding Formulations
  - Smallest Strong Formulations

Strong Extended Formulations for $x \in \bigcup_{i=1}^{n} P_i$

- Balas, Jeroslow and Lowe ’70s early ’80s

$$P_i = \left\{ x \in \mathbb{R}^d \ : \ A^i x \le b^i \right\}$$

$$\begin{aligned} A^i x^i &\le b^i y_i \quad \forall i \\ \sum_{i=1}^{n} x^i &= x \\ \sum_{i=1}^{n} y_i &= 1 \\ y &\in \{0,1\}^n \end{aligned}$$

$\mathcal{H}$-formulation

$$\begin{aligned} \sum_{i=1}^{n} \sum_{v \in \operatorname{ext}(P_i)} v \lambda_v^i &= x \\ \sum_{v \in \operatorname{ext}(P_i)} \lambda_v^i &= y_i \quad \forall i \\ \sum_{i=1}^{n} y_i &= 1 \\ \lambda^i &\in \mathbb{R}_+^{\operatorname{ext}(P_i)} \\ y &\in \{0,1\}^n \end{aligned}$$

$\mathcal{V}$-formulation

- **Convex Hull** (Sharp) = LP relaxation projects to $\operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \right)$
- **Integral** (Locally Ideal) = LP relaxation has integral extreme points (y)

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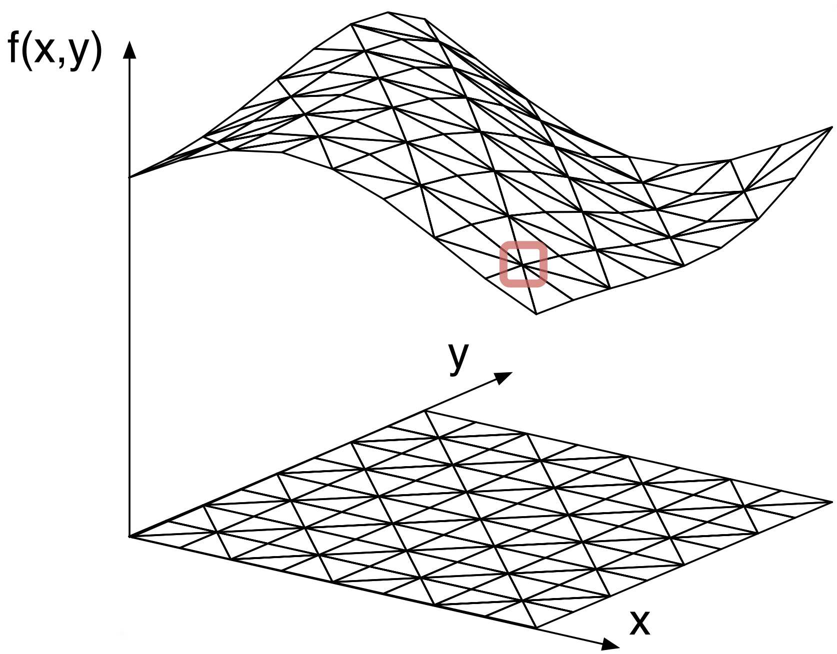

$$
\begin{aligned}
\sum_{i=1}^n \sum_{v \in \operatorname{ext}(P_i)} v\boxed{\lambda_v^i} &= x \\
\sum_{v \in \operatorname{ext}(P_i)} \lambda_v^i &= y_i \quad \forall i \\
\sum_{i=1}^n y_i &= 1 \\
\lambda^i &\in \mathbb{R}_+^{\operatorname{ext}(P_i)} \\
y &\in \{0,1\}^n
\end{aligned}
$$

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“Strong” ~~Extended~~ Formulations for $x \in \bigcup_{i=1}^{n} P_i$

- Balas, Blair and Jeroslow late ‘80s

$$P_i = \left\{ x \in \mathbb{R}^d \;:\; A^{\cancel{i}} x \le b^i \right\}$$

$$\boxed{\begin{aligned} Ax &\le \sum\nolimits_{i=1}^{n} b^i y_i \\ \sum\nolimits_{i=1}^{n} y_i &= 1 \\ y &\in \{0,1\}^n \end{aligned}}$$

$\mathcal{H}$-formulation

- Lee and Wilson late ‘90s

$$V := \bigcup\nolimits_{i=1}^{n} \operatorname{ext}(P_i)$$

$$\boxed{\begin{aligned} \sum\nolimits_{v \in V} v \lambda_v &= x \\ \sum\nolimits_{v \in V} \lambda_v &= 1 \\ \lambda_v &\le \sum\nolimits_{i : v \in \operatorname{ext}(P_i)} y_i \\ \sum\nolimits_{i=1}^{n} y_i &= 1 \\ y \in \{0,1\}^n, \quad \lambda &\in \mathbb{R}_+^V \end{aligned}}$$

$\mathcal{V}$-formulation

Sometimes

- **Convex Hull** (Sharp) = LP relaxation projects to $\operatorname{conv}\left(\bigcup_{i=1}^{n} P_i\right)$
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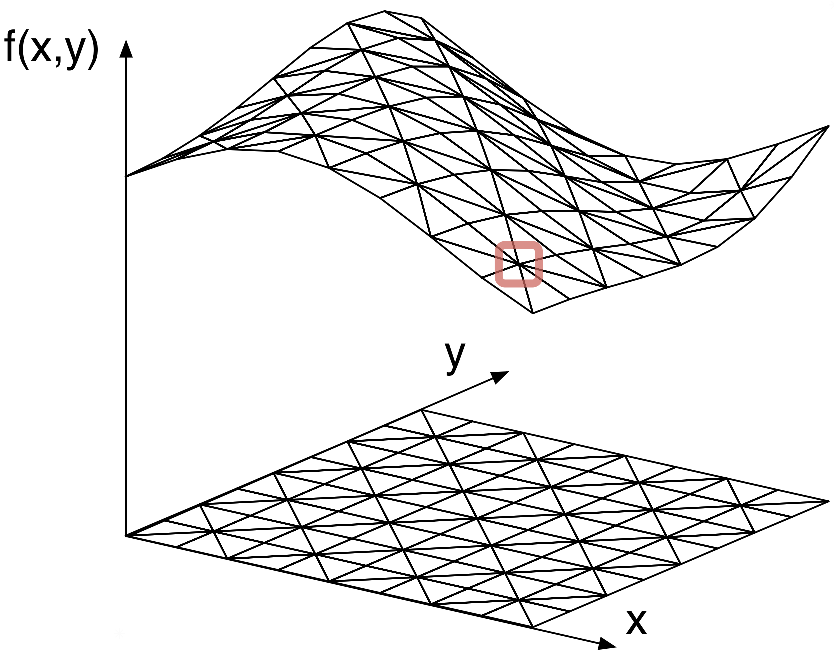

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$$V := \bigcup_{i=1}^{n} \operatorname{ext}(P_i)$$

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\end{aligned}$$

$\mathcal{V}$-formulation

- **Convex Hull** (Sharp) = LP relaxation projects to $\operatorname{conv}\left(\bigcup_{i=1}^{n} P_i\right)$
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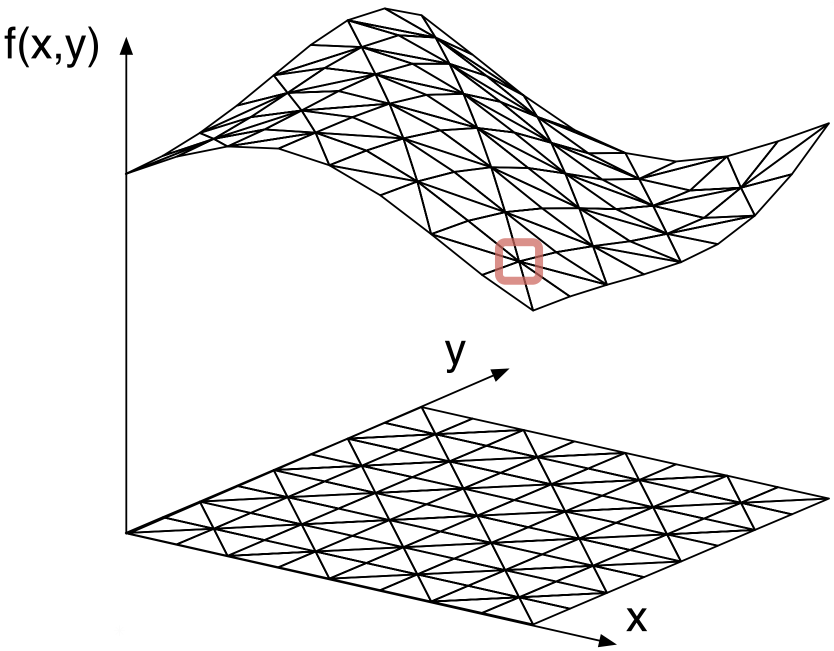
"Strong" Projected Formulations for $x \in \bigcup_{i=1}^{n} P_i$

- Balas, Blair and Jeroslow late '80s



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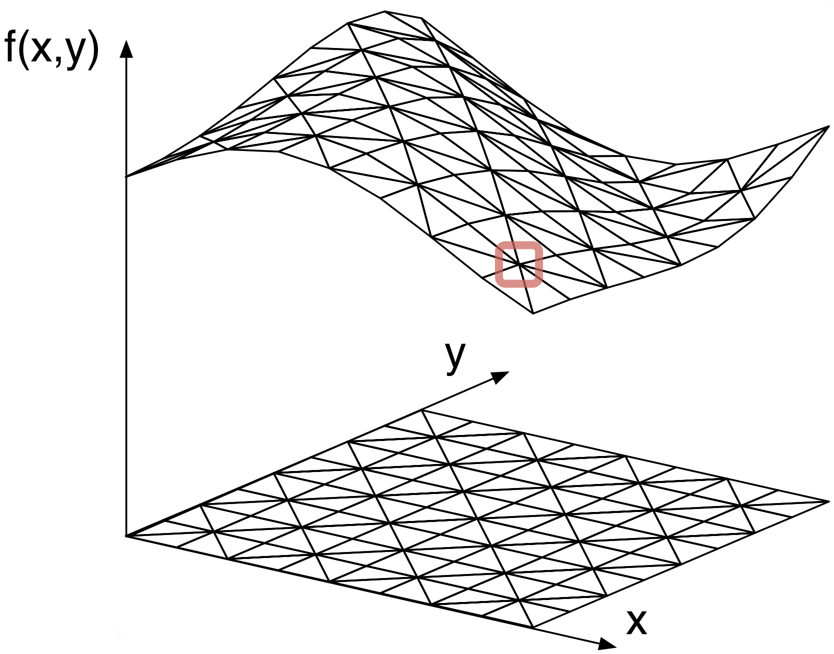

- Lee and Wilson late ‘90s

$$V := \bigcup_{i=1}^{n} \operatorname{ext}(P_i)$$

$$\sum_{v \in V} v \boxed{\lambda_v} = x$$

$$\sum_{v \in V} \lambda_v = 1$$

$$\lambda_v \leq \sum_{i : v \in \operatorname{ext}(P_i)} y_i$$

$$\sum_{i=1}^{n} y_i = 1$$

$$y \in \{0,1\}^n, \quad \lambda \in \mathbb{R}_+^V$$

$\mathcal{V}$-formulation

- **Convex Hull** (Sharp) = LP relaxation projects to $\operatorname{conv}\left(\bigcup_{i=1}^{n} P_i\right)$
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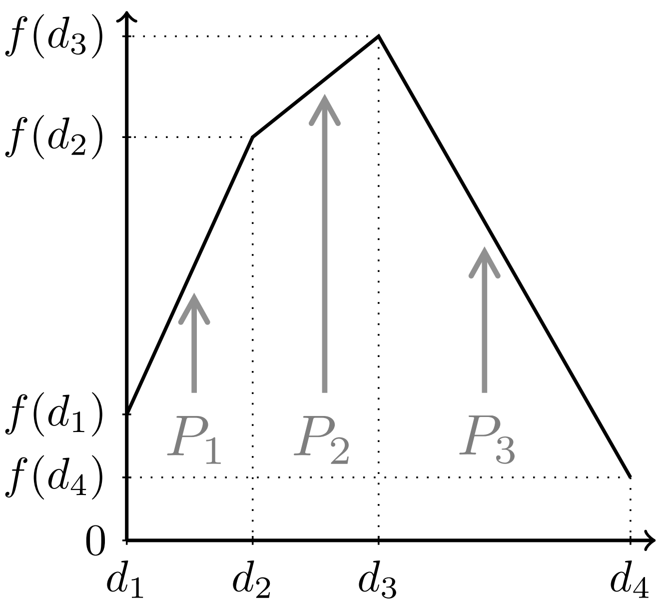
Projected Formulation for Univariate Functions



$$\sum_{j=1}^{4} d_j \lambda_{d_j} = x, \qquad \sum_{j=1}^{4} f(d_j)\lambda_{d_j} = z$$

$$\sum_{j=1}^{4} \lambda_{d_j} = 1, \qquad \lambda_{d_j} \geq 0$$

$$\sum_{i=1}^{3} y_i = 1, \qquad y \in \{0,1\}^3$$

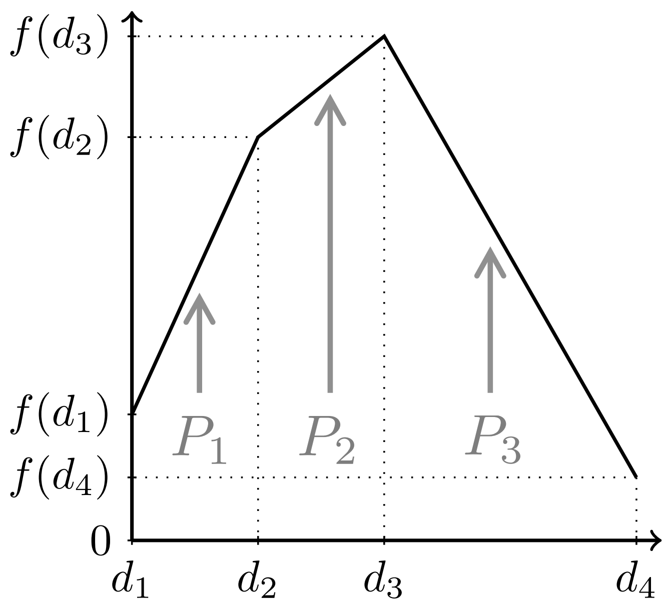
$$\lambda_{d_1} \leq y_1, \qquad \lambda_{d_2} \leq y_1 + y_2$$

$$\lambda_{d_3} \leq y_2 + y_3, \quad \lambda_{d_4} \leq y_3$$

- Convex Hull, but not Integral
- Branching is very ineffective (unbalanced B&B tree)
  - $y_{i_0} = 1 \qquad \Rightarrow \qquad y_i = 0 \quad \forall i \neq i_0$

    $y_{i_0} = 0$ does not imply much (anything)

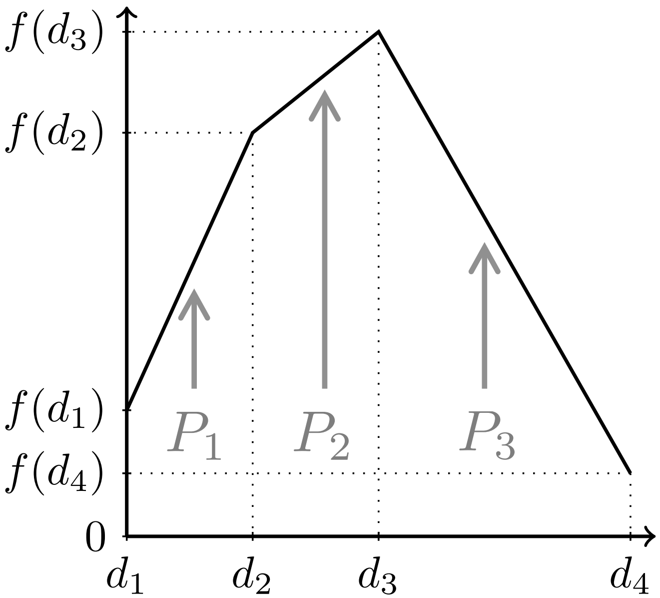
Projected Formulation for Univariate Functions



$$
\begin{aligned}
&\sum\nolimits_{j=1}^{4} d_j \lambda_{d_j} = x, && \sum\nolimits_{j=1}^{4} f(d_j)\lambda_{d_j} = z \\
&\sum\nolimits_{j=1}^{4} \lambda_{d_j} = 1, && \lambda_{d_j} \geq 0 \\
&\cancel{\sum\nolimits_{i=1}^{3} y_i = 1,} && \cancel{y \in \{0,1\}^3} \\
&\cancel{\lambda_{d_1} \leq y_1,} && \cancel{\lambda_{d_2} \leq y_1 + y_2} \\
&\cancel{\lambda_{d_3} \leq y_2 + y_3,} && \cancel{\lambda_{d_4} \leq y_3}
\end{aligned}
$$

- One solution = SOS2 branching (Beale and Tomlin ’70):
  - $\lambda_{d_i} = 0 \quad \forall i \leq i_0 - 1$
  - $\lambda_{d_i} = 0 \quad \forall i \geq i_0 + 1$

Projected Formulation for Univariate Functions



$$\sum_{j=1}^{4} d_j \lambda_{d_j} = x, \qquad \sum_{j=1}^{4} f(d_j)\lambda_{d_j} = z$$
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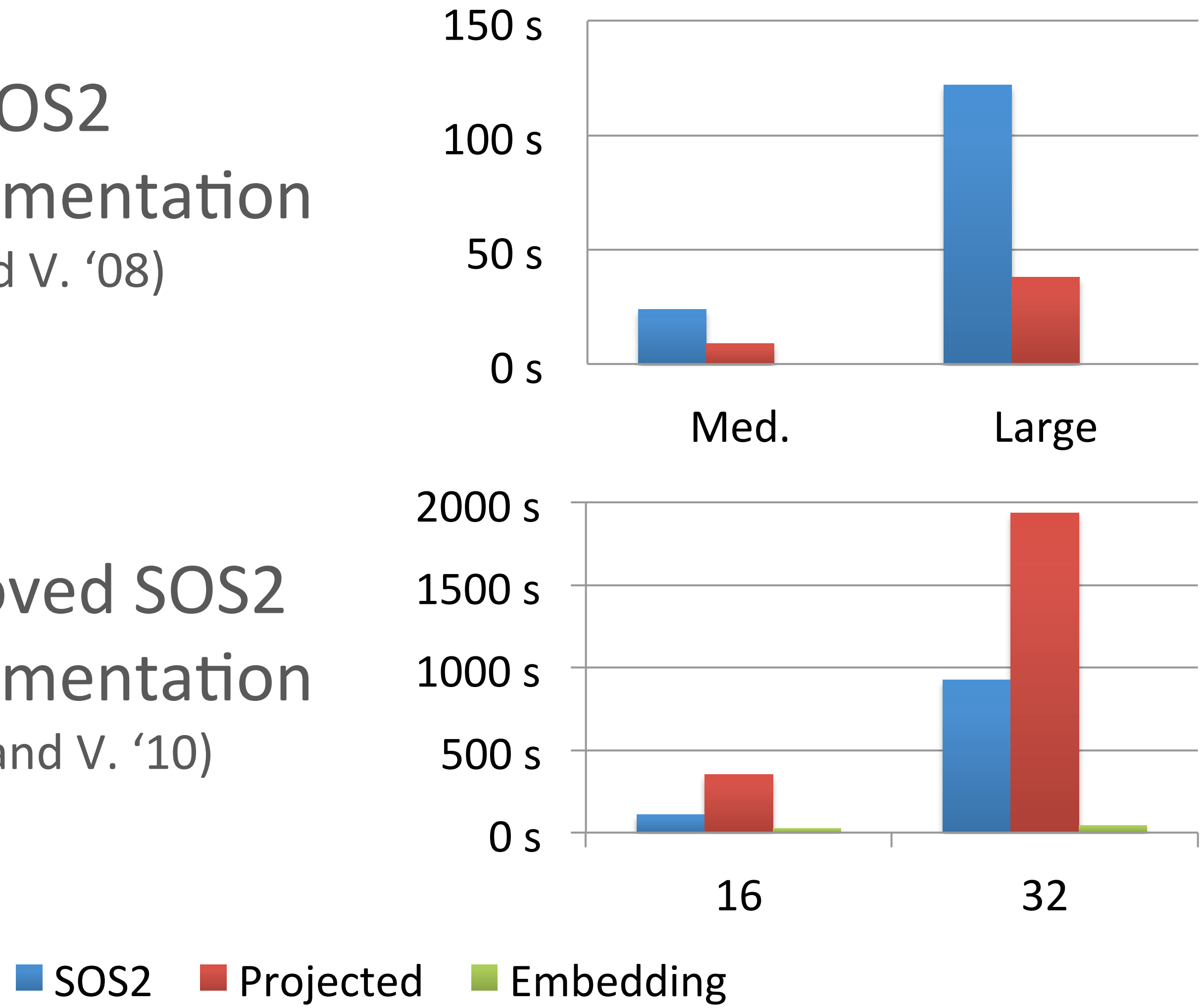
$$\sum_{i=1}^{3} y_i = 1, \qquad y \in \{0,1\}^3$$
$$\lambda_{d_1} \leq y_1, \qquad \lambda_{d_2} \leq y_1 + y_2$$
$$\lambda_{d_3} \leq y_2 + y_3, \qquad \lambda_{d_4} \leq y_3$$

- One solution = SOS2 branching (Beale and Tomlin ’70):
  - $\lambda_{d_i} = 0 \quad \forall i \leq i_0 - 1$ $\qquad y_i = 0 \quad \forall i \leq i_0 - 1$
  - $\lambda_{d_i} = 0 \quad \forall i \geq i_0 + 1$ $\qquad y_i = 0 \quad \forall i \geq i_0$

MIP Formulations v/s Specialized Branching

- CPLEX 9: Basic SOS2 branching implementation (Nemhauser, Keha and V. '08)

- CPLEX 11: Improved SOS2 branching implementation (Nemhauser, Ahmed and V. '10)



Encoding Formulations:
The Role of Binary Variables

Encodings to Induce Specialized Branching

- Discrete alternatives ($P_i = \{v^i\}$) :

$$\sum_{i=1}^{n} y_i v^i = x, \quad \sum_{i=1}^{n} y_i = 1$$
$$y \in \{0,1\}^n$$

Encodings to Induce Specialized Branching

- Discrete alternatives ($P_i = \{v^i\}$) :

$$\sum_{i=1}^{n} y_i v^i = x, \quad \sum_{i=1}^{n} y_i = 1$$

$$\cancel{y \in \{0,1\}^n} \quad y \in \mathbb{R}_+^n$$

$$\sum_{i=1}^{n} y_i h^i = w, \quad w \in \{0,1\}^k$$

- Pick $\{h^i\}_{i=1}^{n} \subseteq \{0,1\}^k$, $\quad h^i \neq h^j$

Encoding

- Li and Lu '09, Adams and Henry '11, V. and Nemhauser '08 for $k = \log_2 n$. Also in the folklore, e.g. Sommer, TIMS '72

Different Encodings = Different Branching

- **Unary** encoding : $\left\{h^i\right\}_{i=1}^n = \left\{e^i\right\}_{i=1}^n$

$$\sum_{i=1}^{8} y_i = 1, \quad y \in \mathbb{R}_+^8$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} y = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{pmatrix}, \quad w \in \{0,1\}^8$$

$$\Rightarrow y_i = w_i$$

Different Encodings = Different Branching

- **Binary** encoding : $\left\{h^i\right\}_{i=1}^{n} = \{0,1\}^{\log_2 n}$

$$\sum_{i=1}^{8} y_i = 1, \quad y \in \mathbb{R}^8_+$$

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$
$$\begin{pmatrix} \boxed{1 & 1 & 1 & 1} & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} y = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \quad w \in \{0,1\}^3$$

$$\boxed{w_1 = 1} \quad \vee \quad w_1 = 0$$
$$\Updownarrow$$
$$\boxed{y_i = 0 \quad \forall i \geq 5} \quad \vee \quad y_i = 0 \quad \forall i \leq 4$$

Discrete Alternatives to Unions of Polyhedra

Adapt extended $\mathcal{V}$-formulation:

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{v \in \operatorname{ext}(P_i)} v\lambda_v^i &= x \\
\sum_{v \in \operatorname{ext}(P_i)} \lambda_v^i &= y_i \ \forall i \\
\sum_{i=1}^{n} y_i &= 1 \\
\lambda^i &\in \mathbb{R}_+^{\operatorname{ext}(P_i)} \\
y &\in \{0,1\}^n
\end{aligned}$$

Discrete Alternatives to Unions of Polyhedra

Adapt extended $\mathcal{V}$-formulation:

$$\sum_{i=1}^{n} \sum_{v \in \operatorname{ext}(P_i)} v \lambda_v^i = x$$

$$\sum_{i=1}^{n} \sum_{v \in \operatorname{ext}(P_i)} \lambda_v^i = 1 \quad \forall i$$

$$\sum_{i=1}^{n} h^i \sum_{v \in \operatorname{ext}(P_i)} \lambda_v^i = w$$

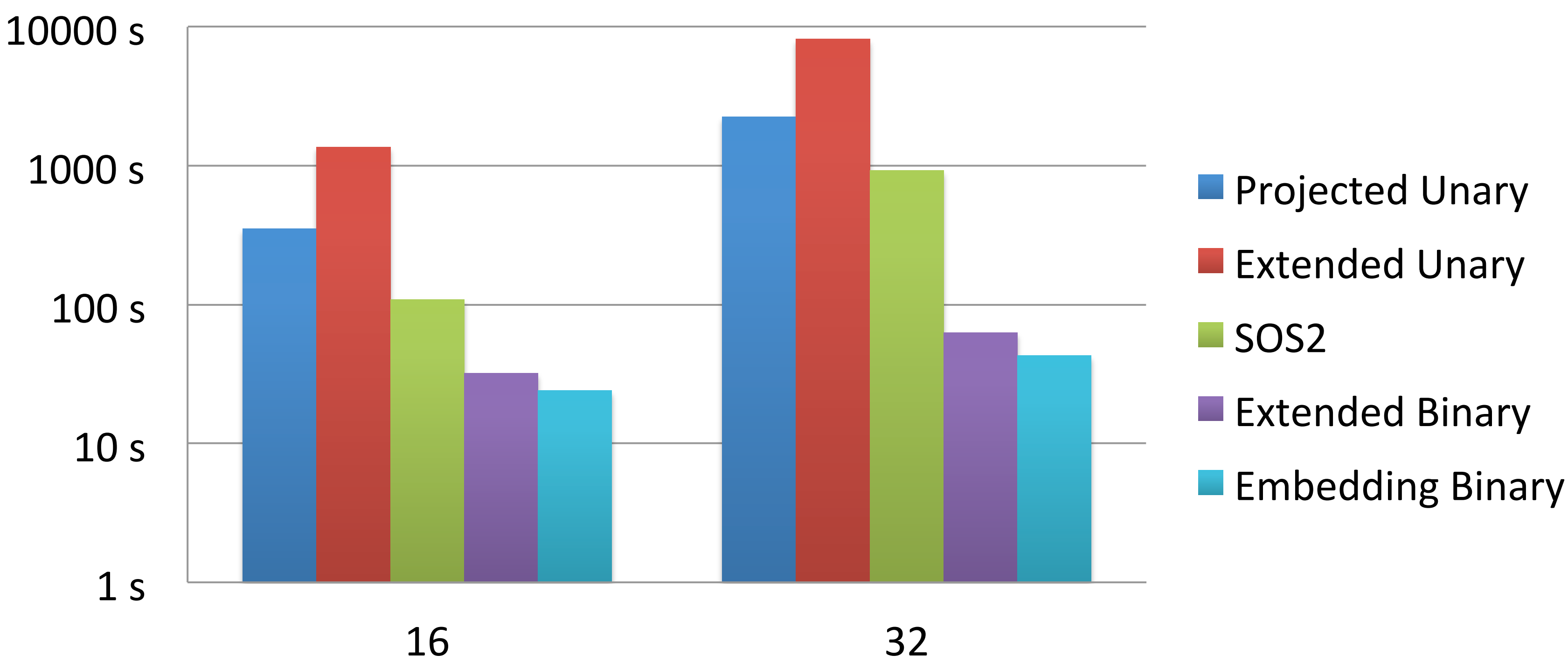
$$\lambda^i \in \mathbb{R}_+^{\operatorname{ext}(P_i)}$$

$$w \in \{0,1\}^k$$

- V., Ahmed and Nemhauser 2010; Yıldız and V. 2013; V. 2014

Performance for Univariate Functions

- Results from Nemhauser, Ahmed and V. ’10 using CPLEX 11



- Multivariate functions : **Embedding Binary** is **6** times faster than **Extended Binary**

Embedding Formulations: Strong Projected Formulations

Polyhedra as MIP Formulations

$$\lambda \in \bigcup_{i=1}^{n} P_i, \qquad P_i = \left\{\lambda \in \mathbb{R}^d \;:\; A^i \lambda \leq b^i\right\}$$

$$Q = \left\{ (\lambda, y) \in \mathbb{R}^d \times \mathbb{R}^n \;:\; \begin{array}{r l} A\lambda \leq & \sum_{i=1}^{n} b^i y_i \\ 1 = & \sum_{i=1}^{n} y_i \\ y_i \geq & 0 \end{array} \right\}$$

$$y \in \mathbb{Z}^n$$

$$\left(\lambda, e^i\right) \in Q \quad \Leftrightarrow \quad \lambda \in P_i$$

Embedding Formulations for Union of Polyhedra

- **Projected** MIP formulation of $\lambda \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^V$:
  - Encoding $\{h^i\}_{i=1}^n \subseteq \{0,1\}^k, \quad h^i \neq h^j$
  - Polyhedron $Q \subseteq \mathbb{R}^V \times \mathbb{R}^k$, s.t.

$$(\lambda, h^i) \in Q \quad \Leftrightarrow \quad \lambda \in P_i$$

- **Embedding formulation** = strongest polyhedron:

Cayley Polytope $\longrightarrow$

$$Q = \operatorname{conv}\Big(\underbrace{\bigcup_{i=1}^{n} P_i \times \{h^i\}}_{\text{Cayley Embedding}}\Big)$$

For unary encoding:

$$h^i = e^i$$

Embedding Formulations for Union of Polyhedra

- **Projected** MIP formulation of $\lambda \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^V$:
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$$(\lambda, h^i) \in Q \quad \Leftrightarrow \quad \lambda \in P_i$$

- **Embedding formulation** = strongest polyhedron:

$$Q = \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \{h^i\}\right)$$

$$\text{size}\,(Q) := \#\text{ of facets of } Q \text{ (usually function of } n)$$

Binary v/s Unary Encodings

$$Q = \mathrm{conv}\left(\bigcup_{i=1}^{n} P_i \times \left\{h^i\right\}\right), \quad \left\{h^i\right\}_{i=1}^{n} \subseteq \left\{0,1\right\}^k$$

- Unary better than Binary ?
  - Formulation contains convex hull through **projection**:
    - $\mathrm{Proj}_\lambda\left(Q\right) = \mathrm{conv}\left(\bigcup_{i=1}^{n} P_i\right)$
    - $\mathrm{size}\left(\mathrm{Proj}_\lambda\left(Q\right)\right) \leq \binom{\mathrm{size}(Q)}{\mathrm{size}(Q)-k-1}$
  - **Binary encoding** has $k = \log_2 n$:
    - Size of projection is at most quasipolynomial in size of formulation
  - **Unary encoding** has $k = n$:
    - Size of projection can be exponential in size of formulation

Binary v/s Unary Encodings

$$Q = \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \{h^i\}\right), \quad \{h^i\}_{i=1}^{n} \subseteq \{0,1\}^k$$

- Binary better than Unary?
  - Formulation contains Minkowski sum through **sections**:
    - For **unary encoding**

$$\left(\lambda, \frac{1}{n}\sum_{i=1}^{n} e^i\right) \in Q \quad \Leftrightarrow \quad \lambda \in \frac{1}{n}P_1 + \ldots + \frac{1}{n}P_n$$

    - **Unary encoding** formulation can be large even if convex hull is simple(x)
    - **Binary encoding** seems to only contain partial sums of $\log_2 n$ polytopes

Simple Case: Combinatorial Part of $\mathcal{V}$-formulation

- $\Delta^V := \left\{ \lambda \in \mathbb{R}^V_+ \; : \; \sum_{v \in V} \lambda_v = 1 \right\}, \operatorname{ext}\left(P_i\right) = T_i \subseteq V$
- $P_i = \left\{ \lambda \in \Delta^V \; : \; \lambda_v \leq 0 \quad \forall v \notin T_i \right\}$

$$
\lambda \in \bigcup_{i=1}^n P_i \qquad \Leftrightarrow \qquad
\boxed{
\begin{aligned}
\sum_{v \in V} v \lambda_v &= x \\
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y \in \{0,1\}^n, \quad & \lambda \in \mathbb{R}^V_+
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}
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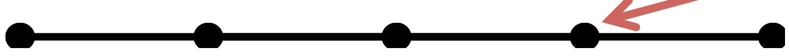
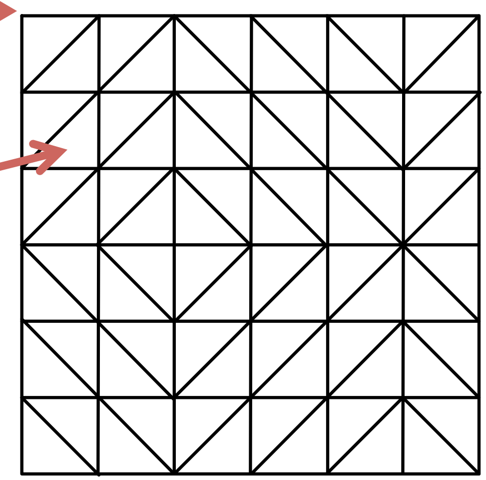


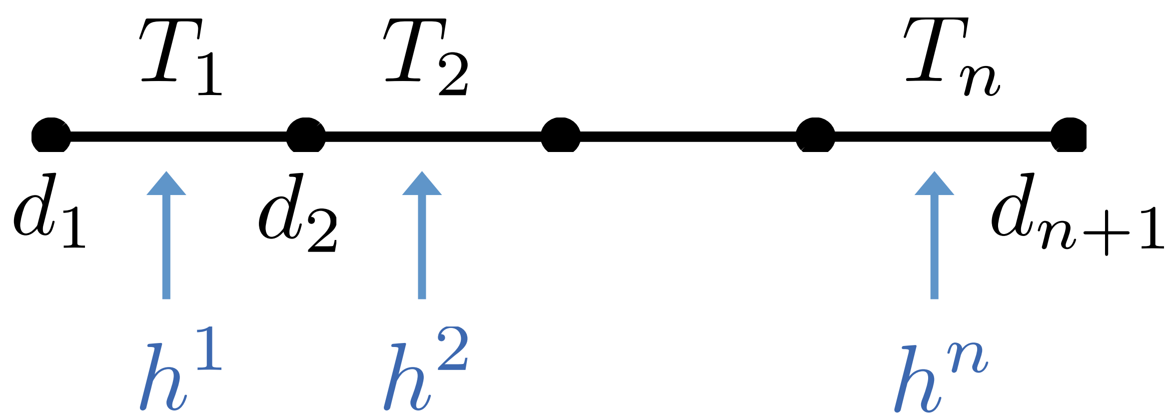
$$\lambda \in \bigcup_{i=1}^n P_i$$



- $\operatorname{conv}\left(\bigcup_{i=1}^n P_i \right) = \Delta^V$

Message 1: The Devil is in the Detail

- Choice of binary encoding is crucial



Formulation Size for Univariate case

- Simple facets: $\lambda_v \geq 0$
  - Only sometimes are facets
  - "Zero" computational cost and at most n of them
- All other facets: $\sum_{v \in V} \alpha_v \lambda_v \leq \sum_{i=1}^{k} \beta_i y_i$
- Unary encoding (Padberg, Lee and Wilson, early 00's):
  - $2n$ facets $(2n + 2$ including bounds$)$
- Binary encoding with Gray code (V. and Nemhauser, 08, 11):
  - $\log_2 n$ facets $(\leq 2 \log_2 n + n$ including bounds$)$

High Binary Complexity? Gray v/s Anti-Gray

- Assumption: $n = 2^k$
  - $\left\{h^i\right\}_{i=1}^{n} = \{0,1\}^k, \quad H := \left\{h^i - h^{i+1}\right\}_{i=1}^{n-1} \subseteq \{-1,0,1\}^k$
  - # facets =twice the # of **linear** hyperplanes spanned by $\boldsymbol{H}$
- Gray code: $\left\{h^i - h^{i+1}\right\}_{i=1}^{n-1} \equiv \left\{e^i\right\}_{i=1}^{k}$
  - # hyperplanes : $k = \log_2 n$, # facets $\leq 2\log_2 n + n$
- One kind of Anti-Gray code: $\left\{h^i - h^{i+1}\right\}_{i=1}^{n-1}$ "$\supseteq$" $\{-1,1\}^k$
  - # hyperplanes = # affine hyperplanes spanned by $\{0,1\}^{k-1}$
  - Using believed growth rate (e.g. Aichholzer and Auremacher '96):
    - # facets = $\Theta\left(n^{\log_2 n}\right)$

Message 2 : Binary Encoding = Smaller Formulation

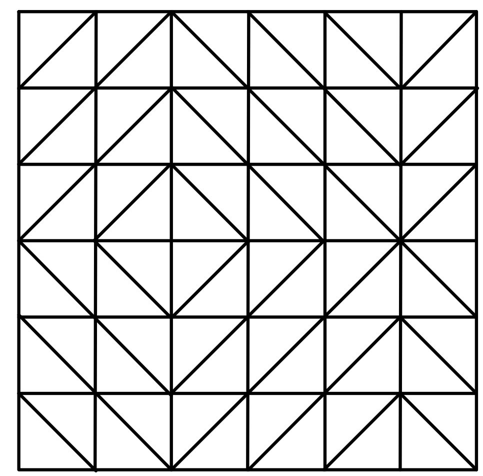
- Size of unary formulation is at least (Lee and Wilson '01):

$$\binom{2\sqrt{n/2}}{\sqrt{n/2}} + \underbrace{\left(\sqrt{n/2}+1\right)^2}_{\text{Non-negativity}}$$

- Size of best binary formulation for union jack triangulation is at most (V. and Nemhauser '08):

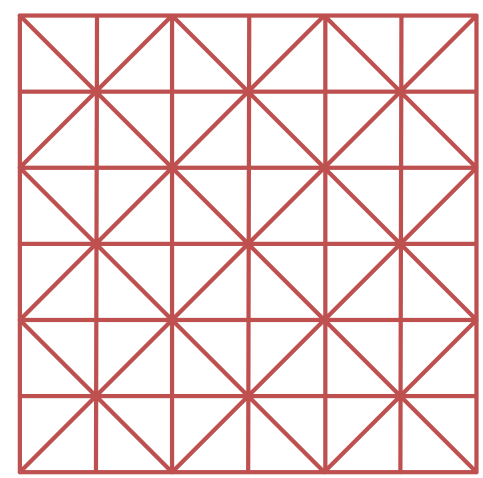
$$4\log_2\sqrt{n/2} + 2 + \underbrace{\left(\sqrt{n/2}+1\right)^2}_{\text{Non-negativity}}$$

$n = 2m^2$



m

m



Beyond Union Jack: Exploit Redundancy

- Embedding-like formulation for triangulations with "even degree outside the boundary"

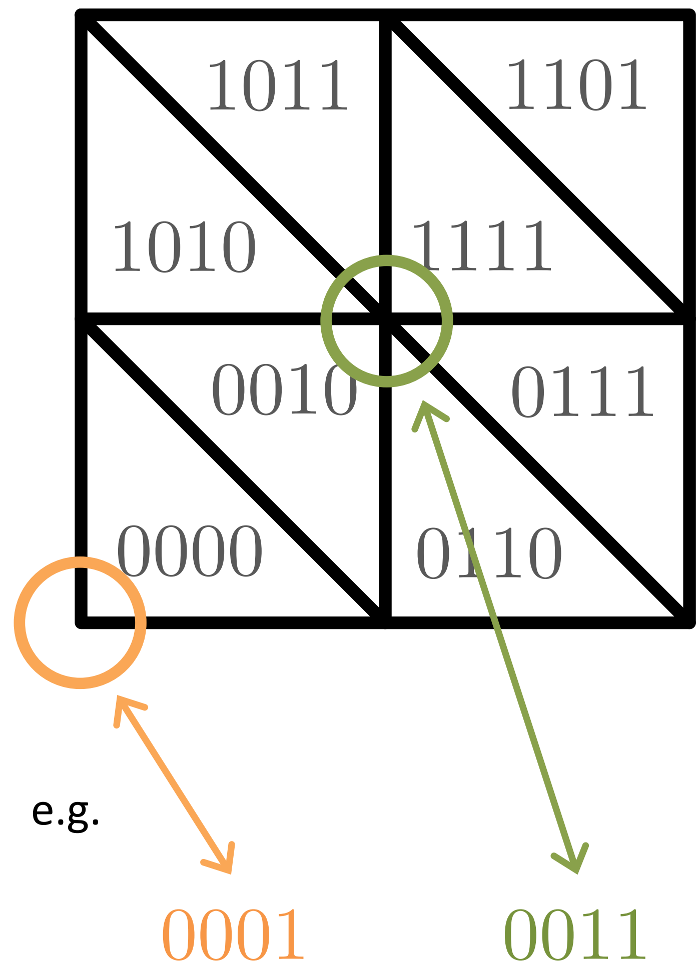
•Formulation size at most two larger than for union jack:

$$4\log_2\sqrt{n/2}+4+\left(\sqrt{n/2}+1\right)^2$$

•Formulation fits **independent branching** framework (V. and Nemhauser '08)

Independent Branching = Embedding + Redundancy

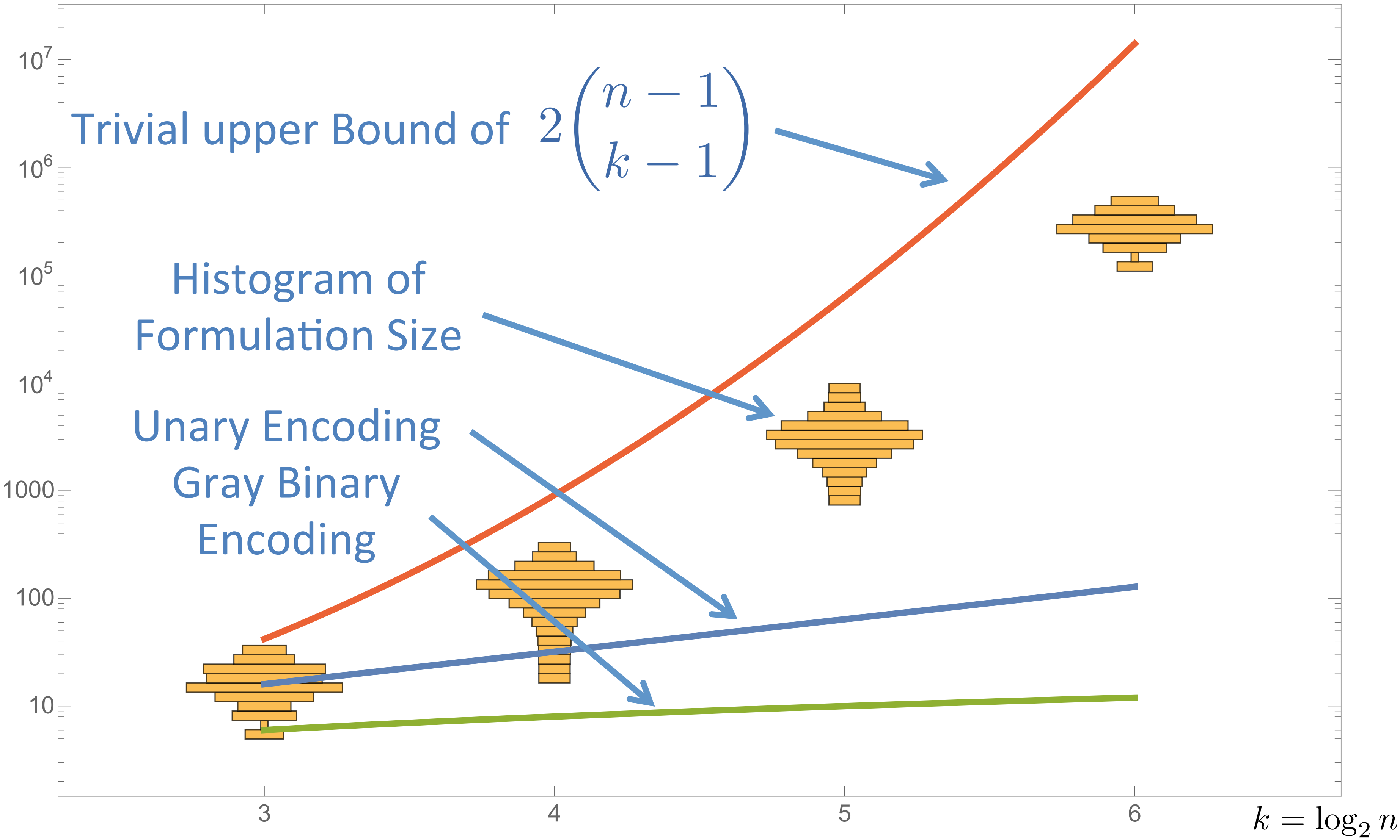
- Triangle $\longleftarrow$ binary vector
- More vectors than triangles
  - Ind. Branch ≠ Embedding
  - Embedding size is larger (17)
- Ind. Branching solution:
  - Add redundant single-vertex polytopes with remaining 8 binary vectors
- Unary cannot reduce size through redundancy



Summary

- Embedding Formulations = Systematic procedure
  - Encoding can significantly affect size
  - Redundancy can help for binary encodings
- Complexity of Union of Polyhedra beyond convex hull
  - Embedding Complexity (Integral Formulation)
  - MIP formulation complexity
- More on encoding properties
  - All gray codes yield the same size, but not combinatorially equivalent polytopes
- Can help discover strong (non-integral) formulations
  - Facility layout problem (Huchette, Dey, V. '14)

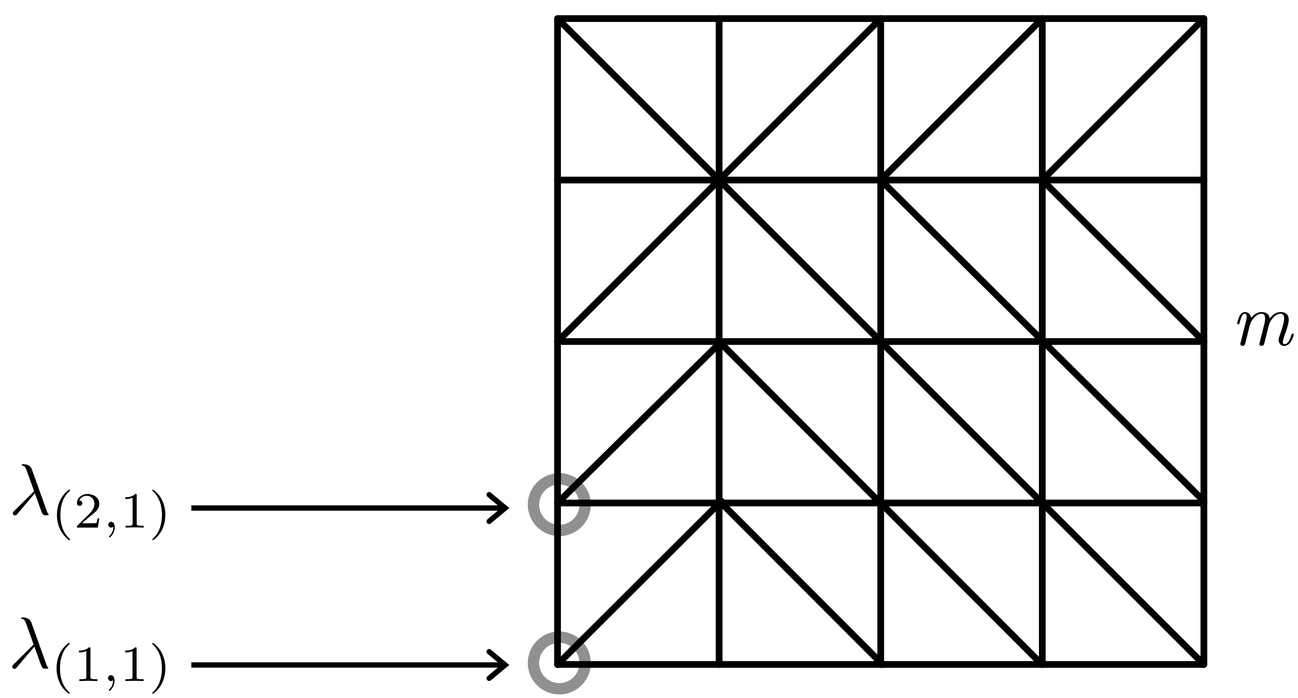
Formulation Size for all Binary Encodings



Beyond Union Jack: Part I = Gray Code for Grid

$\lambda_{(i,j)} \geq 0 \quad i,j \in [m+1]$

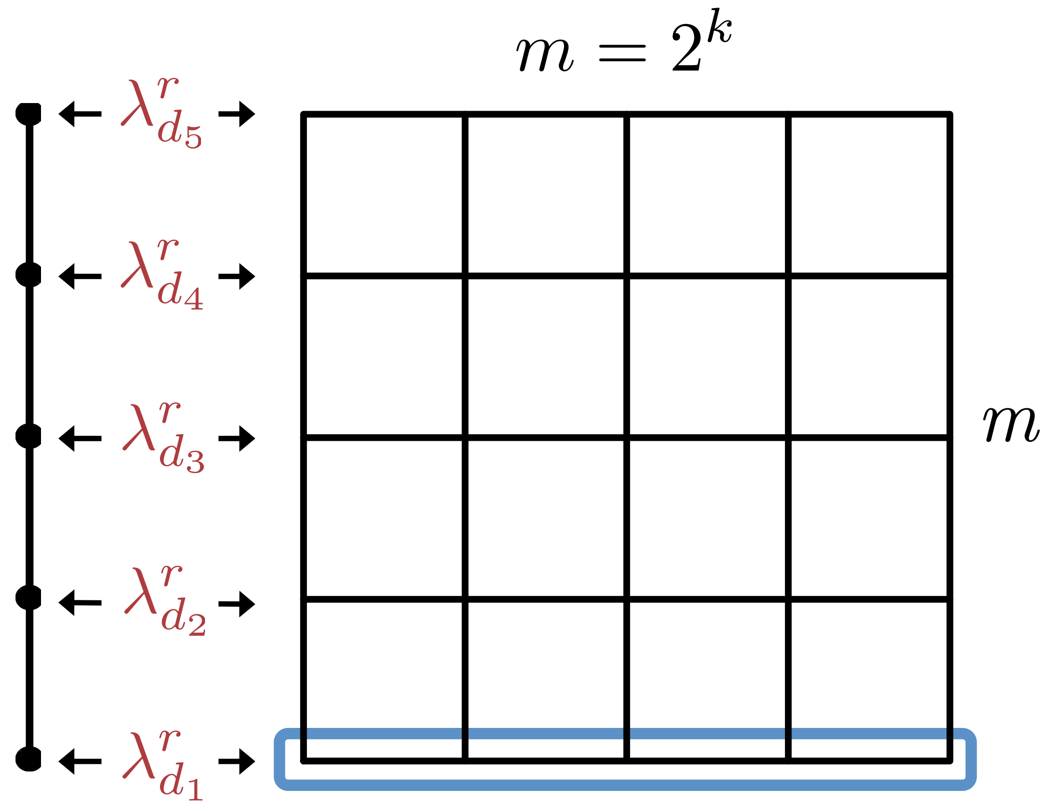
$m = 2^k$



Beyond Union Jack: Part I = Gray Code for Grid

$$\lambda_{(i,j)} \geq 0 \quad i,j \in [m+1]$$

$$\lambda^r_{d_i} = \sum_{j=1}^{m+1} \lambda_{(i,j)}$$



$$(\lambda^r, y^r) \in Q \subseteq \mathbb{R}^m \times \mathbb{R}^k$$

Univariate Gray Code Formulation

Beyond Union Jack: Part I = Gray Code for Grid

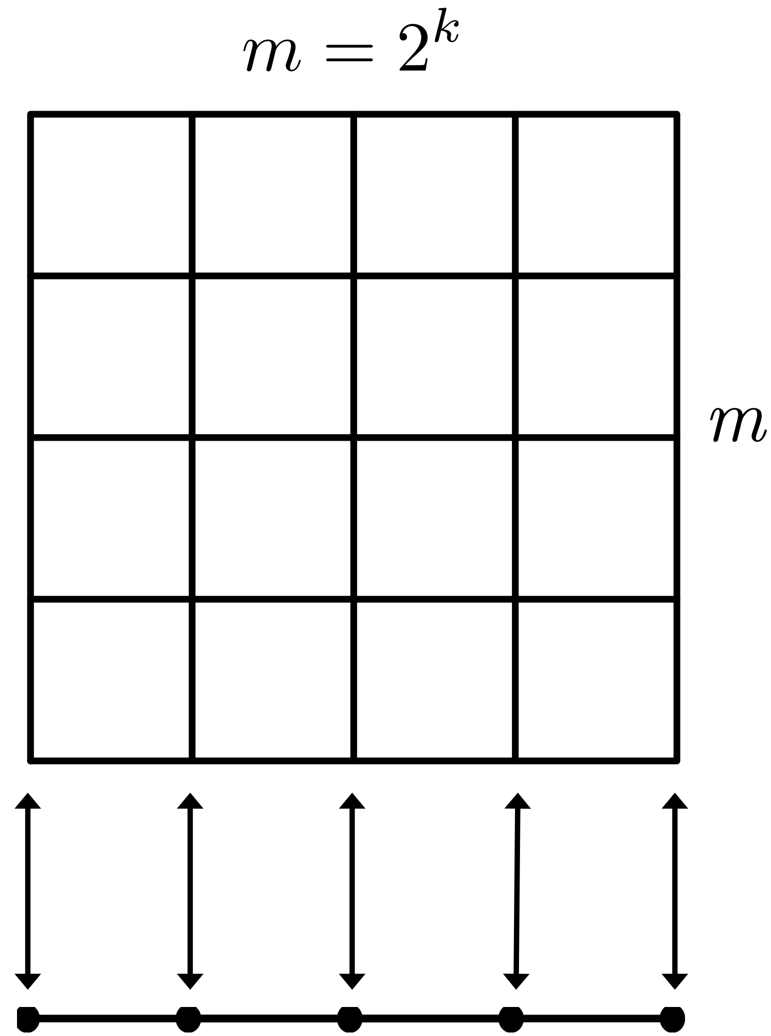
$$\lambda_{(i,j)} \geq 0 \quad i, j \in [m+1]$$

$$\lambda^r_{d_i} = \sum_{j=1}^{m+1} \lambda_{(i,j)}$$

$$\lambda^c_{d_i} = \sum_{i=1}^{m+1} \lambda_{(i,j)}$$

$$(\lambda^r, y^r) \in Q \subseteq \mathbb{R}^m \times \mathbb{R}^k$$

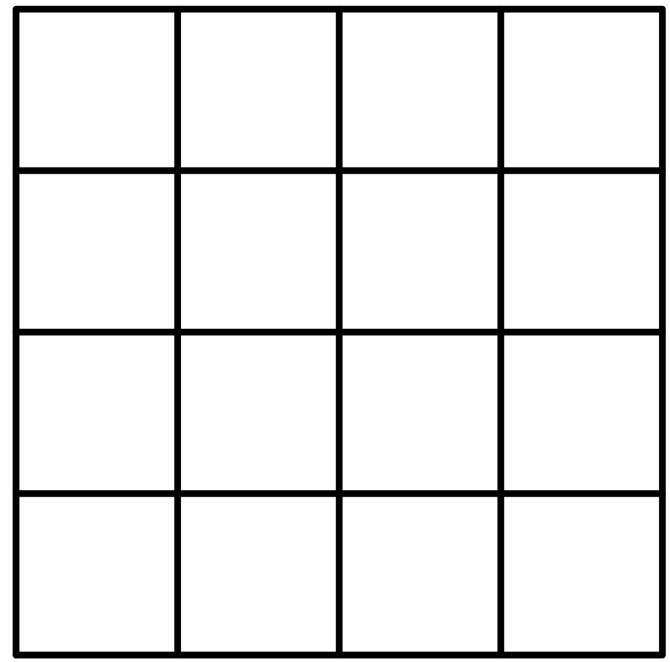
$$(\lambda^c, y^c) \in Q \subseteq \mathbb{R}^m \times \mathbb{R}^k$$



Beyond Union Jack: Part I = Gray Code for Grid

$$\lambda_{(i,j)} \geq 0 \quad i,j \in [m+1]$$

$$m = 2^k$$



m

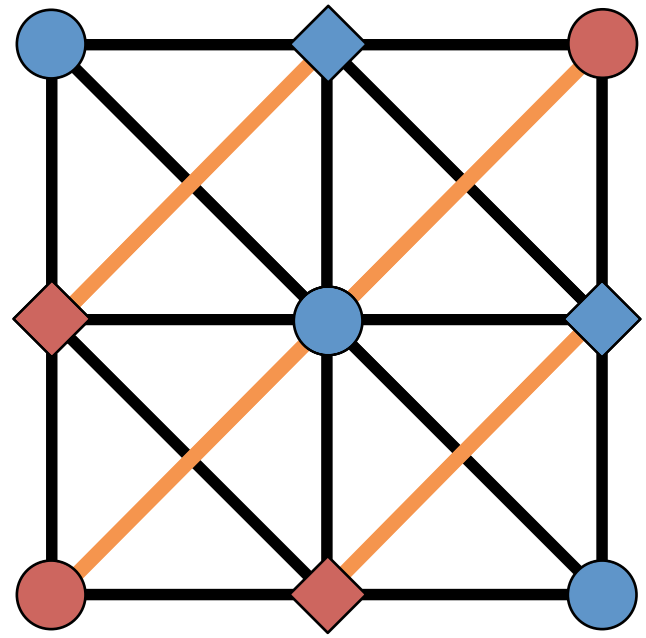
$$\lambda^r_{d_i} = \sum_{j=1}^{m+1} \lambda_{(i,j)}$$

$$\lambda^c_{d_i} = \sum_{i=1}^{m+1} \lambda_{(i,j)}$$

$$\left.\begin{array}{l} (\lambda^r, y^r) \in Q \subseteq \mathbb{R}^m \times \mathbb{R}^k \\ (\lambda^c, y^c) \in Q \subseteq \mathbb{R}^m \times \mathbb{R}^k \end{array}\right\} \; 4 \log_2 m$$

Beyond Union Jack: Part II = Selecting Triangles

1. Add "Dual" Triangulation
2. Color vertices following diagonal arcs:
   - Keep color for original arcs
   - Change color for dual arcs
3. Add binary y_1^t and constraints:



$$\sum_{(i,j) \text{ colored red}} \lambda_{(i,j)} \leq y_1^t \quad \text{and} \quad \sum_{(i,j) \text{ colored blue}} \lambda_{(i,j)} \leq 1 - y_1^t$$

4. May need to repeat coloring once more

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