# Embedding Formulations and Complexity for Unions of Polyhedra

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Massachusetts Institute of Technology

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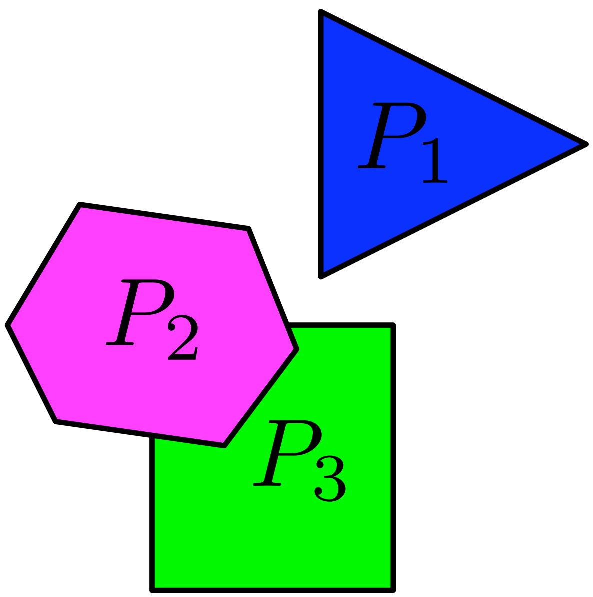
# (Linear) Mixed 0-1 Integer Formulations

- Modeling Finite Alternatives = Unions of Polyhedra

$$x \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^d$$



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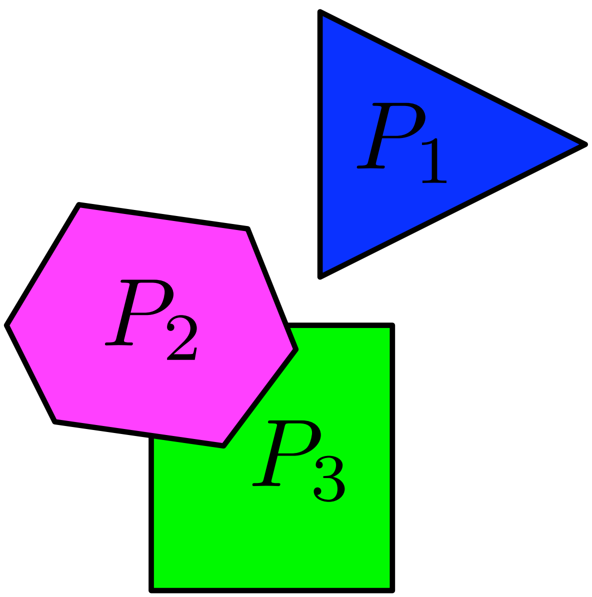


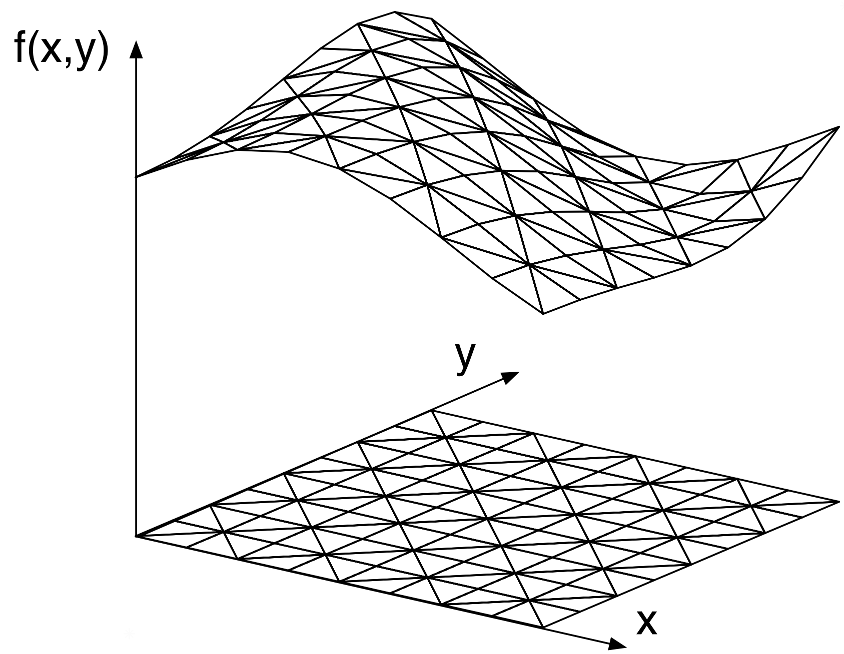


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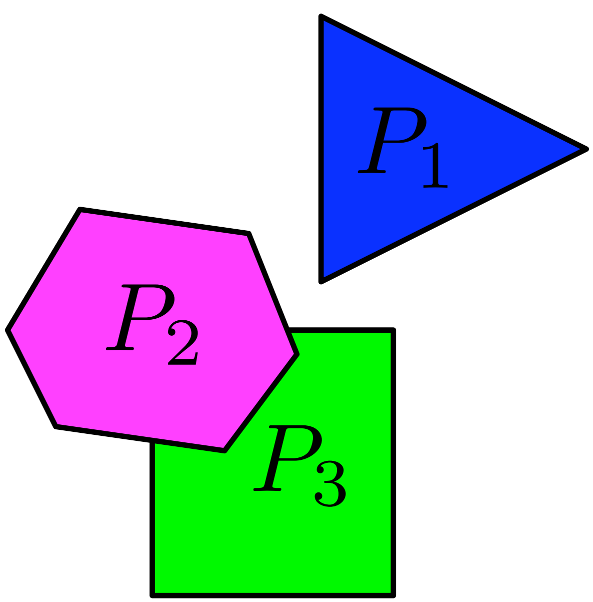


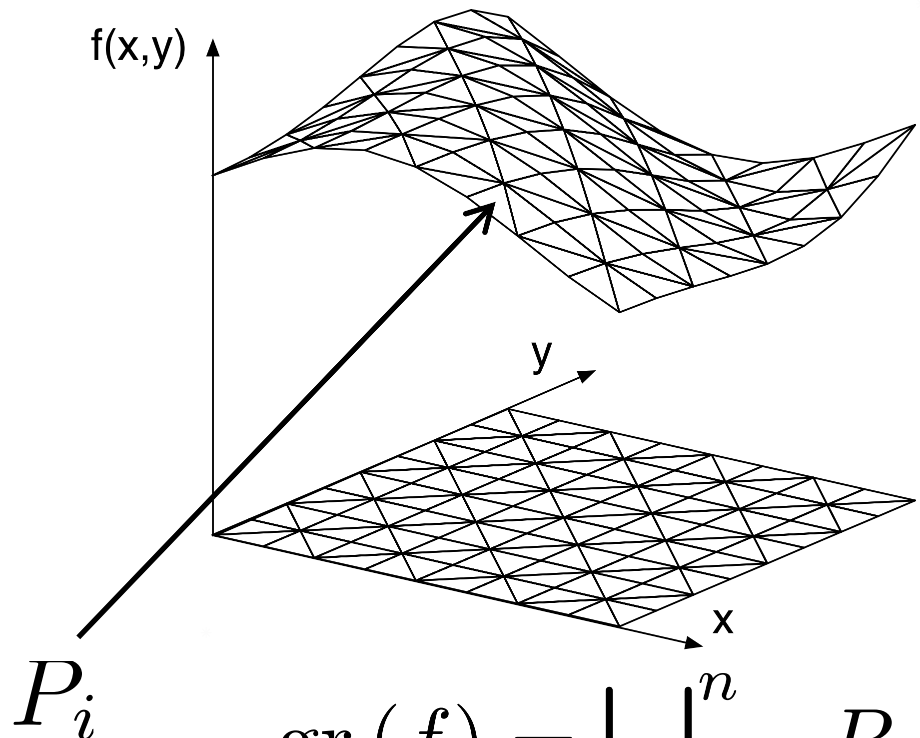


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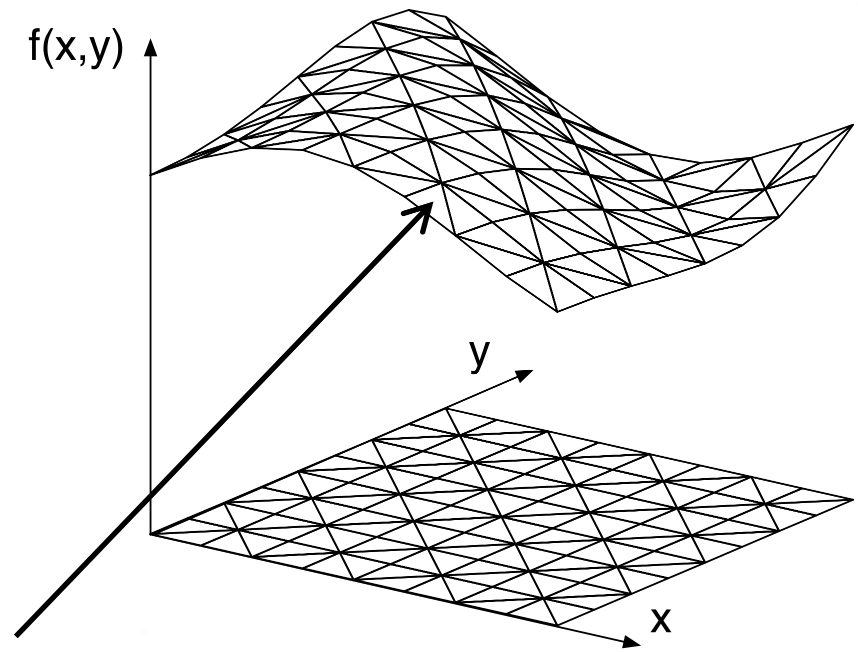
# (Linear) Mixed <u>0-1</u> Integer Formulations

- Modeling Finite Alternatives = Unions of Polyhedra

$$\min \quad \sum_{j=1}^{m} f_j(x_j, y_j)$$

$$s.t.$$

$$(x, y) \in X$$




$P_i$

$$\operatorname{gr}(f) = \bigcup_{i=1}^{n} P_i$$

# Size of Smallest 0-1 Formulation for $x \in \bigcup_{i=1}^{n} P_i$

- Standard **ideal (integral)** **extended** formulation for $P_i = \left\{x \in \mathbb{R}^d \ : \ A^i x \leq b^i\right\}$ (Balas, Jeroslow and Lowe):

$$
\begin{array}{lll}
A^i x^i \leq b^i y_i & & \forall i \in \{1,\ldots,n\} \\
\sum_{i=1}^{n} x^i = x, & x^i \in \mathbb{R}^d & \forall i \in \{1,\ldots,n\} \\
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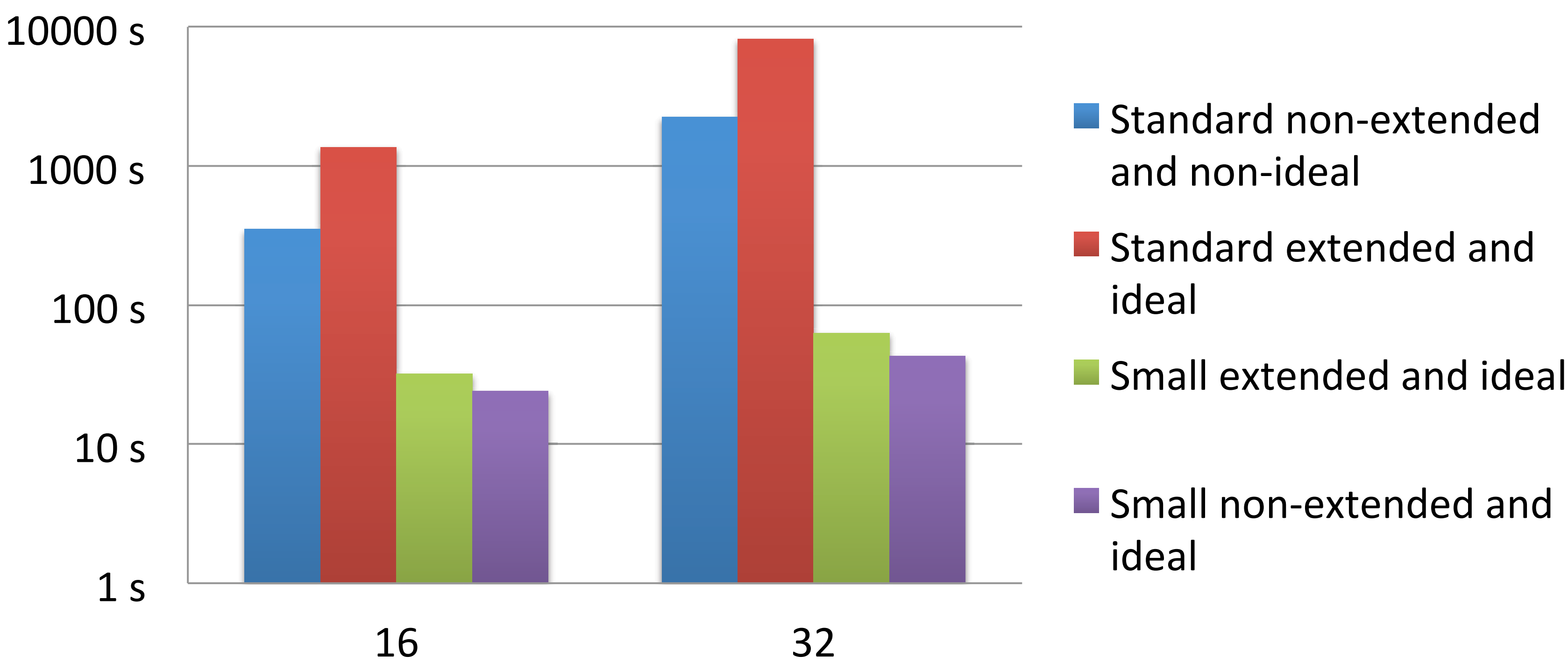
- What about non-extended (i.e. no variables copies) ?
- What about non-ideal? (i.e. **some** fractional extreme pts.)?
- What about **precise** lower/upper bounds on size?

# Performance for Univariate Functions

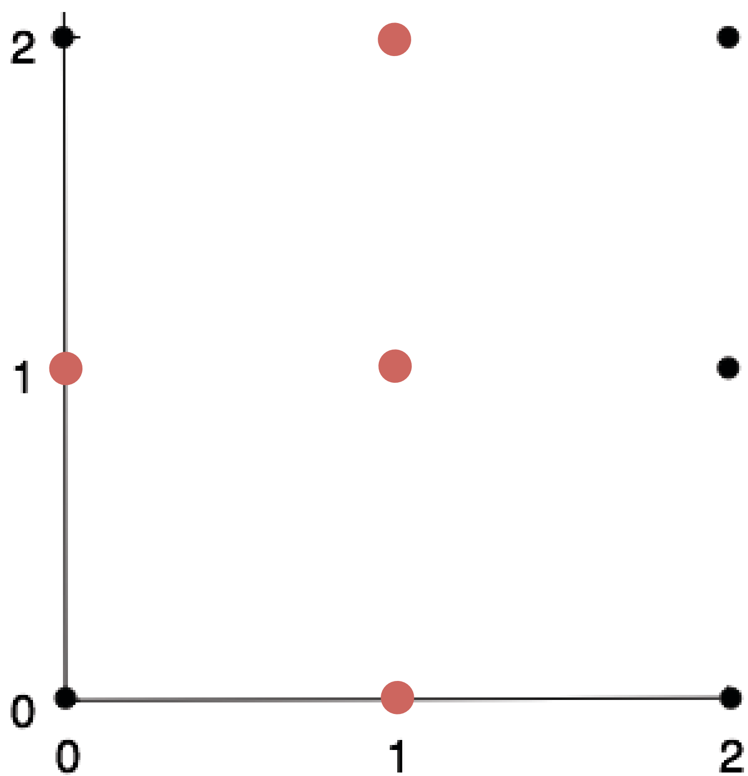
- Results from Nemhauser, Ahmed and V. '10 using CPLEX 11




- Non-extended and ideal formulations provide a significant computational advantage

# Constructing Non-extended Ideal Formulations

- Pure Integer :



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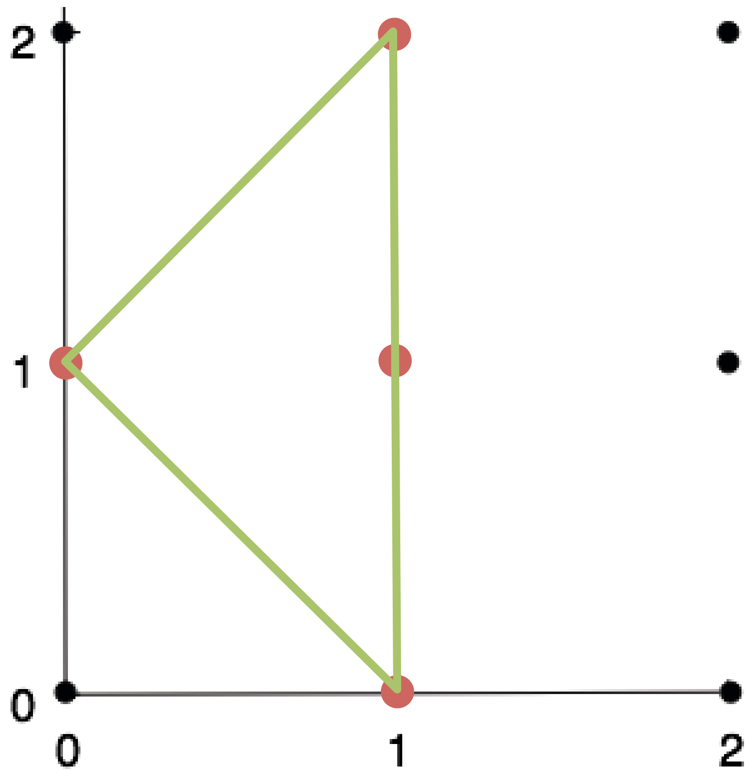
$$Q := \operatorname{conv}\left(\left\{p^i\right\}_{i=1}^{n}\right)$$



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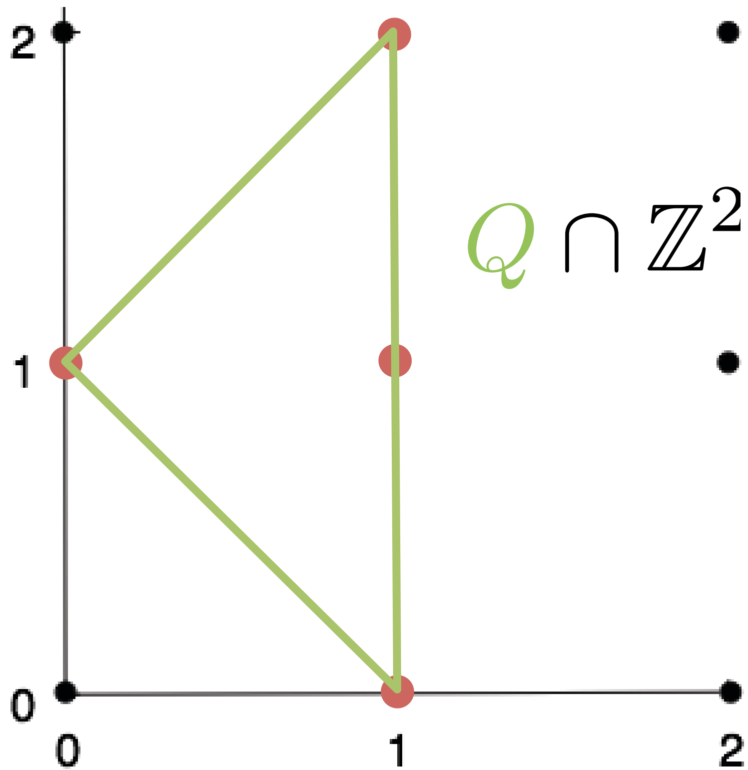

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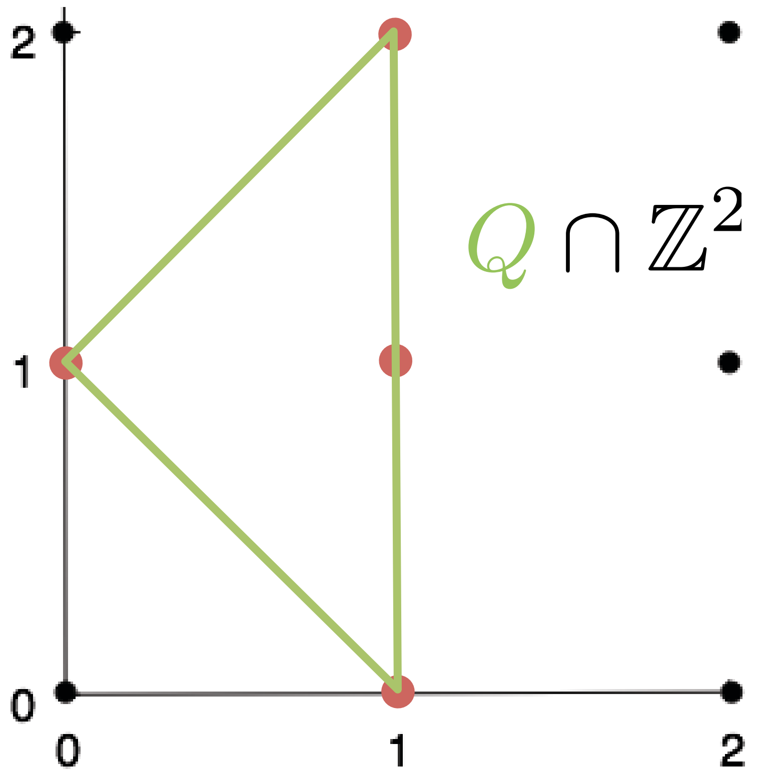
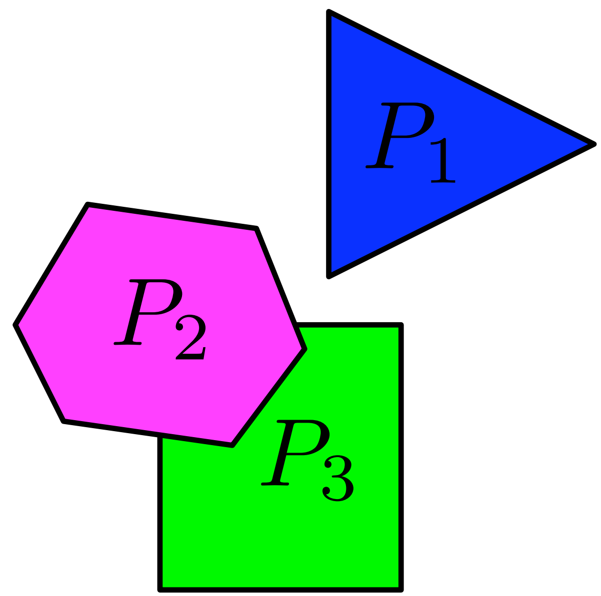



- Mixed Integer:



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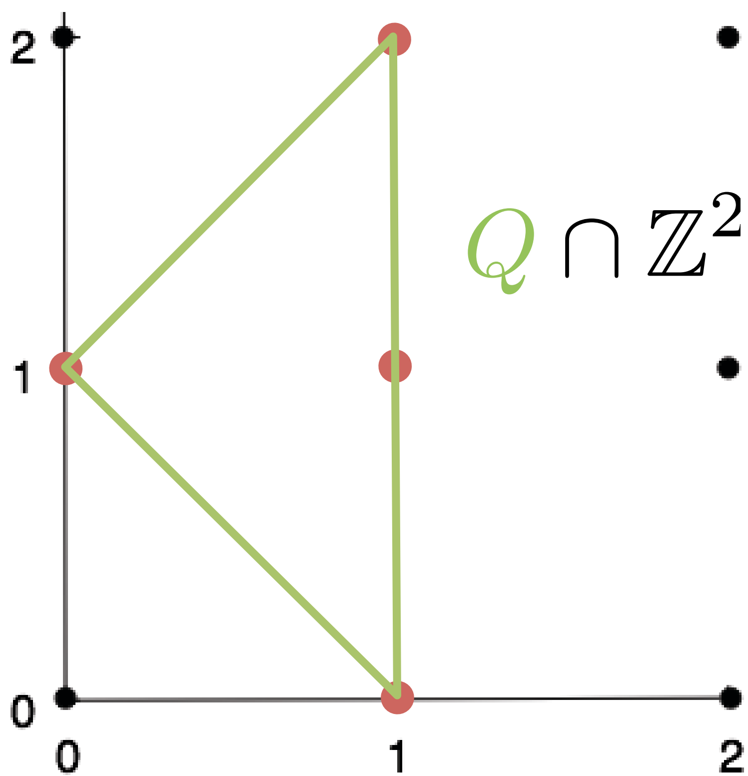


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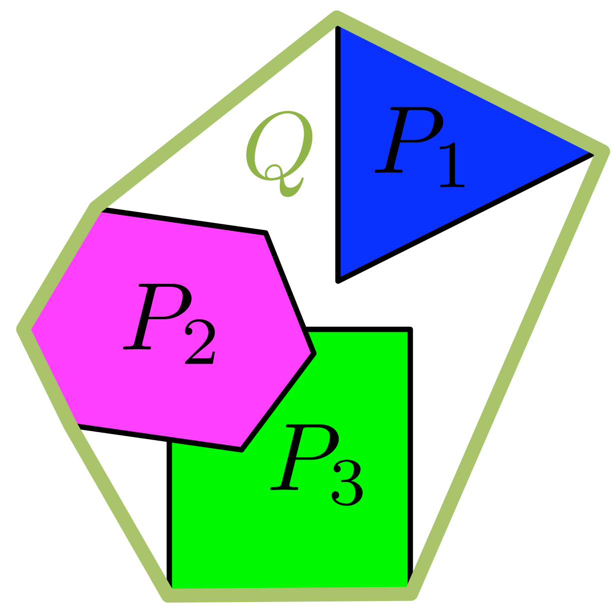
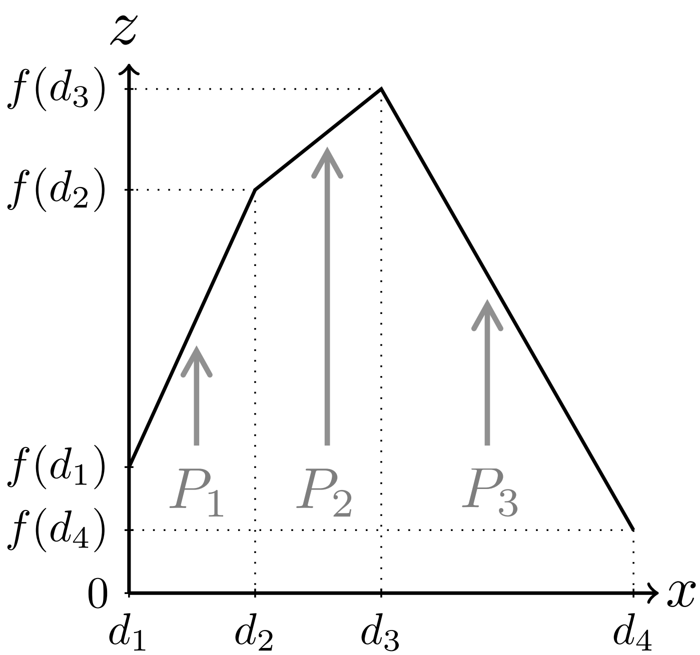


# “Simple” Family of Polyhedra

$$(x, z) \in \mathrm{gr}\,(f) = \bigcup_{i=1}^{3} P_i$$



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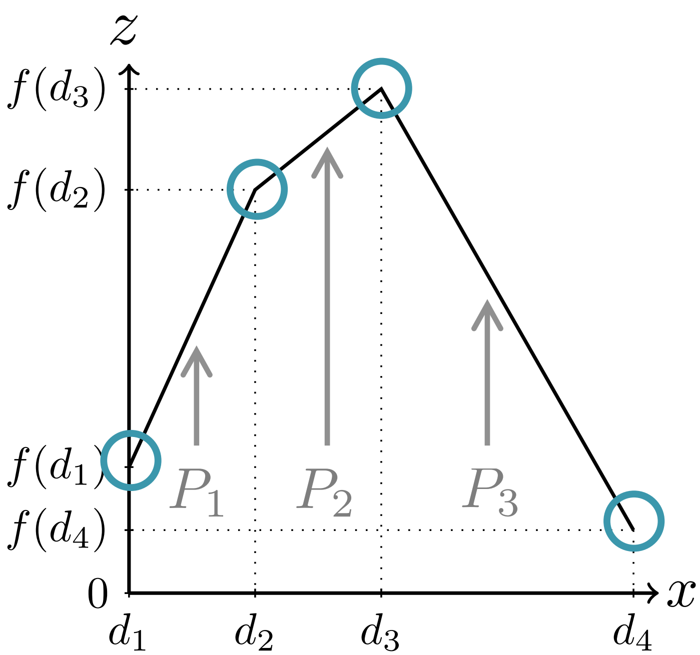


$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{4} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_{d_j}$$

$$\lambda \in \Delta^4 := \left\{ \lambda \in \mathbb{R}_+^4 \;:\; \sum_{i=1}^{4} \lambda_i = 1 \right\}$$

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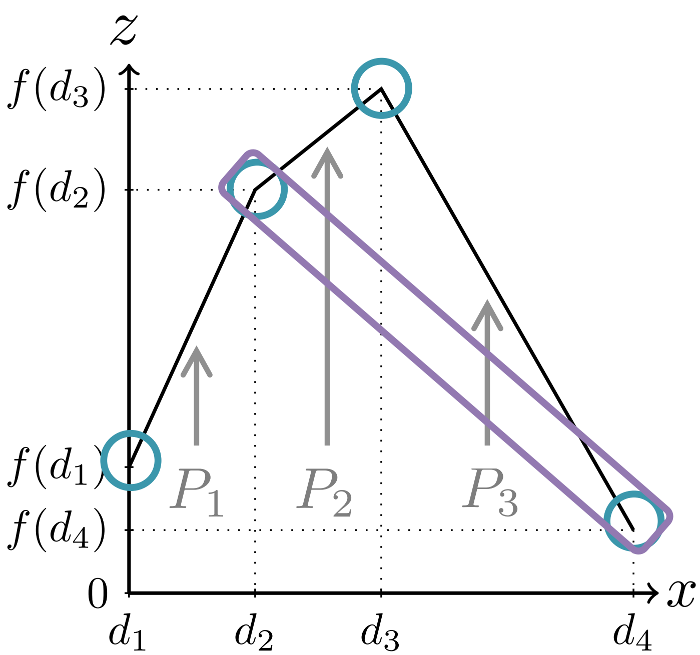


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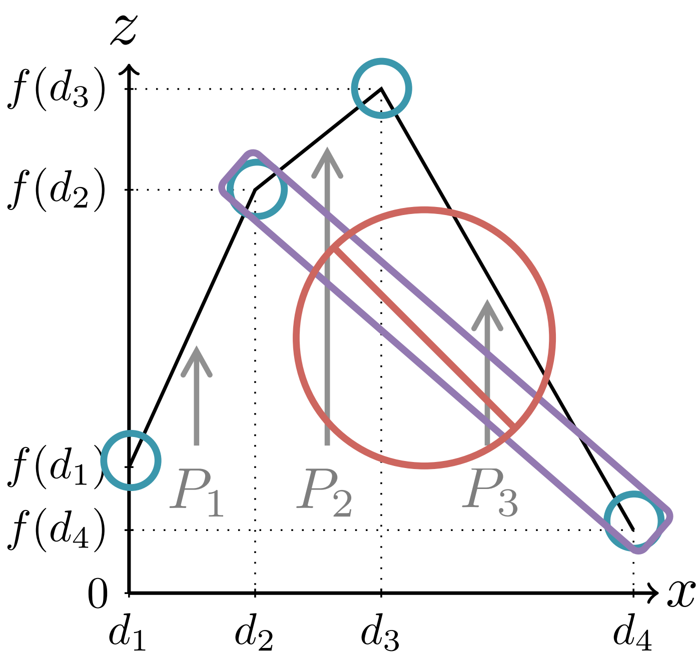


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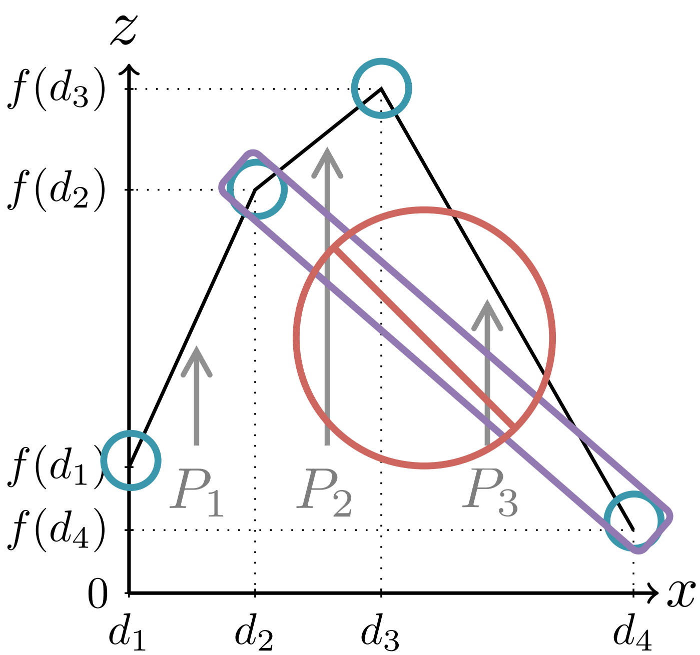


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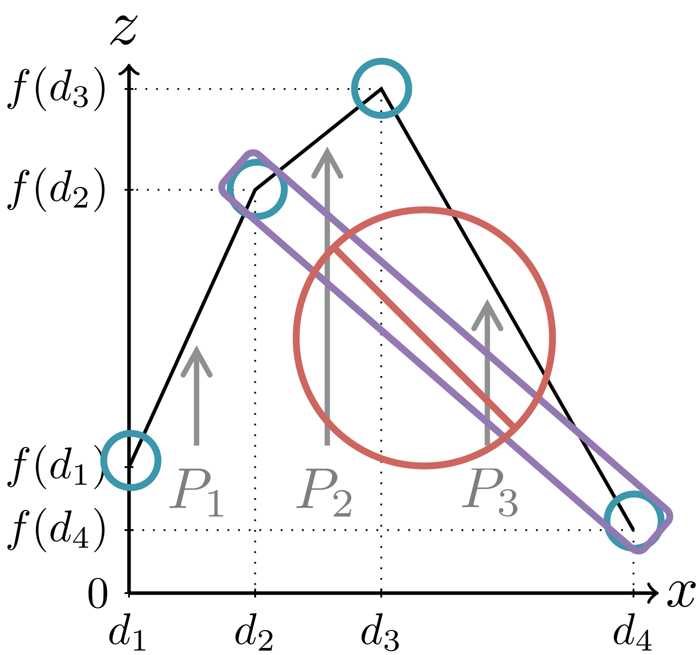


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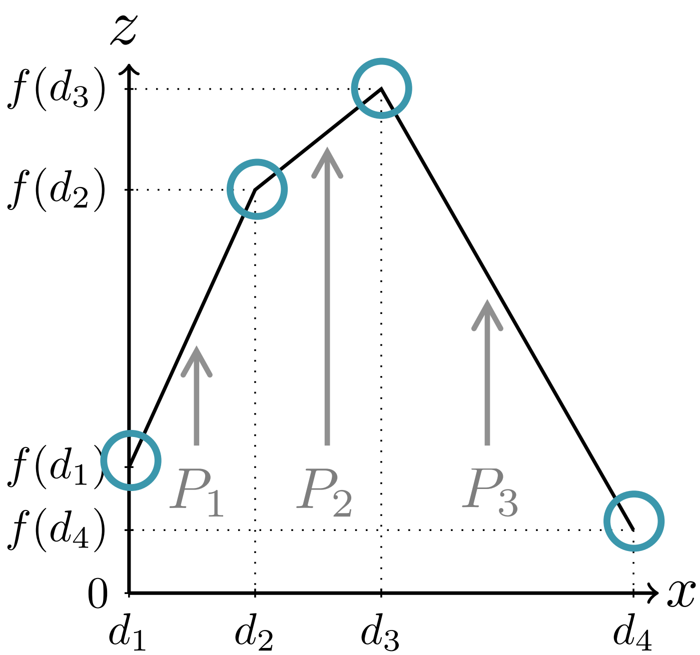


SOS2 Constraints

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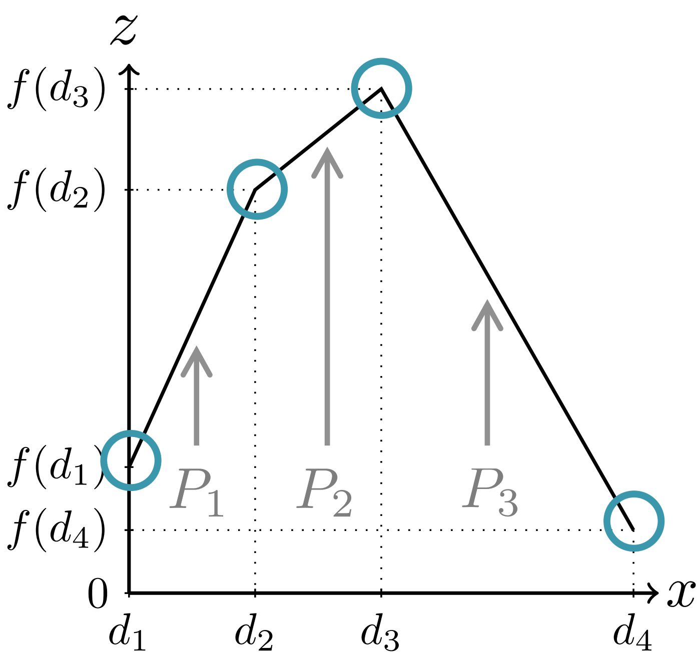


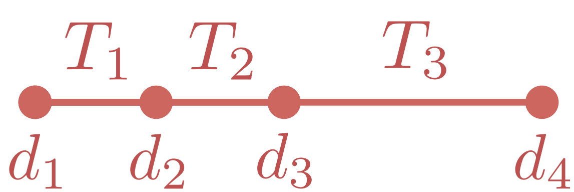


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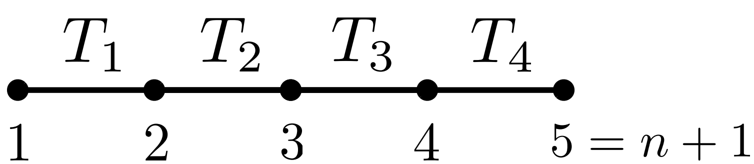


$$\sum_{i=1}^{5} \lambda_i = 1$$

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$$\overbrace{\begin{aligned}
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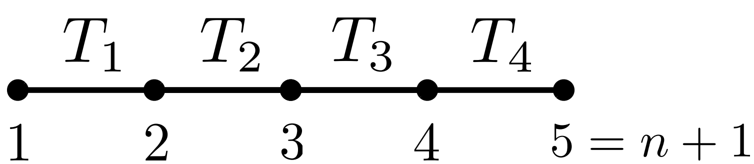


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General Inequalities

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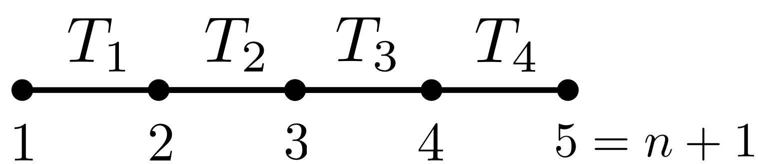


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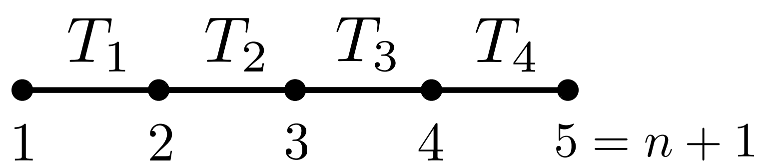


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  - Ideal formulation:
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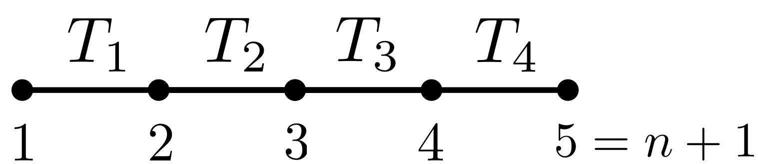


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$$2\lceil \log_2 n \rceil$$

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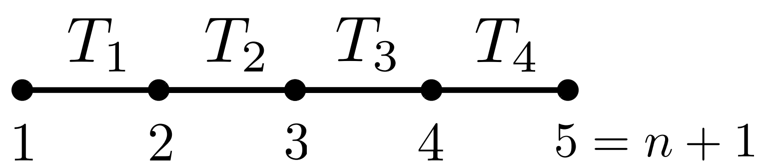


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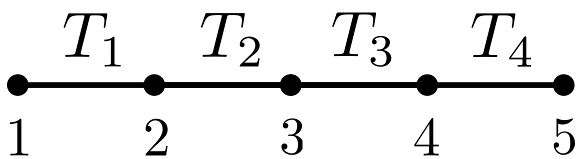
    $$2 \leq \ldots \leq 5+2n$$

# What is a Formulation?

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$$P_i := \{\lambda \in \Lambda^5 \,:\, \lambda_j = 0 \quad j \notin \{i, i+1\}\}$$

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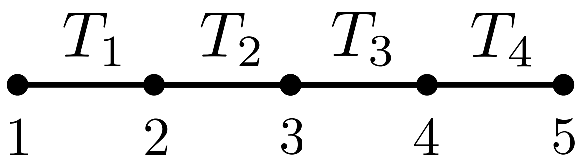


$Q$ = LP relaxation

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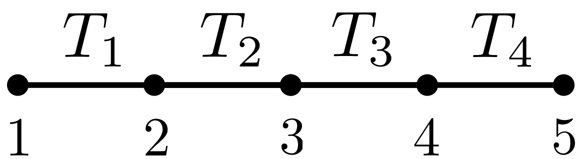


$Q$ = LP relaxation → $\sum_{i=1}^{5} \lambda_i = 1$, $\sum_{i=1}^{4} y_i = 1$

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$$(\lambda, y) \in Q \cap \left(\mathbb{R}^5 \times \mathbb{Z}^4\right)$$
$$\Updownarrow$$
$$y = e^i \;\wedge\; \lambda \in P_i$$

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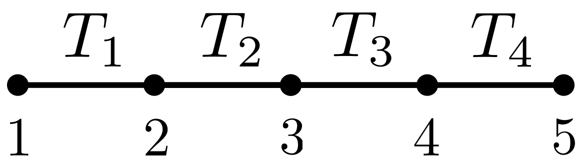


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$$y \in \{0,1\}^4, \quad \sum_{i=1}^{5} \lambda_i = 1 \quad \sum_{i=1}^{4} y_i = 1$$

$$\begin{aligned}
0 &\le \lambda_1 \le y_1 \\
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\end{aligned}$$

$$(\lambda, y) \in Q \cap \left(\mathbb{R}^5 \times \mathbb{Z}^4\right)$$

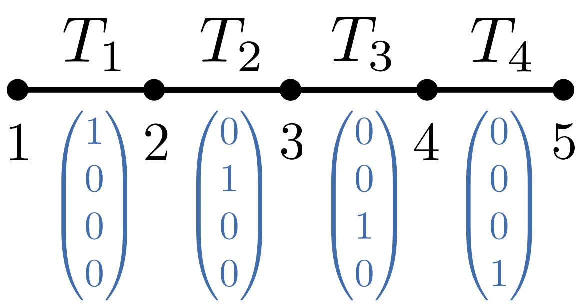
$$\Updownarrow$$

$$y = e^i \;\wedge\; \lambda \in P_i$$

Unary Encoding

$$P_i := \{\lambda \in \Lambda^5 \,:\, \lambda_j = 0 \quad j \notin \{i, i+1\}\}$$

# What is a Formulation?

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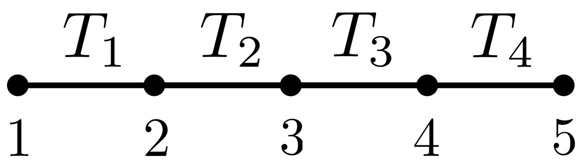
# Alternate Meaning of 0-1 Variables




$Q$ = LP relaxation $\rightarrow$ $\boxed{\sum_{i=1}^{5} \lambda_i = 1}$

- V. and Nemhauser '08.

$$
\boxed{
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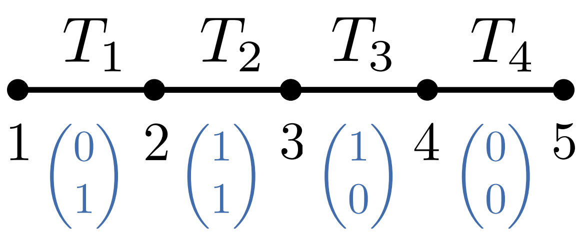


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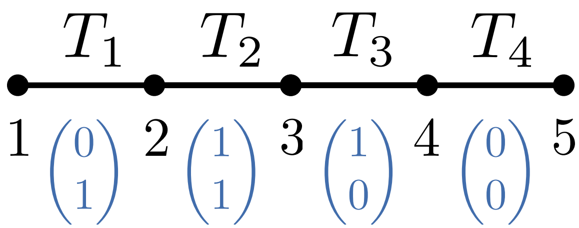


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Binary Encoding

$$P_i := \{\lambda \in \Lambda^5 \, : \, \lambda_j = 0 \quad j \notin \{i, i+1\}\}$$

# Embedding Formulations for Union of Polyhedra

- **Non-Extended** formulation of $\lambda \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^V$:
  - Encoding $H := \{h^i\}_{i=1}^{n} \subseteq \{0,1\}^k, \quad h^i \neq h^j$
  - Polyhedron $Q \subseteq \mathbb{R}^V \times \mathbb{R}^k$, s.t.

$$(\lambda, y) \in Q \cap \left(\mathbb{R}^V \times \mathbb{Z}^k\right) \quad \Leftrightarrow \quad y = h^i \ \wedge \ \lambda \in P_i$$

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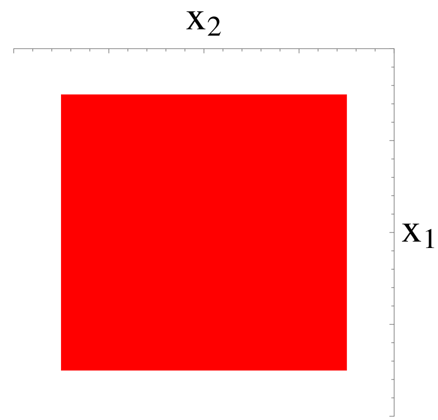
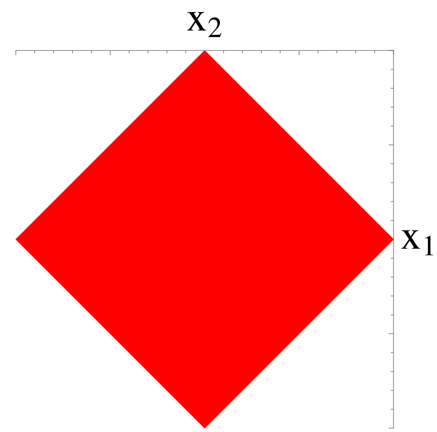
For unary encoding:

$$h^i = e^i$$

# Embedding Formulation = Ideal non-Extended

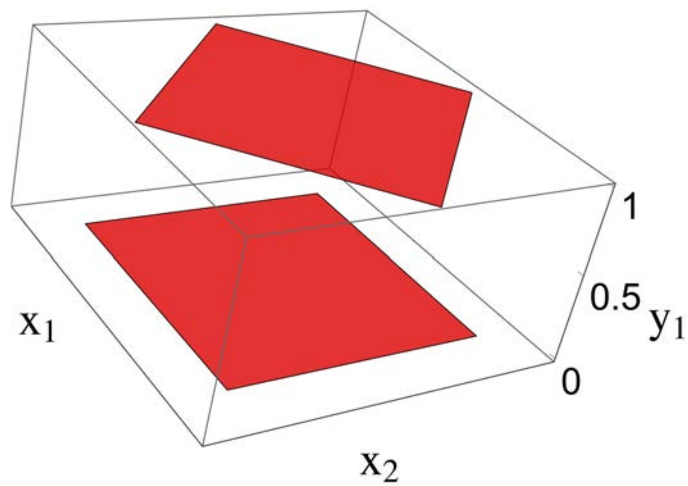


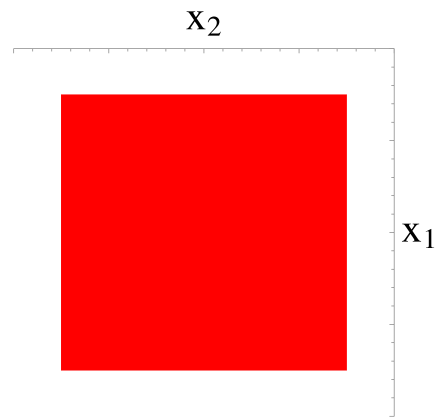

$P_1$




$P_2$

# Embedding Formulation = Ideal non-Extended

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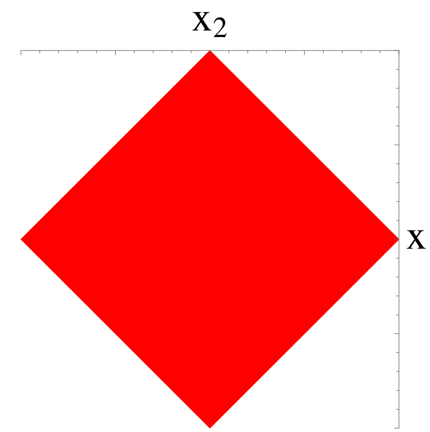


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$$\left(P_1 \times \{0\}\right) \cup \left(P_2 \times \{1\}\right)$$

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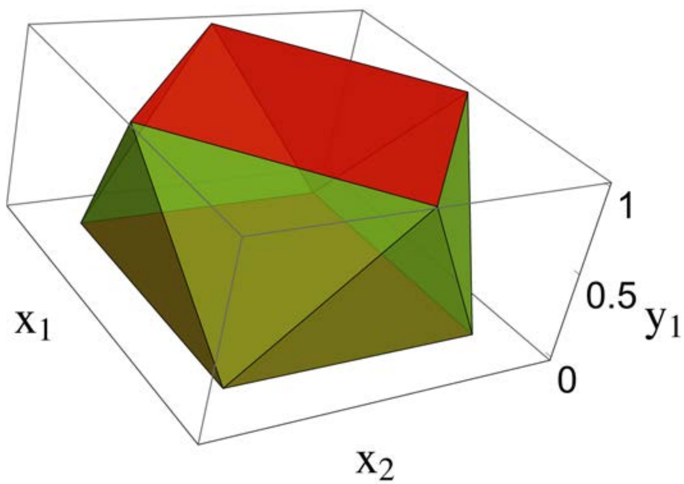


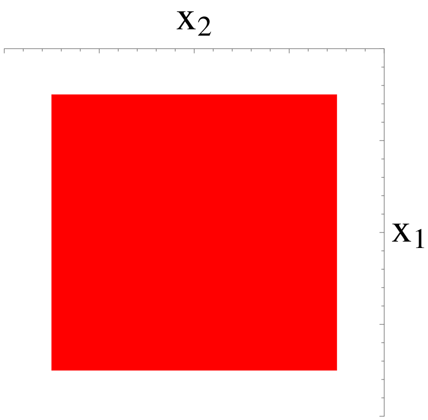


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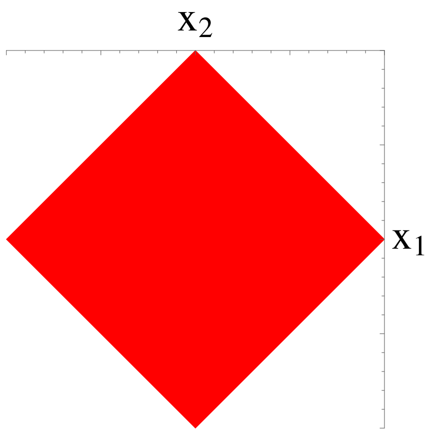


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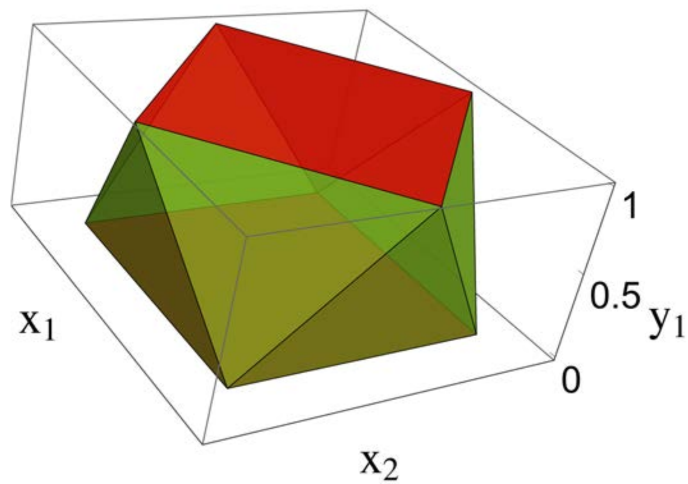


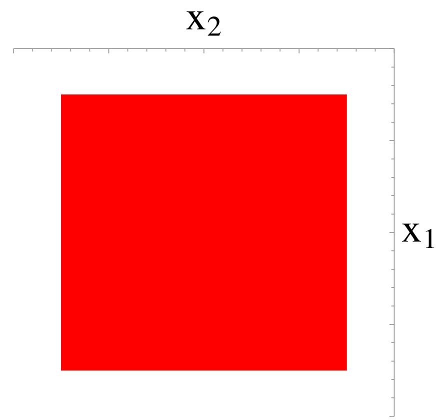


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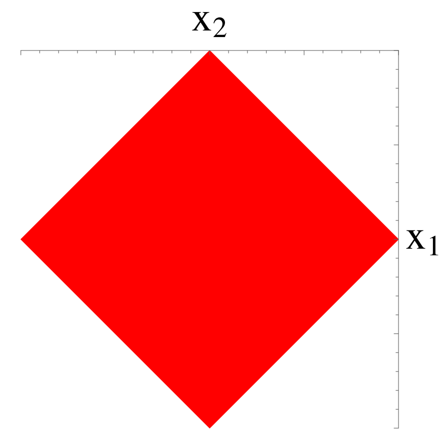


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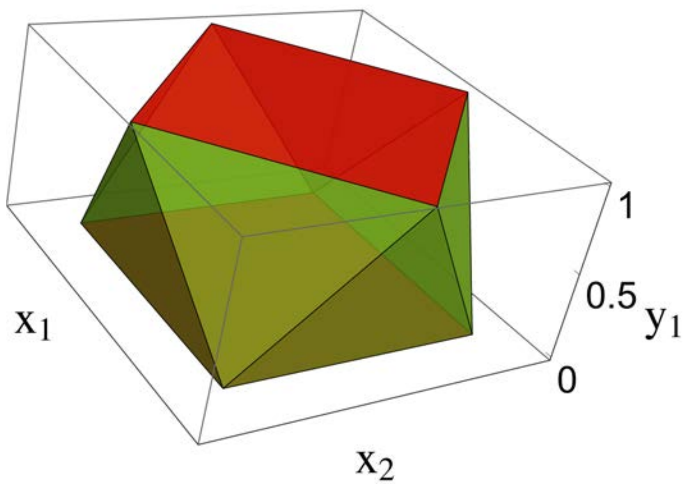


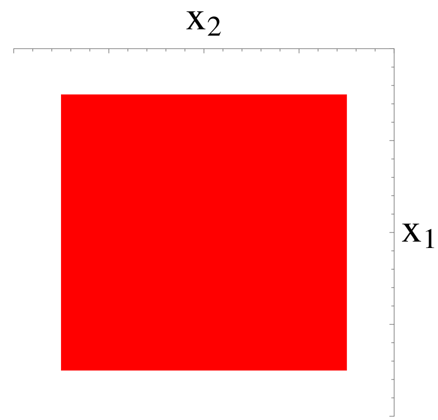


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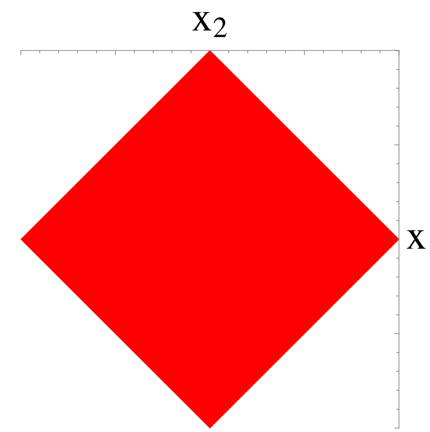


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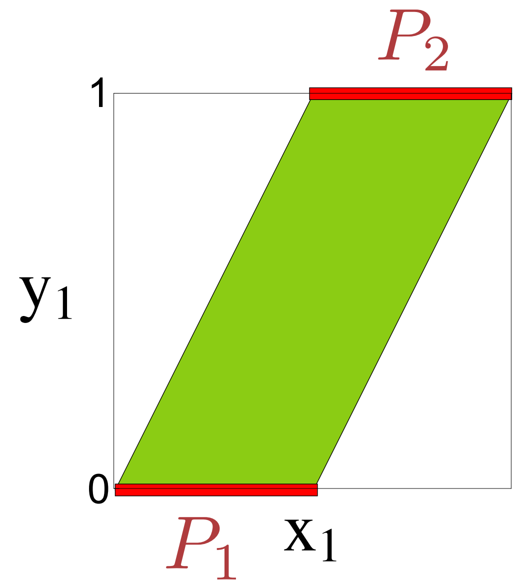
$$\operatorname{ext}(Q(H)) \subseteq \mathbb{R}^2 \times \mathbb{Z}$$

# Complexity of Family of Polyhedra $\mathcal{P} := \{P_i\}_{i=1}^{n}$

- Embedding complexity = smallest **ideal** formulation




$$\mathrm{mc}(\mathcal{P}) := \min_H \{\mathrm{size}(Q(H))\}$$

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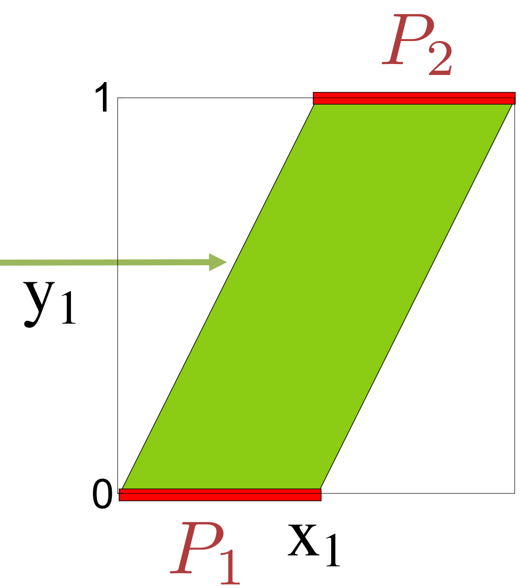


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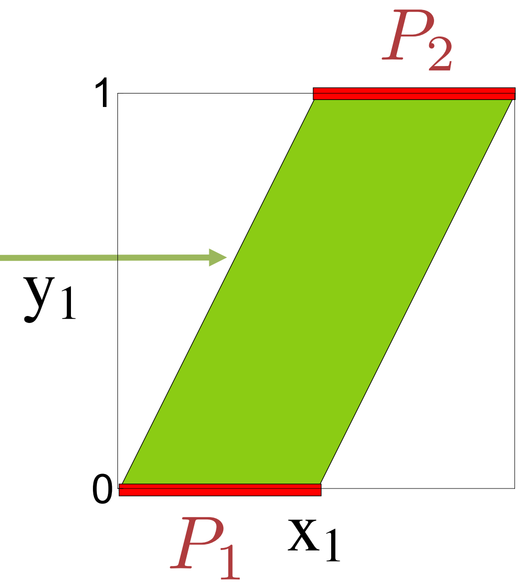


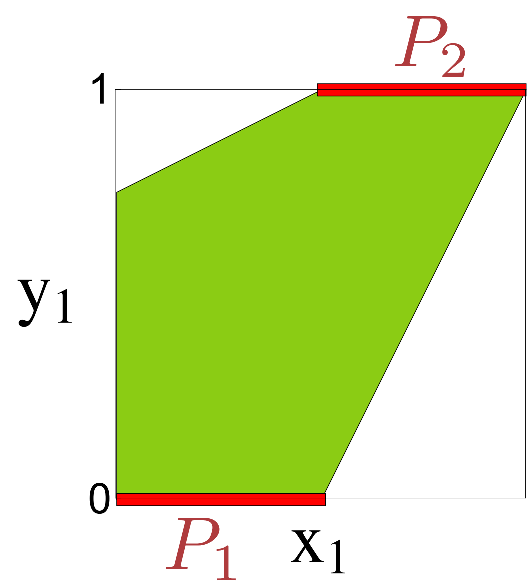


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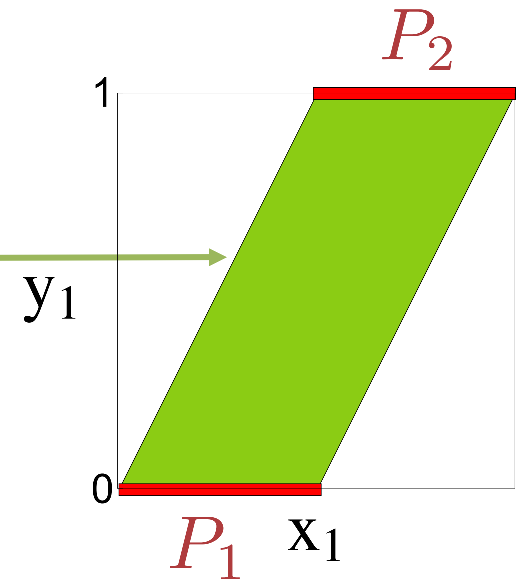
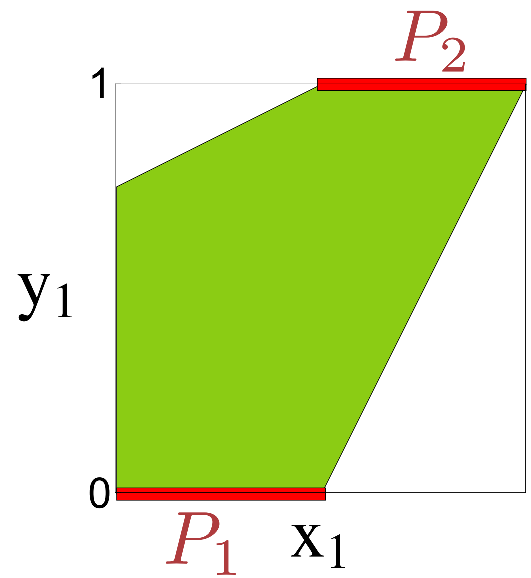



- Relaxation complexity = smallest formulation

$$\mathrm{rc}(\mathcal{P}) := \min_{Q,H} \{\mathrm{size}(Q)\}$$

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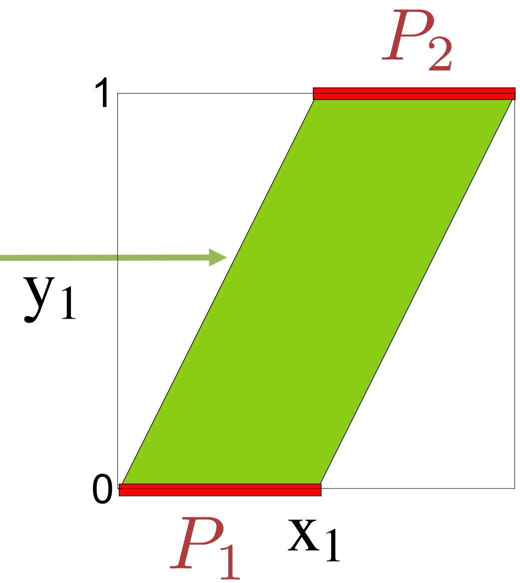


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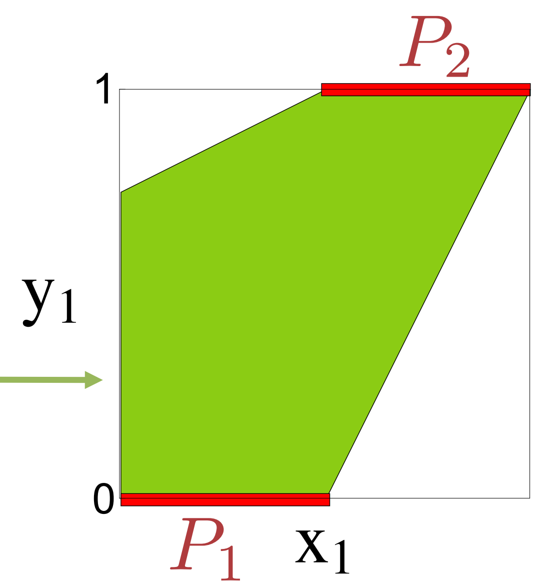


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# Summary of Results

- Lower and Upper bounds for special structures:
  - e.g. for Special Order Sets of Type 2 (SOS2) on $n$ variables
    - Embedding complexity (ideal)

$$2\lceil \log_2 n \rceil \longleftarrow \text{General Inequalities}$$

$$n+1 \leq \ldots \leq n+1+2\lceil \log_2 n \rceil \longleftarrow \text{Total}$$

    - Relaxation complexity (non-ideal)

$$2 \leq \ldots \leq 4 \longleftarrow \text{General Inequalities}$$

$$2 \leq \ldots \leq 5+2n \longleftarrow \text{Total}$$

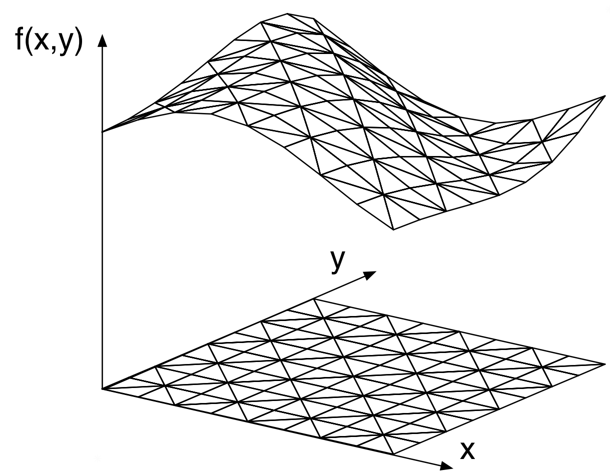
- Relation to other complexity measures

$$\operatorname{hc}(\mathcal{P}) := \operatorname{size}\left(\operatorname{conv}\left(\bigcup_{i=1}^{n} P_i\right)\right)$$

$$\operatorname{xc}(\mathcal{P}) := \min_R \left\{\operatorname{size}(R) \;:\; \operatorname{proj}_x(R) = \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i\right)\right\}$$

- Still open questions (see V. 2015)

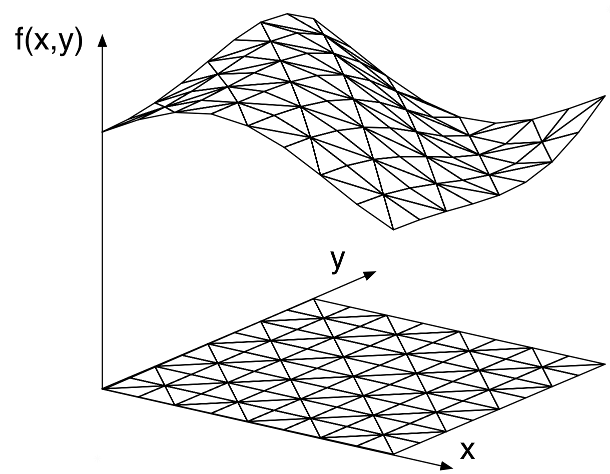
# Why bounds? Encoding Selection Matters



# Why bounds? Encoding Selection Matters

- Size of unary formulation is:
  (Lee and Wilson '01)

$$\binom{2\sqrt{n/2}}{\sqrt{n/2}} + \left(\sqrt{n/2}+1\right)^2$$



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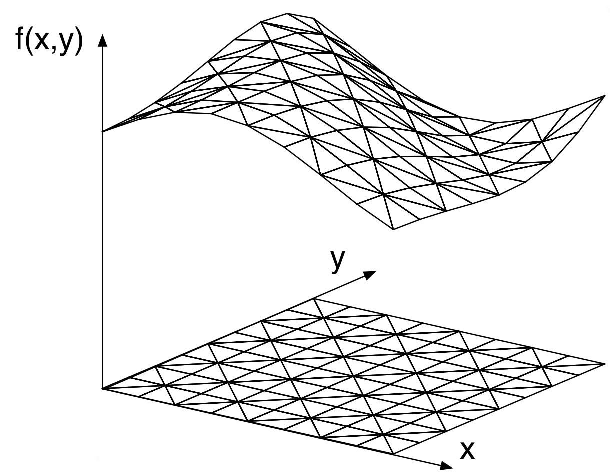

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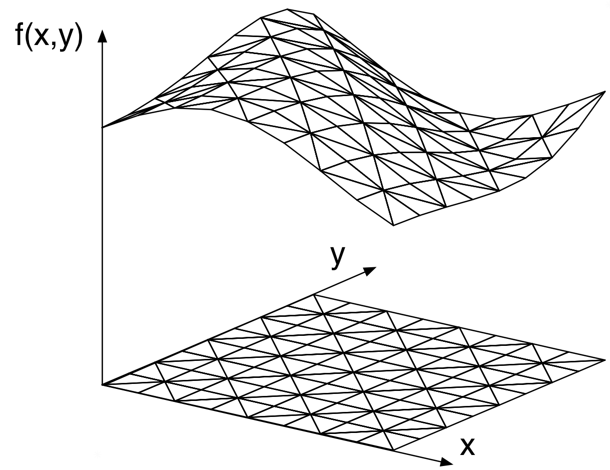
General Inequalities

Variable Bounds



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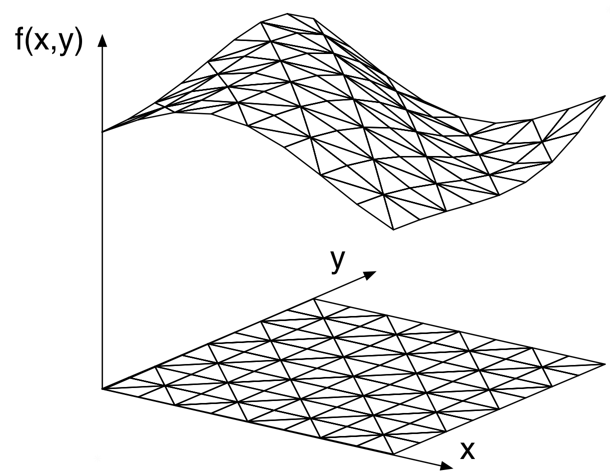


- Size of one binary formulation:
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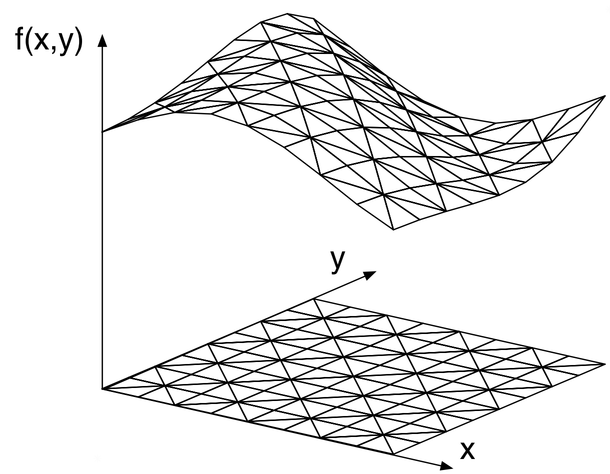


- Size of one binary formulation:
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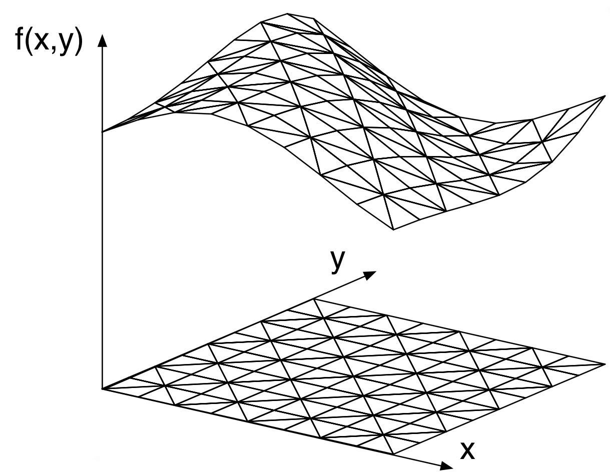


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$$4\log_2\sqrt{n/2} + 2 + \left(\sqrt{n/2}+1\right)^2$$

- Right embedding = significant computational advantage over alternatives (Extended, Big-M, etc.)

# Summary

- Embedding Formulations = Systematic procedure
  - Encoding can significantly affect size
- Complexity of Union of Polyhedra beyond convex hull
  - Embedding Complexity (non-extended ideal formulation)
  - Relaxation Complexity (any non-extended formulation)
  - Still open questions on relations between complexity
- More details (practical formulation construction)
  - Embedding Formulations and Complexity for Unions of Polyhedra, arXiv:1506.01417
- Application to facility layout problem (Huchette, Dey, V. '14)
  - INFORMS 2015, Philadelphia, Monday, Nov $2^{nd}$, 13:30 - 15:00
  - MC11, 11-Franklin 1, Marriott