Embedding Formulations for Unions of Convex Sets

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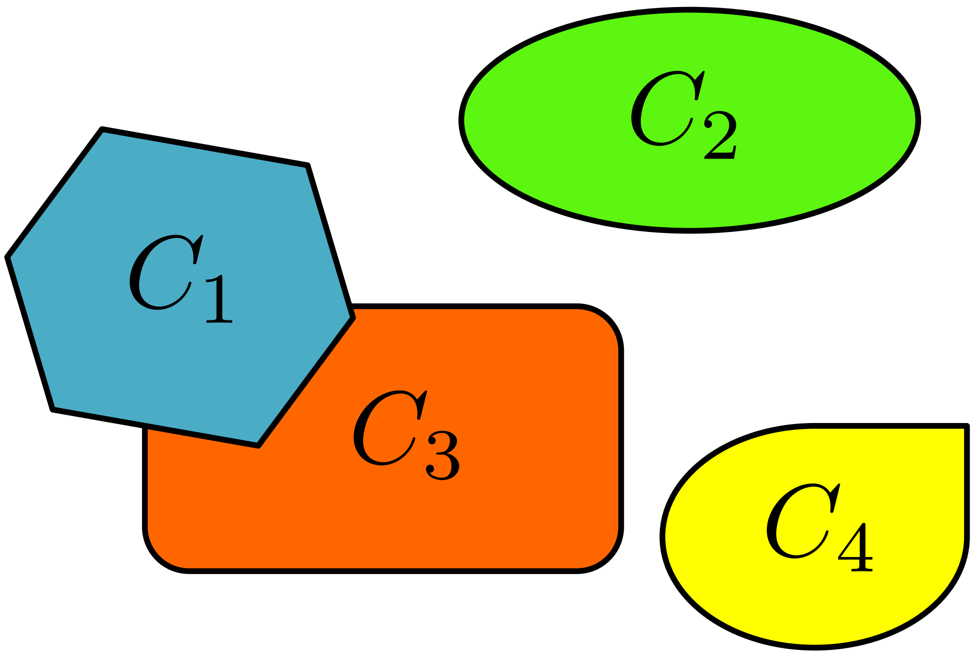
Nonlinear Mixed 0-1 Integer Formulations

- Modeling Finite Alternatives = Unions of Convex Sets

$$x \in \bigcup_{i=1}^{n} C_i \subseteq \mathbb{R}^d$$



Extended and Non-Extended Formulations for $\bigcup_{i=1}^{n} C_i$

$$C_i = \{x \in \mathbb{R}^d \,:\, A^i x \leq b^i\}$$

Extended

$$\begin{aligned} A^i x^i &\leq b^i y_i & \forall i \in [n] \\ \sum_{i=1}^{n} x^i &= x & \\ \sum_{i=1}^{n} y_i &= 1 & \\ y &\in \{0,1\}^n & \\ x,\; x^i &\in \mathbb{R}^d & \forall i \in [n] \end{aligned}$$

Small? and strong (ideal*)

Non-Extended

$$\begin{aligned} A^i x - b^i &\leq M_i (1 - y_i) & \forall i \in [n] \\ \sum_{i=1}^{n} y_i &= 1 & \\ y &\in \{0,1\}^n & \\ x &\in \mathbb{R}^d & \forall i \in [n] \end{aligned}$$

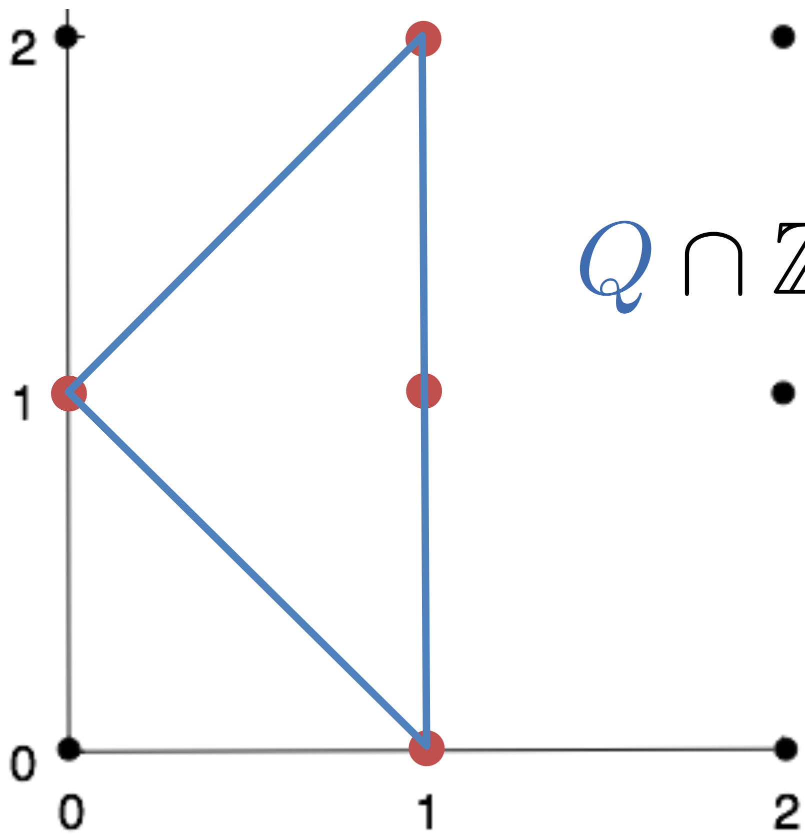
Small, but weak?

*Integral y in extreme points of LP relaxation

Constructing Non-extended Ideal Formulations

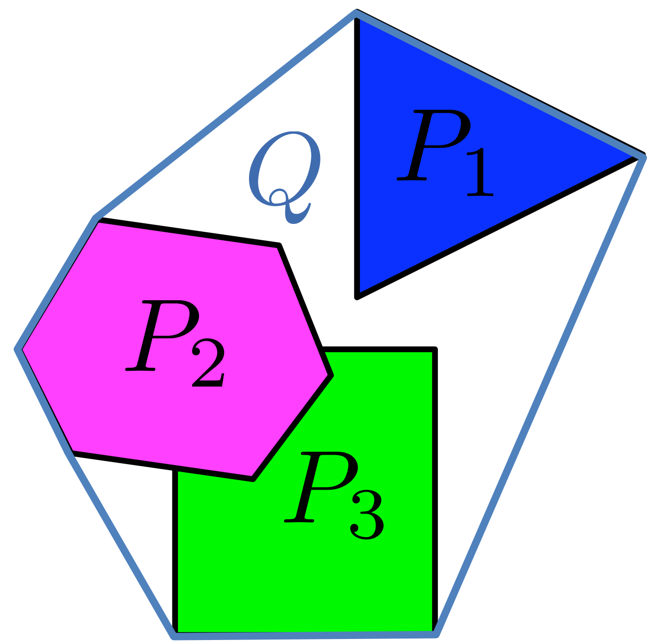
- Pure Integer :

$$Q := \text{conv}\left(\{p^i\}_{i=1}^n\right)$$

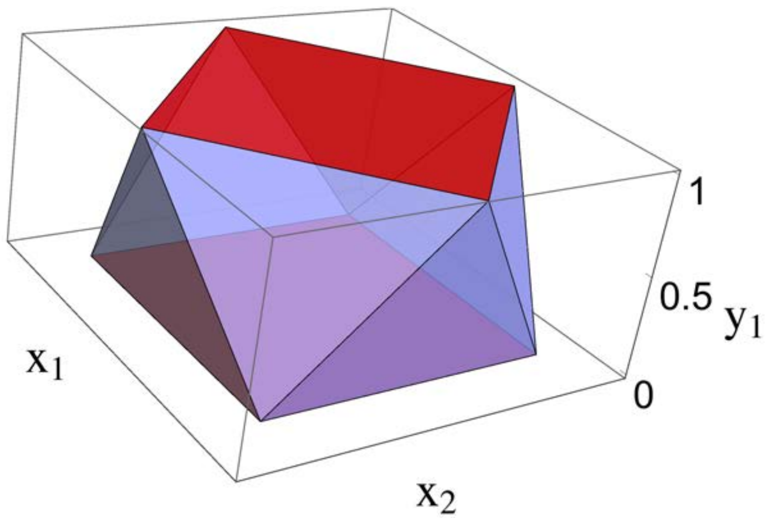


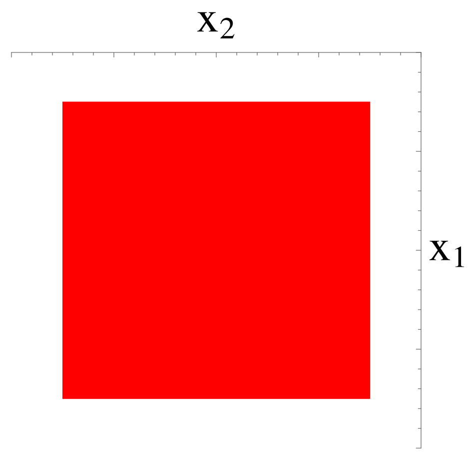
$$Q \cap \mathbb{Z}^2$$

- Mixed Integer:

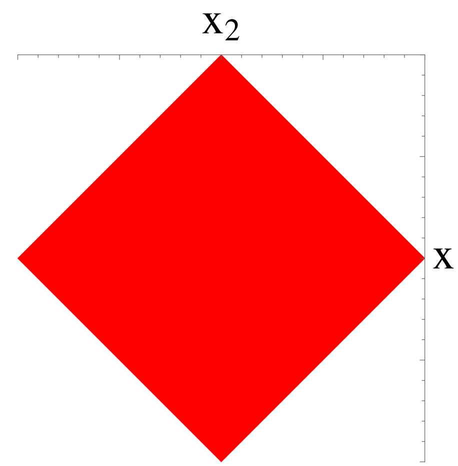


Embedding Formulation = Ideal non-Extended





P_1



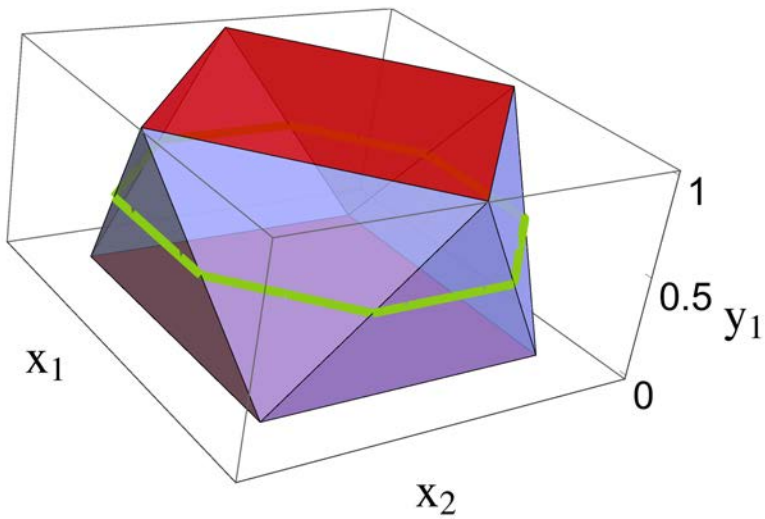
P_2

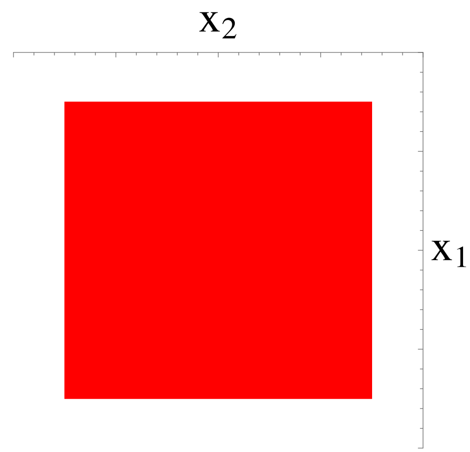
$$Q(H) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \{h^i\}\right)$$

$$(x, y) \in Q \cap \left(\mathbb{R}^d \times \mathbb{Z}^k\right) \quad \Leftrightarrow \quad y = h^i \;\wedge\; x \in P_i$$

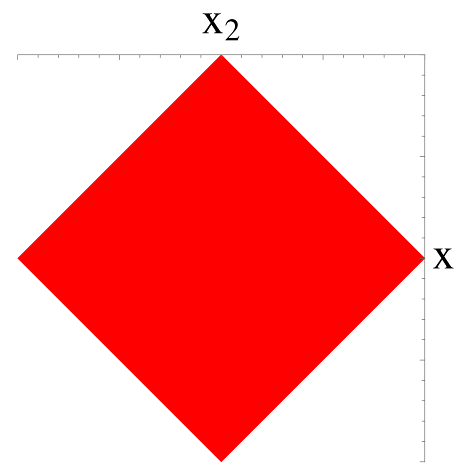
$$\operatorname{ext}(Q) \subseteq \mathbb{R}^d \times \mathbb{Z}^k \qquad H := \{h^i\}_{i=1}^{n} \subseteq \{0,1\}^k, \quad h^i \neq h^j$$

Unary Encoding, Minkowski Sum and Cayley Trick



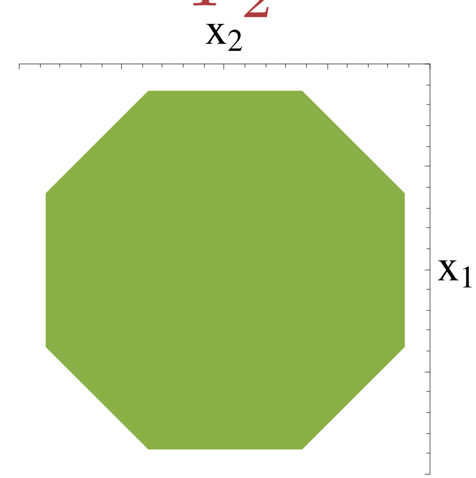


P_1



P_2

$$Q \cap \left(\mathbb{R}^2 \times \{0.5\}\right) \equiv P_1 + P_2 =$$



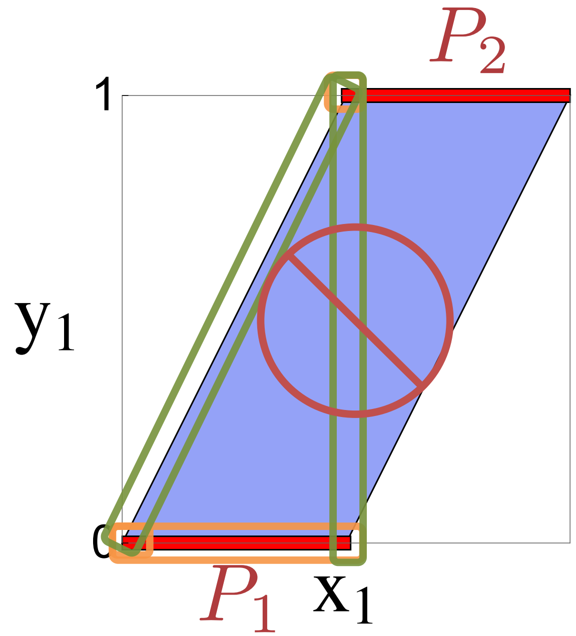
$$H = \left\{e^i\right\}_{i=1}^n$$

$$Q\left(H\right) \cap \left(\mathbb{R}^d \times \left\{\frac{1}{n}\sum_{i=1}^n \mathbf{e}^i\right\}\right) \equiv \sum_{i=1}^n P_i$$

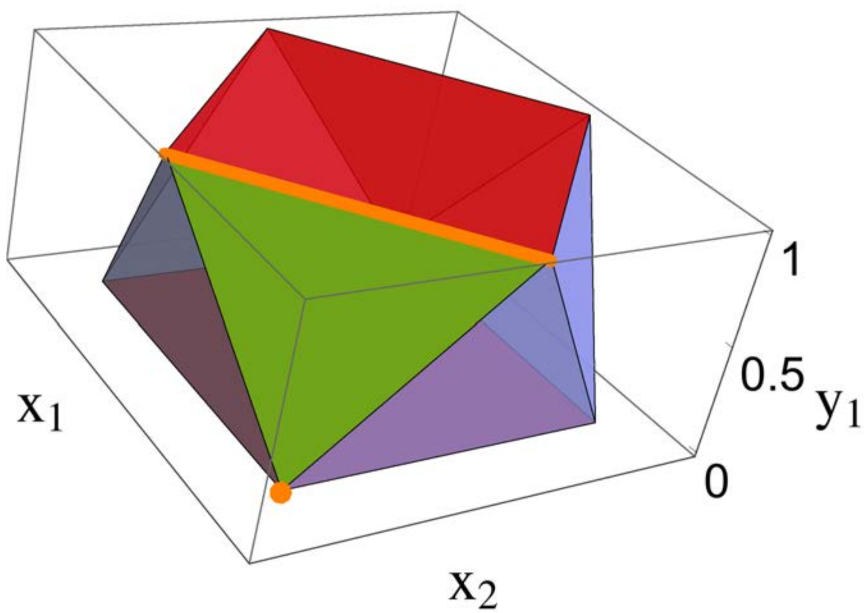
Faces of Cayley Embedding

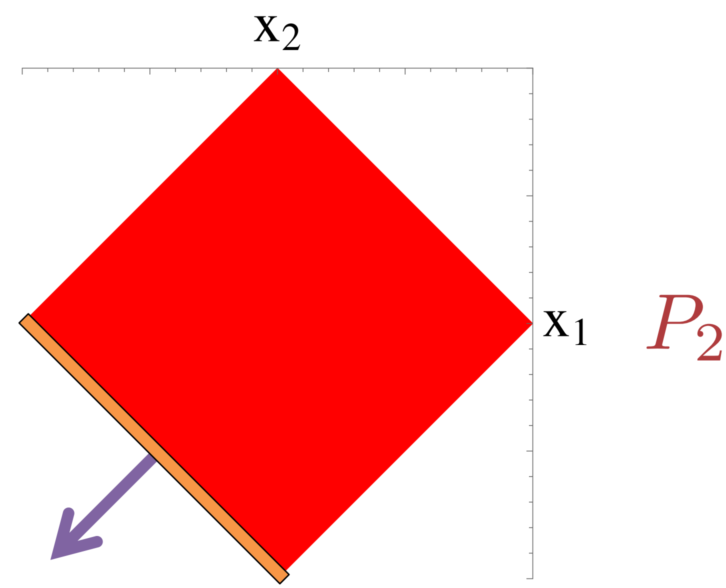
- Two types of facets (or faces):
  - $P_1 \times \{0\} \equiv y_i \geq 0$
  - $\operatorname{conv}\left(\left(F_1 \times 0\right) \cup \left(F_2 \times 1\right)\right)$

    F_i proper face of P_i
  - Not all combinations of faces
  - Which ones are valid?



Valid Combinations = Common Normals



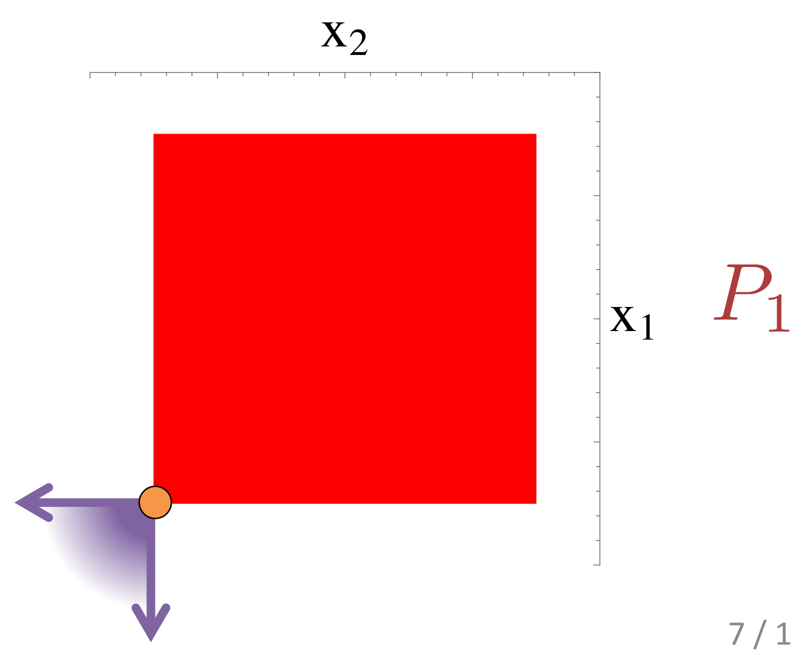


$$N(F_1) \cap N(F_2) \neq \emptyset$$

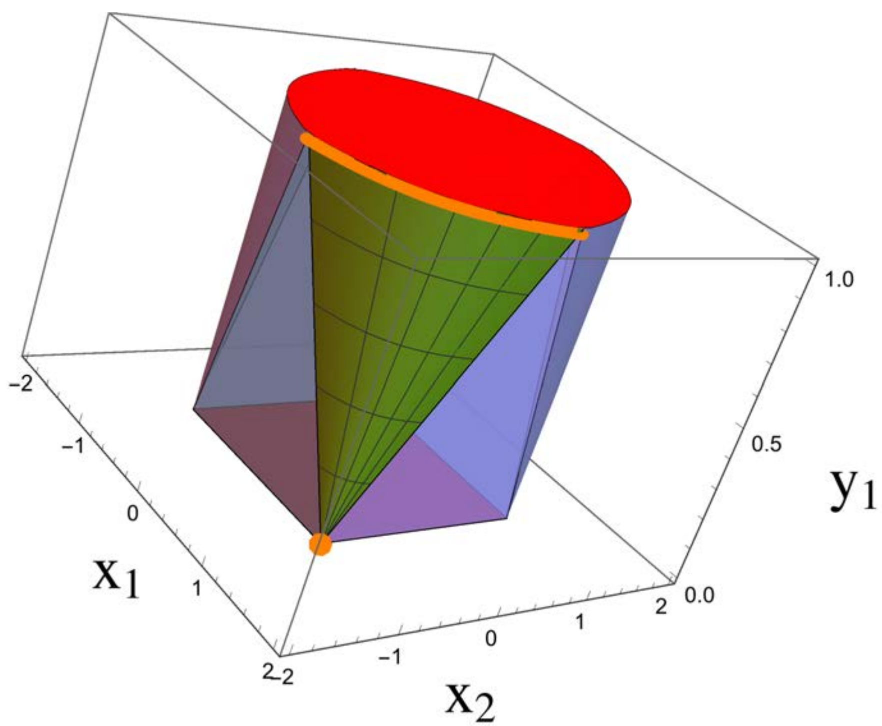
$$\Updownarrow$$

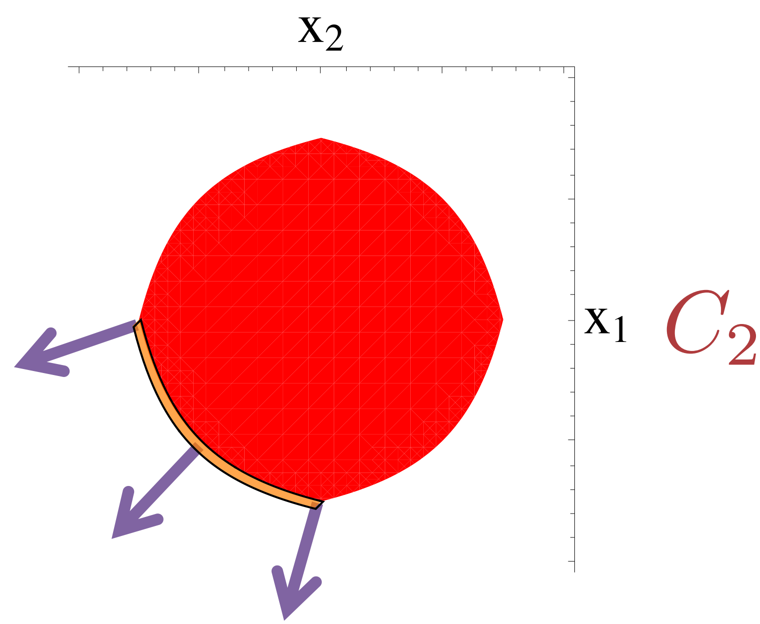
$$\mathrm{conv}\left(\left(F_1 \times 0\right) \cup \left(F_2 \times 1\right)\right)$$

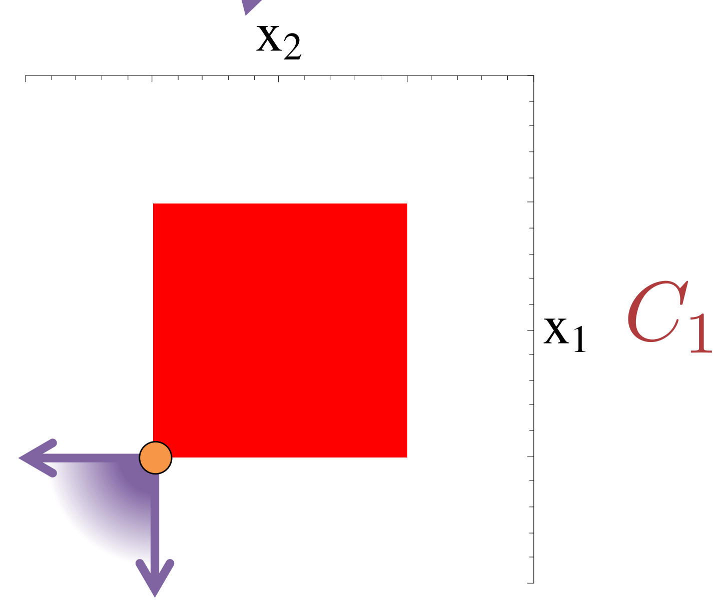
is face of $Q(H)$



Characterization Extends to Closed Convex Sets







Easy Case: Nearly-Homothetic Convex Sets (NHC)

- $C_i = \lambda_i C + b^i + C_\infty$

$$\operatorname{conv}\left(\bigcup_{i=1}^n \left(C_i \times \{\mathbf{e}^i\}\right)\right) =$$

$$\gamma_C\left(x - \sum_{i=1}^n y_i b^i\right) \leq \sum_{i=1}^n \lambda_i y_i$$

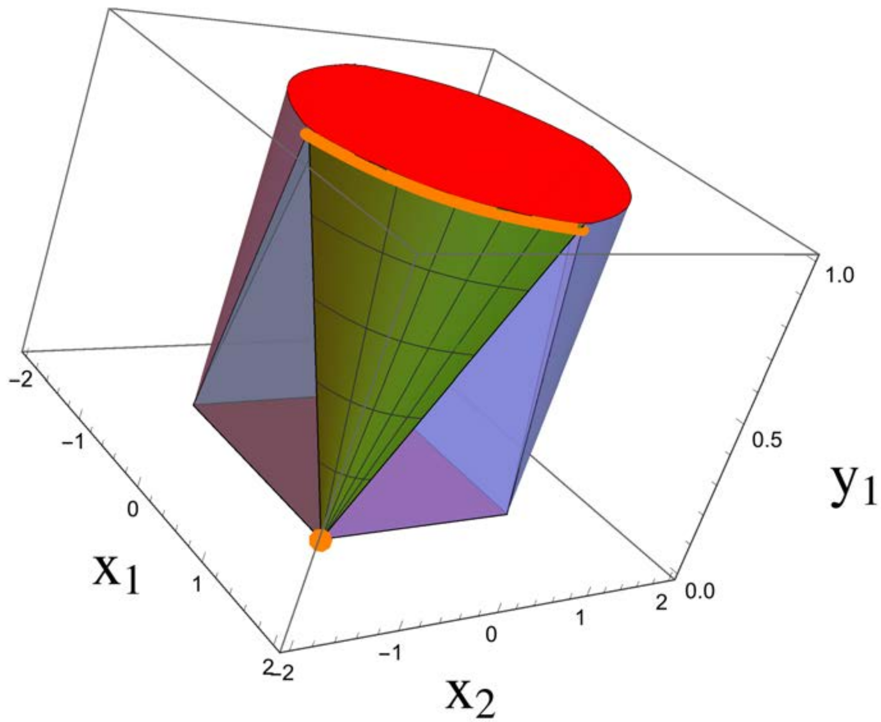
$$\sum_{i=1}^n y_i = 1$$

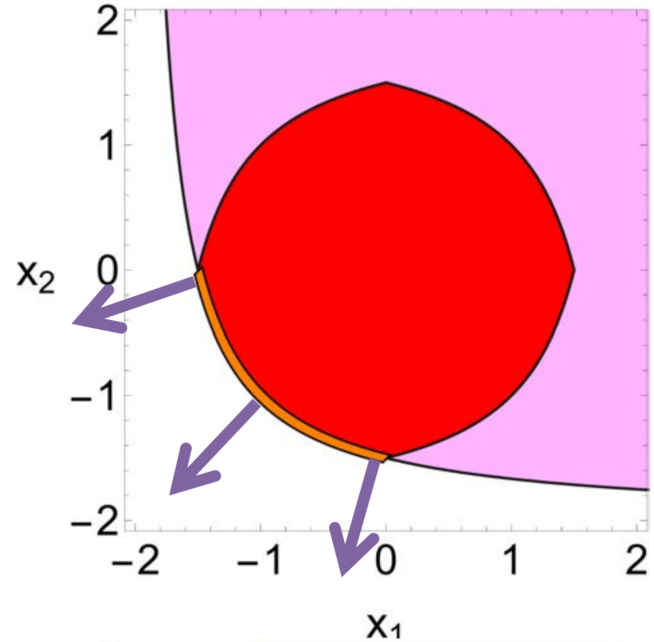
$$y \geq 0 \qquad \forall i \in [n]$$

$$\gamma_C(x) := \inf\{\lambda > 0 \,:\, x \in \lambda C\}$$

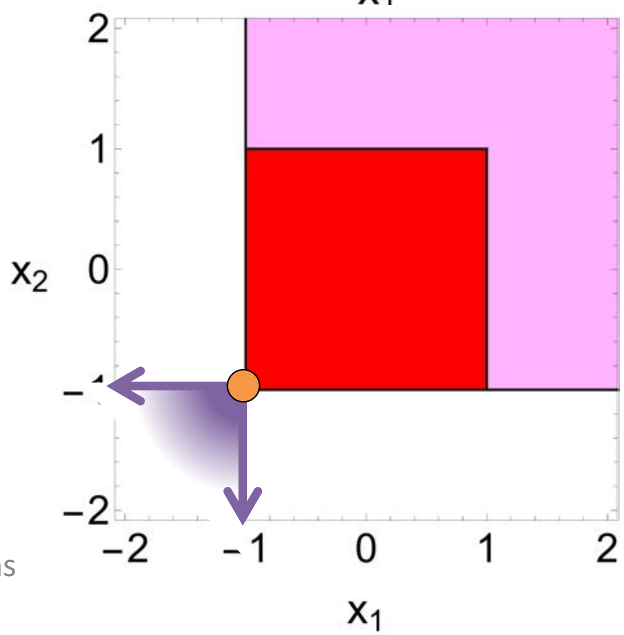
- Generalizes polyhedral results from Balas ‘85, Jeroslow ’88 and Blair ‘90

General Sets: Reduce to Intersection of NHCs





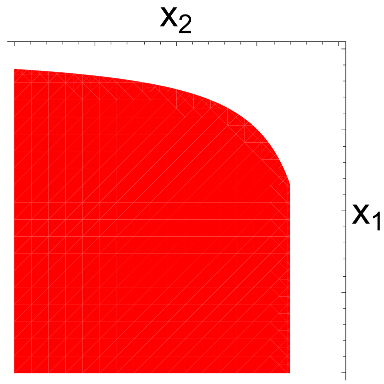
C_2



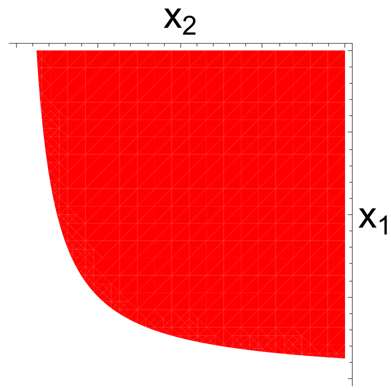
C_1

Small Formulations for Isotone Sets

- Studied by Hijazi et al. '12 and Bonami et al. '15 (n=1, 2):
  - $C_i = \{x \in \mathbb{R}^d : l^i \leq x \leq u^i, \quad f_i(x) \leq 0\}$
- $f_i(x)$ component-wise monotonous (i=1,2 opposite).



C_1



C_2

- Ideal Formulation

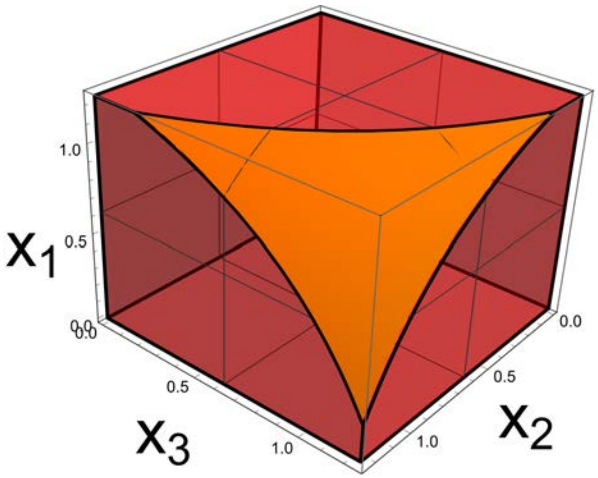
$$y_1 l^1 + y_2 l^2 \leq x \leq y_1 u^1 + y_2 u^2$$

$$f_J^i(x, y) \leq 0 \qquad \forall J \subseteq [d], i \in [2]$$

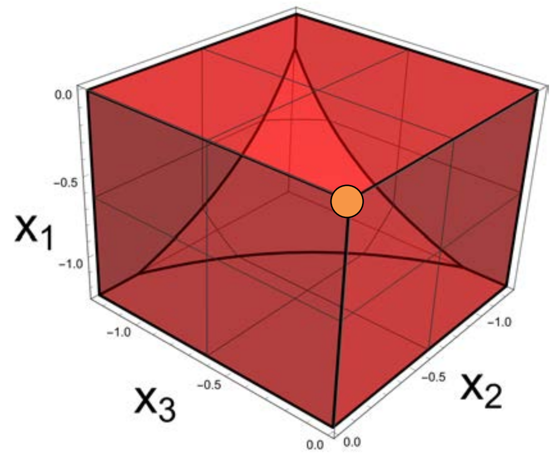
$$y_1 + y_2 = 1$$

$$y_i \in \{0, 1\} \qquad i \in [2]$$

Boundary Structure = Redundancy Detection



C_1



C_2

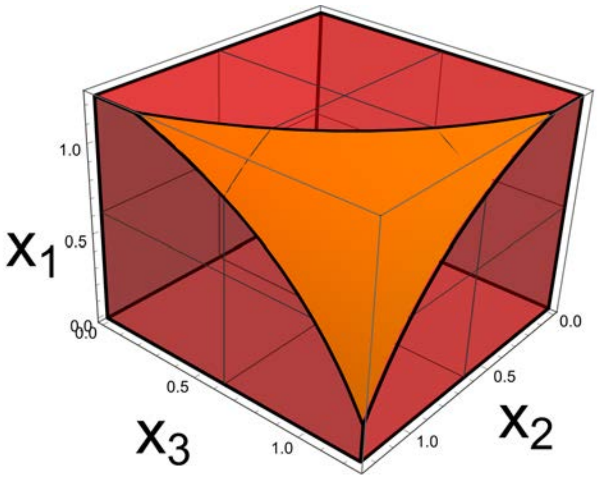
$$y_1 l^1 + y_2 l^2 \le x \le y_1 u^1 + y_2 u^2$$

$$f^i(x, y) \le 0 \qquad \forall i \in [2]$$

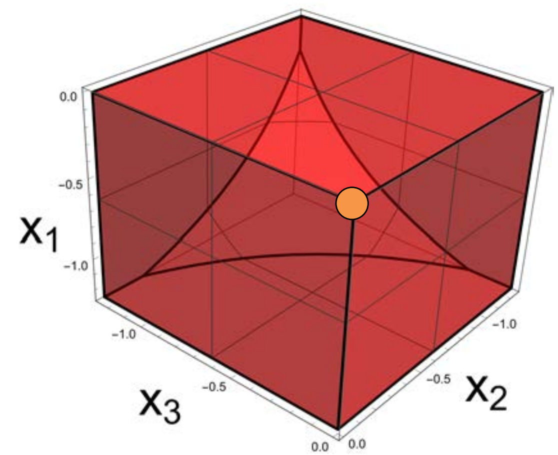
$$y_1 + y_2 = 1$$

$$y_i \ge \{0, 1\} \qquad i \in [2]$$

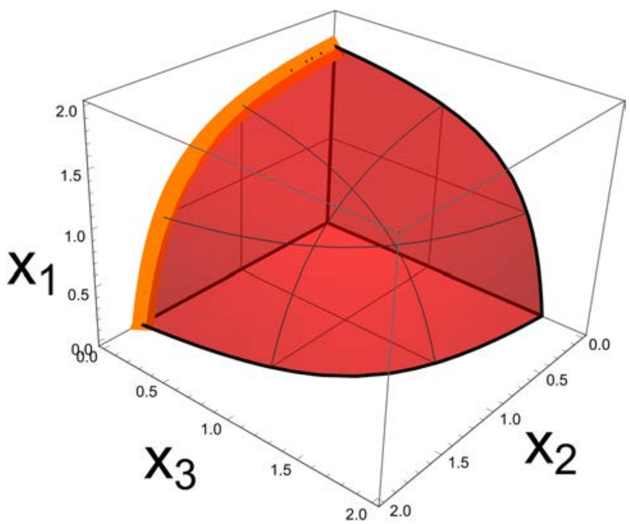
Boundary Structure = Redundancy Detection



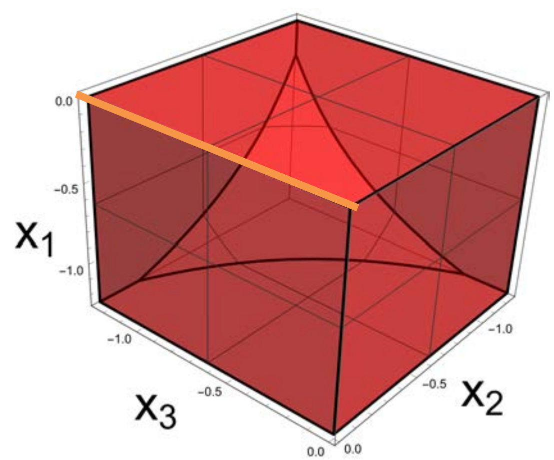
C_1



C_2

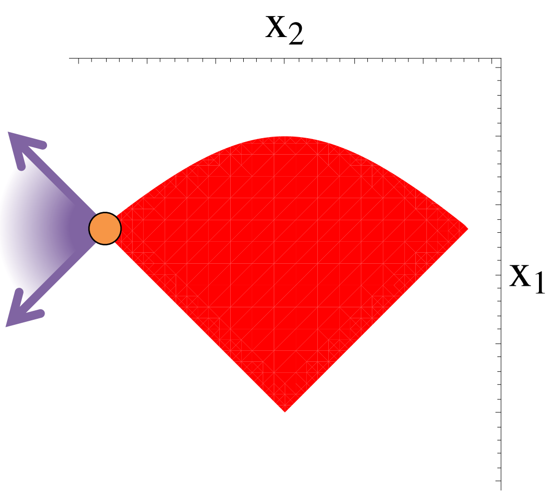


C_1



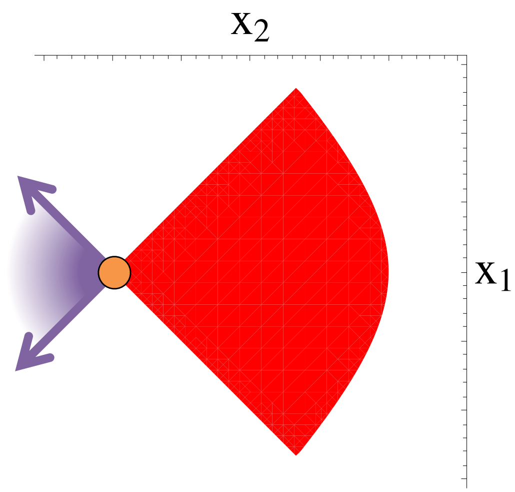
C_2

Also Non-isotone Sets and n>2 : Pizza Slices



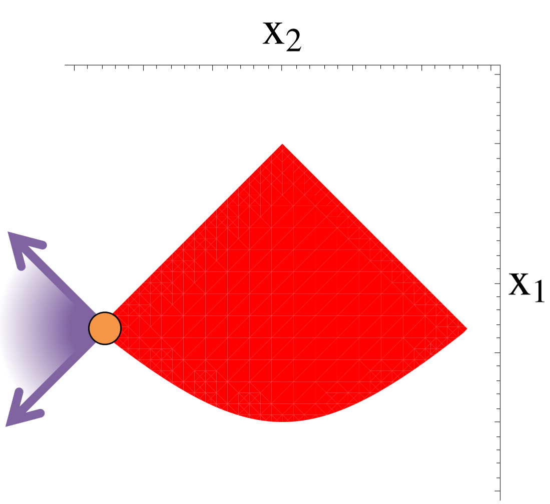
$C_1:$

$$\sqrt{1+x_1^2} \leq 2 - x_2$$
$$x_1 - x_2 \leq 1$$
$$-x_2 - x_1 \leq 1$$



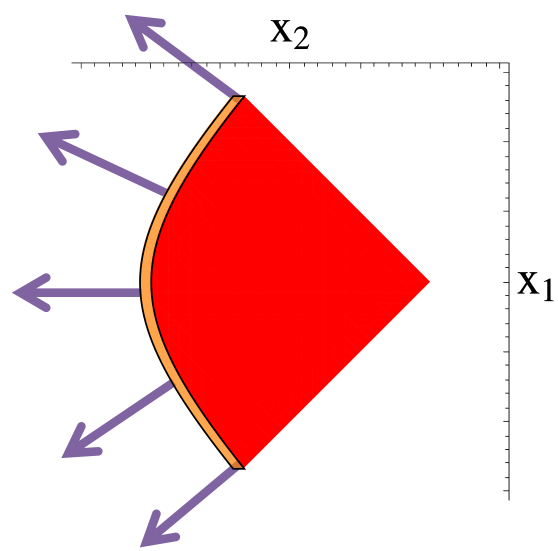
$C_2:$

$$\sqrt{1+x_2^2} \leq 2 - x_1$$
$$x_2 - x_1 \leq 1$$
$$-x_2 - x_1 \leq 1$$



$C_3:$

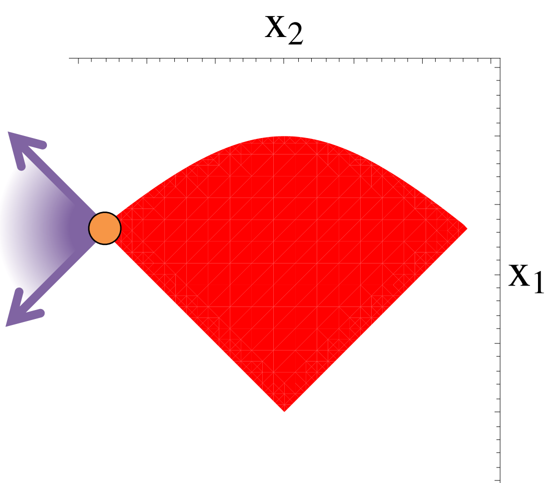
$$\sqrt{1+x_1^2} \leq 2 + x_2$$
$$x_2 - x_1 \leq 1$$
$$x_1 + x_2 \leq 1$$



$C_4:$

$$\sqrt{1+x_2^2} \leq 2 + x_1$$
$$x_1 - x_2 \leq 1$$
$$x_1 + x_2 \leq 1$$

Also Non-isotone Sets and n>2 : Pizza Slices



C_1 :

$$\sqrt{1+x_1^2} \leq 2 - x_2$$
$$x_1 - x_2 \leq 1$$
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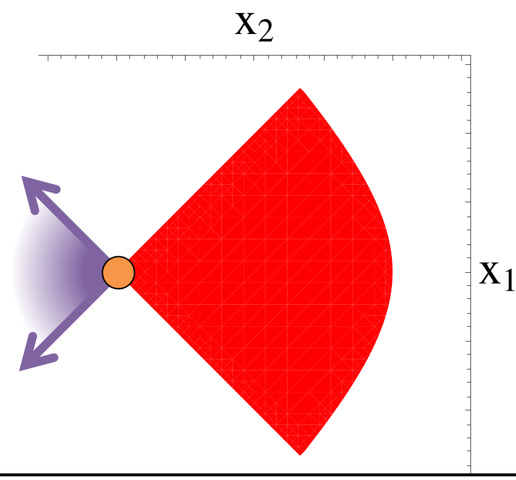


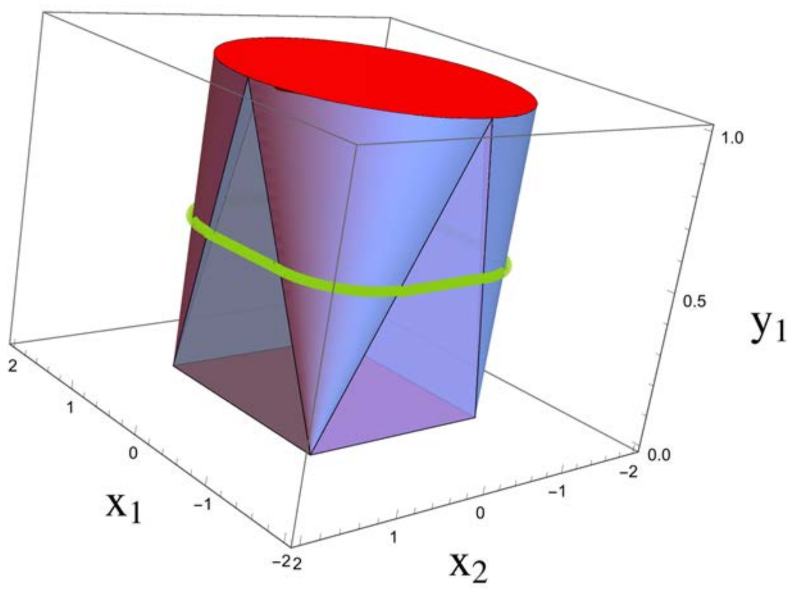
C_2 :

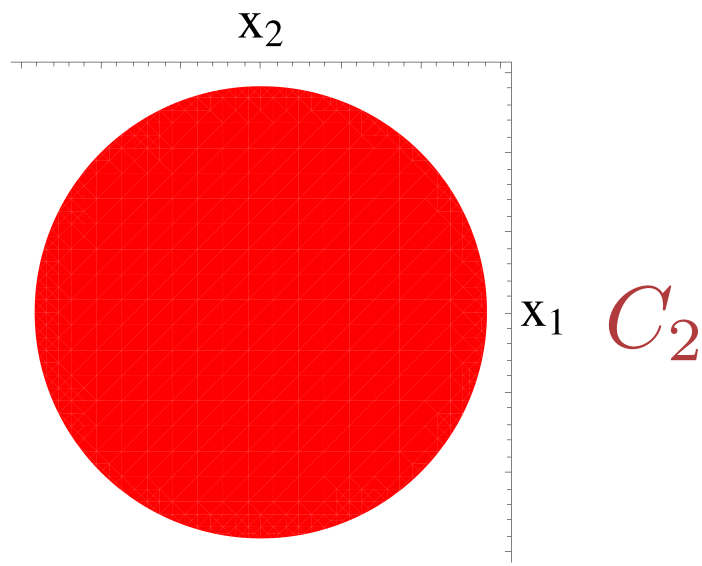
$$\sqrt{1+x_2^2} \leq 2 - x_1$$
$$x_2 - x_1 \leq 1$$
$$-x_2 - x_1 \leq 1$$

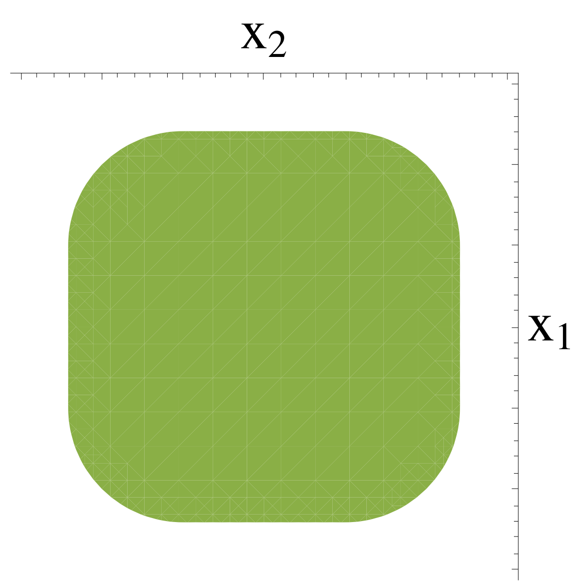
$$\sqrt{y_4^2 + \left(x_2 - \frac{1}{3}y_1 + \frac{1}{3}y_3\right)^2} \leq 2y_4 + x_1 + \frac{4}{3}y_1 + y_2 + \frac{4}{3}y_3$$

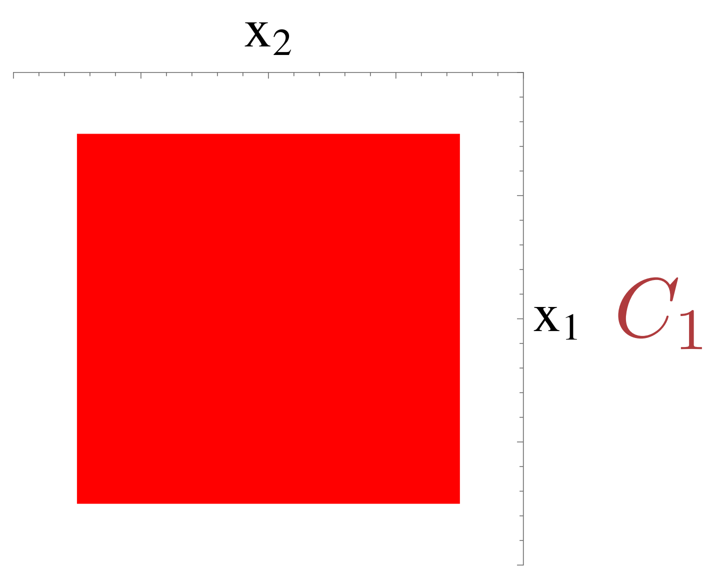
$\operatorname{conv}\left(\bigcup_{i=1}^{4}\left(C_i \times \{\mathbf{e}^i\}\right)\right)$ = 4 conic + 4 linear inequalities

Bad Example: Representability Issues

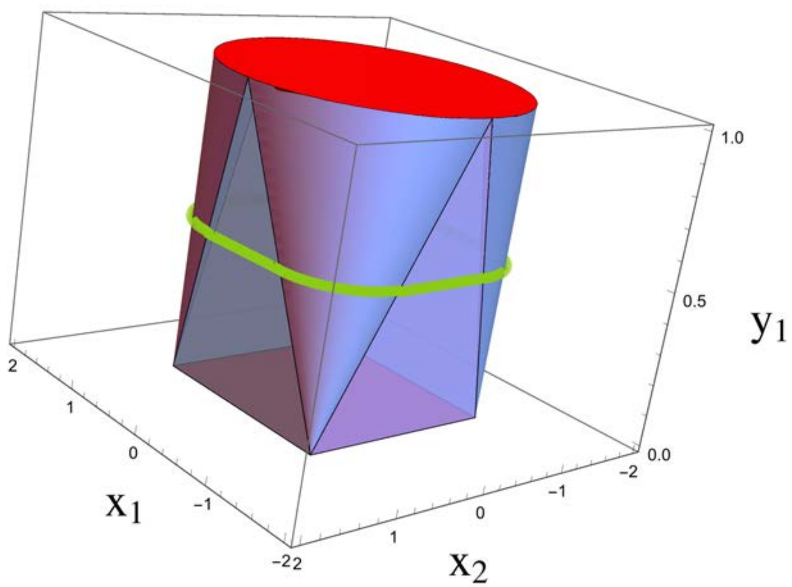


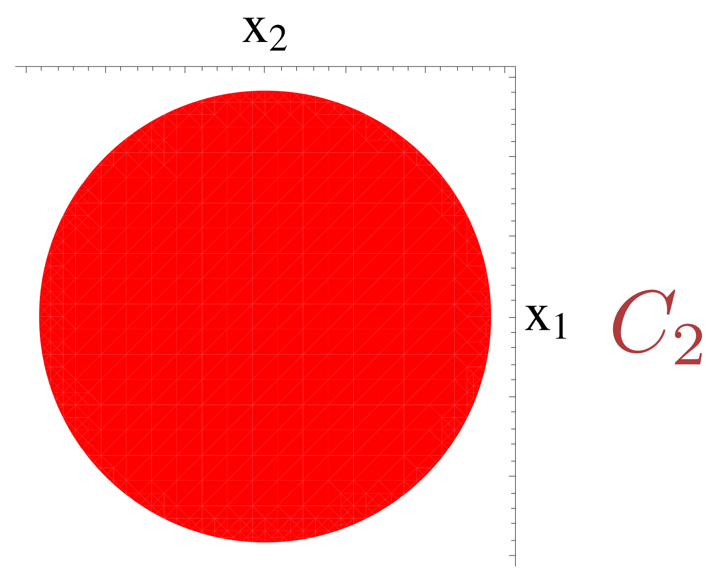




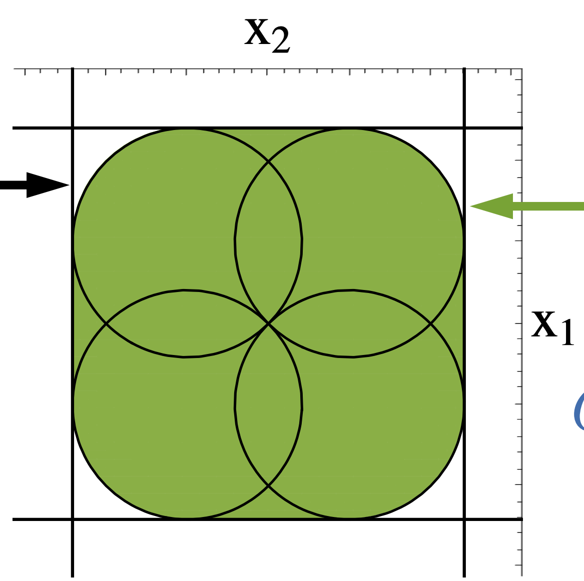


Bad Example: Representability Issues





Zariski closure of boundary



Description with finite number of (quadratic) polynomial inequalities?

$$Q := \operatorname{conv}\left(\left(C_1 \times \{0\}\right) \cup \left(C_2 \times \{1\}\right)\right)$$

can fail to be basic semi-algebraic

Summary

- Embedding Formulations = Systematic procedure for ideal non-extended formulations
- Can yield small practical formulations
- Not always practical (basic semi-algebraic representability)