Embedding Formulations, Complexity and Representability for Unions of Convex Sets

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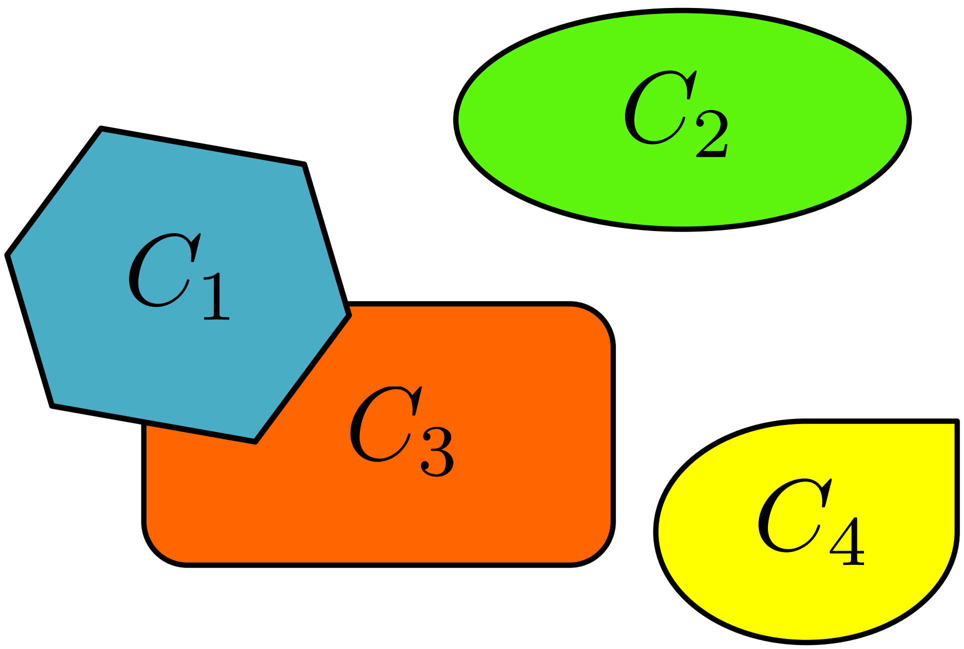
Nonlinear Mixed 0-1 Integer Formulations

- Modeling Finite Alternatives = Unions of Convex Sets

$$x \in \bigcup_{i=1}^{n} C_i \subseteq \mathbb{R}^d$$



Extended and Non-Extended Formulations for $\bigcup_{i=1}^{n} C_i$

$$C_i = \{x \in \mathbb{R}^d \; : \; A^i x \leq b^i\}$$

Extended

$$\begin{aligned} A^i x^i &\leq b^i y_i & \forall i \in [n] \\ \sum_{i=1}^{n} x^i &= x & \\ \sum_{i=1}^{n} y_i &= 1 & \\ y &\in \{0,1\}^n & \\ x, \; x^i &\in \mathbb{R}^d & \forall i \in [n] \end{aligned}$$

Large, but strong (ideal*)

Non-Extended

$$\begin{aligned} A^i x - b^i &\leq M_i (1 - y_i) & \forall i \in [n] \\ \sum_{i=1}^{n} y_i &= 1 & \\ y &\in \{0,1\}^n & \\ x &\in \mathbb{R}^d & \forall i \in [n] \end{aligned}$$

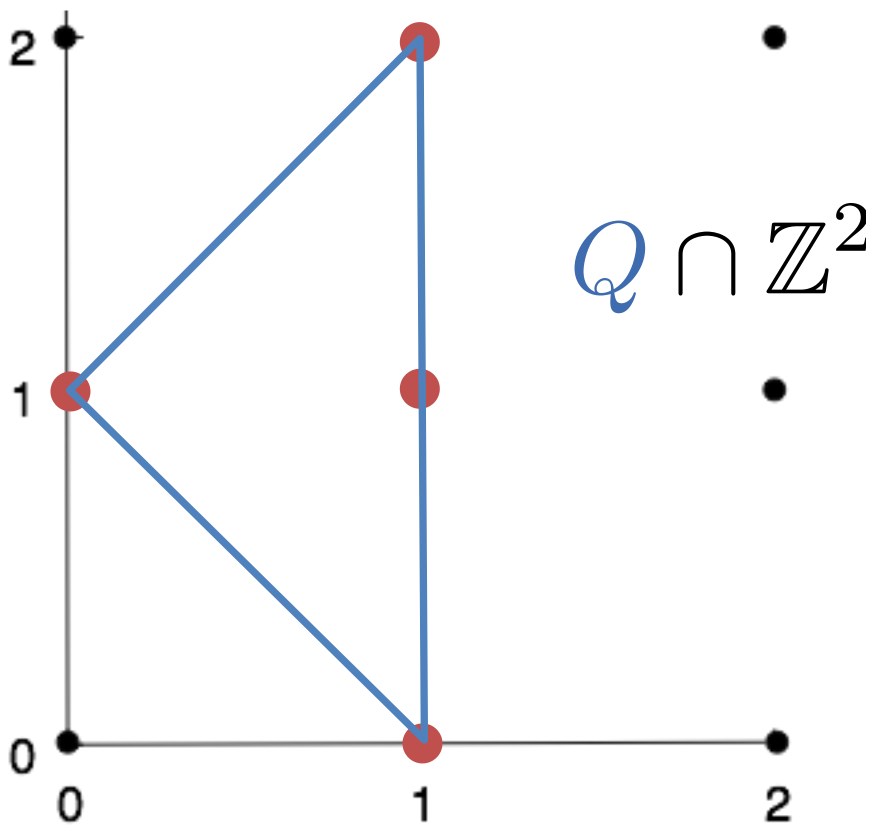
Small, but weak?

*Integral y in extreme points of LP relaxation

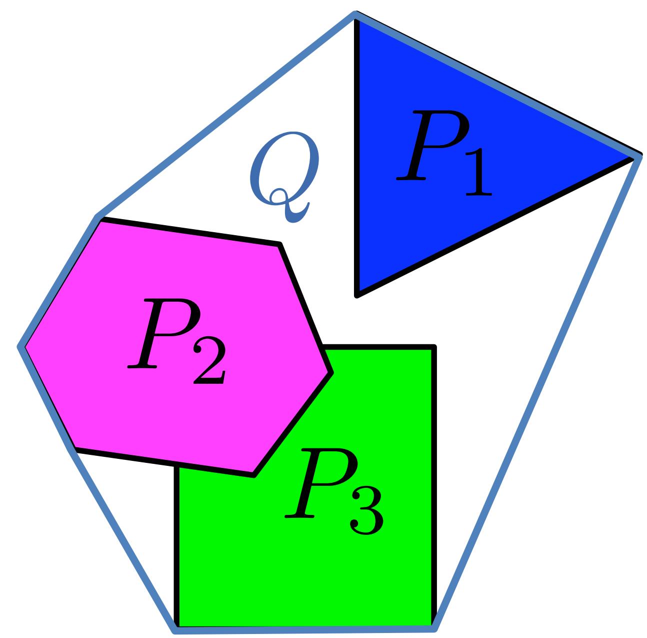
Constructing Non-extended Ideal Formulations

- Pure Integer :

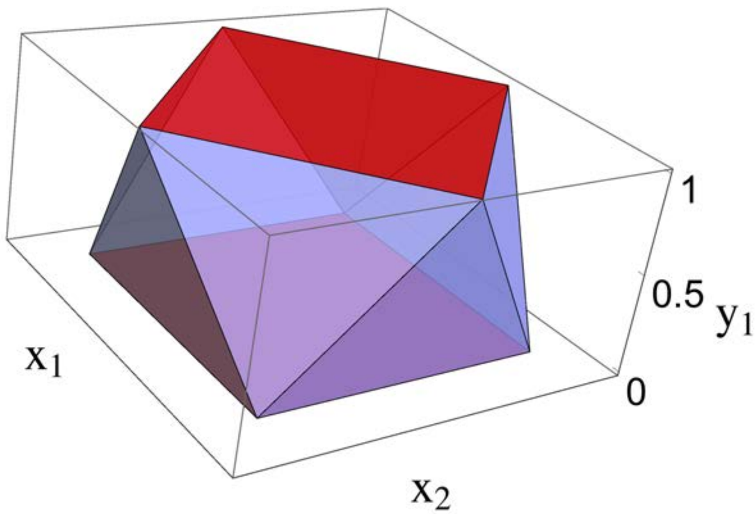
$$Q := \operatorname{conv}\left(\{p^i\}_{i=1}^n\right)$$

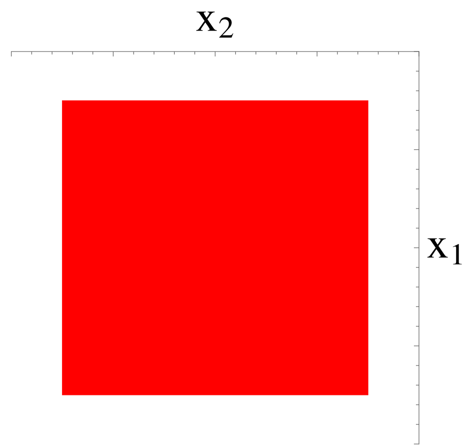


- Mixed Integer:

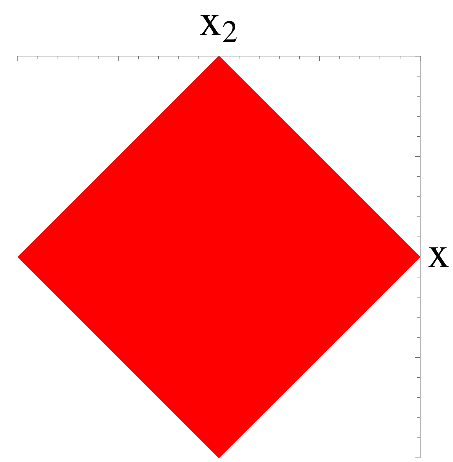


Embedding Formulation = Ideal non-Extended





P_1



P_2

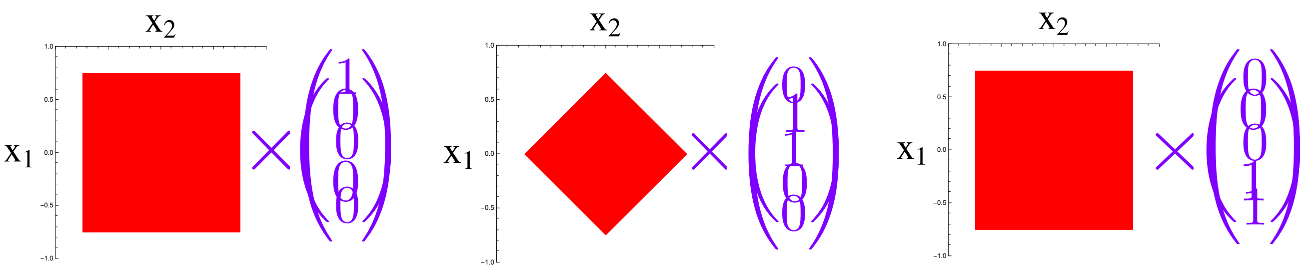
$$Q(H) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \{h^i\}\right)$$

$$(x, y) \in Q \cap \left(\mathbb{R}^d \times \mathbb{Z}^k\right) \quad \Leftrightarrow \quad y = h^i \ \wedge \ x \in P_i$$

$$\operatorname{ext}(Q) \subseteq \mathbb{R}^d \times \mathbb{Z}^k \qquad H := \{h^i\}_{i=1}^{n} \subseteq \{0,1\}^k, \quad h^i \neq h^j$$

Alternative Encodings

- 0-1 encodings guarantee validity



P_1 P_2 P_3

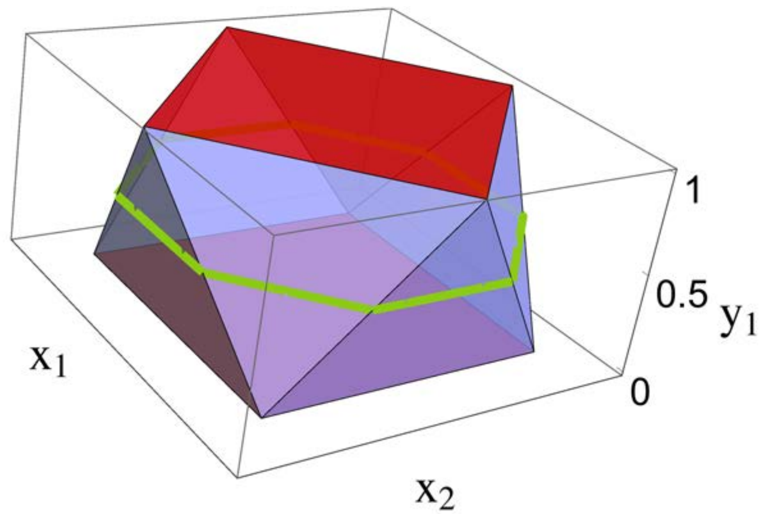
- Options for 0-1 encodings:
  - Traditional or **Unary** encoding

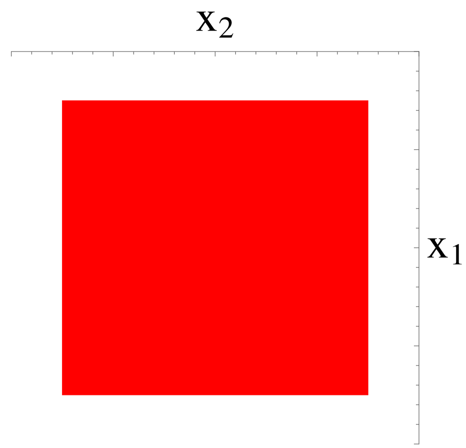
$$H = \left\{ y \in \{0,1\}^n \ : \ \sum_{i=1}^{n} y_i = 1 \right\}$$
$$= \left\{ \mathbf{e}^i \right\}_{i=1}^{n}$$

$$\mathbf{e}^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

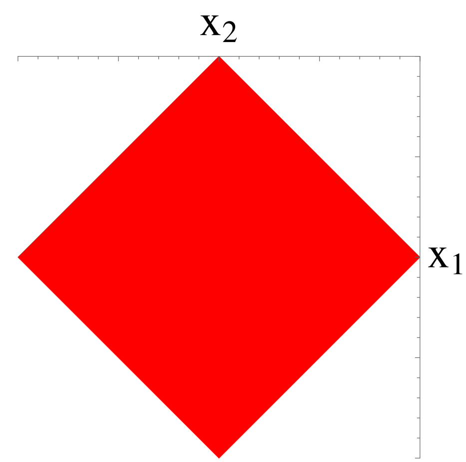
  - **Binary** encodings: $H \equiv \{0,1\}^{\log_2 n}$
  - Others (e.g. **incremental** encoding $\equiv$ unary)

Unary Encoding, Minkowski Sum and Cayley Trick



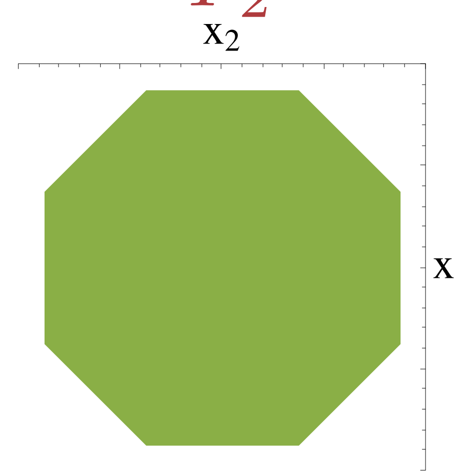


P_1



P_2

$$Q \cap \left(\mathbb{R}^2 \times \{0.5\}\right) \equiv P_1 + P_2 =$$



For traditional or unary encoding:

$$Q\left(H\right) \cap \left(\mathbb{R}^d \times \left\{\frac{1}{n}\sum_{i=1}^{n} \mathbf{e}^i\right\}\right) \equiv \sum_{i=1}^{n} P_i$$

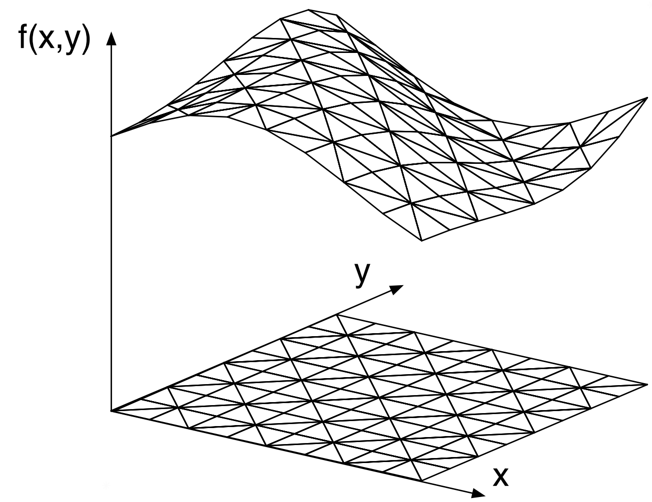
Encoding Selection Matters

- Size of unary formulation is:
  (Lee and Wilson '01)

$$\binom{2\sqrt{n/2}}{\sqrt{n/2}} + \left(\sqrt{n/2}+1\right)^2$$

General Inequalities

Variable Bounds



- Size of one binary formulation:
  (V. and Nemhauser '08)

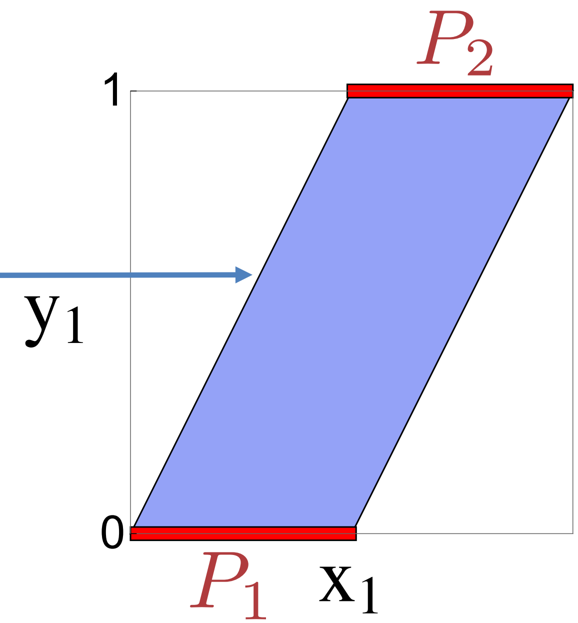
$$4\log_2\sqrt{n/2}+2+\left(\sqrt{n/2}+1\right)^2$$

- Right embedding = significant computational advantage over alternatives (Extended, Big-M, etc.)

Complexity of Family of Polyhedra $\mathcal{P} := \{P_i\}_{i=1}^n$

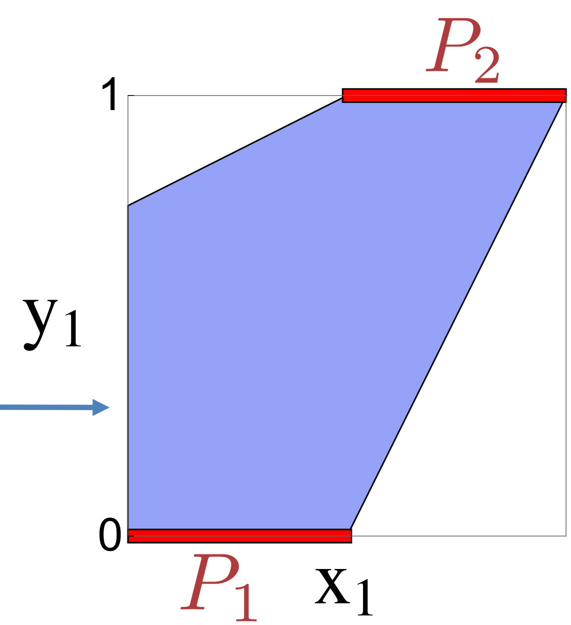
- Embedding complexity = smallest **ideal** formulation

$$\mathrm{mc}(\mathcal{P}) := \min_H \{\mathrm{size}(Q(H))\}$$



- Relaxation complexity = smallest formulation

$$\mathrm{rc}(\mathcal{P}) := \min_{Q,H} \{\mathrm{size}(Q)\}$$



$$\mathrm{size}(Q) := \#\text{ of facets of } Q$$

Complexity Results

- Lower and Upper bounds for special structures:
  - e.g. for Special Order Sets of Type 2 (SOS2) on n variables
    - Embedding complexity (ideal)

      $2\lceil \log_2 n \rceil$ ← General Inequalities

      $n+1 \leq \ldots \leq n+1+2\lceil \log_2 n \rceil$ ← Total

    - Relaxation complexity (non-ideal)

      $2 \leq \ldots \leq 4$ ← General Inequalities

      $2 \leq \ldots \leq 5+2n$ ← Total

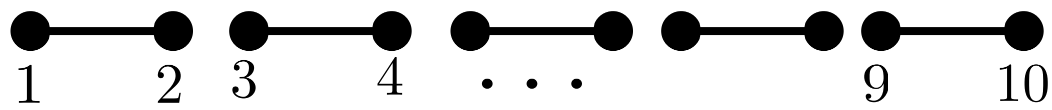
- Relation to other complexity measures

$$\mathrm{hc}\left(\mathcal{P}\right) := \mathrm{size}\left(\mathrm{conv}\left(\bigcup\nolimits_{i=1}^{n} P_i\right)\right)$$

$$\mathrm{xc}\left(\mathcal{P}\right) := \min\nolimits_R \left\{\mathrm{size}\left(R\right) \;:\; \mathrm{proj}_x\left(R\right) = \mathrm{conv}\left(\bigcup\nolimits_{i=1}^{n} P_i\right)\right\}$$

- Still open questions (see V. 2015)

Example of Constant Sized Non-Ideal Formulation



$$x \in \bigcup_{i=1}^{n} P_i$$

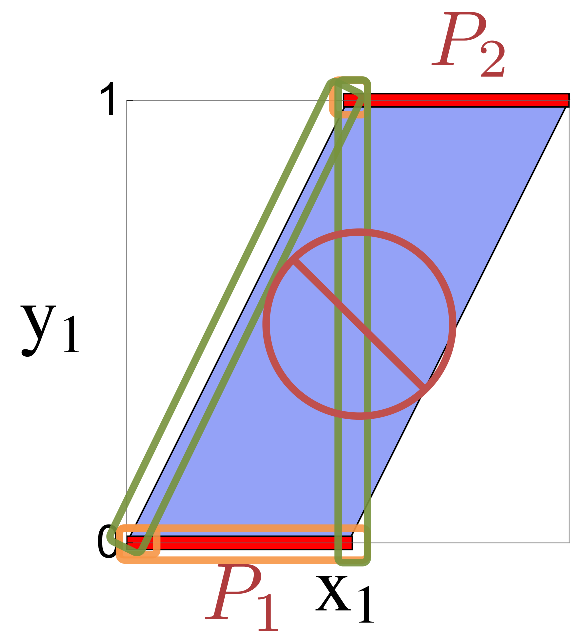
$$P_i \subseteq \Delta^{2n} := \left\{ x \in \mathbb{R}_{+}^{2n} \; : \; \sum_{i=1}^{2n} x_i = 1 \right\}$$

$$P_i = \left\{ x \in \Delta^{2n} \; : \; x_j = 0 \quad \forall j \notin \{2i-1, 2i\} \right\}$$

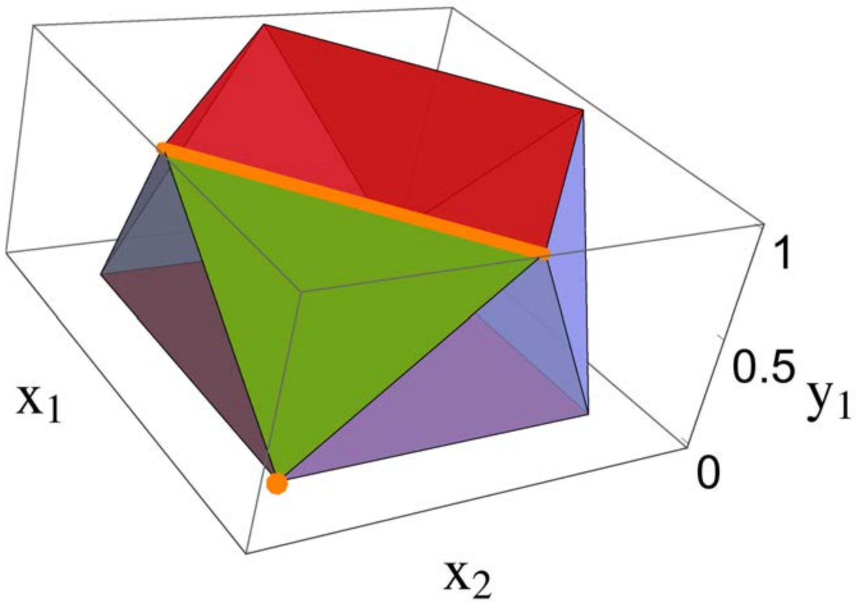
Faces for Ideal Formulation with Unary Encoding

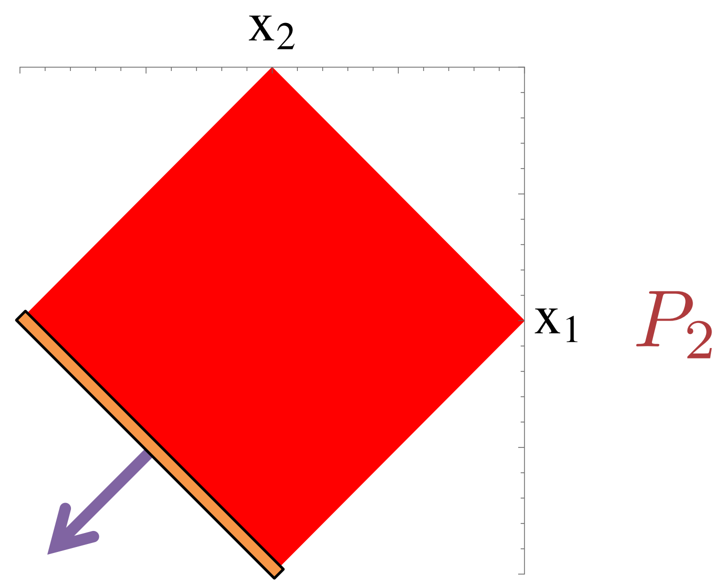
- Two types of facets (or faces):
  - $P_1 \times \{0\} \equiv y_i \geq 0$
  - $\text{conv}\left(\left(F_1 \times 0\right) \cup \left(F_2 \times 1\right)\right)$

    F_i proper face of P_i
  - Not all combinations of faces
  - Which ones are valid?



Valid Combinations = Common Normals



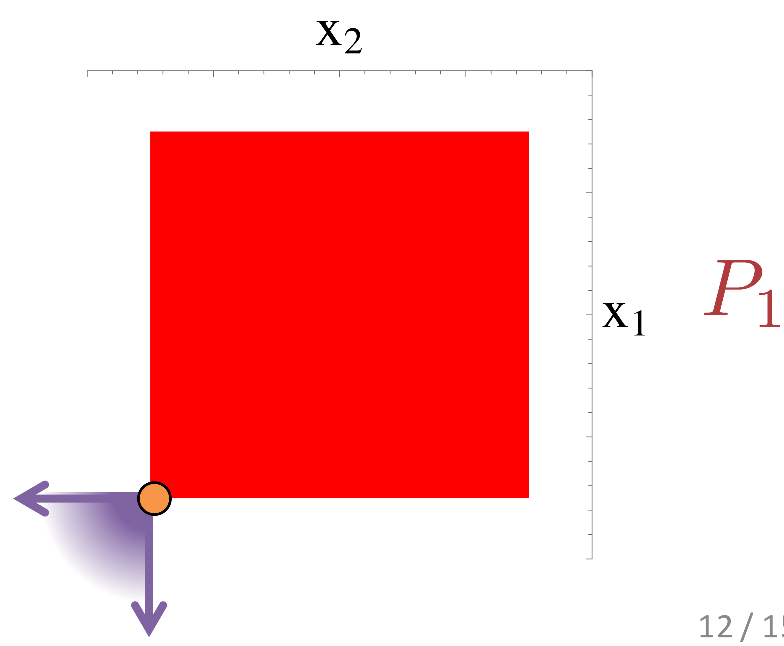


$$N\left(F_1\right) \cap N\left(F_2\right) \neq \emptyset$$

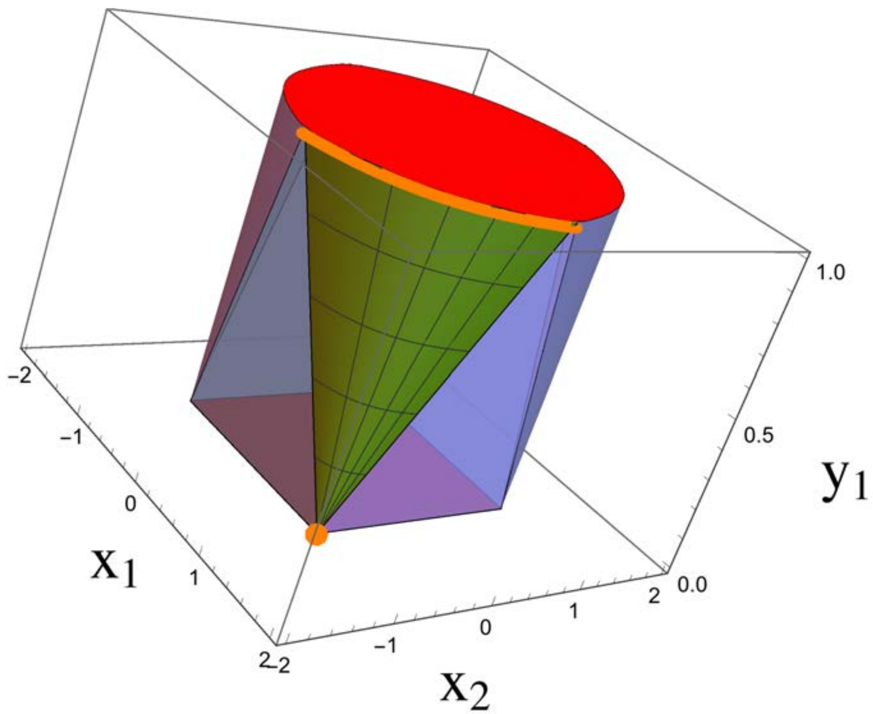
$$\Updownarrow$$

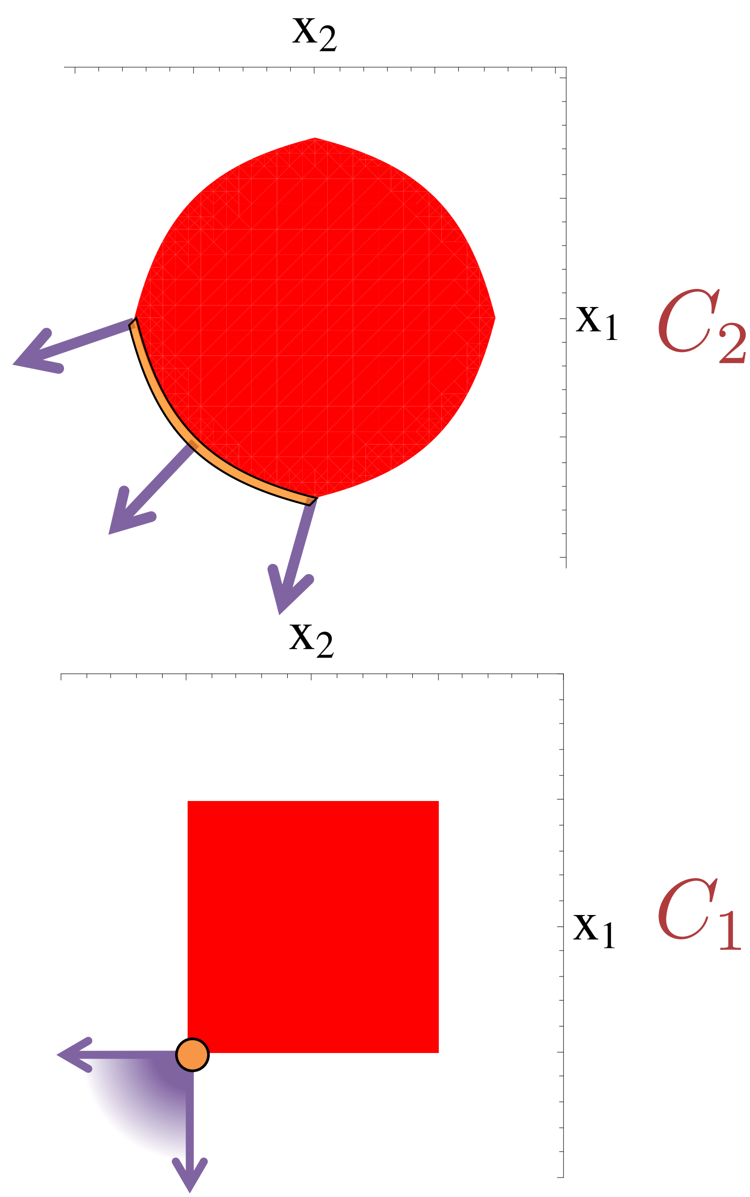
$$\operatorname{conv}\left(\left(F_1 \times 0\right) \cup \left(F_2 \times 1\right)\right)$$

is face of $Q\left(H\right)$



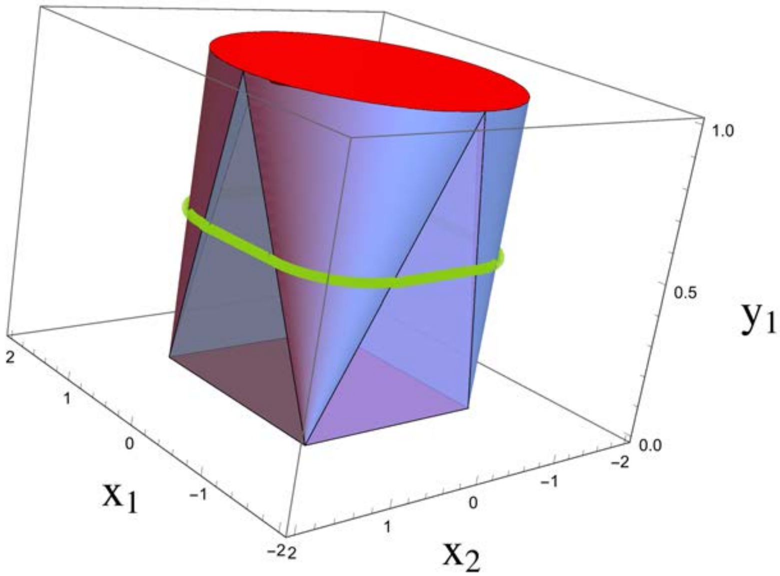
Unary Embedding for Unions of Convex Sets

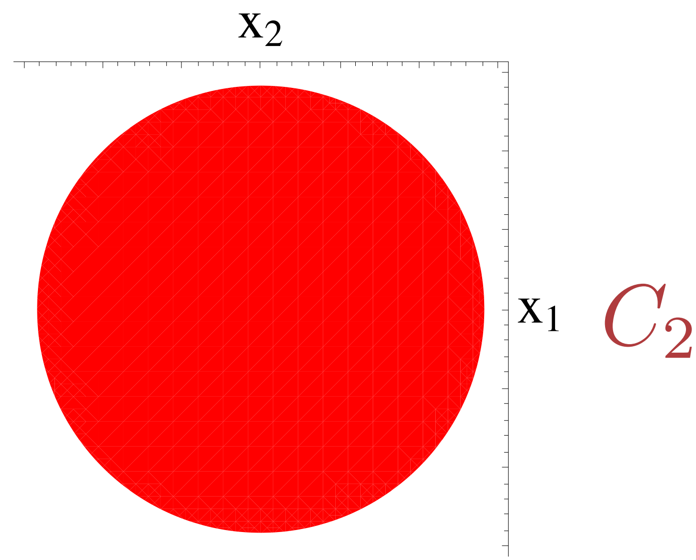




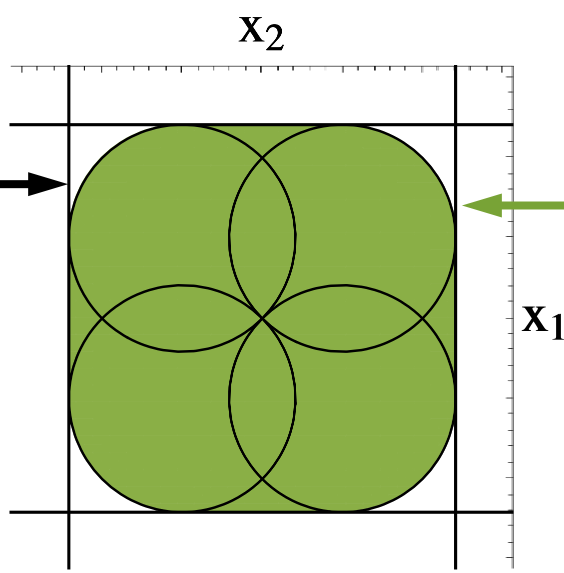
- Description of boundary of $Q(H)$ is easy if "normals condition" yields convex hull of 1 nonlinear constraint and point(s)

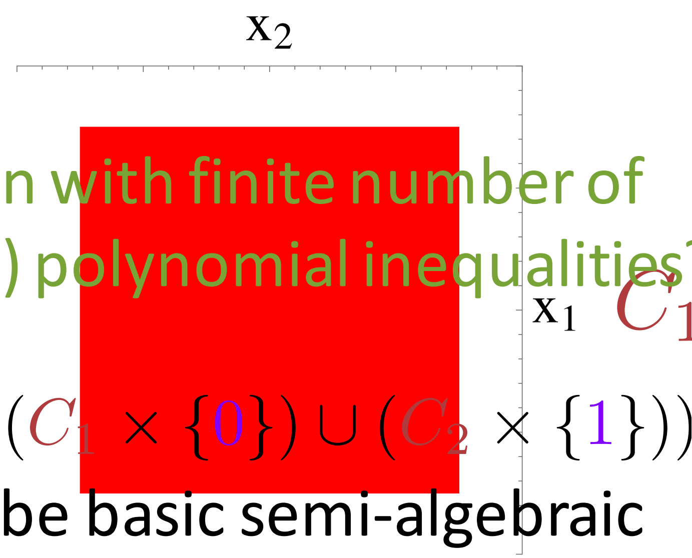
Bad Example: Representability Issues





Zariski closure of boundary





Description with finite number of (quadratic) polynomial inequalities?

$$Q := \operatorname{conv}\left((C_1 \times \{0\}) \cup (C_2 \times \{1\})\right)$$

can fail to be basic semi-algebraic

Summary

- Embedding Formulations = Systematic procedure for strong (ideal) non-extended formulations
  - Encoding can significantly affect size
- Complexity of Union of Polyhedra beyond convex hull
  - Embedding Complexity (non-extended ideal formulation)
  - Relaxation Complexity (any non-extended formulation)
  - Still open questions on relations between complexity (Embedding Formulations and Complexity for Unions of Polyhedra, arXiv:1506.01417)
- Embedding Formulations for Convex Sets
  - MINLP formulations
  - Can have representability issues
    - Open question: minimum number of auxiliary variables for fixing this