Small and Strong Formulations for Unions of Convex Sets from the Cayley Embedding

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“Extended” / Non-Extended Formulations for $\bigcup_{i=1}^{n} C_i$

$$C_i = \{x \in \mathbb{R}^d \;:\; A^i x \le b^i\}$$

"Extended” ≡ Variable Copies

$$\begin{aligned} A^i x^i &\le b^i y_i && \forall i \in [n] \\ \sum_{i=1}^{n} x^i &= x \\ \sum_{i=1}^{n} y_i &= 1 \\ y &\in \{0,1\}^n \\ x,\; x^i &\in \mathbb{R}^d && \forall i \in [n] \end{aligned}$$

Small? and strong (ideal*)

Speed: worse than expected

Non-Extended

$$\begin{aligned} A^i x - b^i &\le M_i (1 - y_i) && \forall i \in [n] \\ \sum_{i=1}^{n} y_i &= 1 \\ y &\in \{0,1\}^n \\ x &\in \mathbb{R}^d && \forall i \in [n] \end{aligned}$$

Small, but weak?

Speed: better than expected

*Integral y in extreme points of LP relaxation

Non-Polyhedral = Different Representations

e.g. Ceria and Soares '99

$$C_i = \{x \in \mathbb{R}^d \;:\; f_i(x) \leq 0\}$$

$$\tilde{f}(x,y) = \begin{cases} y f(x/y) & \text{if } y > 0 \\ \lim_{\alpha \downarrow 0} \alpha f(x' - x + x/\alpha) & \text{if } y = 0 \\ +\infty & \text{if } y < 0 \end{cases}$$

$$\begin{aligned} \tilde{f}_i\left(x^i, y_i\right) &\leq 0 & \forall i \in [n] \\ \sum_{i=1}^n x^i &= x & \\ \sum_{i=1}^n y_i &= 1 & \\ y &\in \{0,1\}^n & \\ x,\; x^i &\in \mathbb{R}^d & \forall i \in [n] \end{aligned}$$

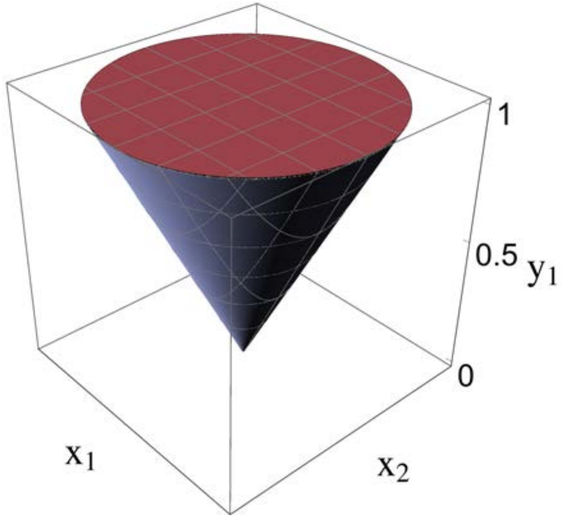
e.g. Ben-tal and Nemirovski '01

$$C_i = \left\{ x \in \mathbb{R}^d \;:\; \begin{array}{r} \exists u \in \mathbb{R}^{p_i} \text{ s.t.} \\ A^i x + D^i u - b \in K^i \end{array} \right\}$$

K^i closed convex cone

$$\begin{aligned} A^i x^i + D^i u^i - b y_i &\in K^i & \forall i \in [n] \\ \sum_{i=1}^n x^i &= x & \\ \sum_{i=1}^n y_i &= 1 & \\ y &\in \{0,1\}^n & \\ x,\; x^i &\in \mathbb{R}^d & \forall i \in [n] \\ u^i &\in \mathbb{R}^{p_i} & \forall i \in [n] \end{aligned}$$

Generic Formulation Through Gauge Functions



- For C such that $\mathbf{0} \in \operatorname{int}(C)$ let:

  $\gamma_C(x) := \inf\{\lambda > 0 \ : \ x \in \lambda C\}$

  $\operatorname{epi}(\gamma_C) = \operatorname{cone}(C \times \{1\})$

- If $b^i \in C_i$ then ideal formulation:

$$
\begin{aligned}
\gamma_{C^i - \{b^i\}}\left(x^i - y_i b^i\right) &\le y_i & \forall i \in [n] \\
\sum_{i=1}^n x^i &= x \\
\sum_{i=1}^n y_i &= 1 \\
y &\in \{0,1\}^n \\
x,\ x^i &\in \mathbb{R}^d & \forall i \in [n]
\end{aligned}
$$

Simple Non-Extended Ideal Formulation

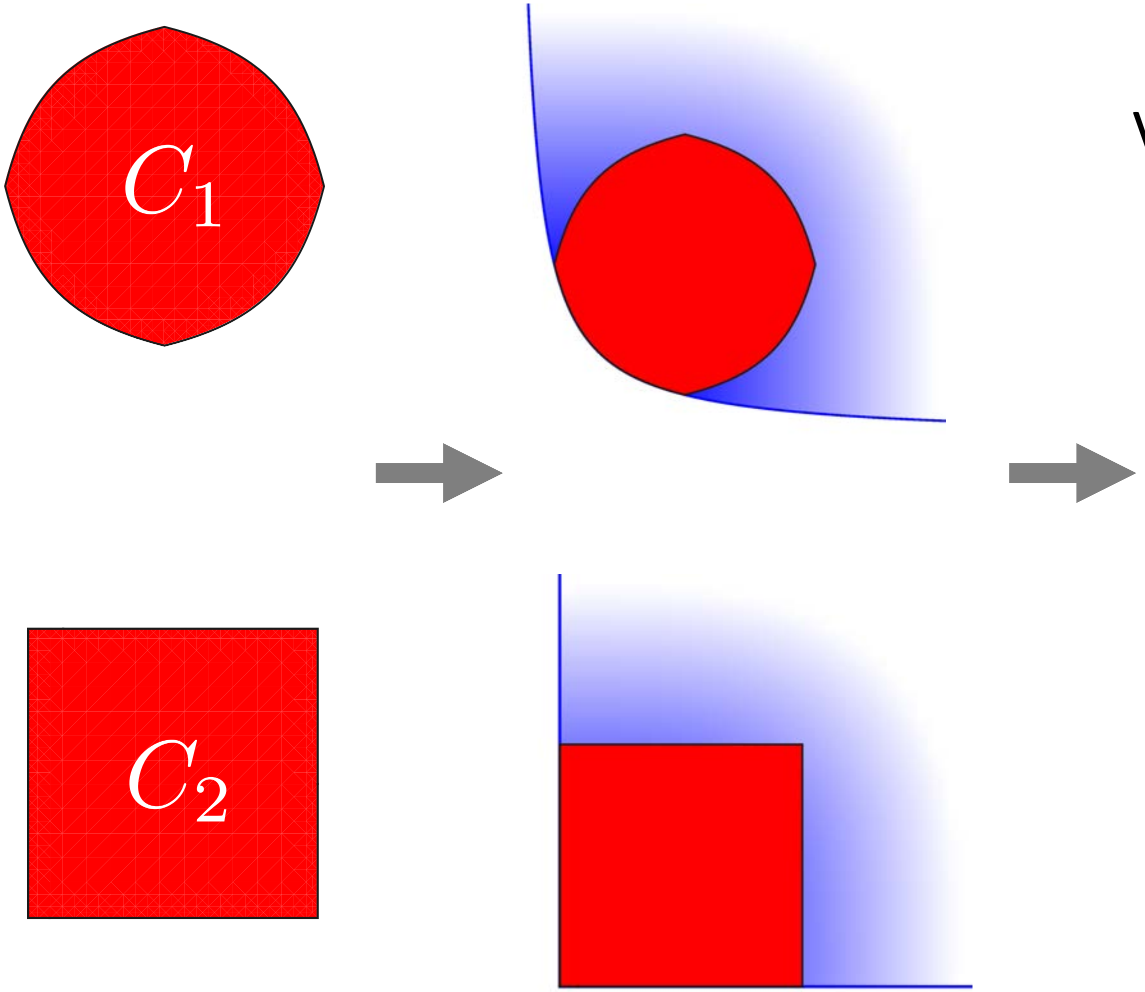
- Unions of (nearly) Homothetic Closed Convex Sets:

$$C_i = \lambda_i C + b^i + C_\infty$$

$$\gamma_C\left(x - \sum_{i=1}^n y_i b^i\right) \leq \sum_{i=1}^n \lambda_i y_i$$

$$\sum_{i=1}^n y_i = 1,\ y \in \{0,1\}^n$$

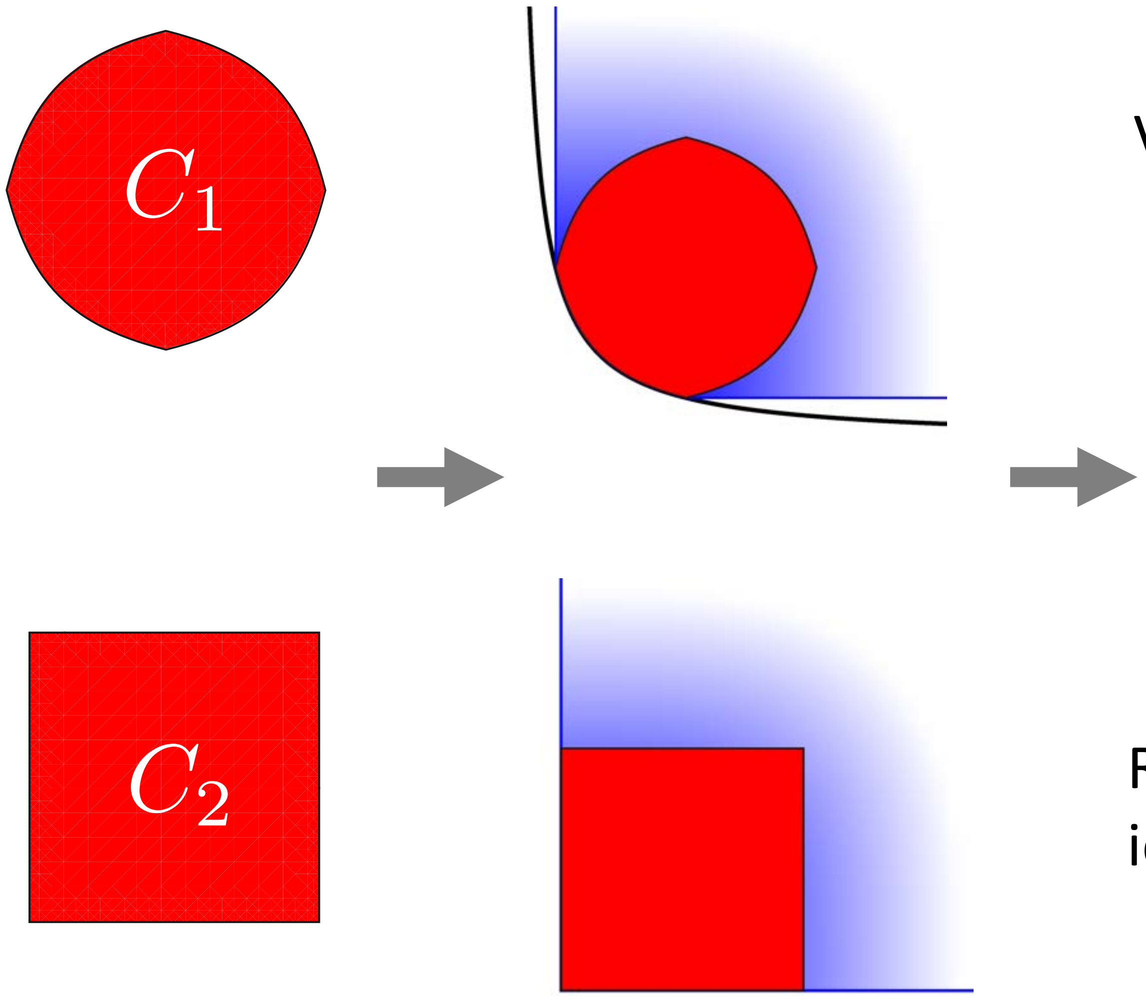
Sticking Homothetic Formulations Together



Valid, but not ideal!

Combine 4 homothetic formulations

Sticking Homothetic Formulations Together



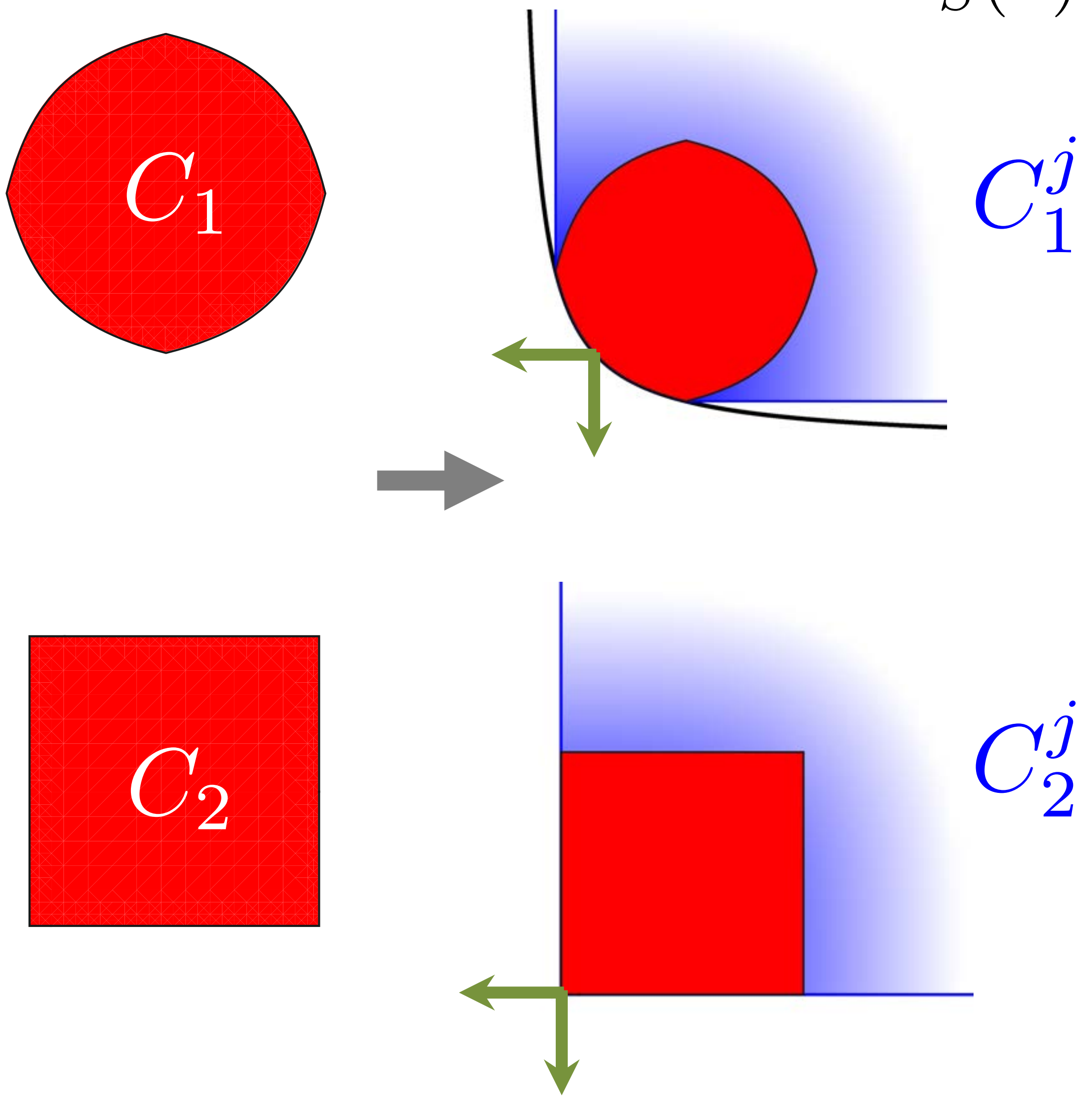
Valid, but not ideal!

Combine 4 homothetic formulations

Right relaxations yield ideal formulation

Sufficient Conditions For Ideal Formulation

$$\sigma_S(u) := \sup\{u \cdot x \;:\; x \in S\}$$



$$\forall u \in \mathbb{R}^n \quad \exists j$$

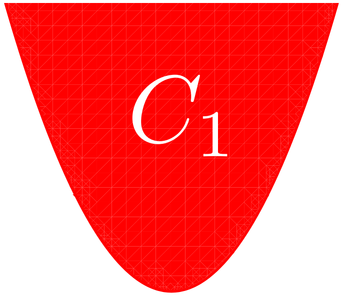
$$s.t.$$

$$\sigma_{C_i}(u) = \sigma_{C_i^j}(u)$$

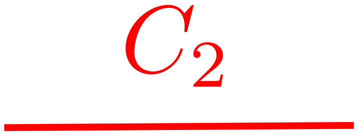
$$\forall i \in \{1, 2\}$$

Similar to “lifting” of e.g. Tawarmalani et al. ‘10

May Need to "Find" Homothetic Constraints

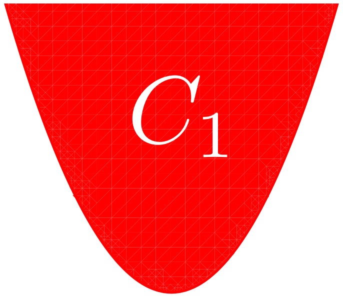


$$x_1^2 \leq x_2 \leq 1$$



$$[-1, 1] \times 0$$

May Need to “Find” Homothetic Constraints



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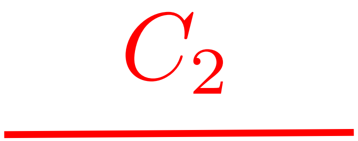
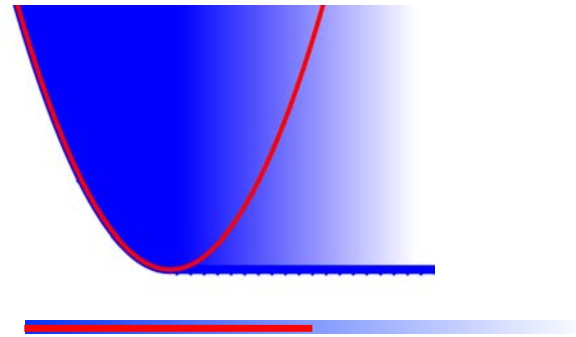
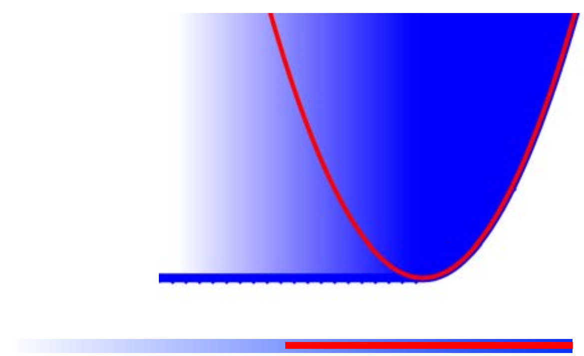


$$[-1, 1] \times 0$$

$$C_i + (\mathbb{R}_+ \times \{0\})$$



$$C_i + (\mathbb{R}_- \times \{0\})$$



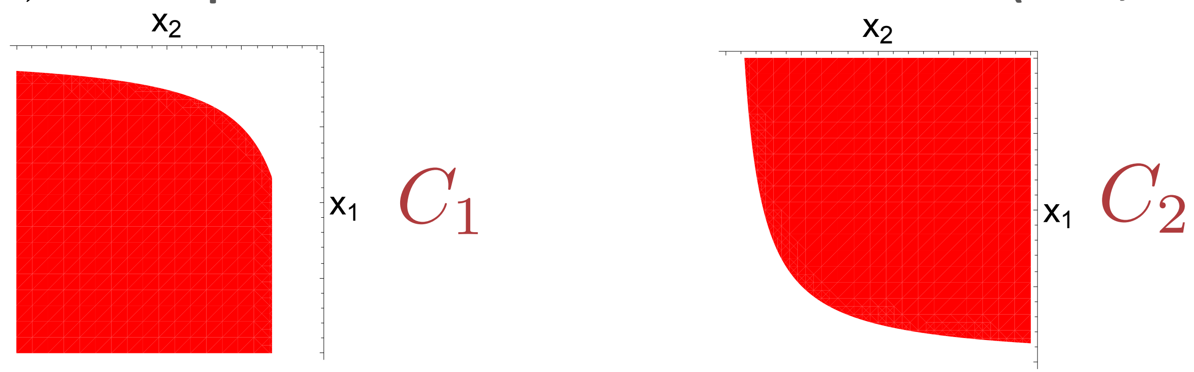
$$C_1 + (\mathbb{R}_+ \times \{0\}) :$$

$$(\max\{x_1, 0\})^2 \le x_2 \le 1$$

Similar to Bestuzheva et al. ‘16 who divide sets in two.

Existing Small Ideal Formulations (Isotone Sets)

- Studied by Hijazi et al. ‘12 and Bonami et al. ’15 (n=1, 2):
  - $C_i = \{x \in \mathbb{R}^d \, : \, l^i \leq x \leq u^i, \quad f_i(x) \leq 0\}$
- $f_i(x)$ component-wise monotonous (i=1,2 opposite).



- Ideal Formulation

$$
\begin{aligned}
y_1 l^1 + y_2 l^2 \leq x \leq y_1 u^1 + y_2 u^2 & \\
f^i_J(x, y) \leq 0 & \quad \forall J \subseteq [d], i \in [2] \\
y_1 + y_2 = 1 & \\
y_i \in \{0, 1\} & \quad i \in [2]
\end{aligned}
$$

Generalization and Simplification

- More than 2 sets (with general "opposite condition").
- Generalization of the monotone/isotone condition (beyond affine transformation)
- Significantly smaller formulation: One non-linear constraint per set.

$$y_1 l^1 + y_2 l^2 \leq x \leq y_1 u^1 + y_2 u^2$$

$$f_J^i(x, y) \leq 0 \qquad \forall J \subseteq [d], i \in [2]$$

$$y_1 + y_2 = 1$$

$$y_i \in \{0, 1\} \qquad i \in [2]$$

$$\hat{f}^i(x, y) \leq 0 \quad \forall i \in [2]$$

Details of Size Reduction

$$C_i = \{x \in \mathbb{R}^d \,:\, l^i \le x \le u^i, \quad f_i(x) \le 0\}$$

$$G_i = \{x \in \mathbb{R}^d \,:\, f_i(x) \le 0\}$$

- Original formulation:

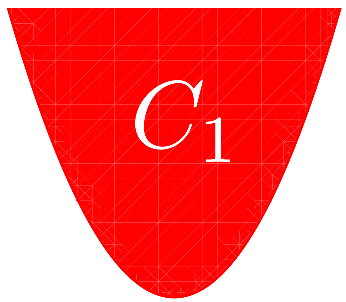
$$\gamma_{G_i}\left([x]_J\right) \le y_i, \boxed{\forall J \subseteq [d]} \qquad \left([x]_J\right)_j := \begin{cases} x_j & j \in J \\ 0 & o.w. \end{cases}$$

- Smaller formulation:

$$\gamma_{G_i}\left([x]^+\right) \le y_i \qquad \left([x]^+\right)_j := \max\{x_j, 0\}$$

  - max can cause representability issues.

Algebraic Representation Issues

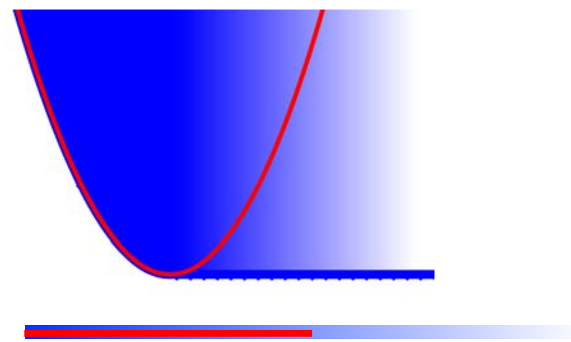


$$x_1^2 \le x_2 \le 1$$

C_2

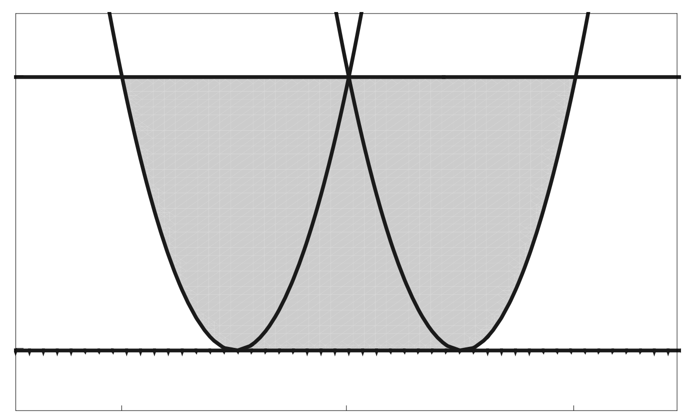
$$[-1,1] \times 0$$

$C_i + (\mathbb{R}_+ \times \{0\})$



$$C_1 + (\mathbb{R}_+ \times \{0\}) : \ (\max\{x_1, 0\})^2 \le x_2 \le 1$$

- Non-basic semi-algebraic set contained in formulation.
- Finite polynomial inequalities requires max or auxiliary vars.



Summary

- Small ideal formulations without "variable copies".
  - Piecewise representation by (nearly) homothetic sets.
  - Representation of gauge formulation = gauge calculus.
- More on the paper (arXiv:1704.03954):
  - More examples and generalizations:
    - Orthogonal sets, polyhedral formulations by Balas, Blair and Jeroslow, and "truly" non-polyhedral sets.
  - More construction techniques, gauge calculus, etc.
  - Necessary and sufficient conditions for piecewise. formulation being ideal (more geometric conditions).
- Support function matching / "Lifting" for more general non-convex sets: Tawarmalani et al. '10