Discrete Geometry for Small and Strong Mixed Integer Programming Formulations

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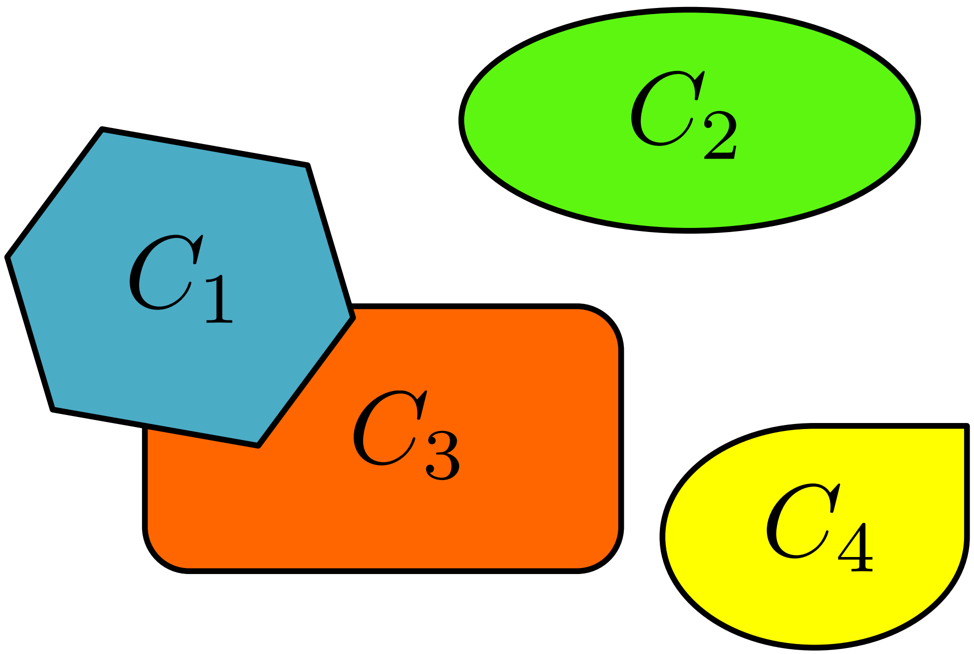
Nonlinear Mixed 0-1 Integer Formulations

- Modeling Finite Alternatives = Unions of Convex Sets

$$x \in \bigcup_{i=1}^{n} C_i \subseteq \mathbb{R}^d$$



Extended and Non-Extended Formulations for $\bigcup_{i=1}^{n} C_i$

$$C_i = \{x \in \mathbb{R}^d \; : \; A^i x \leq b^i\}$$

Extended	Non-Extended
$A^i x^i \leq b^i y_i \quad \forall i \in [n]$ $\sum_{i=1}^{n} x^i = x$ $\sum_{i=1}^{n} y_i = 1$ $y \in \{0,1\}^n$ $x, \; x^i \in \mathbb{R}^d \quad \forall i \in [n]$	$A^i x - b^i \leq M_i (1 - y_i) \quad \forall i \in [n]$ $\sum_{i=1}^{n} y_i = 1$ $y \in \{0,1\}^n$ $x \in \mathbb{R}^d \quad \forall i \in [n]$
Small? and strong (ideal*)	Small, but weak?

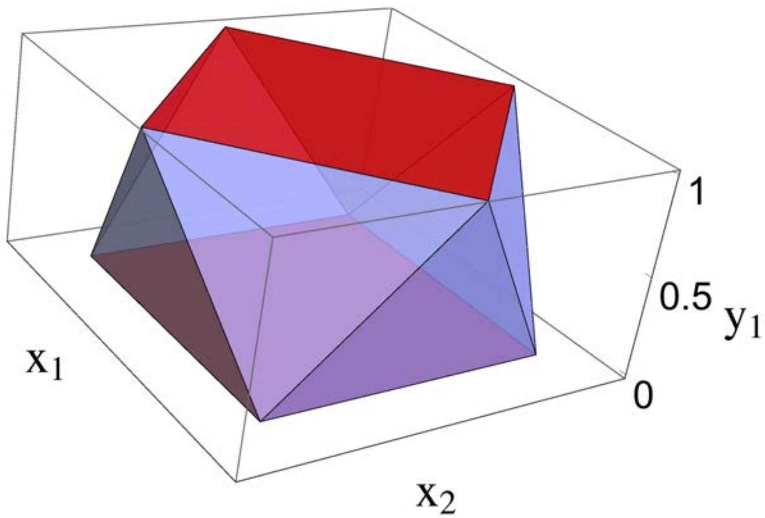
*Integral y in extreme points of LP relaxation

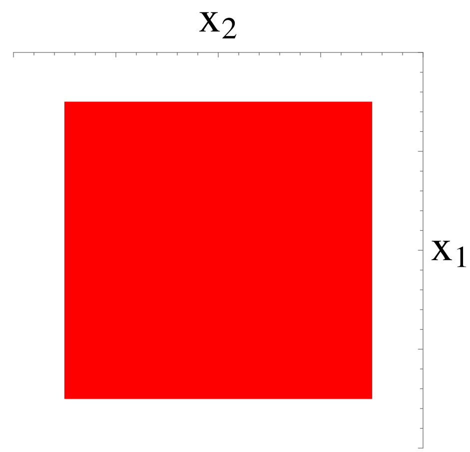
Outline

- Smallest ideal non-extended linear formulation
  - Generalization of Cayley Embedding of Polytopes
  - Hyperplane arrangements
- Small ideal non-extended nonlinear formulations
  - Boundary structure of Cayley Embedding: from polytopes to closed convex sets
- "Formulations" with a general integer variables (possibly a fixed number)

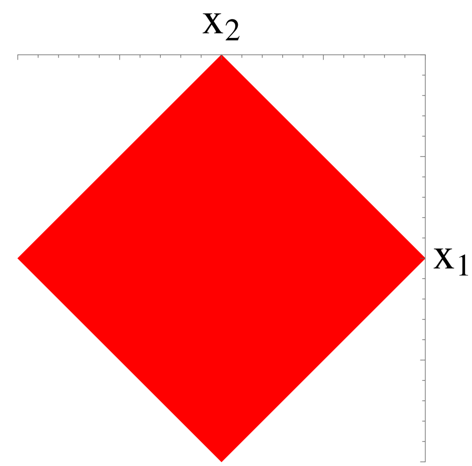
Ideal Non-Extended Formulations and Hyperplane Arrangements

Embedding Formulation = Ideal non-Extended





P_1



P_2

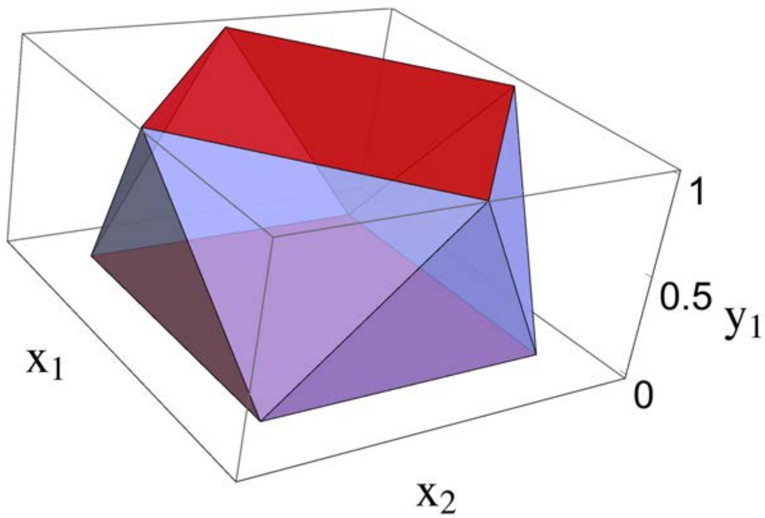
$$Q(H) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \{h^i\}\right)$$

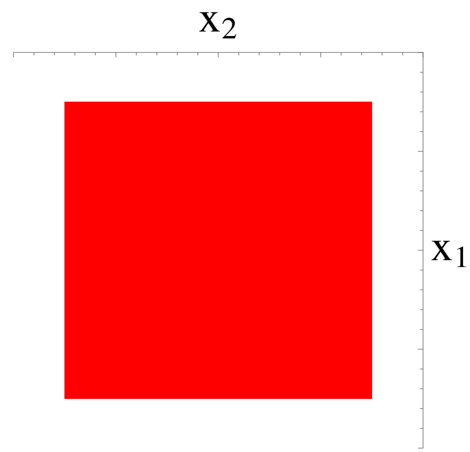
$$(x, y) \in Q \cap \left(\mathbb{R}^d \times \mathbb{Z}^k\right) \quad \Leftrightarrow \quad y = h^i \ \wedge \ x \in P_i$$

$$\operatorname{ext}(Q) \subseteq \mathbb{R}^d \times \mathbb{Z}^k \qquad H := \{h^i\}_{i=1}^{n} \subseteq \{0,1\}^k, \quad h^i \neq h^j$$

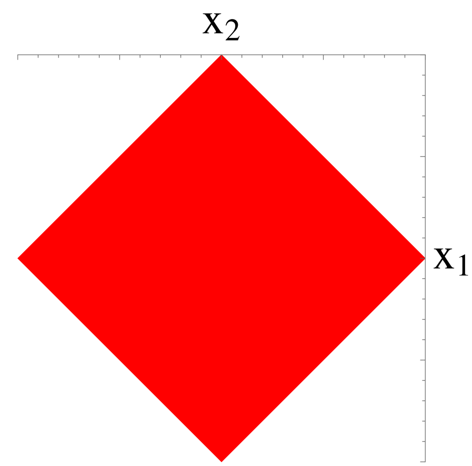
$$\text{Cayley } \equiv h^i = e^i, \quad k = n$$

Embedding Formulation = Ideal non-Extended





P_1



P_2

$$Q\left(H\right) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \left\{h^i\right\}\right)$$

$$(x, y) \in Q \cap \left(\mathbb{R}^d \times \mathbb{Z}^k\right) \quad \Leftrightarrow \quad y = h^i \ \wedge \ x \in P_i$$

$$\operatorname{ext}(Q) \subseteq \mathbb{R}^d \times \mathbb{Z}^k \qquad H := \left\{h^i\right\}_{i=1}^{n} \subseteq \{0,1\}^k, \quad h^i \neq h^j$$

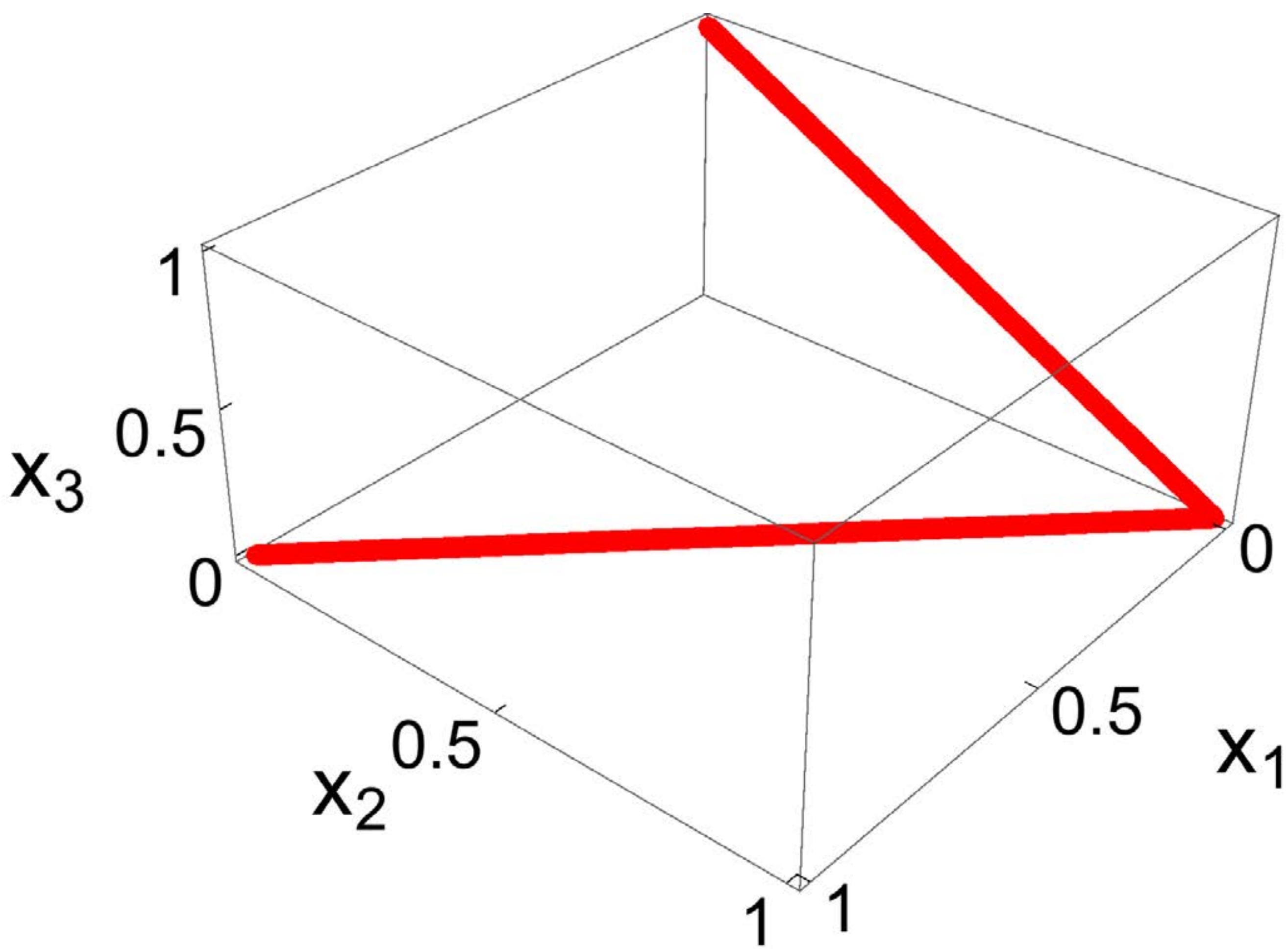
$$\mathcal{P} := \{P_i\}_{i=1}^{n} \longrightarrow \operatorname{mc}\left(\mathcal{P}\right) := \min_H \left\{\operatorname{size}\left(Q\left(H\right)\right)\right\}$$

Special Ordered Sets = Simplex Faces = $\mathcal{P} := \{P_i\}_{i=1}^{n}$

- $$\Delta^{d+1} := \left\{ x \in \mathbb{R}_+^{d+1} \,:\, \sum_{i=1}^{d+1} x_i = 1 \right\} = \operatorname{conv}\left(\{e^i\}_{i=1}^{d+1} \right)$$
  $$P_i := \operatorname{conv}\left(\{e^j\}_{j \in T_i} \right) = \left\{ x \in \Delta^{d+1} \,:\, \sum_{j \notin T_i} x_i \le 0 \right\}$$
  $$T_i \subseteq \{1, \ldots, d+1\}$$
- $$\operatorname{mc}(\mathcal{P}) := \min_H \{\operatorname{size}(Q(H))\},$$
  $$\operatorname{size}(Q(H)) := \#\ \text{facets}$$
- $$\operatorname{mc}_G(\mathcal{P}) := \min_H \{\operatorname{size}_G(Q(H))\},$$
  $$\operatorname{size}_G(Q(H)) := \#\ \text{non-bound facets}$$

Special Ordered Sets of Type 2 (SOS2) = $\mathcal{P} := \{P_i\}_{i=1}^n$

- $P_i := \operatorname{conv}\left(\{e^i, e^{i+1}\}\right) \subseteq \Delta^{n+1}, \quad i \in [n]$



Special Ordered Sets of Type 2 (SOS2) = $\mathcal{P} := \{P_i\}_{i=1}^n$

- $P_i := \operatorname{conv}\left(\left\{e^i, e^{i+1}\right\}\right) \subseteq \Delta^{n+1}, \quad i \in [n]$

Claim:

$$\operatorname{mc}_G(\mathcal{P}) = 2\lceil \log_2 n \rceil,$$

$$n+1 \leq \operatorname{mc}(\mathcal{P}) \leq n+1+2\lceil \log_2 n \rceil$$

- $(x, y) \in Q(H) = \operatorname{conv}\left(\bigcup_{i=1}^n P_i \times \left\{h^i\right\}\right)$

$$= \operatorname{conv}\left(\bigcup_{i=1}^n \left\{e^i, e^{i+1}\right\} \times \left\{h^i\right\}\right)$$

- $a \cdot x \leq b \cdot y$

$$a_{i+1} \leq \min\left\{b \cdot h^i, b \cdot h^{i+1}\right\}$$

$b \in L(H) := \operatorname{aff}(H) - h^1$

$$\underbrace{b \cdot \left(h^{i+1} - h^i\right)} = 0$$

Embedding Formulation for SOS2: Part 1

- From encodings (H) to hyperplanes:

$$\{h^i\}_{i=1}^{n}$$

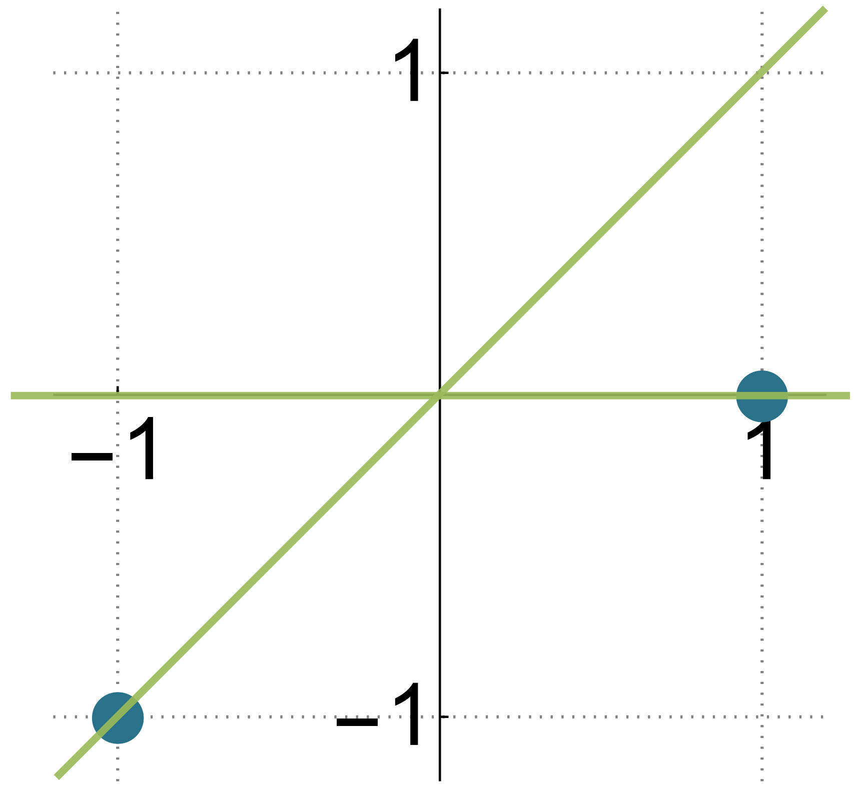
$$c^i = h^{i+1} - h^i$$

$$\{c^i\}_{i=1}^{n-1}$$

Hyperplanes spanned by

$$\{b^i \cdot y = 0\}_{j=1}^{L}$$

$$h^1 = \begin{pmatrix}1\\1\end{pmatrix},\ h^2 = \begin{pmatrix}0\\0\end{pmatrix},\ h^3 = \begin{pmatrix}1\\0\end{pmatrix}$$



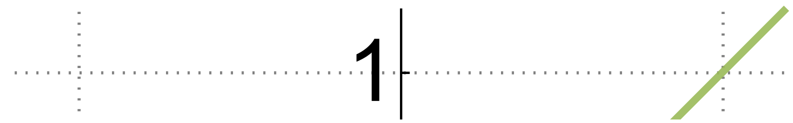
Embedding Formulation for SOS2: Part 1

- From encodings (H) to hyperplanes:

$$\{h^i\}_{i=1}^n$$

$$h^1 = \begin{pmatrix}1\\1\end{pmatrix}, h^2 = \begin{pmatrix}0\\0\end{pmatrix}, h^3 = \begin{pmatrix}1\\0\end{pmatrix}$$

$$c^i = h^{i+1} - h^i$$

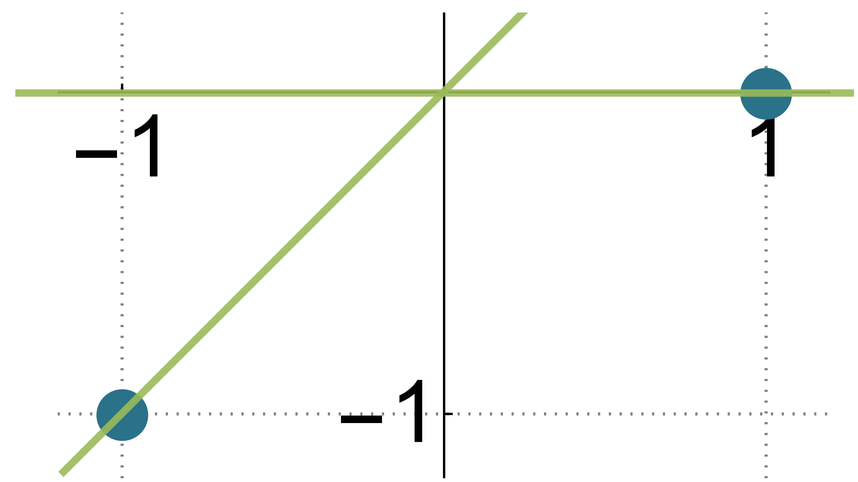


non-bound facets = 2 × # of hyperplanes

$i=1$

Hyperplanes spanned by

$$\{b^i \cdot y = 0\}_{j=1}^L$$



Embedding Complexity for SOS2

- Lower Bound: $L(H) := \operatorname{aff}(H) - h^1$

$$\mathrm{mc}_G(\mathcal{P}) \geq 2 \times \min \# \text{ of hyperplanes}$$

$$\min \# \text{ of hyperplanes} \geq \dim(L(H))$$

$$\dim(L(H)) \geq \lceil \log_2 n \rceil$$

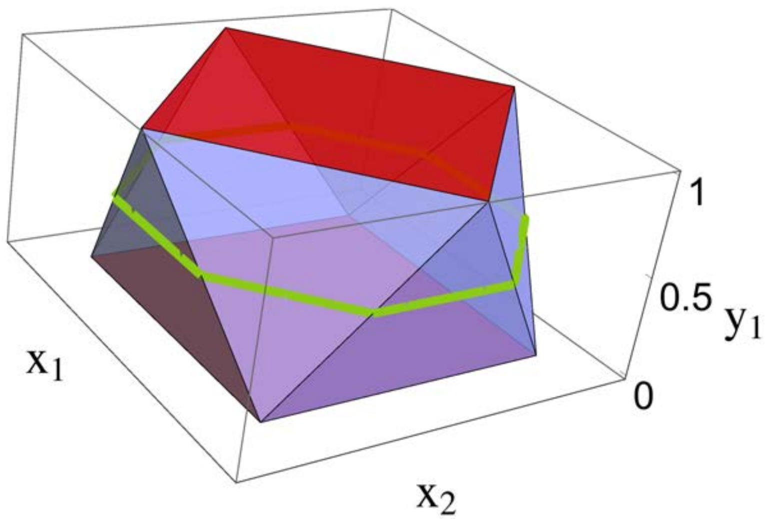
- Upper Bound: $H = \{0,1\}^{\lceil \log_2 n \rceil}$
  - Gray code: $\{h^i - h^{i+1}\}_{i=1}^{n-1} \equiv \{e^i\}_{i=1}^{\lceil \log_2 n \rceil}$

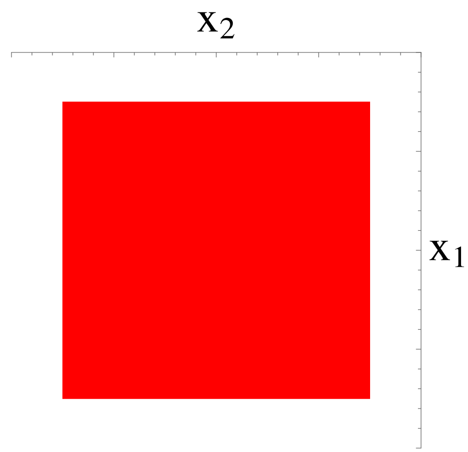
$$\mathrm{size}_G(Q(H)) = 2\lceil \log_2 n \rceil$$

$$n + 1 \leq \mathrm{mc}(\mathcal{P}) \leq n + 1 + 2\lceil \log_2 n \rceil$$

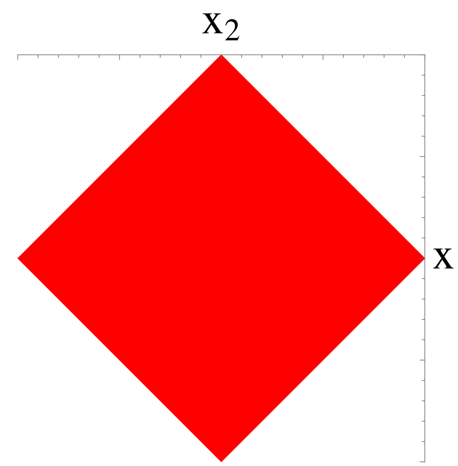
Minkowski Sums and Nonlinear MIP Formulations

Unary Encoding, Minkowski Sum and Cayley Trick

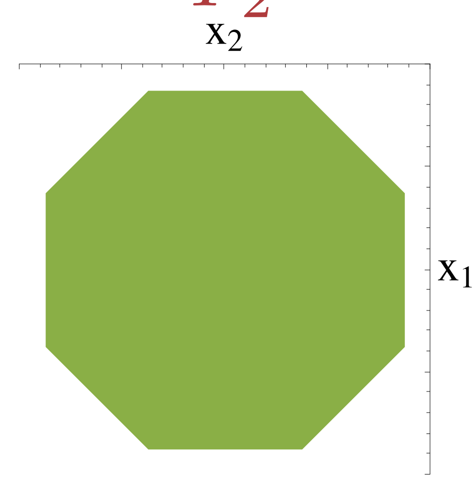




P_1



P_2



$$Q \cap \left(\mathbb{R}^2 \times \{0.5\}\right) \equiv P_1 + P_2 =$$

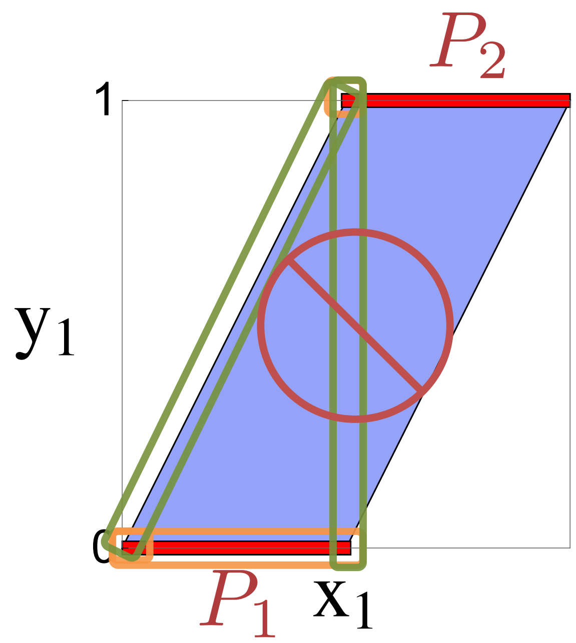
$$H = \left\{e^i\right\}_{i=1}^n$$

$$Q\,(H) \cap \left(\mathbb{R}^d \times \left\{\tfrac{1}{n}\textstyle\sum_{i=1}^n \mathbf{e}^i\right\}\right) \equiv \textstyle\sum_{i=1}^n P_i$$

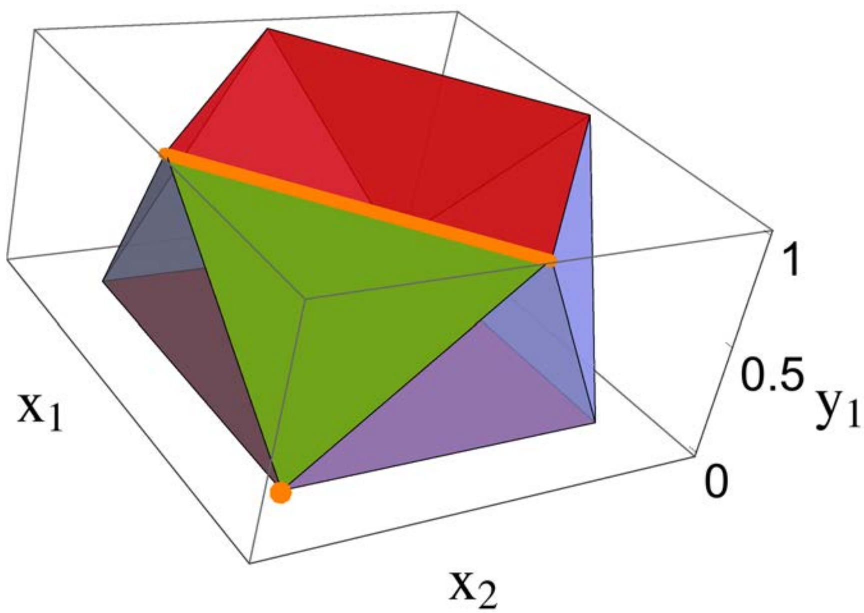
Faces of Cayley Embedding

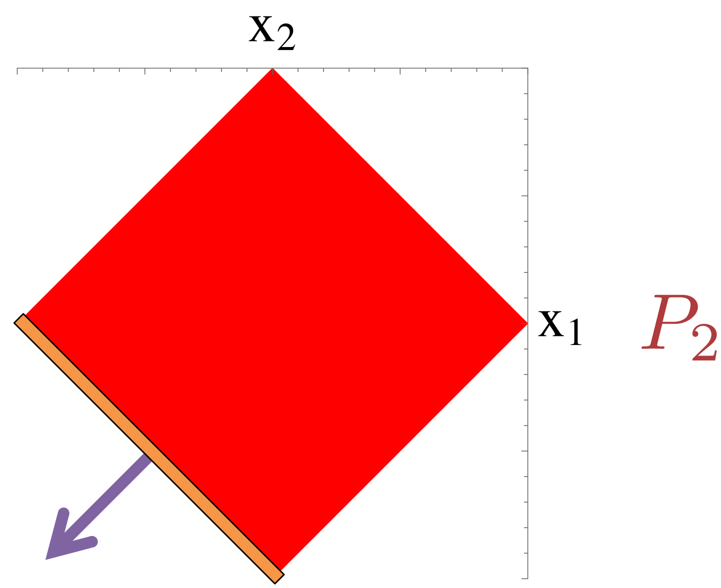
- Two types of facets (or faces):
  - $P_1 \times \{0\} \equiv y_i \geq 0$
  - $\operatorname{conv}\left((F_1 \times 0) \cup (F_2 \times 1)\right)$

    F_i proper face of P_i
  - Not all combinations of faces
  - Which ones are valid?



Valid Combinations = Common Normals



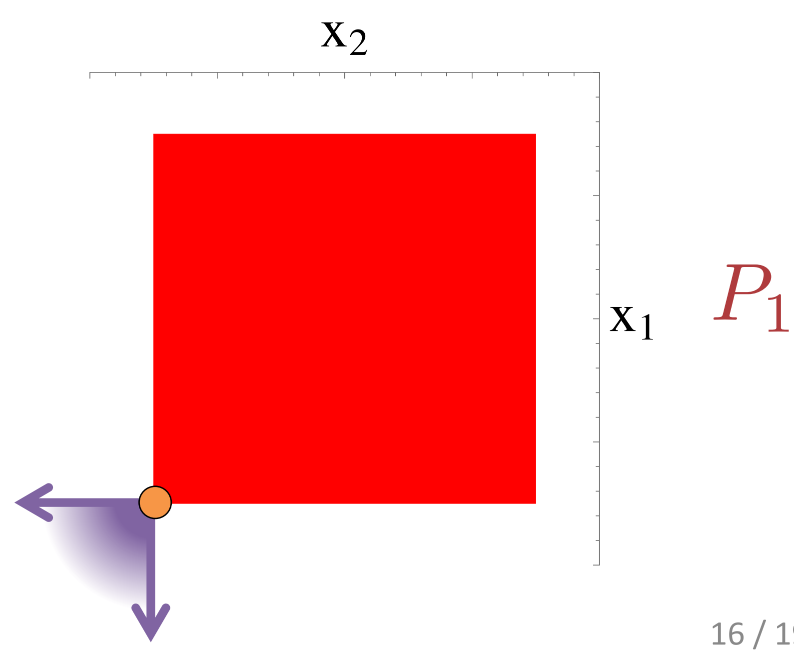


$$N(F_1) \cap N(F_2) \neq \emptyset$$

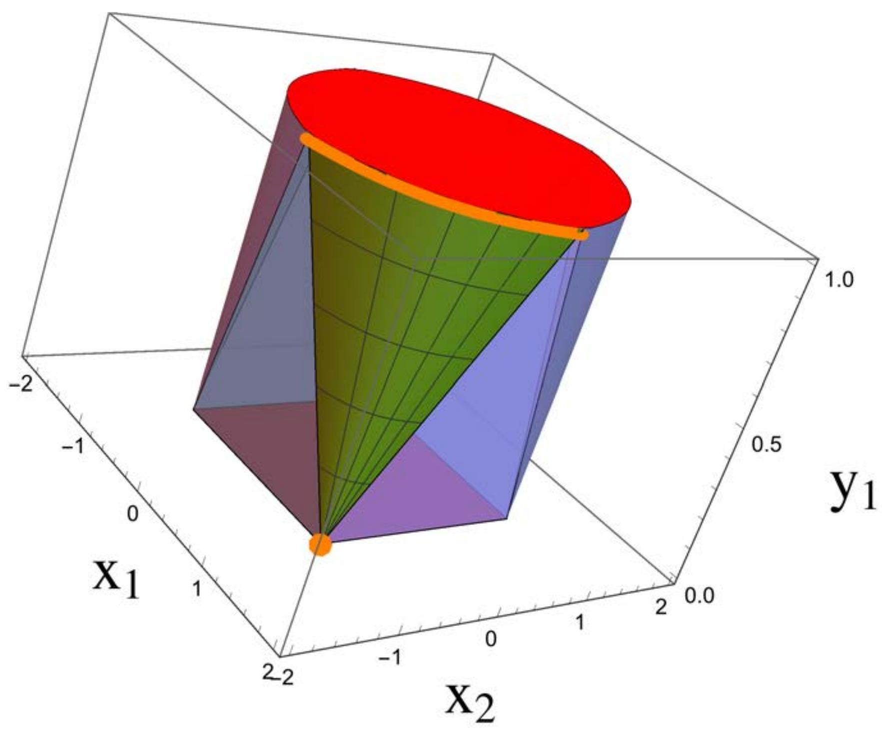
$$\Updownarrow$$

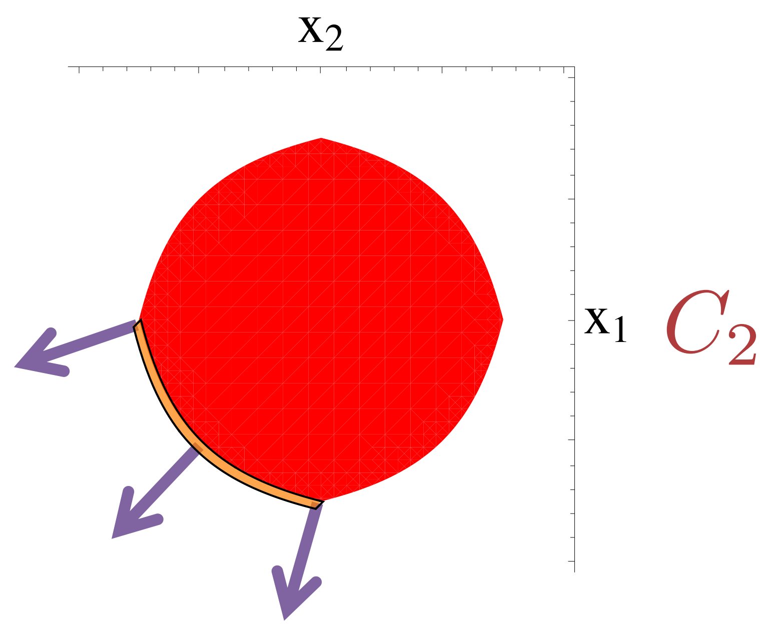
$$\text{conv}\left((F_1 \times 0) \cup (F_2 \times 1)\right)$$

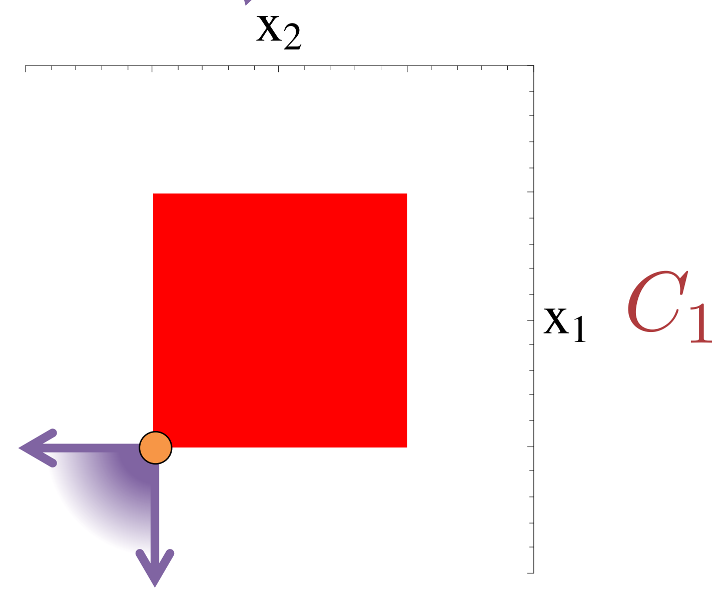
is face of $Q(H)$



Characterization Extends to Closed Convex Sets

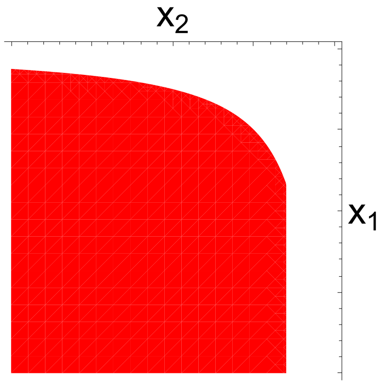




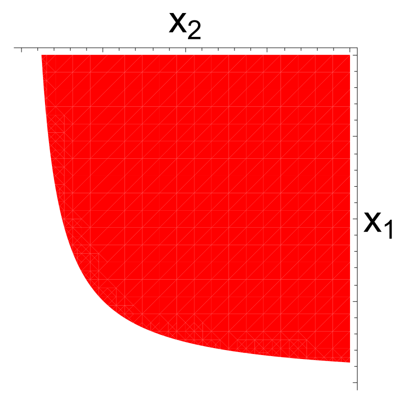


Small Formulations for Isotone Sets

- Studied by Hijazi et al. '12 and Bonami et al. '15 (n=1, 2):
  - $C_i = \{x \in \mathbb{R}^d \,:\, l^i \leq x \leq u^i, \quad f_i(x) \leq 0\}$
- $f_i(x)$ component-wise monotonous (i=1,2 opposite).



C_1



C_2

- Ideal Formulation

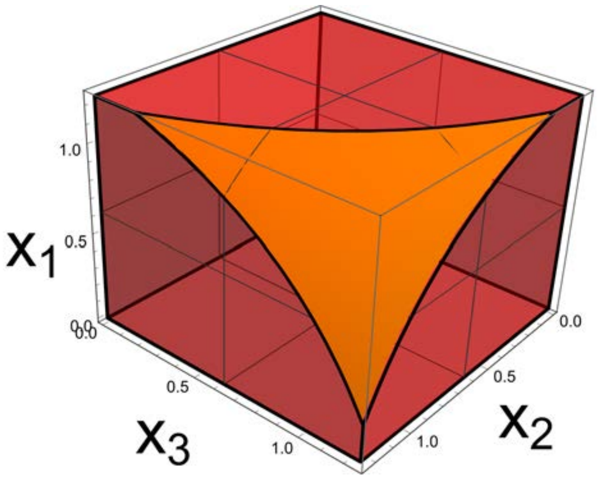
$$y_1 l^1 + y_2 l^2 \leq x \leq y_1 u^1 + y_2 u^2$$

$$f_J^i(x, y) \leq 0 \qquad \forall J \subseteq [d], i \in [2]$$

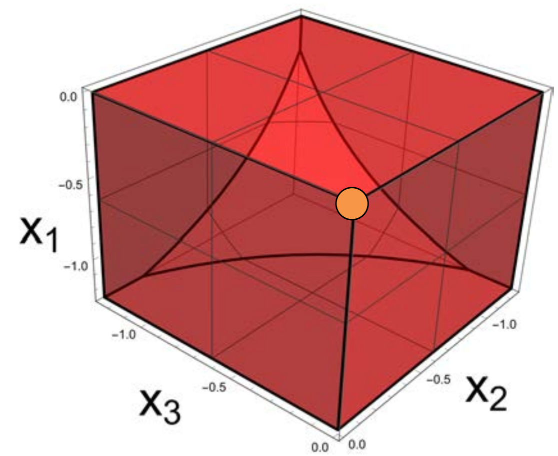
$$y_1 + y_2 = 1$$

$$y_i \in \{0, 1\} \qquad i \in [2]$$

Boundary Structure = Redundancy Detection



C_1



C_2

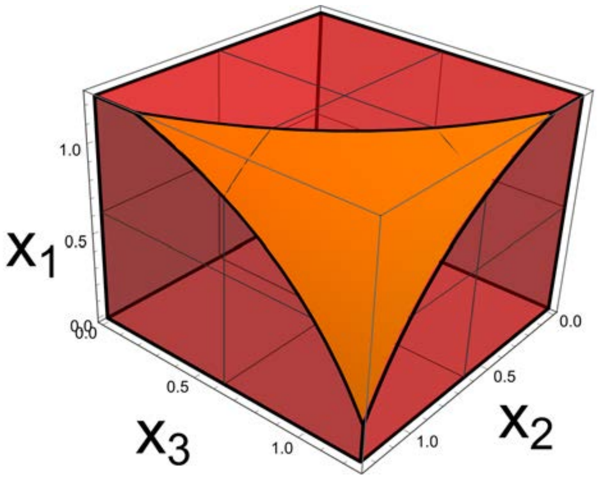
$$y_1 l^1 + y_2 l^2 \le x \le y_1 u^1 + y_2 u^2$$

$$f^i(x, y) \le 0 \qquad \forall i \in [2]$$

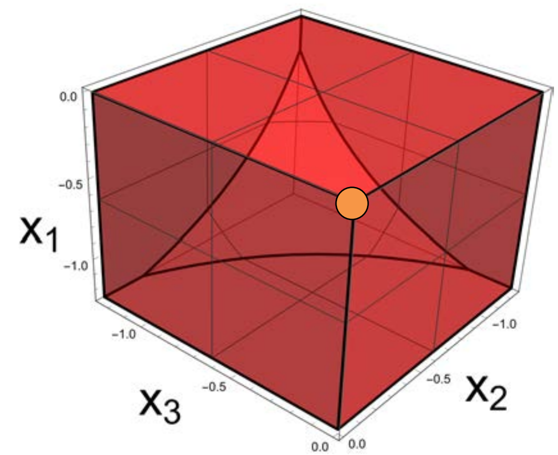
$$y_1 + y_2 = 1$$

$$y_i \ge \{0, 1\} \qquad i \in [2]$$

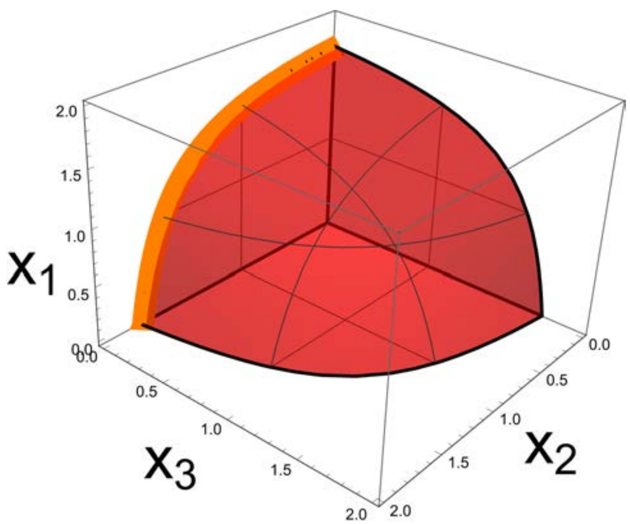
Boundary Structure = Redundancy Detection



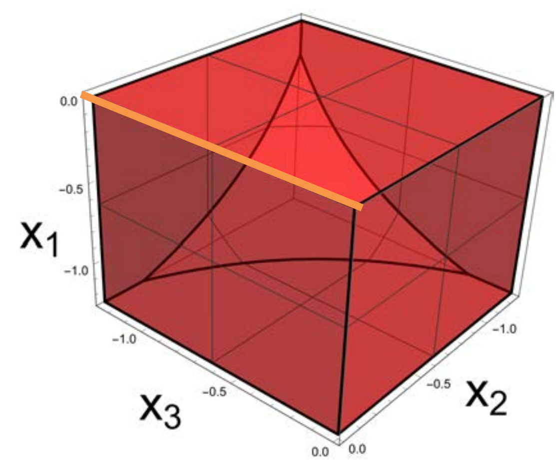
C_1



C_2



C_1

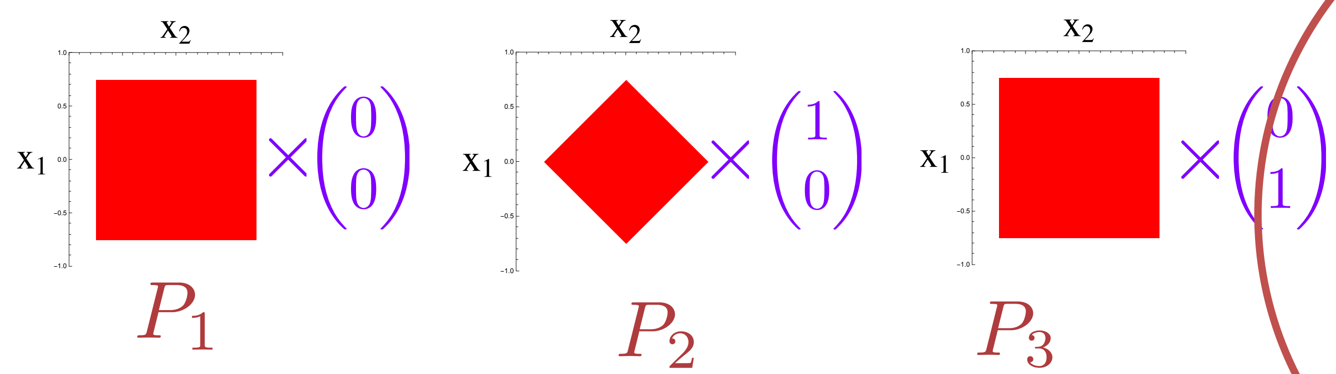


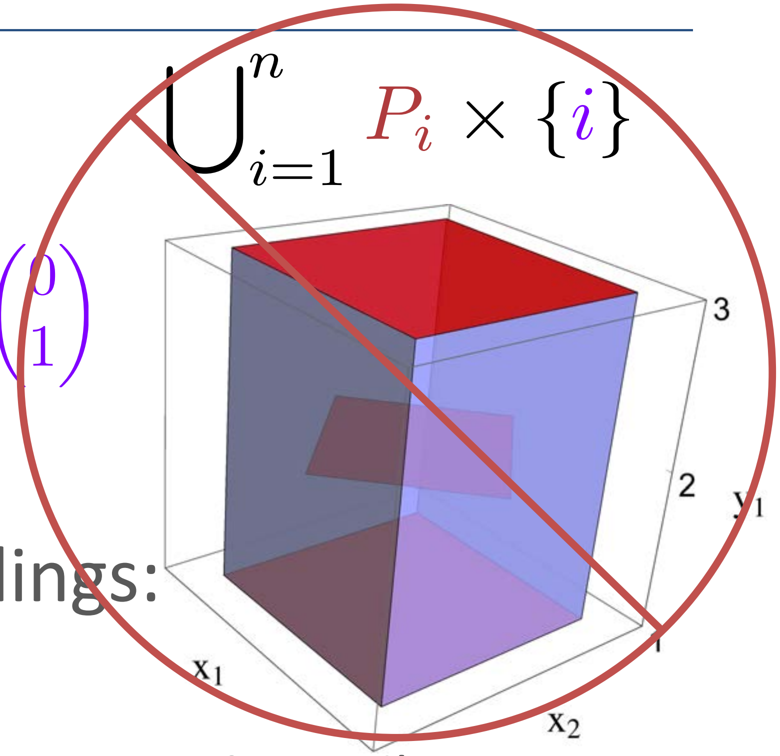
C_2

"Formulations" With a Fixed Number of General Integer Variables

Alternative Encodings

- “Only” use 0-1 encodings ?





- General integer (rational?) encodings:
  - Points in convex position
  - Recover convex sets by (possibly non-axis aligned) sections
  - Example: Integers in moment curve
  - For SOS2: Hyperplane characterization still works
    - 2-dim moment curve = 2(n-1) general inequalities
    - More bounds soon (with Joey Huchette)

Summary

- Embedding Formulations = Systematic procedure for ideal non-extended formulations
  - Encoding can significantly affect size
  - Results beyond SOS2, but many open questions
- Extension to General Convex Sets
  - Can yield practical formulations
  - Not always practical (basic semi-algebraic representability)
- Using General Integer Variables
  - Smaller formulations not likely (in general)
  - General convex MIP representability (w. M. Lubin and I. Zadik):
    - The set of prime numbers is not convex MIP representable