Embedding Formulations and Complexity for Unions of Polyhedra

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Massachusetts Institute of Technology

DOS Optimization Seminars,
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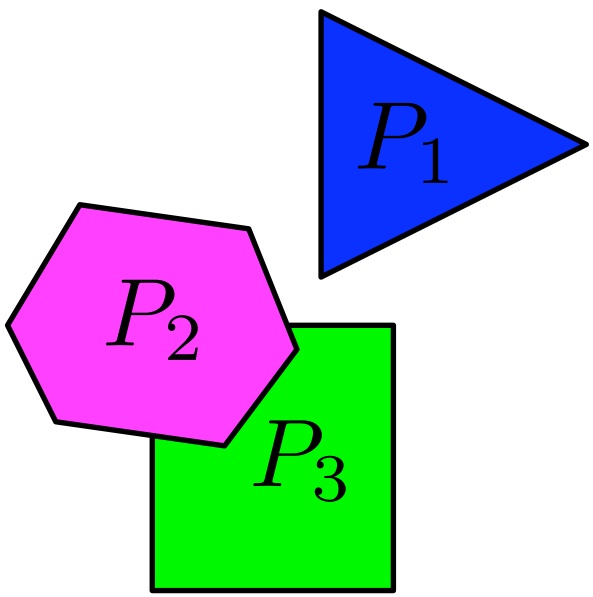
(Linear) Mixed 0-1 Integer Formulations

- Modeling Finite Alternatives = Unions of Polyhedra

$$x \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^d$$



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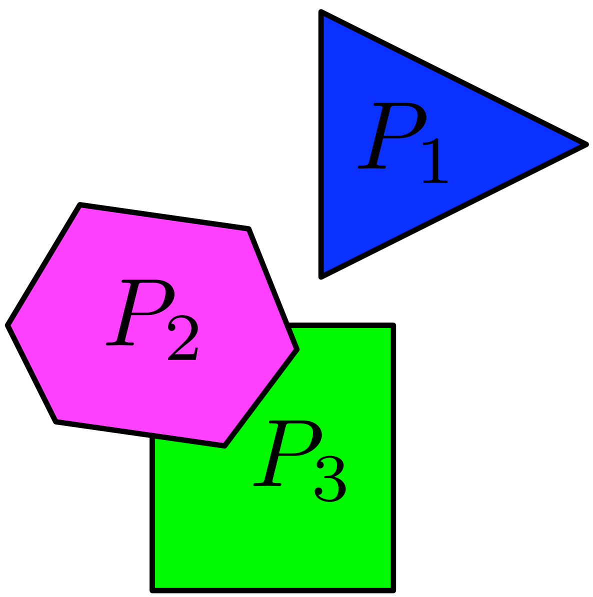

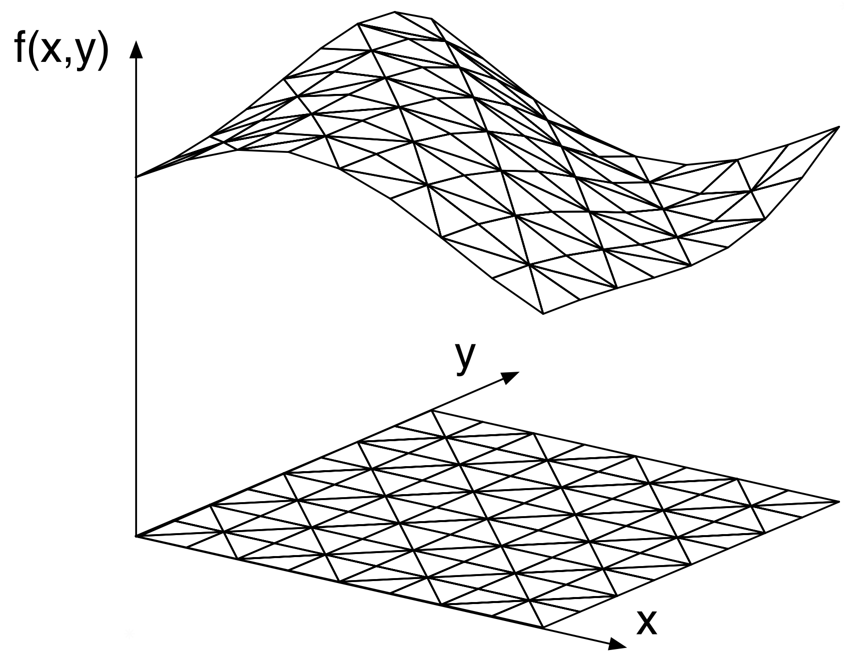

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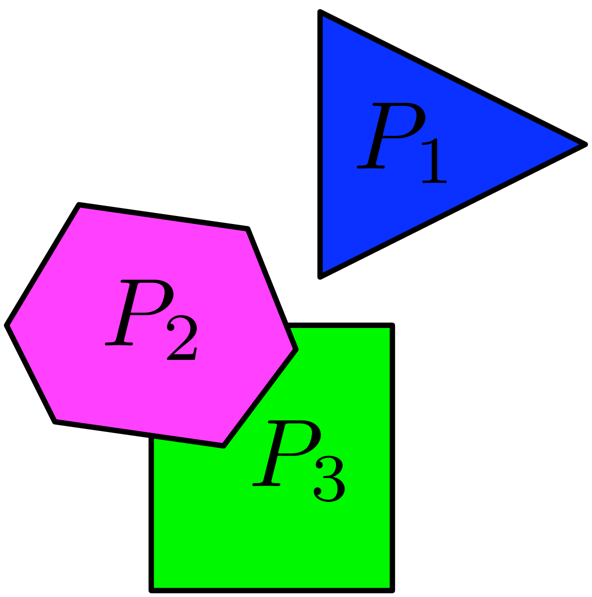

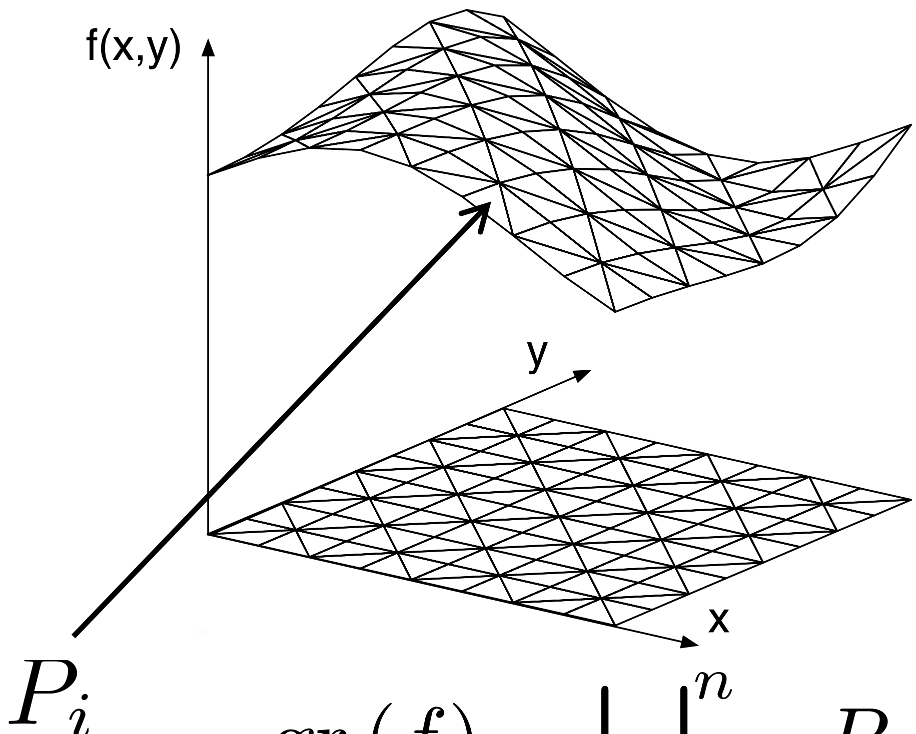

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(Linear) Mixed 0-1 Integer Formulations

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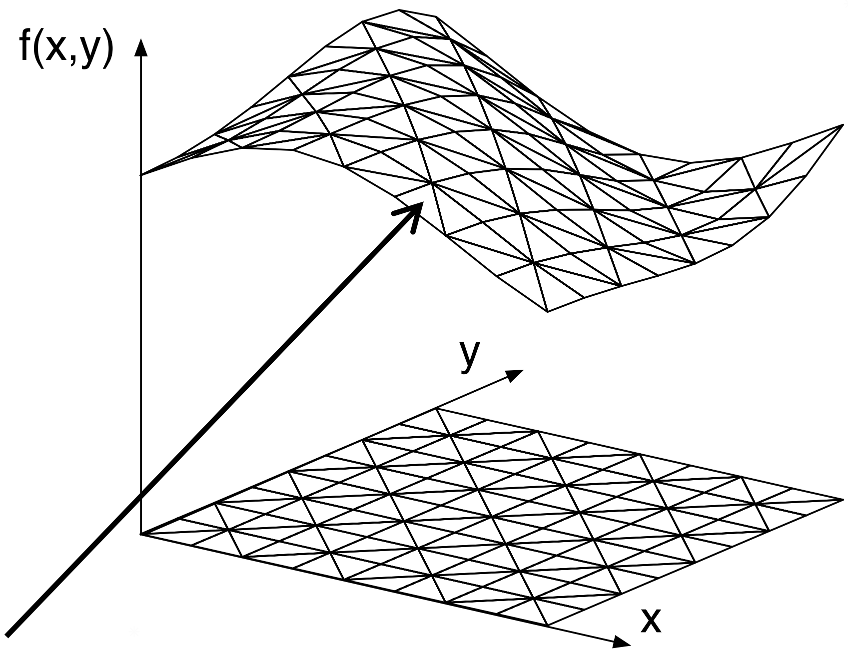
$$\min \quad \sum_{j=1}^{m} f_j(x_j, y_j)$$

$$s.t.$$

$$(x, y) \in X$$



$$P_i$$

$$\mathrm{gr}\,(f) = \bigcup_{i=1}^{n} P_i$$

Size of Smallest 0-1 Formulation for $x \in \bigcup_{i=1}^{n} P_i$

- Standard **ideal (integral)** **extended** formulation for $P_i = \left\{ x \in \mathbb{R}^d \;:\; A^i x \le b^i \right\}$ (Balas, Jeroslow and Lowe):

$$
\begin{aligned}
A^i x^i &\le b^i y_i & & \forall i \in \{1,\ldots,n\} \\
\sum\nolimits_{i=1}^{n} x^i &= x, & x^i \in \mathbb{R}^d & \quad \forall i \in \{1,\ldots,n\} \\
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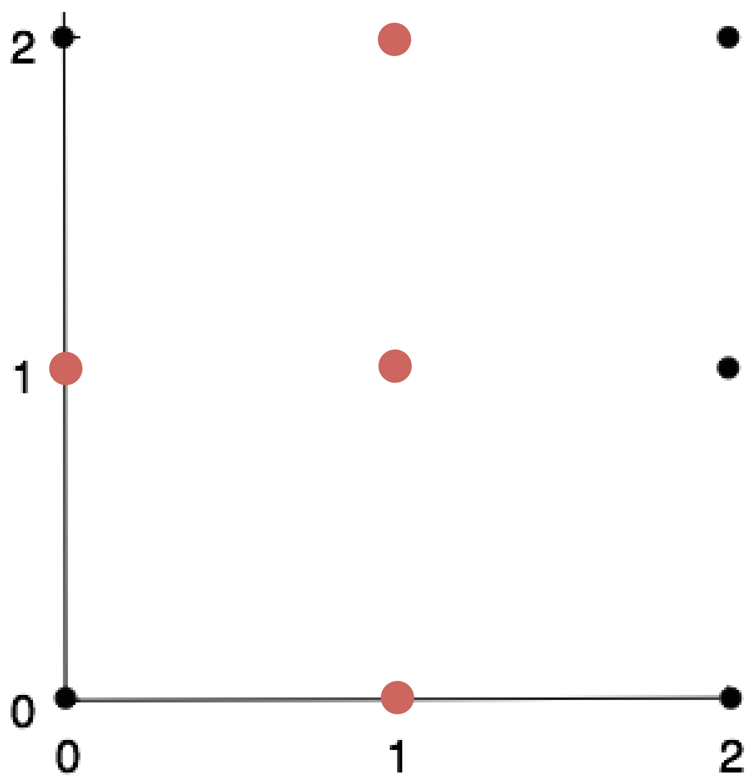
- What about non-extended (i.e. no variables copies) ?
- What about non-ideal? (i.e. **some** fractional extreme pts.)?
- What about **precise** lower/upper bounds on size?

Constructing Non-extended Ideal Formulations

- Pure Integer :



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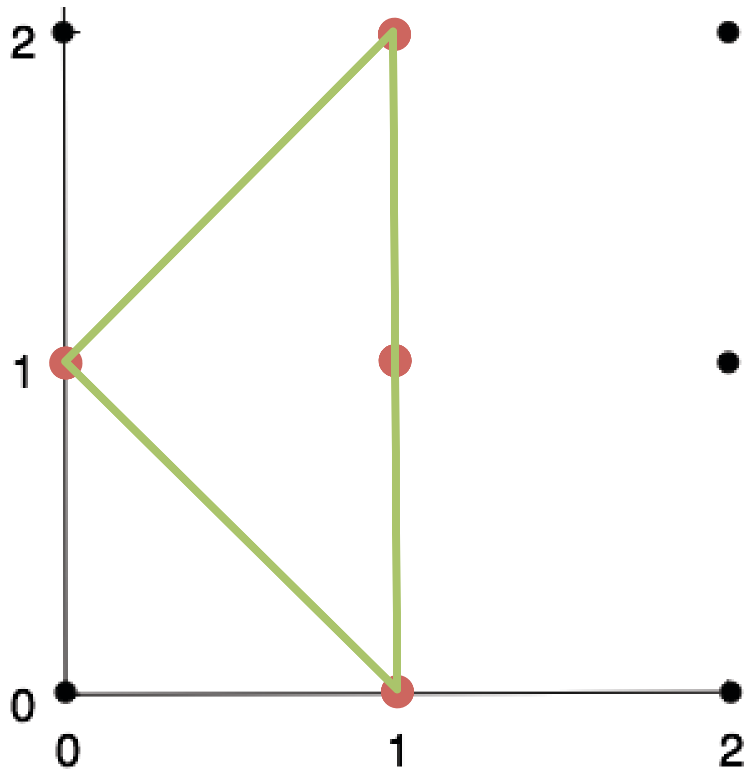
$$Q := \operatorname{conv}\left(\left\{p^i\right\}_{i=1}^{n}\right)$$



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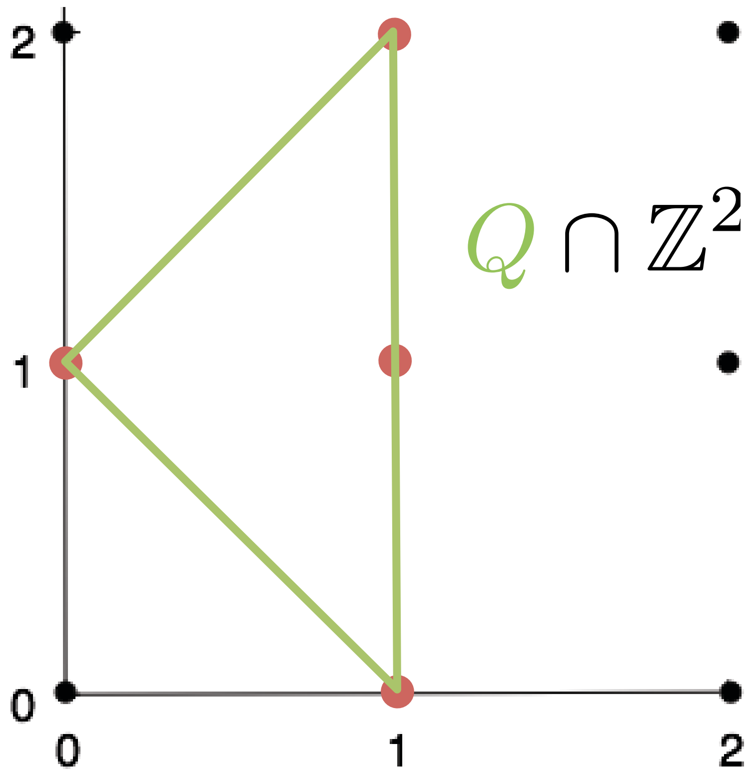

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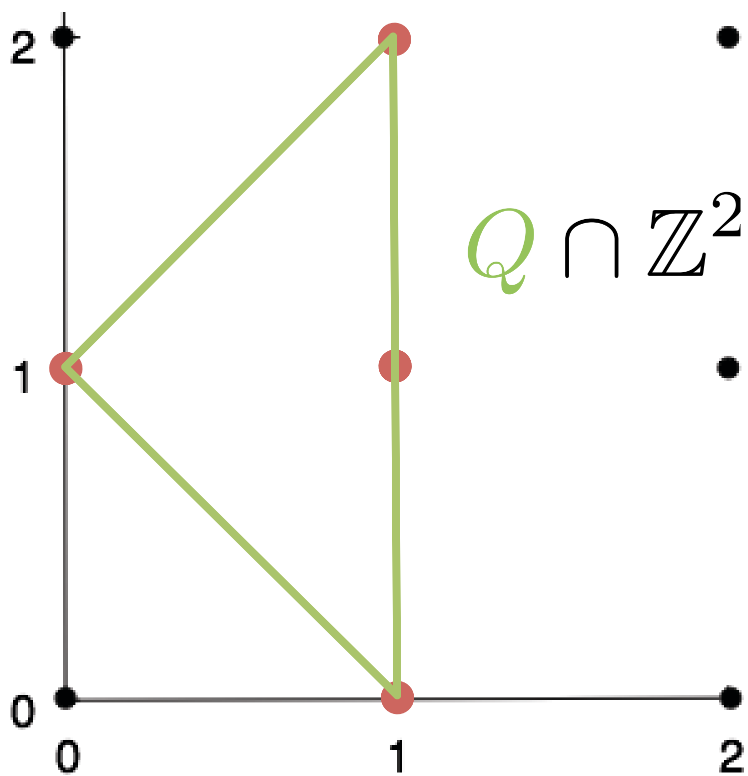

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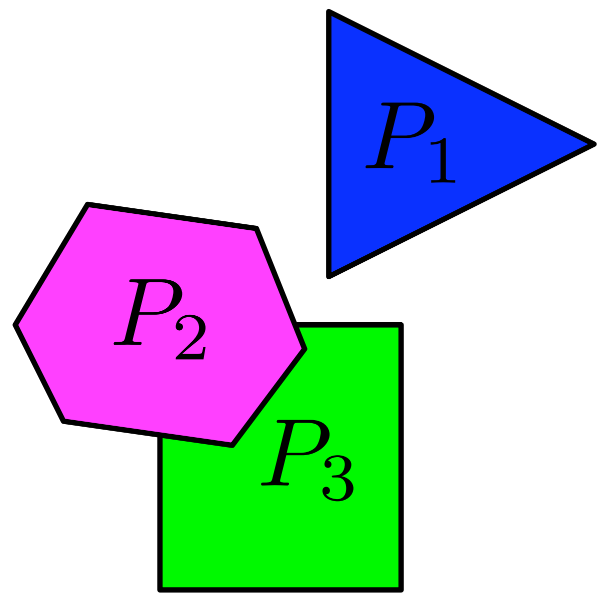

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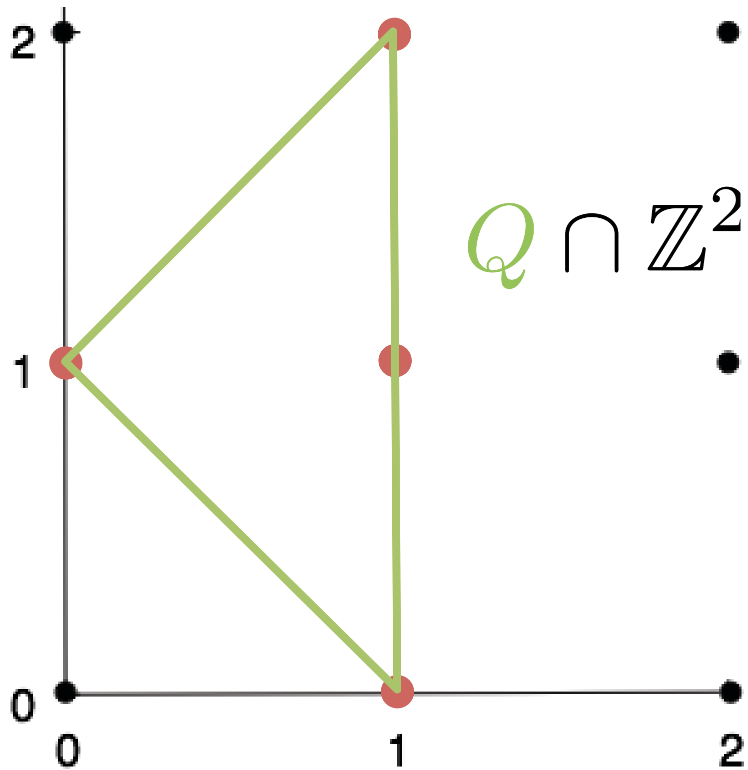

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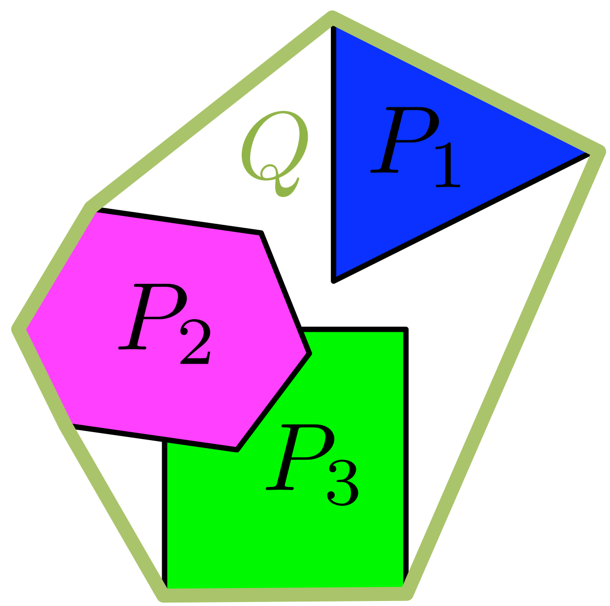
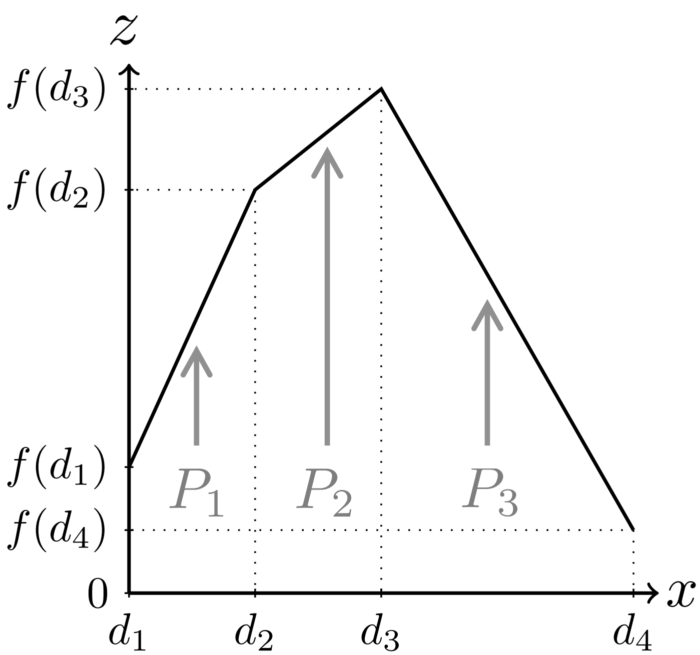


Outline

- Introduction
  - Simple class of polyhedra, formulations and complexity
- Smallest non-extended formulations (ideal or not)
  - Relaxation complexity
- Smallest non-extended ideal formulations
  - Embedding complexity
- Constructing formulations in practice
  - Multivariate piecewise linear functions
- Conclusions

“Simple” Family of Polyhedra

$$(x, z) \in \mathrm{gr}\,(f) = \bigcup_{i=1}^{3} P_i$$



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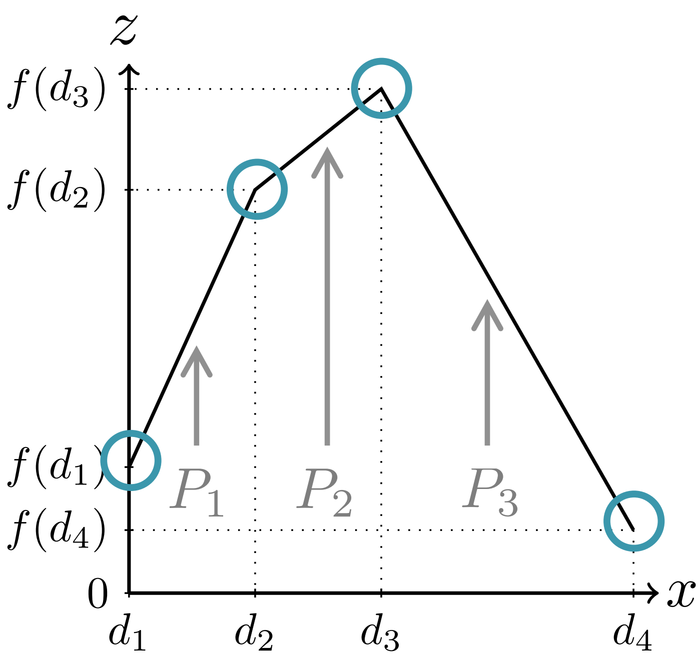

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{4} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_{d_j}$$

$$\lambda \in \Delta^4 := \left\{ \lambda \in \mathbb{R}^4_+ \;:\; \sum_{i=1}^{4} \lambda_i = 1 \right\}$$

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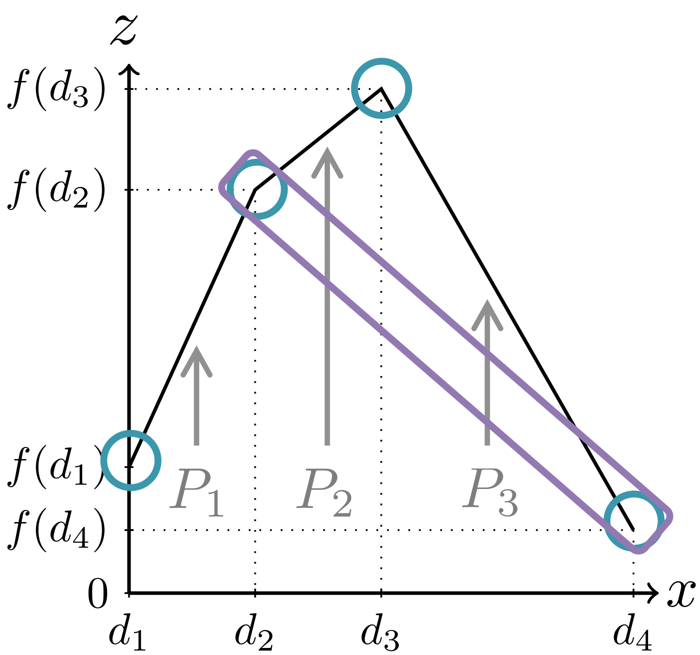

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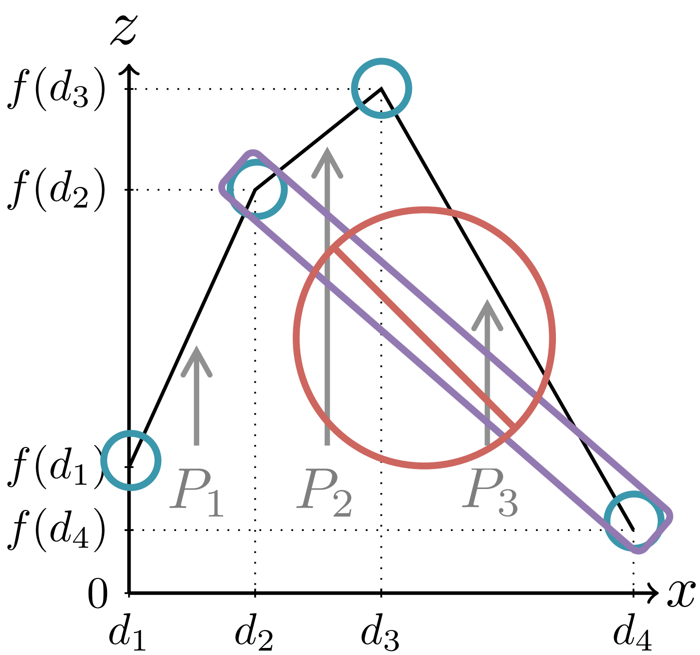

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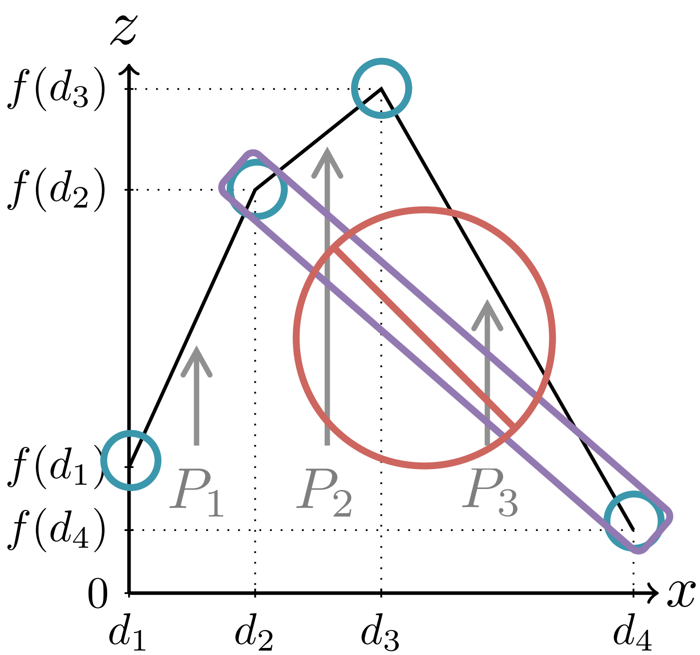

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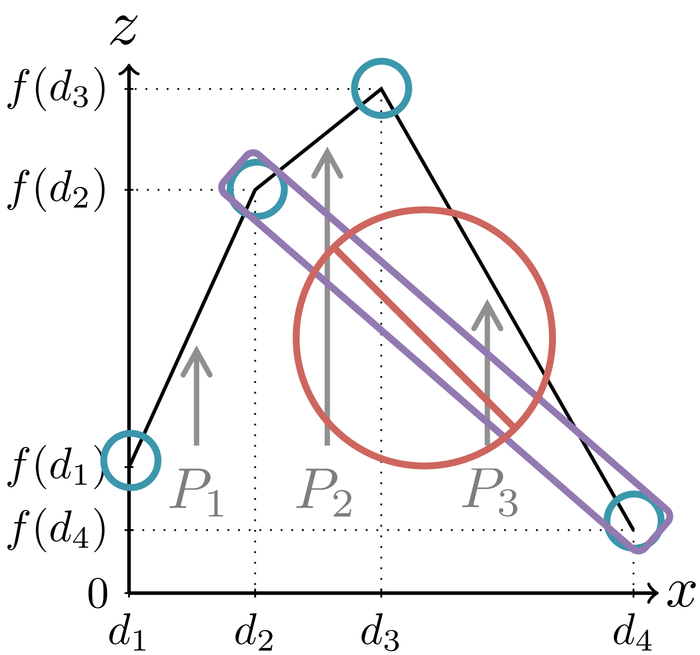

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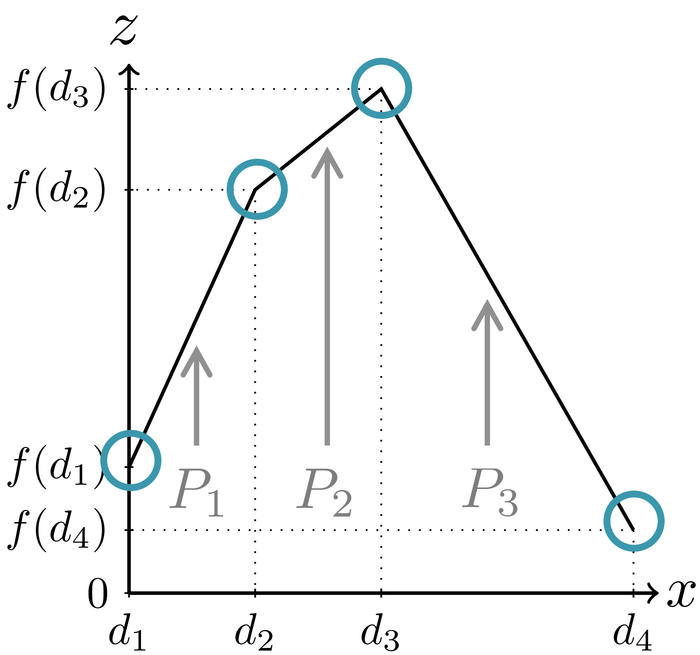

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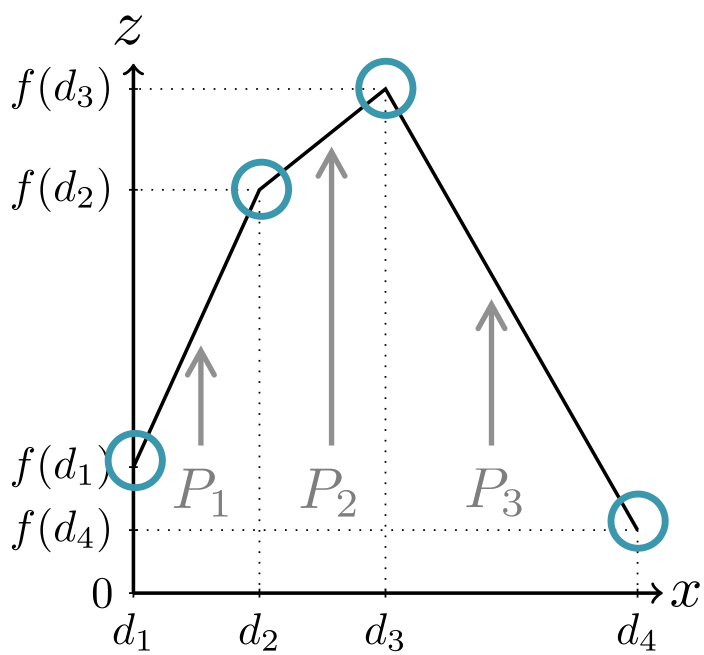

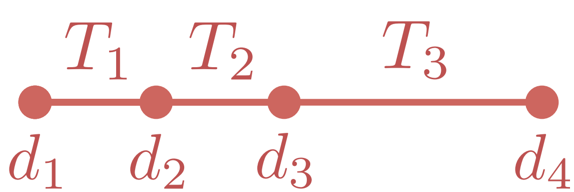

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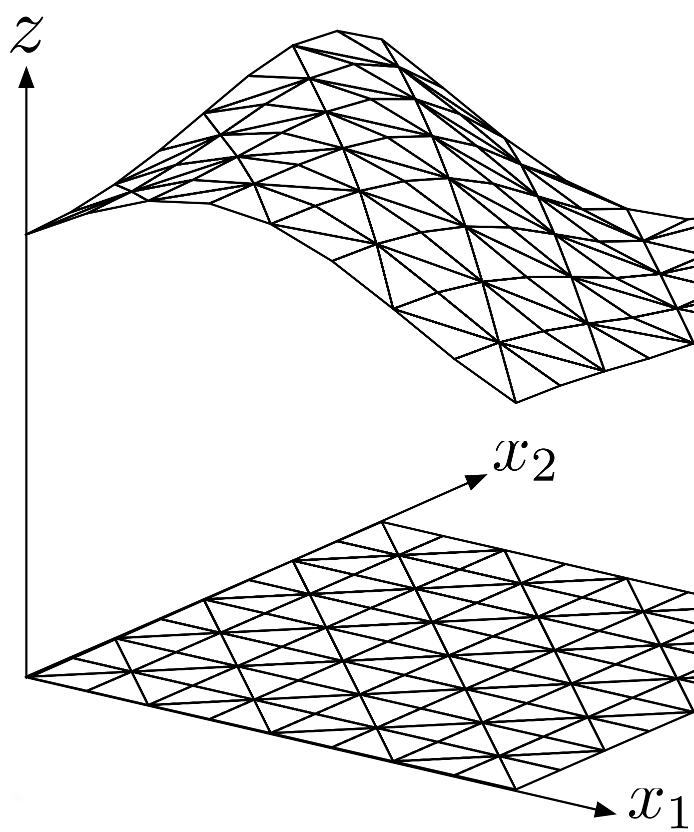
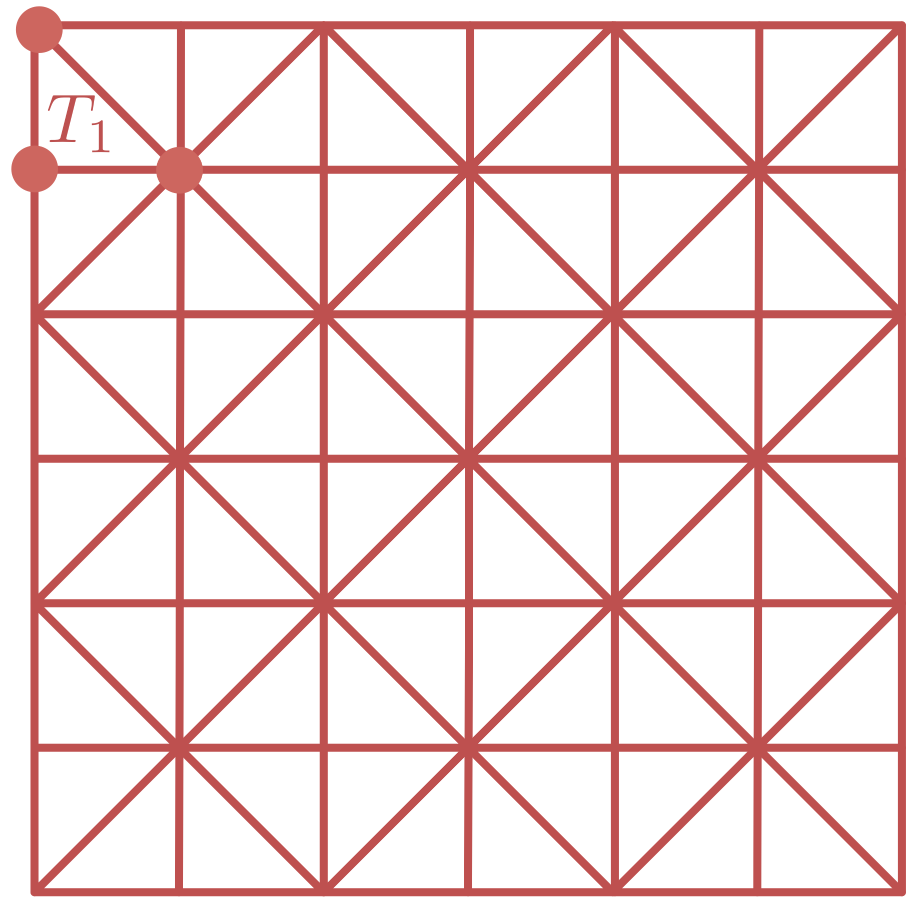
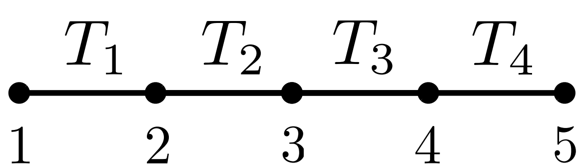
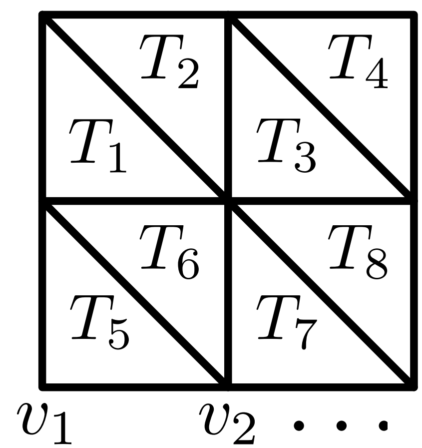




"Simple" Family of Polyhedra: Faces of a Simplex

- $\Delta^V := \left\{\lambda \in \mathbb{R}^V_+ \;:\; \sum_{v \in V} \lambda_v = 1\right\}$,
- $P_i = \{\lambda \in \Delta^V \;:\; \lambda_v = 0 \quad \forall v \notin T_i\}$
- $\lambda \in \bigcup_{i=1}^{n} P_i$:





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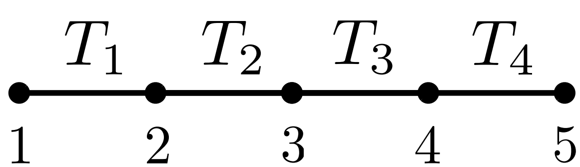
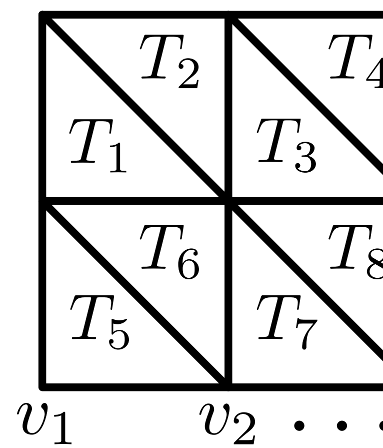
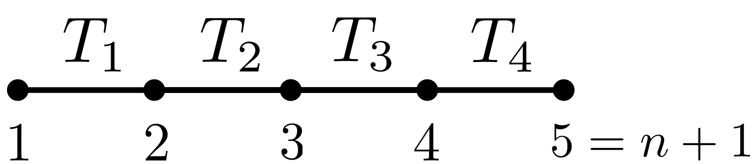
- $\lambda \in \bigcup_{i=1}^n P_i :$





- $\operatorname{conv}\left(\bigcup_{i=1}^n P_i \right) = \Delta^V$

Standard **Non-ideal** Formulation for SOS2



$$\sum_{i=1}^{5} \lambda_i = 1$$

$$y \in \{0,1\}^4, \quad \sum_{i=1}^{4} y_i = 1$$

$$\overbrace{\begin{aligned}
0 &\leq \lambda_1 \leq y_1 \\
0 &\leq \lambda_2 \leq y_1 + y_2 \\
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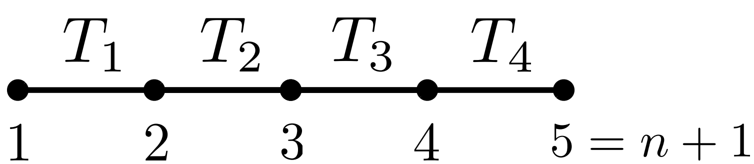

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General Inequalities

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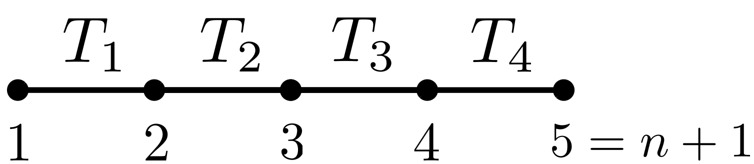

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Bounds

General Inequalities

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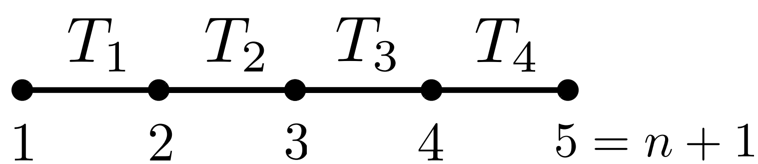

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Bounds

General Inequalities

- Minimum # of (general) inequalities?
  - Ideal formulation:
  - Non-ideal formulation:

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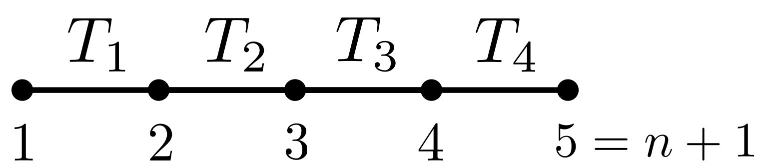

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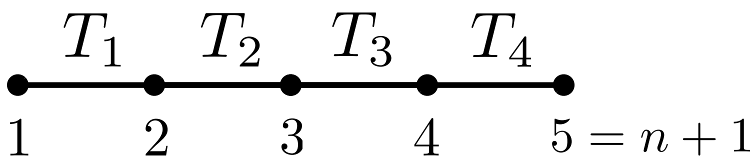
General Inequalities

- Minimum # of (general) inequalities?
  - Ideal formulation:

    $$2\lceil \log_2 n \rceil$$

    $$n + 1 \le \ldots \le n + 1 + 2\lceil \log_2 n \rceil$$
  - Non-ideal formulation:

Standard **Non-ideal** Formulation for SOS2



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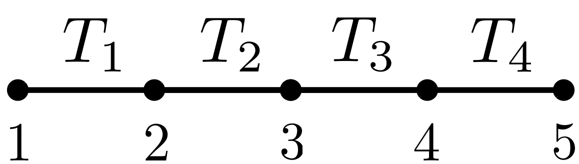
    $$n+1 \leq \ldots \leq n+1+2\lceil \log_2 n \rceil$$
  - Non-ideal formulation:

    $$2 \leq \ldots \leq 4$$

    $$2 \leq \ldots \leq 5+2n$$

What is a Formulation?



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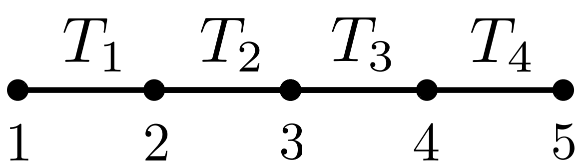
$$0 \leq \lambda_4 \leq y_3 + y_4$$

$$0 \leq \lambda_5 \leq y_4$$

$$P_i := \left\{ \lambda \in \Lambda^5 \, : \, \lambda_j = 0 \quad j \notin \{i, i+1\} \right\}$$

What is a Formulation?



Q = LP relaxation

$$\boxed{\begin{aligned}\sum_{i=1}^{5} \lambda_i &= 1\\ \sum_{i=1}^{4} y_i &= 1\end{aligned}}$$

$y \in \{0,1\}^4,$

$$\boxed{\begin{aligned}0 \leq \lambda_1 &\leq y_1\\ 0 \leq \lambda_2 &\leq y_1 + y_2\\ 0 \leq \lambda_3 &\leq y_2 + y_3\\ 0 \leq \lambda_4 &\leq y_3 + y_4\\ 0 \leq \lambda_5 &\leq y_4\end{aligned}}$$

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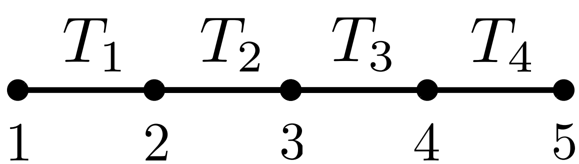

Q = LP relaxation $\longrightarrow$

$$\sum_{i=1}^{5} \lambda_i = 1$$
$$y \in \{0,1\}^4, \quad \sum_{i=1}^{4} y_i = 1$$

$$\begin{aligned}
0 &\le \lambda_1 \le y_1 \\
0 &\le \lambda_2 \le y_1 + y_2 \\
0 &\le \lambda_3 \le y_2 + y_3 \\
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\end{aligned}$$

$$(\lambda, y) \in Q \cap \left(\mathbb{R}^5 \times \mathbb{Z}^4\right)$$
$$\Updownarrow$$
$$y = e^i \ \wedge \ \lambda \in P_i$$

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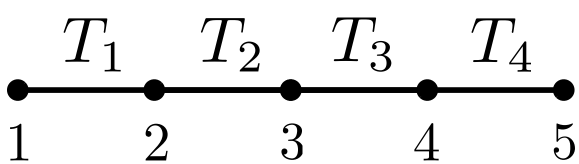

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$$(\lambda, y) \in Q \cap \left(\mathbb{R}^5 \times \mathbb{Z}^4\right)$$

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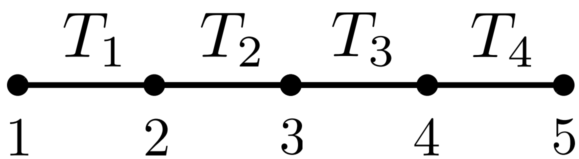
Unary Encoding

$$P_i := \{\lambda \in \Lambda^5 \ : \ \lambda_j = 0 \quad j \notin \{i, i+1\}\}$$

Alternate Meaning of 0-1 Variables



Q = LP relaxation $\longrightarrow$ $\boxed{\sum_{i=1}^{5} \lambda_i = 1}$

- V. and Nemhauser ’08.

$$
\boxed{
\begin{aligned}
0 &\le \lambda_1 + \lambda_5 \le 1 - y_1 \\
0 &\le \lambda_3 \qquad\quad\;\; \le y_1 \\
0 &\le \lambda_4 + \lambda_5 \le 1 - y_2 \\
0 &\le \lambda_1 + \lambda_2 \le y_2
\end{aligned}
}
$$

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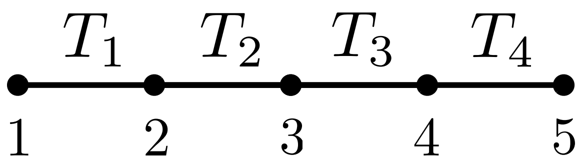
Alternate Meaning of 0-1 Variables



Q = LP relaxation $\longrightarrow$ $\boxed{\sum_{i=1}^{5} \lambda_i = 1}$

$$h^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; h^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; h^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; h^4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$\begin{aligned} 0 \leq \lambda_1 + \lambda_5 &\leq 1 - y_1 \\ 0 \leq \lambda_3 \qquad &\leq y_1 \\ 0 \leq \lambda_4 + \lambda_5 &\leq 1 - y_2 \\ 0 \leq \lambda_1 + \lambda_2 &\leq y_2 \end{aligned}$$

$$(\lambda, y) \in Q \cap \left(\mathbb{R}^5 \times \mathbb{Z}^2\right)$$

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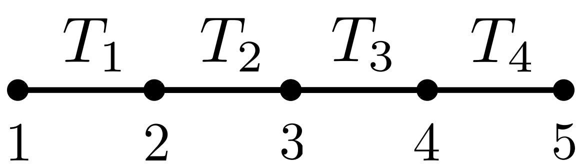
Alternate Meaning of 0-1 Variables



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$$\boxed{\begin{gathered} (\lambda, y) \in Q \cap \left(\mathbb{R}^5 \times \mathbb{Z}^2\right) \\ \Updownarrow \\ y = h^i \ \wedge \ \lambda \in P_i \end{gathered}}$$

Binary Encoding

$$P_i := \left\{\lambda \in \Lambda^5 \ : \ \lambda_j = 0 \quad j \notin \{i, i+1\}\right\}$$

Embedding Formulations for Union of Polyhedra

- **Non-Extended** formulation of $\lambda \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^V$:
  - Encoding $H := \{h^i\}_{i=1}^{n} \subseteq \{0,1\}^k, \quad h^i \neq h^j$
  - Polyhedron $Q \subseteq \mathbb{R}^V \times \mathbb{R}^k$, s.t.

$$(\lambda, y) \in Q \cap \left(\mathbb{R}^V \times \mathbb{Z}^k\right) \quad \Leftrightarrow \quad y = h^i \ \wedge \ \lambda \in P_i$$

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- **Embedding formulation** = strongest polyhedron (ideal):

$$Q(H) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \{h^i\}\right)$$

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For unary encoding:

$$h^i = e^i$$

Embedding Formulations for Union of Polyhedra

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$$\text{size}\,(Q) := \#\ \text{of facets of}\ Q\ \text{(usually function of}\ n)$$

Complexity of Family of Polyhedra $\mathcal{P} := \{P_i\}_{i=1}^n$

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$$\mathrm{hc}\left(\left\{P_i \times h^i\right\}_{i=1}^n\right)$$

- Hull complexity

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Complexity of Family of Polyhedra $\mathcal{P} := \{P_i\}_{i=1}^n$

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$$\mathrm{rc}_G(\mathcal{P}) := \min_{Q,H} \{\mathrm{size}_G(Q) \;:\; (Q, H) \text{ is formulation}\}$$

- Embedding complexity = smallest **ideal** formulation

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General Inequalities

- Hull complexity

$$\mathrm{hc}(\mathcal{P}) := \mathrm{size}\left(\mathrm{conv}\left(\bigcup_{i=1}^n P_i\right)\right)$$

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Relaxation Complexity

Bounds on Relaxation Complexity

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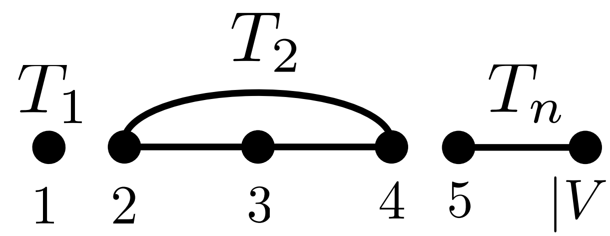
- Disjoint Case : $T_i \cap T_j = \emptyset$

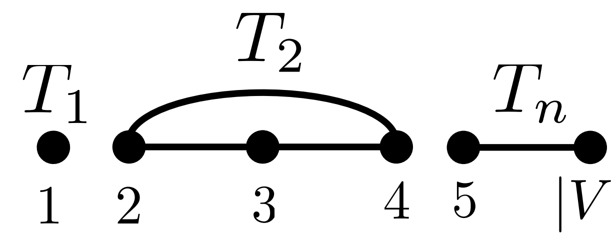


Bounds on Relaxation Complexity

- Disjoint Case : $T_i \cap T_j = \emptyset$

$$\mathrm{rc}_G\left(\mathcal{P}\right) = 2$$

$$2 \leq \mathrm{rc}\left(\mathcal{P}\right) \leq 2 + |V| + n$$



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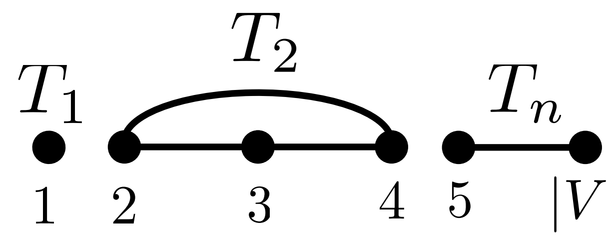
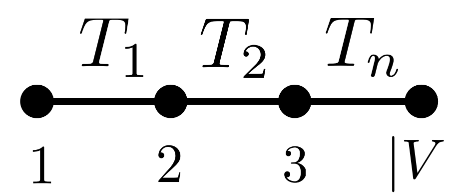


- SOS2 constraints : $T_i = \{i, i+1\}$



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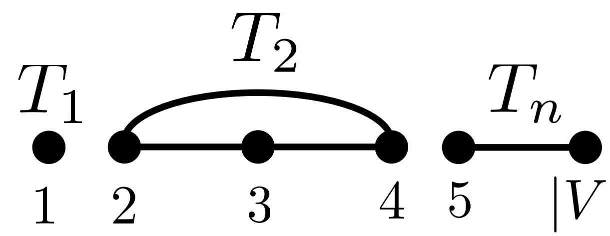
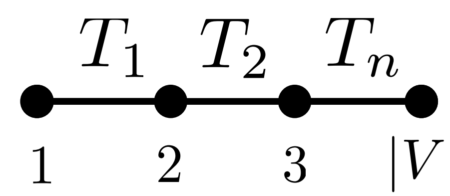


- SOS2 constraints : $T_i = \{i, i+1\}$

$$2 \leq \mathrm{rc}_G\,(\mathcal{P}) \leq 4$$

$$2 \leq \mathrm{rc}\,(\mathcal{P}) \;\; \leq 5 + 2n$$



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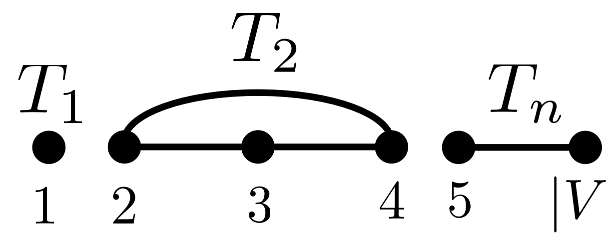

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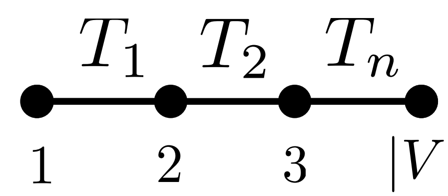
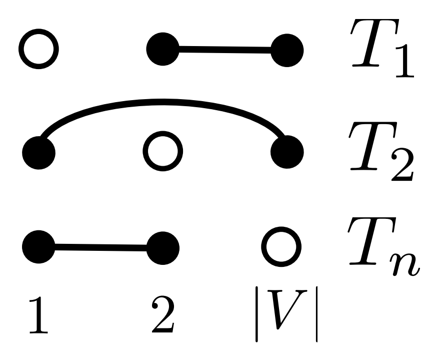


- SOS(-1) constraints : $T_i = V \setminus \{i\}$



Bounds on Relaxation Complexity

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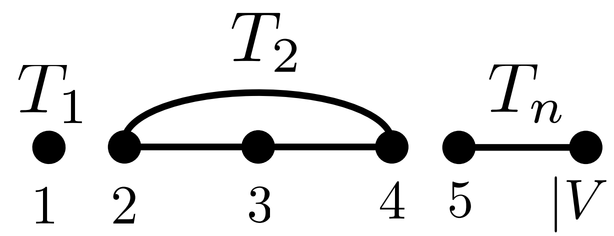

- SOS2 constraints : $T_i = \{i, i+1\}$

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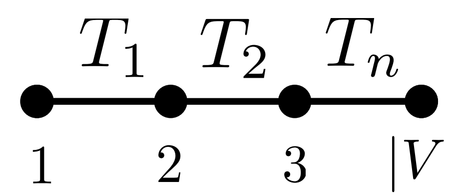
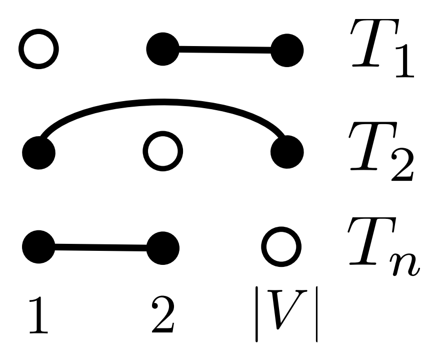


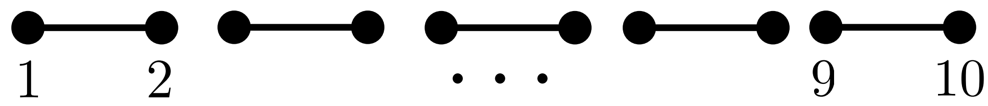
- SOS(-1) constraints : $T_i = V \setminus \{i\}$

$$\mathrm{mc}_G\left(\mathcal{P}\right) = \mathrm{rc}_G\left(\mathcal{P}\right) = n$$

$$n \leq \mathrm{rc}\left(\mathcal{P}\right) \leq \mathrm{mc}\left(\mathcal{P}\right) \leq 3n$$



Formulation for Disjoint Case



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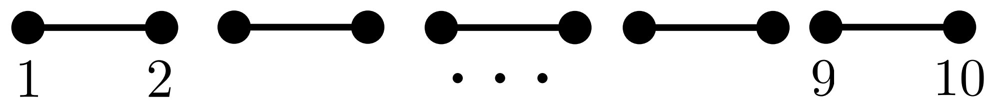

$$\sum_{i=1}^{n} p^{i}\left(\lambda_{2i-1}+\lambda_{2i}\right) \leq \sum_{i=1}^{n} p^{i} y_i$$

$$\sum_{i=1}^{2n} \lambda_i = 1, \quad \lambda \in \mathbb{R}_{+}^{2n}$$

$$\sum_{i=1}^{n} y_i = 1, \quad y \in \mathbb{R}_{+}^{n}$$

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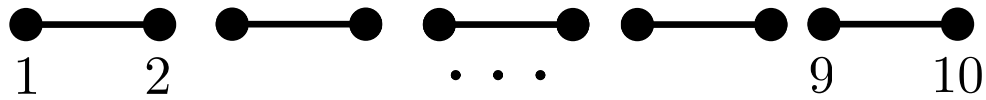
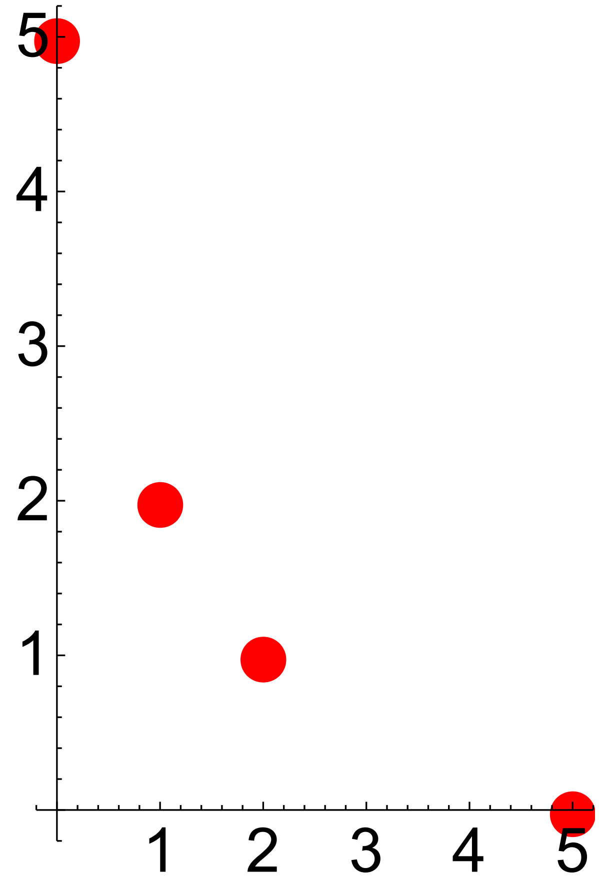


$$\sum_{i=1}^{n} p^i \left(\lambda_{2i-1} + \lambda_{2i}\right) \leq \sum_{i=1}^{n} p^i y_i$$

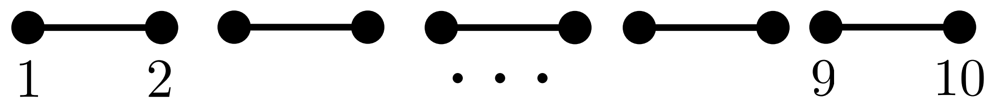
$$\sum_{i=1}^{2n} \lambda_i = 1, \quad \lambda \in \mathbb{R}_+^{2n}$$

$$\sum_{i=1}^{n} y_i = 1, \quad y \in \mathbb{R}_+^{n}$$

$$p^i \in \mathbb{R}_+^2, \quad \operatorname{conv}\left(\left\{p^j\right\}_{j \neq i}\right) \not\ni p^i$$



Formulation for Disjoint Case



$$\sum_{i=1}^{n} p^i \left(\lambda_{2i-1} + \lambda_{2i}\right) \le \sum_{i=1}^{n} p^i y_i$$

$$\sum_{i=1}^{2n} \lambda_i = 1, \quad \lambda \in \mathbb{R}^{2n}_+$$

$$\sum_{i=1}^{n} y_i = 1, \quad y \in \mathbb{R}^{n}_+$$

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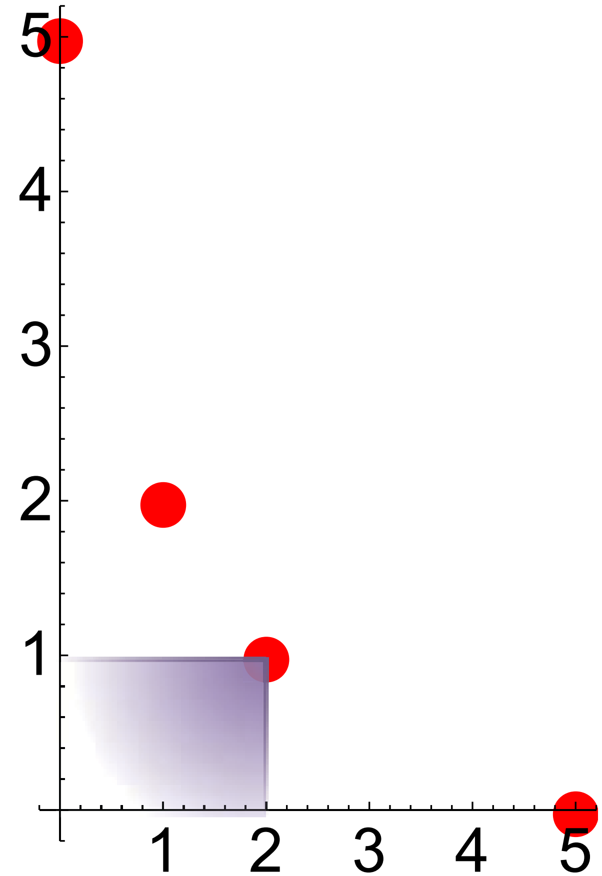
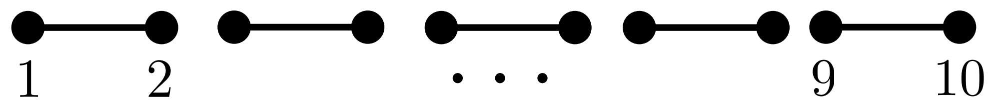


Formulation for Disjoint Case



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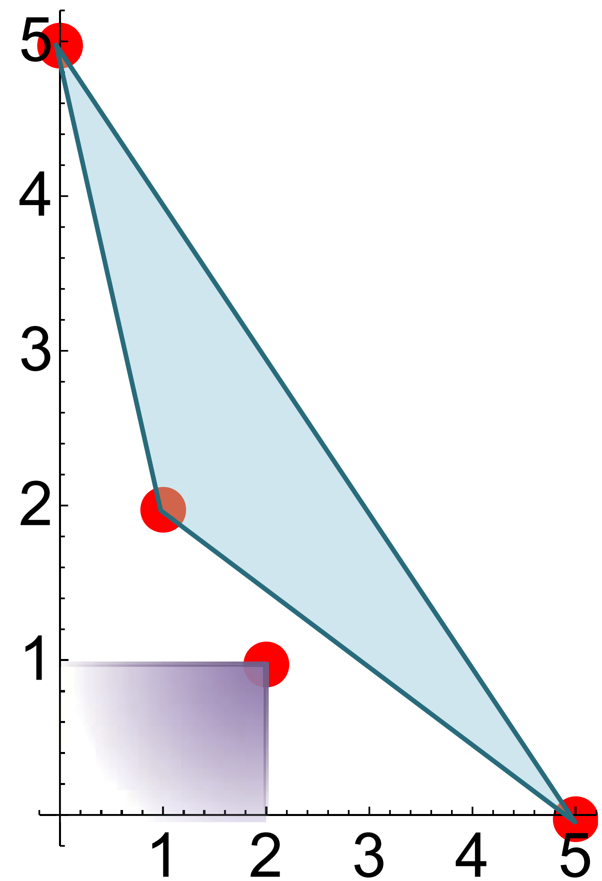
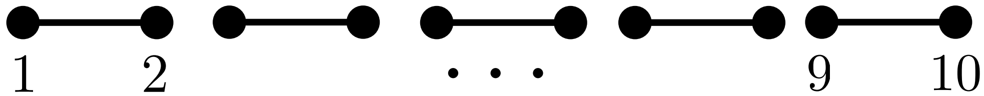


Formulation for Disjoint Case



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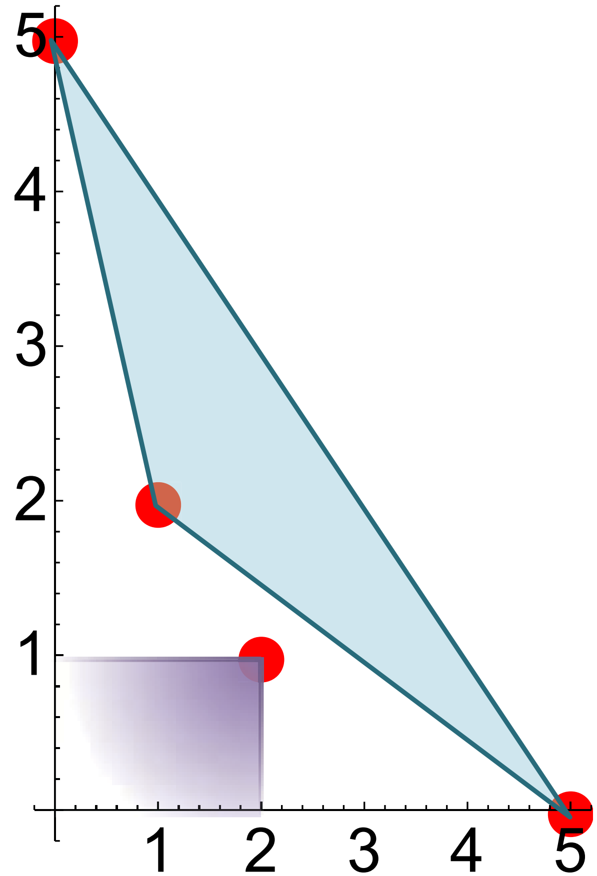

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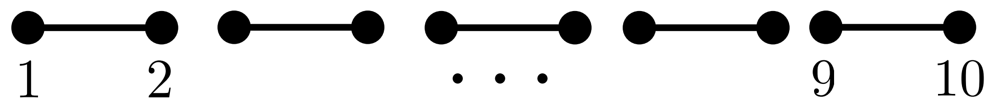

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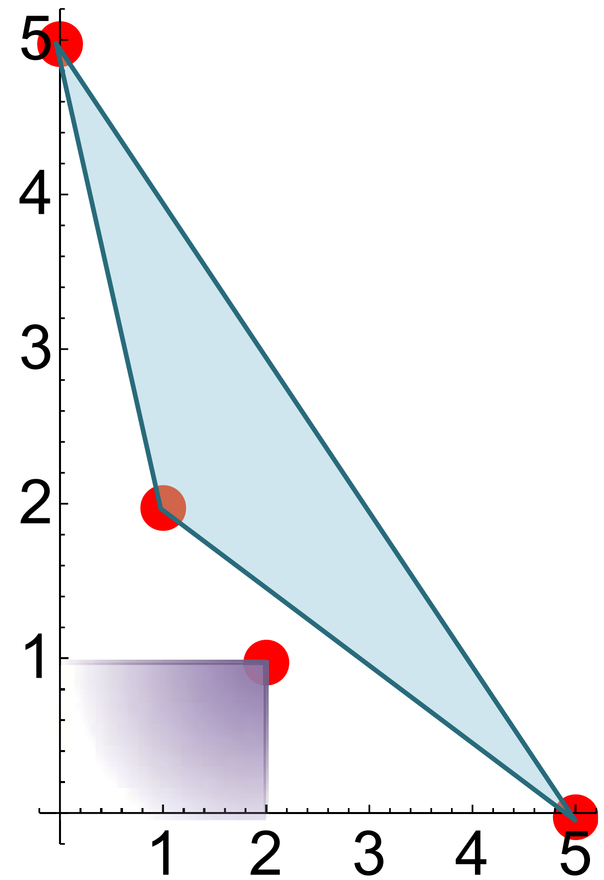

- Polynomial sized coefficients:
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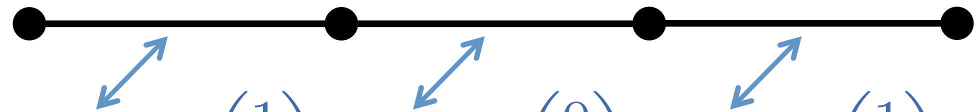
- 80 fractional extreme points for n = 5

Embedding Complexity: $\mathrm{size}\left(Q\left(H\right)\right)$ for SOS2

Embedding Formulation for SOS2: Part 1

- From encodings to hyperplanes:

$$\{h^i\}_{i=1}^{n}$$



$$h^1 = \begin{pmatrix}1\\1\end{pmatrix},\ h^2 = \begin{pmatrix}0\\0\end{pmatrix},\ h^3 = \begin{pmatrix}1\\0\end{pmatrix}$$

Embedding Formulation for SOS2: Part 1

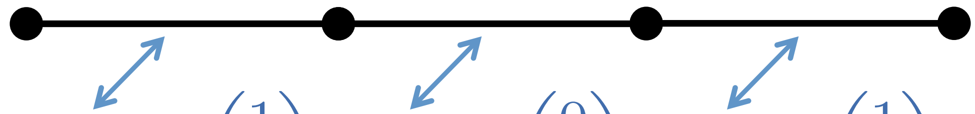
- From encodings to hyperplanes:

$$\left\{h^i\right\}_{i=1}^{n}$$

$$c^i = h^{i+1} - h^i$$

$$\downarrow$$

$$\left\{c^i\right\}_{i=1}^{n-1}$$



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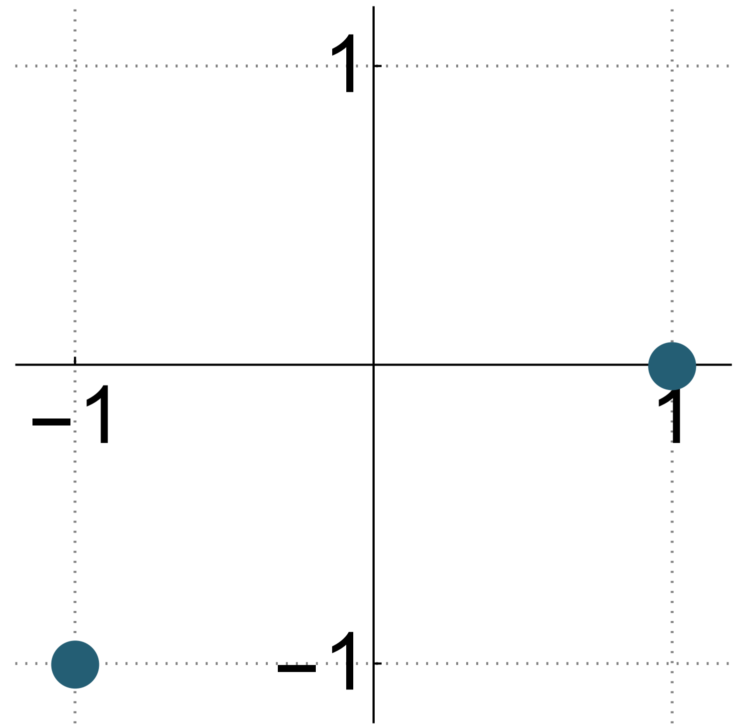

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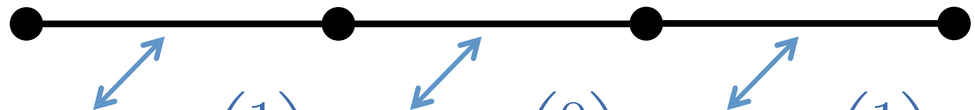
Hyperplanes spanned by

$$\left\{b^i \cdot y = 0\right\}_{j=1}^{L}$$



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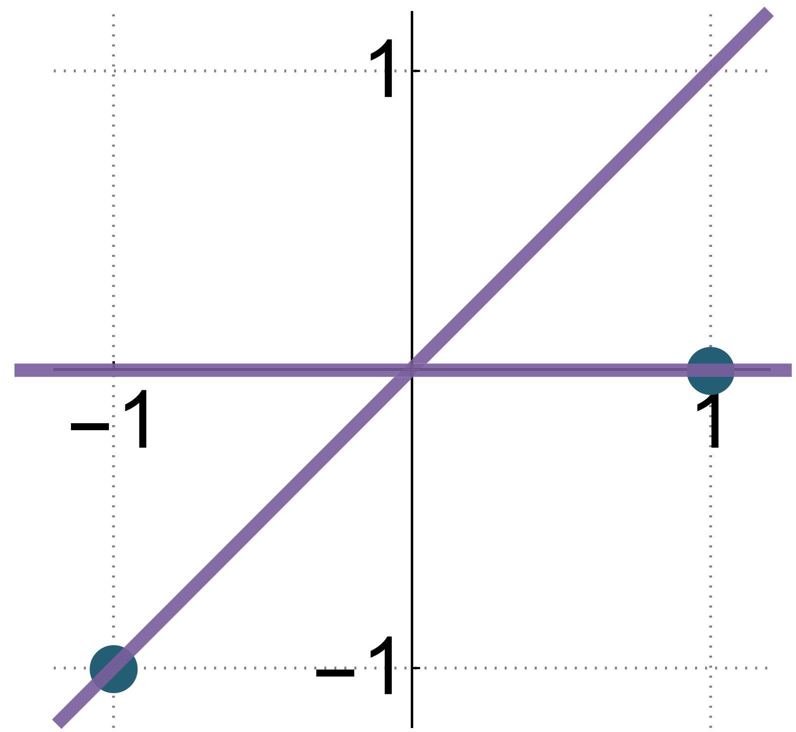

Embedding Formulation for SOS2: Part 2

$$\left\{b^i \cdot y = 0\right\}_{j=1}^{L}$$

$$Q(H) =$$

$$L\left(H\right) := \operatorname{aff}\left(H\right) - h^1$$

$$\left(b^j \cdot h^1\right)\lambda_1 + \sum_{i=2}^{n} \min\left\{b^j \cdot h^i, b^j \cdot h^{i-1}\right\}\lambda_i + \left(b^j \cdot h^n\right)\lambda_{n+1} \leq \quad b^j \cdot y \quad \forall j$$

$$-\left(b^j \cdot h^1\right)\lambda_1 - \sum_{i=2}^{n} \max\left\{b^j \cdot h^i, b^j \cdot h^{i-1}\right\}\lambda_i - \left(b^j \cdot h^n\right)\lambda_{n+1} \leq -b^j \cdot y \quad \forall j$$

$$\sum_{i=1}^{n+1} \lambda_i = 1, \quad \lambda \in \mathbb{R}_{+}^{n+1}$$

$$y \in L(H)$$

- # general inequalities = 2 × # of hyperplanes

Embedding Complexity for SOS2

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- Unary encoding (Padberg / Lee and Wilson, early 00's):

$$\mathrm{size}_G\left(Q\left(H\right)\right) = 2(n-1), \quad \mathrm{size}\left(Q\left(H\right)\right) = 2n$$

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$$2 + 2\left\lceil \log_2 n \right\rceil \leq \text{size}\left(Q\left(H\right)\right) \leq n + 1 + 2\left\lceil \log_2 n \right\rceil$$

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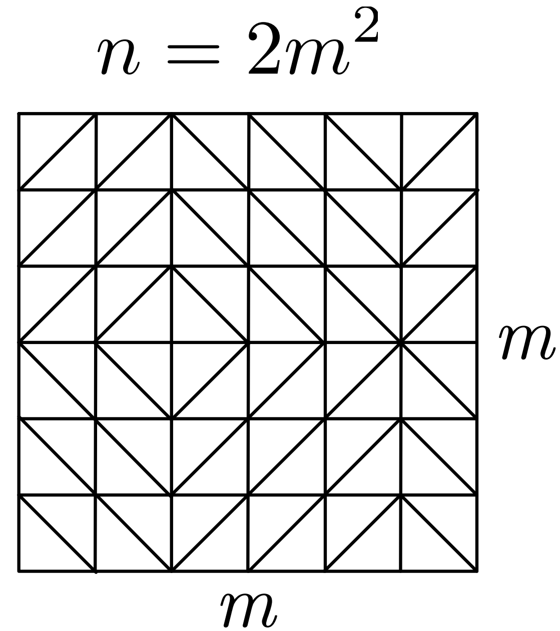
- Adding lower bounds (# hyperplanes ≥ dimension):

$$\mathrm{mc}_G\left(\mathcal{P}\right) = 2\left\lceil \log_2 n \right\rceil,$$

$$n + 1 \leq \mathrm{xc}\left(\mathcal{P}\right) \leq \mathrm{mc}\left(\mathcal{P}\right) \leq n + 1 + 2\left\lceil \log_2 n \right\rceil$$

Practical Constructions for Multivariate Piecewise Linear Functions

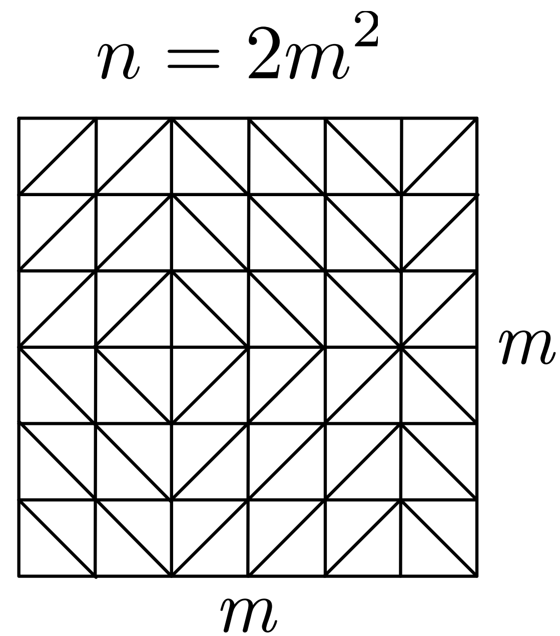
Formulations and Complexity for Triangulations



Formulations and Complexity for Triangulations

- Lower bound:

$$\left(\sqrt{n/2}+1\right)^2 \leq \mathrm{mc}\left(\mathcal{P}\right)$$



Formulations and Complexity for Triangulations

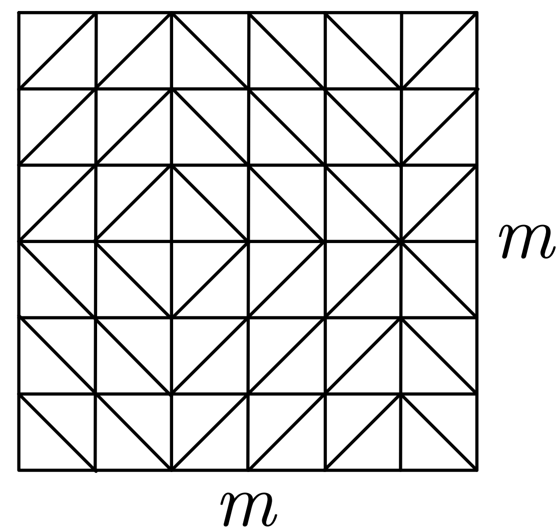
- Lower bound:

$$\left(\sqrt{n/2}+1\right)^2 \leq \mathrm{mc}\left(\mathcal{P}\right)$$

- Size of unary formulation is:
  (Lee and Wilson '01)

$$\mathrm{mc}\left(\mathcal{P}\right) \leq \binom{2\sqrt{n/2}}{\sqrt{n/2}} + \left(\sqrt{n/2}+1\right)^2$$

$n = 2m^2$



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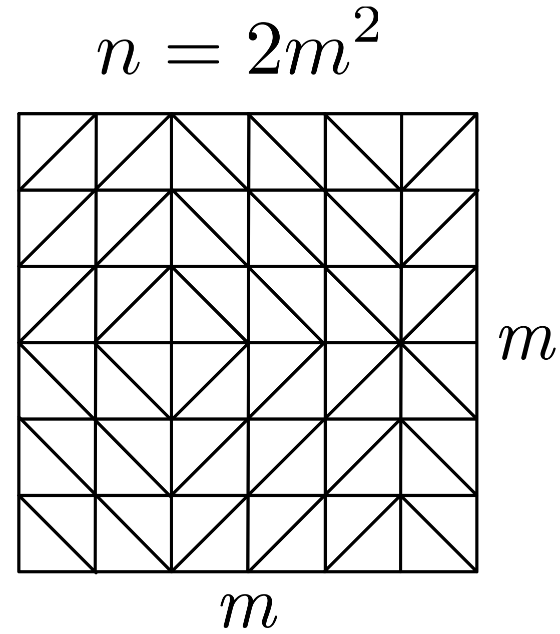
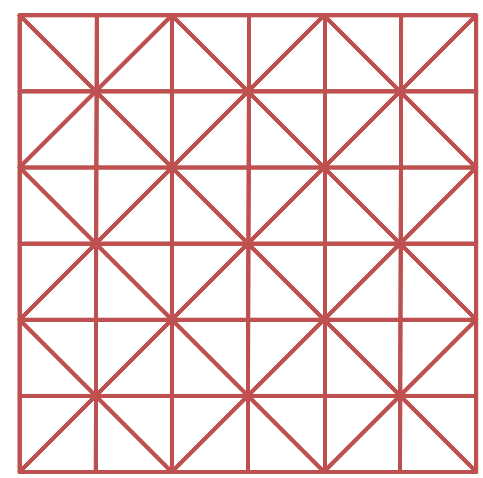
$$\mathrm{mc}\left(\mathcal{P}\right) \leq \binom{2\sqrt{n/2}}{\sqrt{n/2}} + \left(\sqrt{n/2}+1\right)^2$$

- Small binary formulation for union jack triangulation of size:
  (V. and Nemhauser '08)

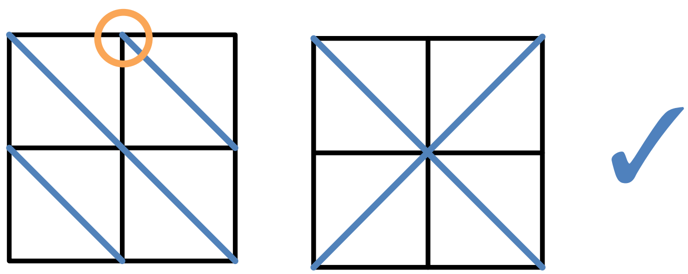
$$\mathrm{mc}\left(\mathcal{P}\right) \leq 4\log_2\sqrt{n/2}+2+\left(\sqrt{n/2}+1\right)^2$$

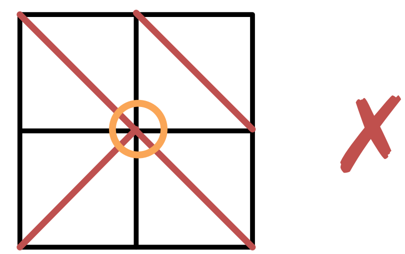




Beyond Union Jack: Exploit Redundancy

- **Embedding-like** formulation for triangulations with "even degree outside the boundary"





- Formulation size slightly larger than for union jack:

$$4\log_2\sqrt{n/2}+4+\left(\sqrt{n/2}+1\right)^2$$

- Formulation fits **independent branching** framework (V. and Nemhauser '08)

Summary

- Embedding Formulations = Systematic procedure
  - Encoding can significantly affect size
- Complexity of Union of Polyhedra beyond convex hull
  - Embedding Complexity (non-extended ideal formulation)
  - Relaxation Complexity (any non-extended formulation)
  - Still open questions on relations between complexity
- More details (practical formulation construction)
  - Embedding Formulations and Complexity for Unions of Polyhedra, arXiv:1506.01417
- Application to facility layout problem (Huchette, Dey, V. '14)
  - INFORMS 2015, Philadelphia, Nov 2nd
- Extension to unions of convex sets = representability (Soon ☺)
- More on independent branching = SOSK (Huchette, V. '??)