Cuts for Nonlinear MIP: Closed Form Expressions and Extended Formulations

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Outline

- Introduction
- Two Techniques
- Summary

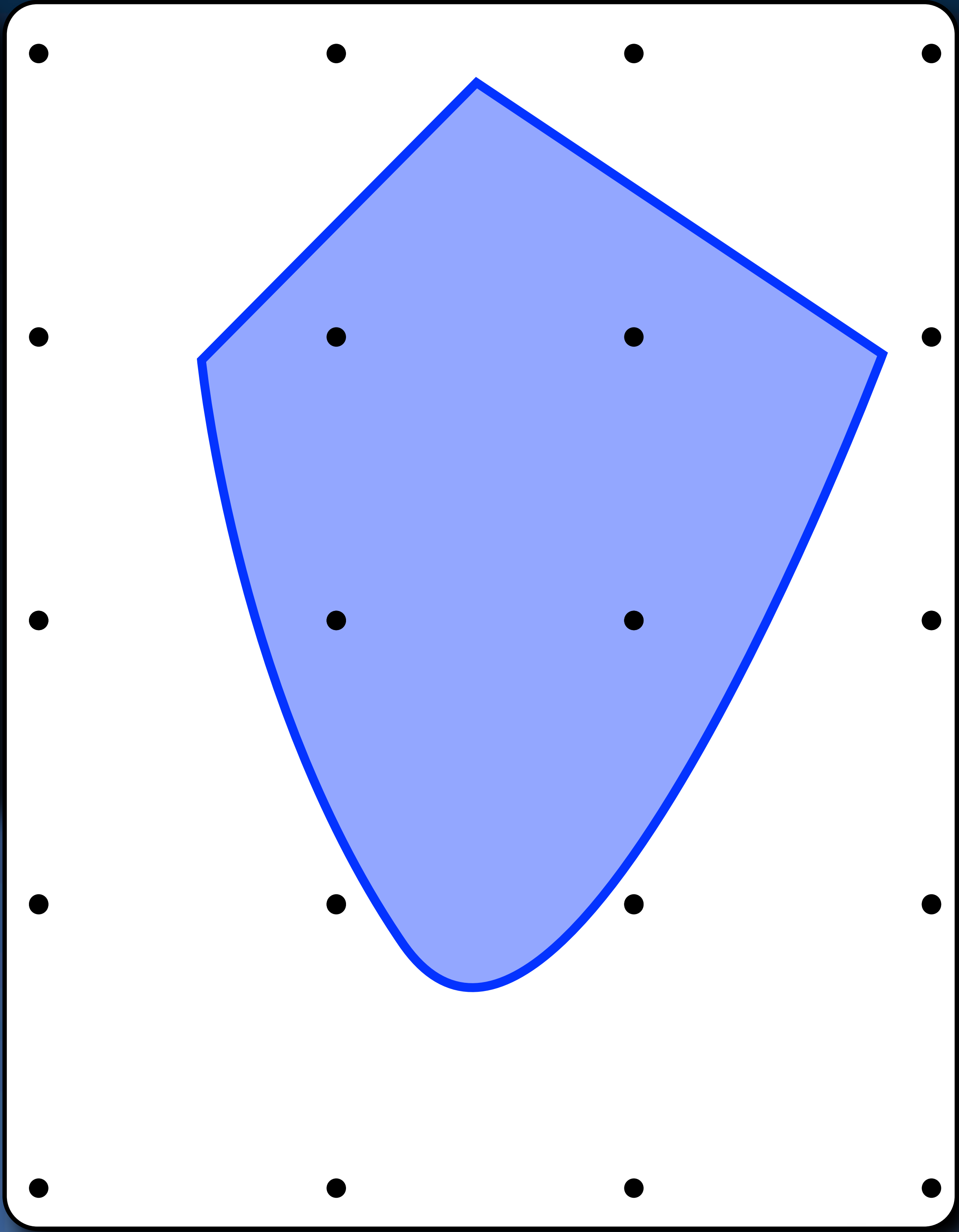
"Intersection" Cuts:

Continuous Relaxation

$C := \{x \in \mathbb{R}^n \; : \; f(x) \leq 0\}$

Open Set S without "interesting" points e.g. $S \cap \mathbb{Z}^n = \emptyset$

$$\operatorname{conv}(C \setminus S) = \{x \; : \; f(x) \leq 0, \; g_j(x) \leq 0, \; j \in J\}$$



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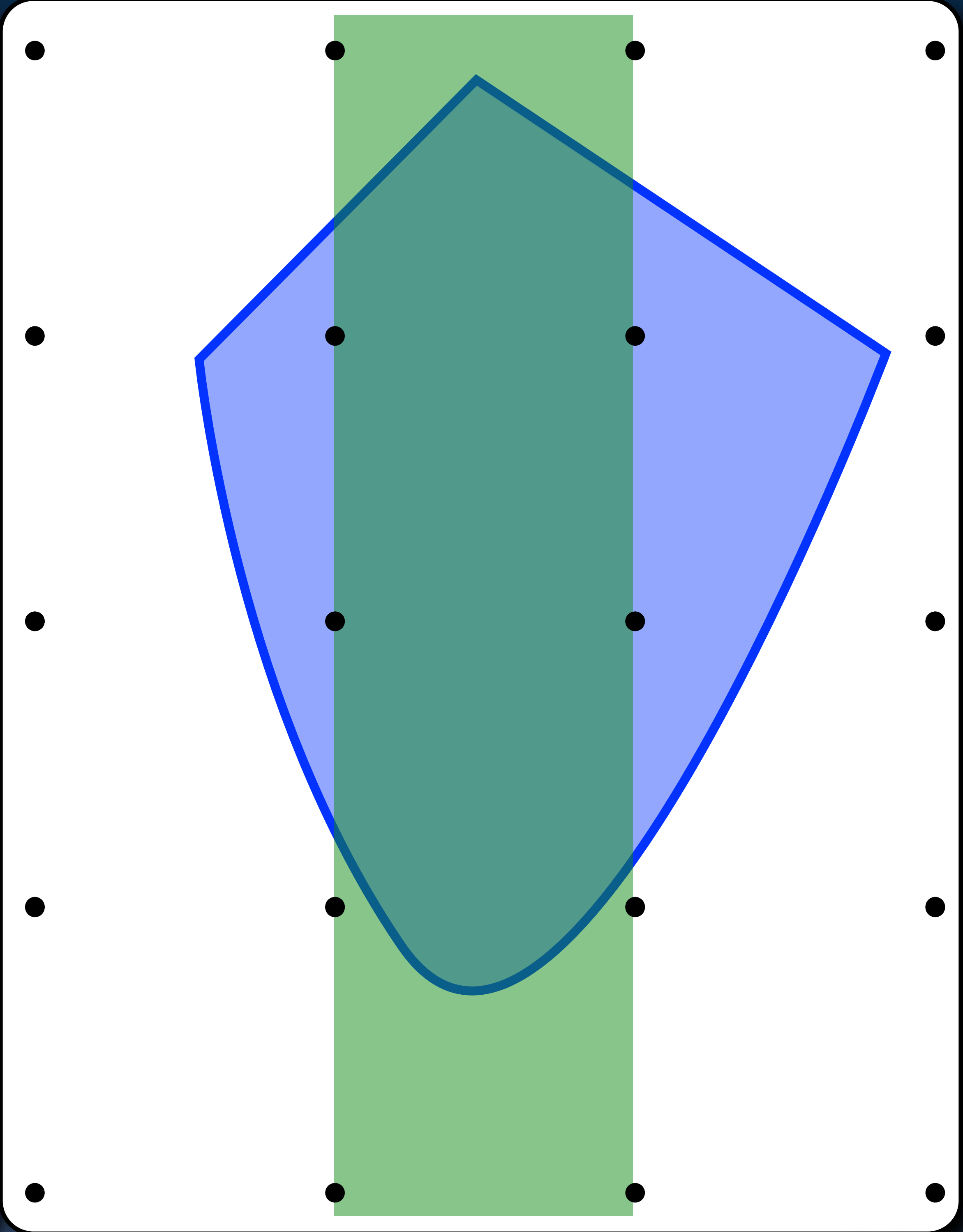
Split Disjunction

$$S = \{x \in \mathbb{R}^n \ : \ \pi_0 < \pi^T x < \pi_1\}$$

$$\pi \in \mathbb{Z}^n, \quad \pi_0 \in \mathbb{Z}, \quad \pi_1 = \pi_0 + 1$$



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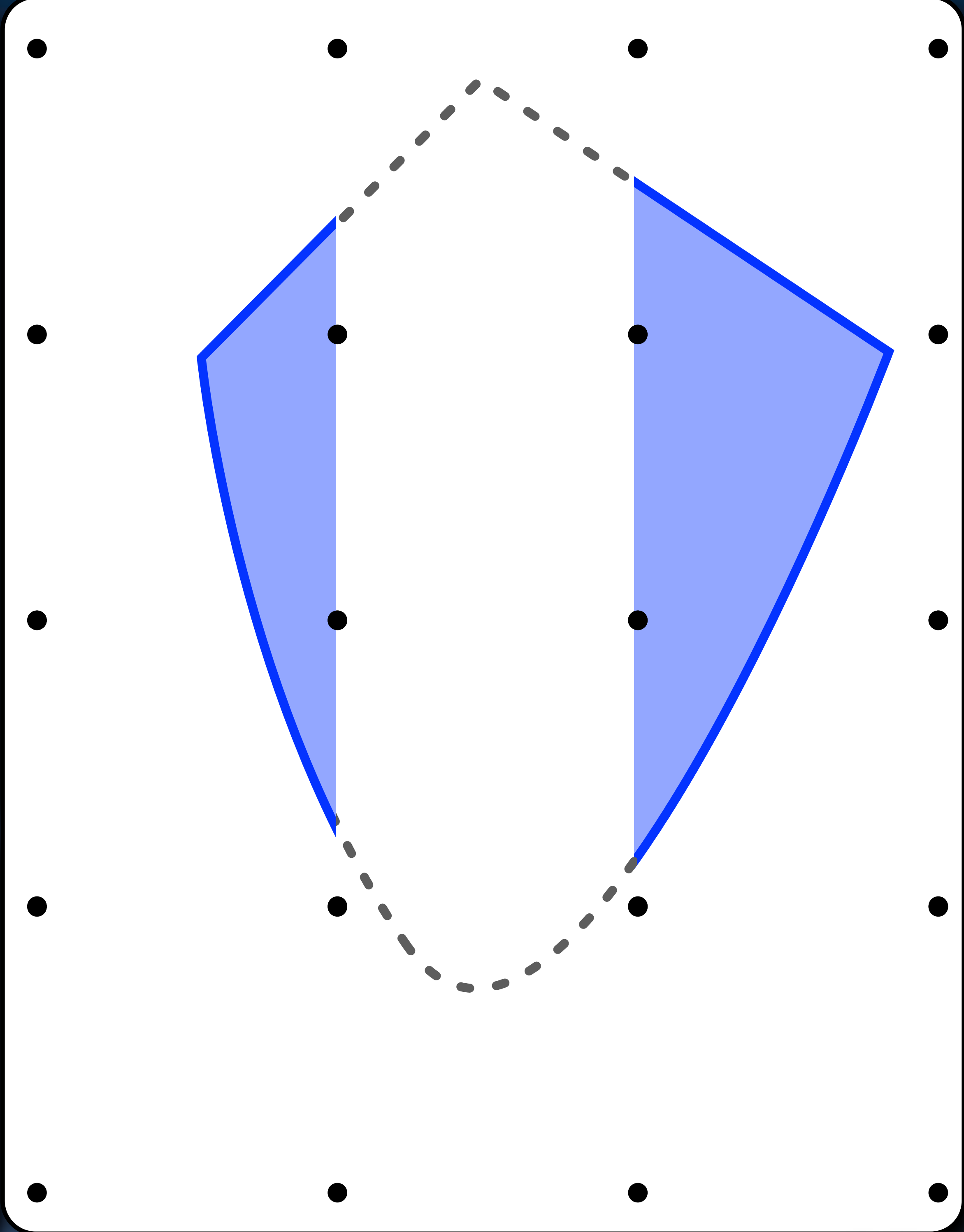

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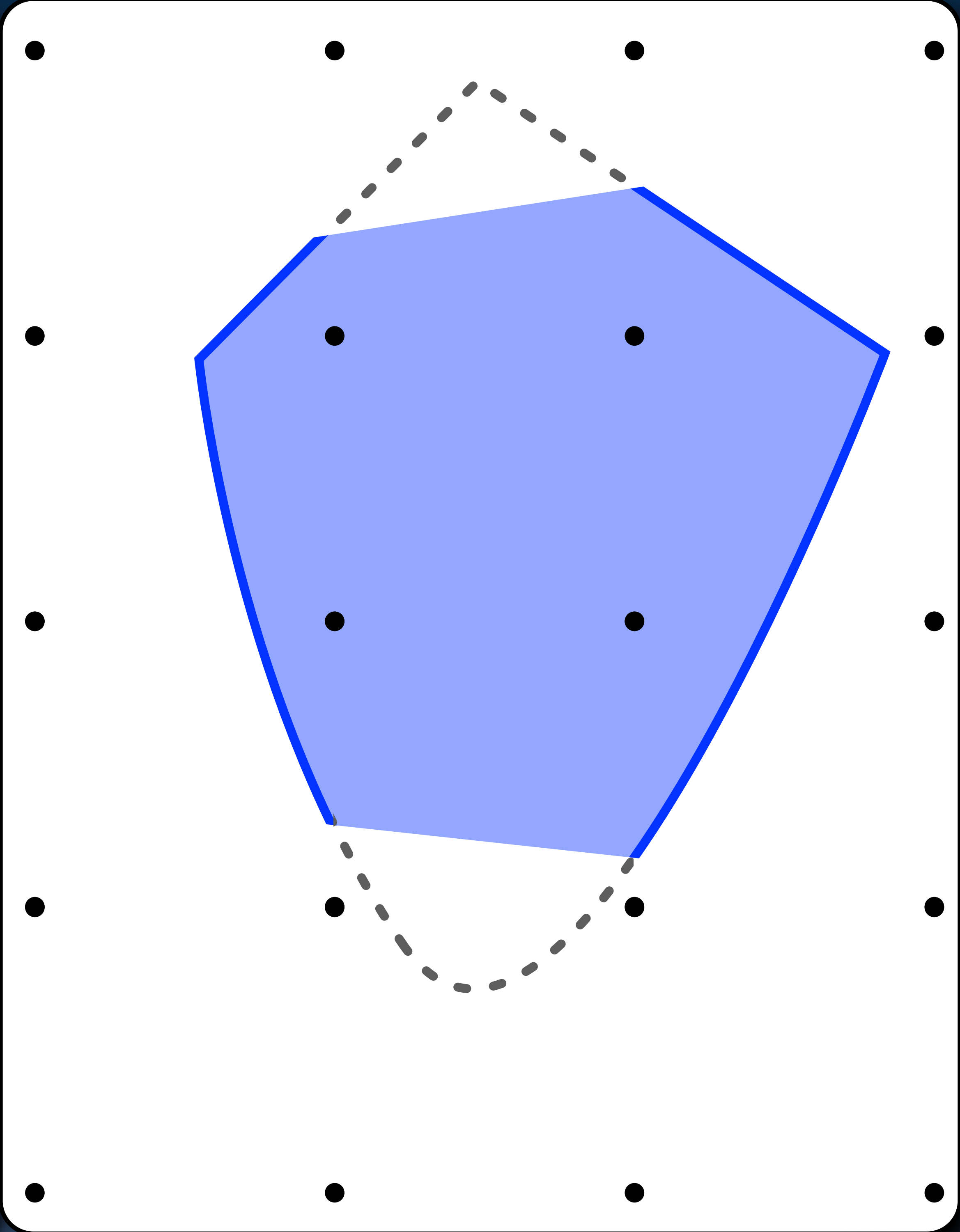

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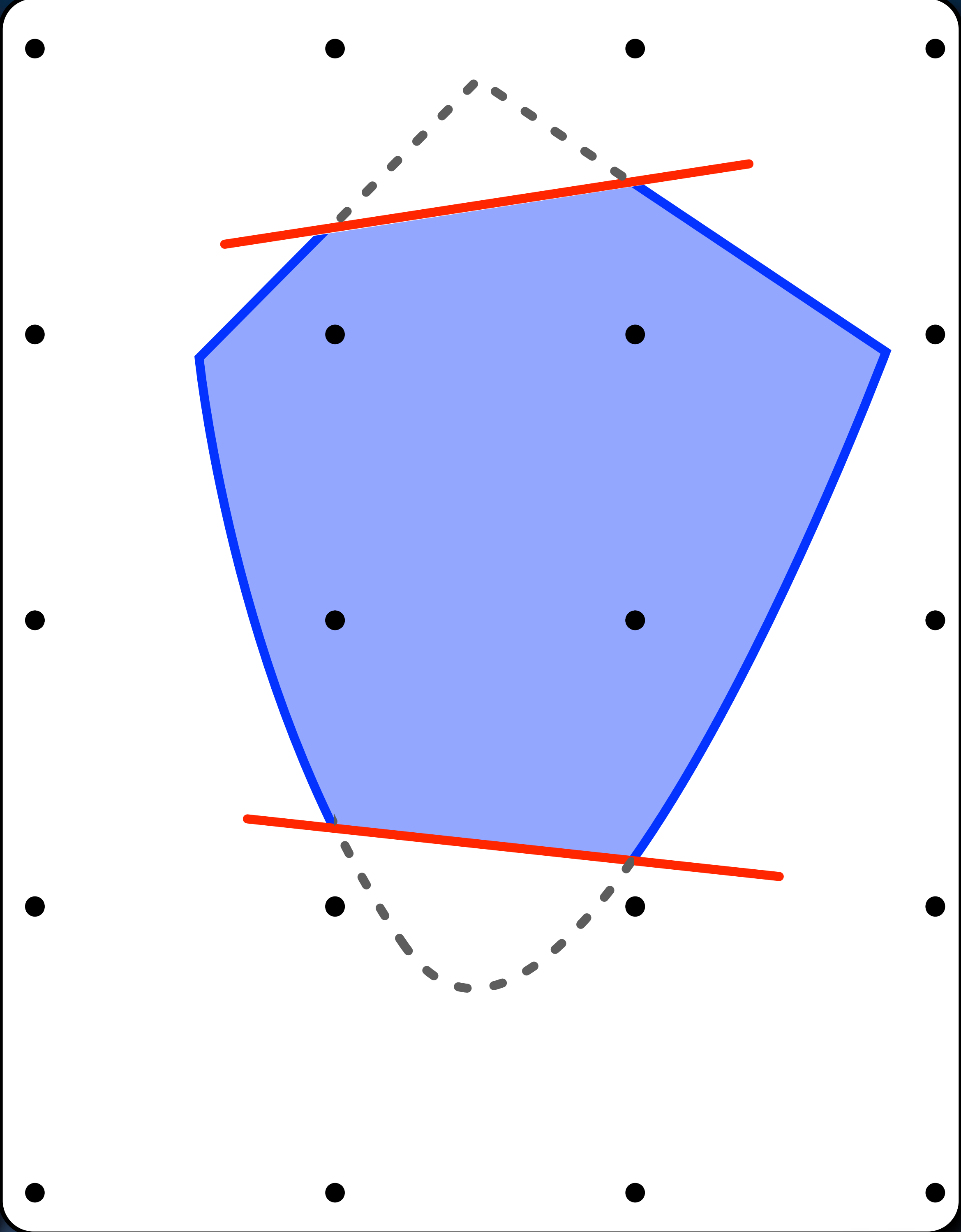

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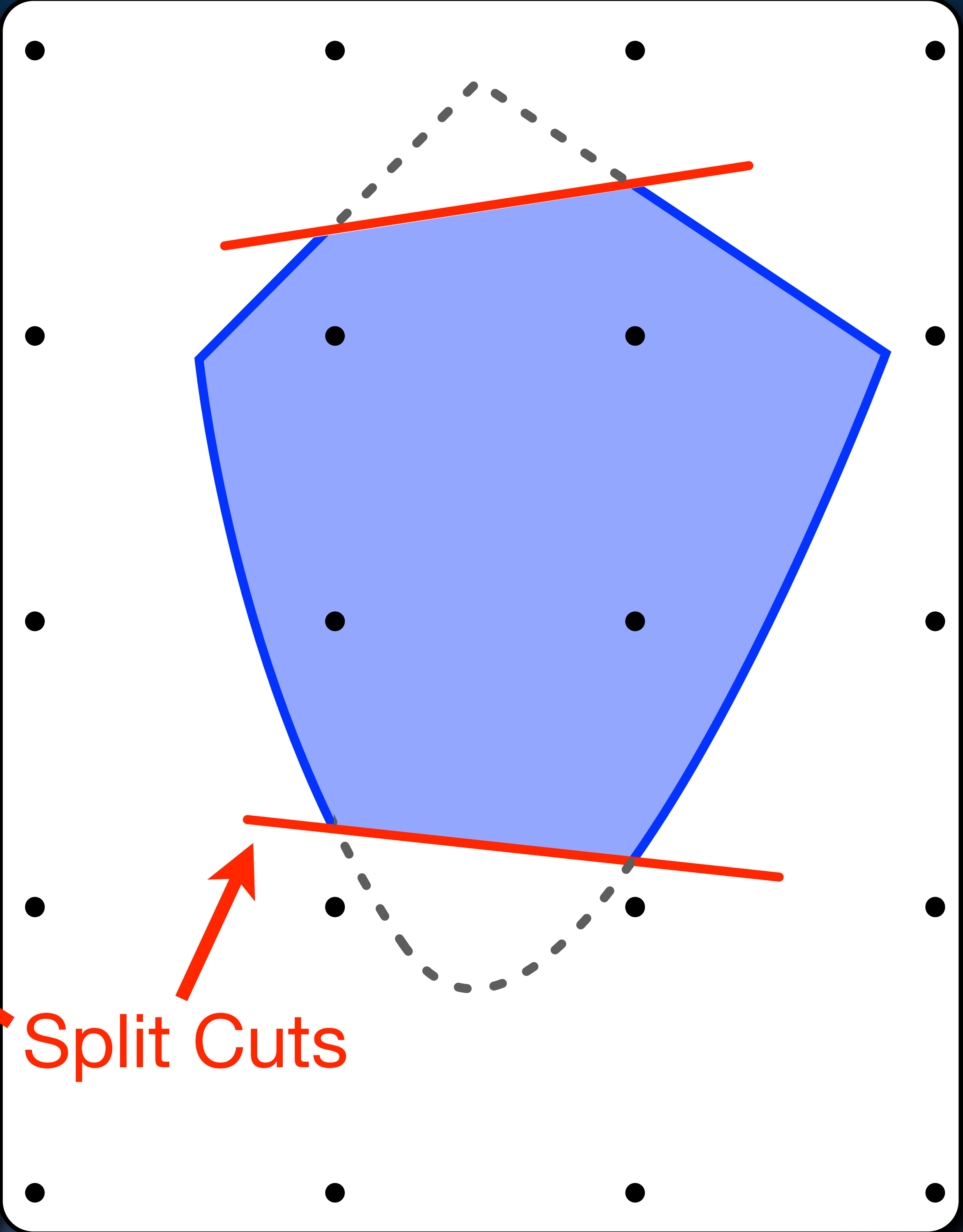

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$\operatorname{conv}(C \setminus S) =$

- "$S = \operatorname{int} S$"
- $C_{\pi,\pi_0} := \operatorname{conv}(C \setminus S)$

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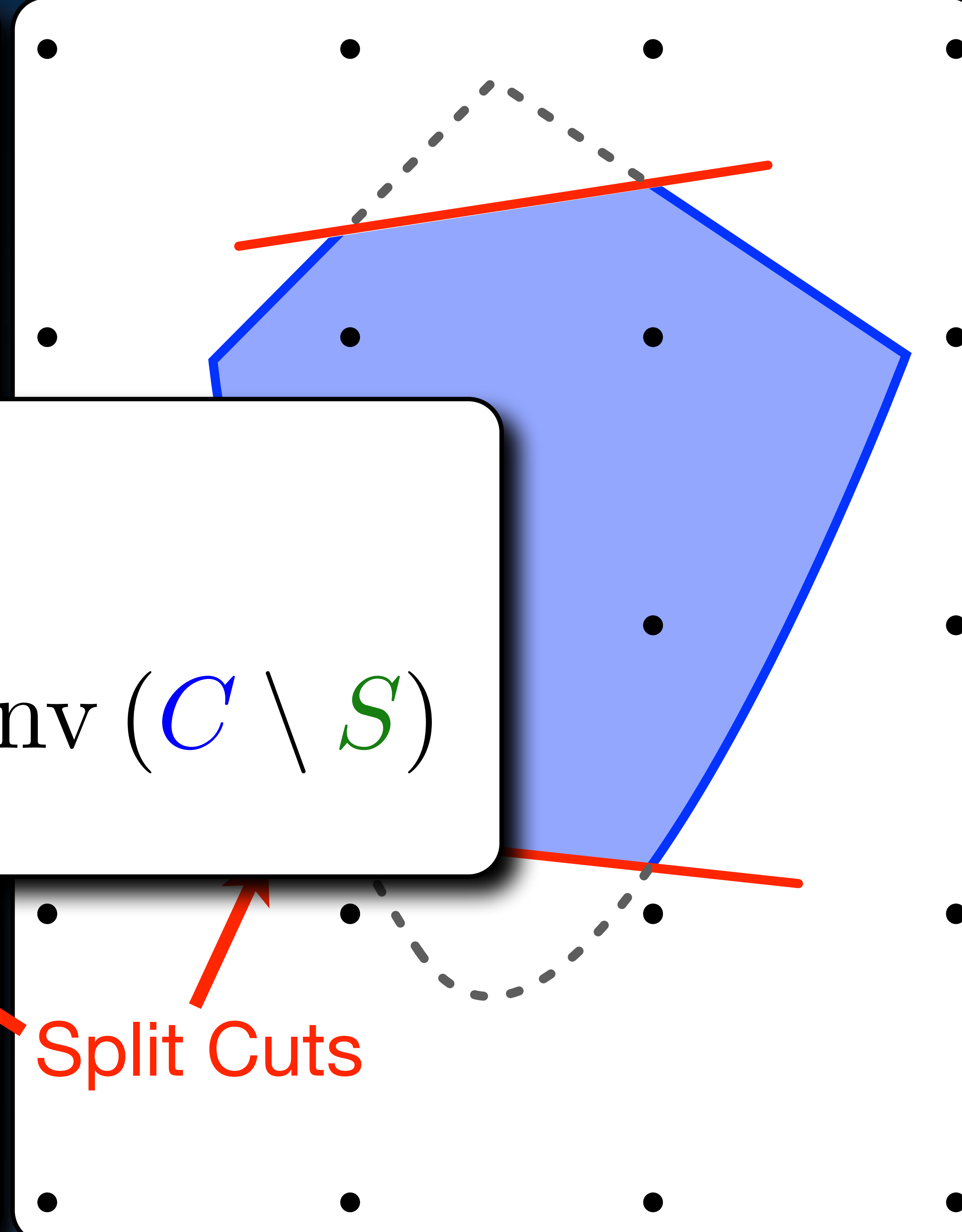

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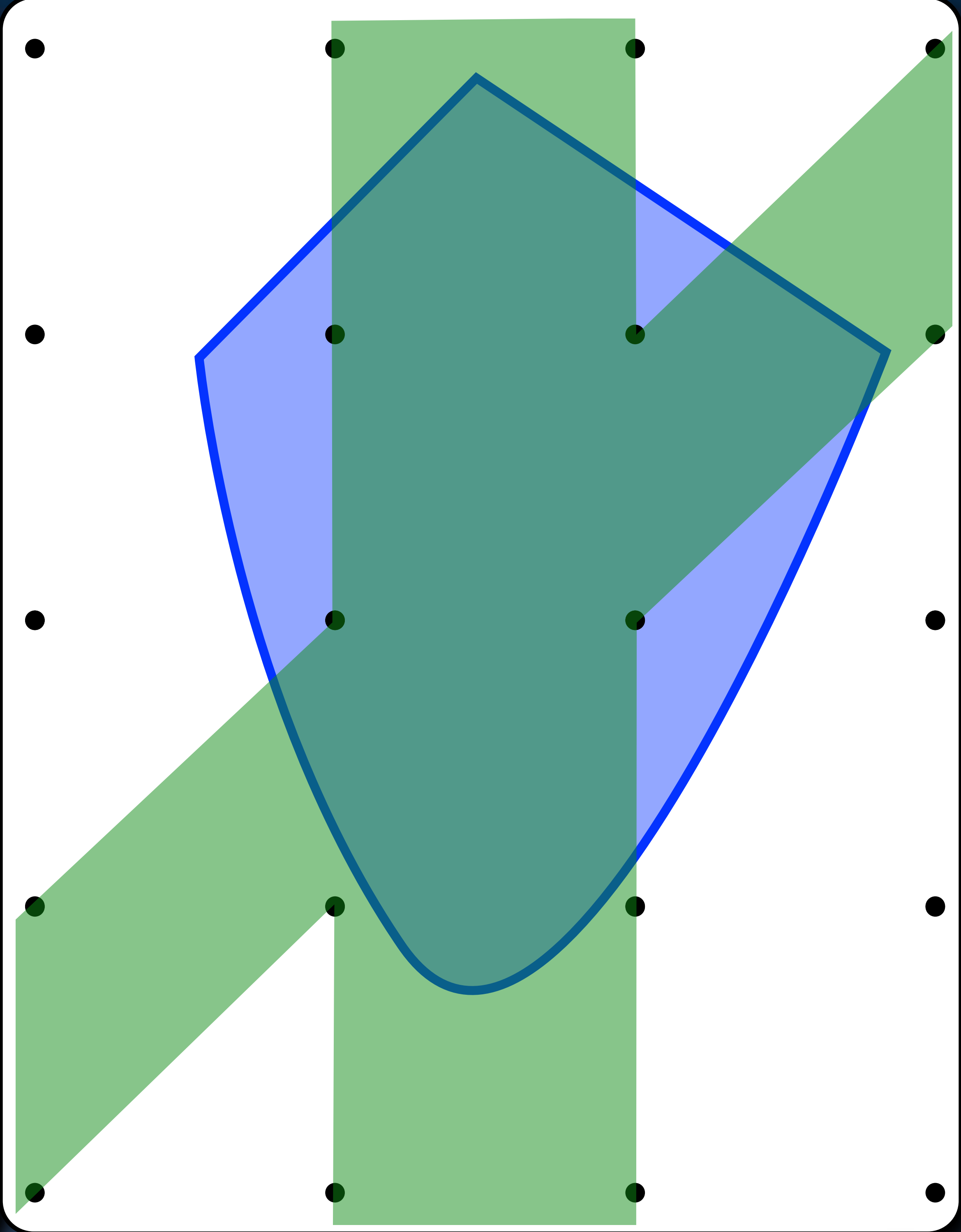
Multi-branch Split Disjunctions

$$S = \bigcup_{i \in I} \{x \;:\; \pi_0^i < \pi_i^T x < \pi_1^i\}$$

$$I \subseteq [n] := \{1, \ldots, n\}$$



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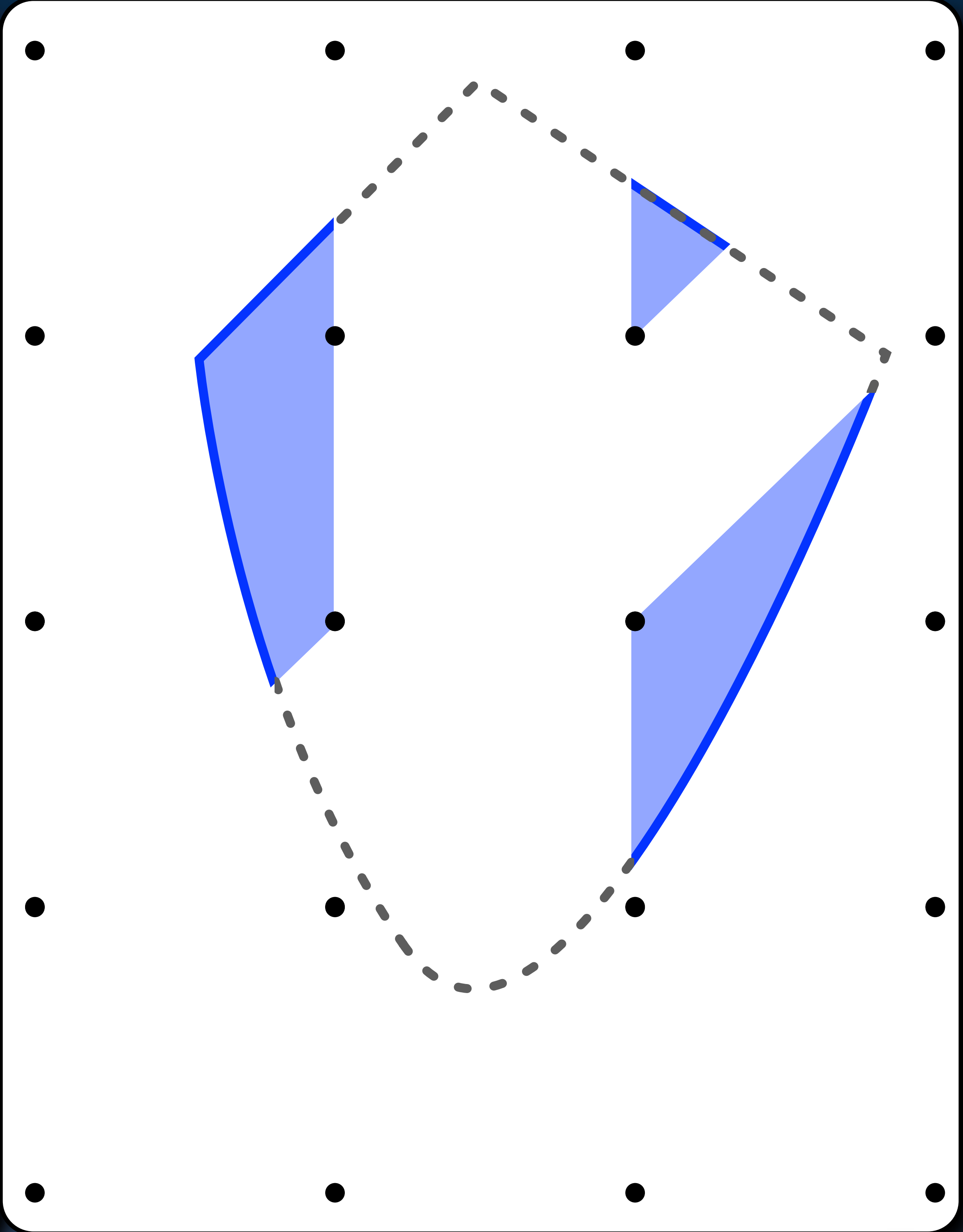

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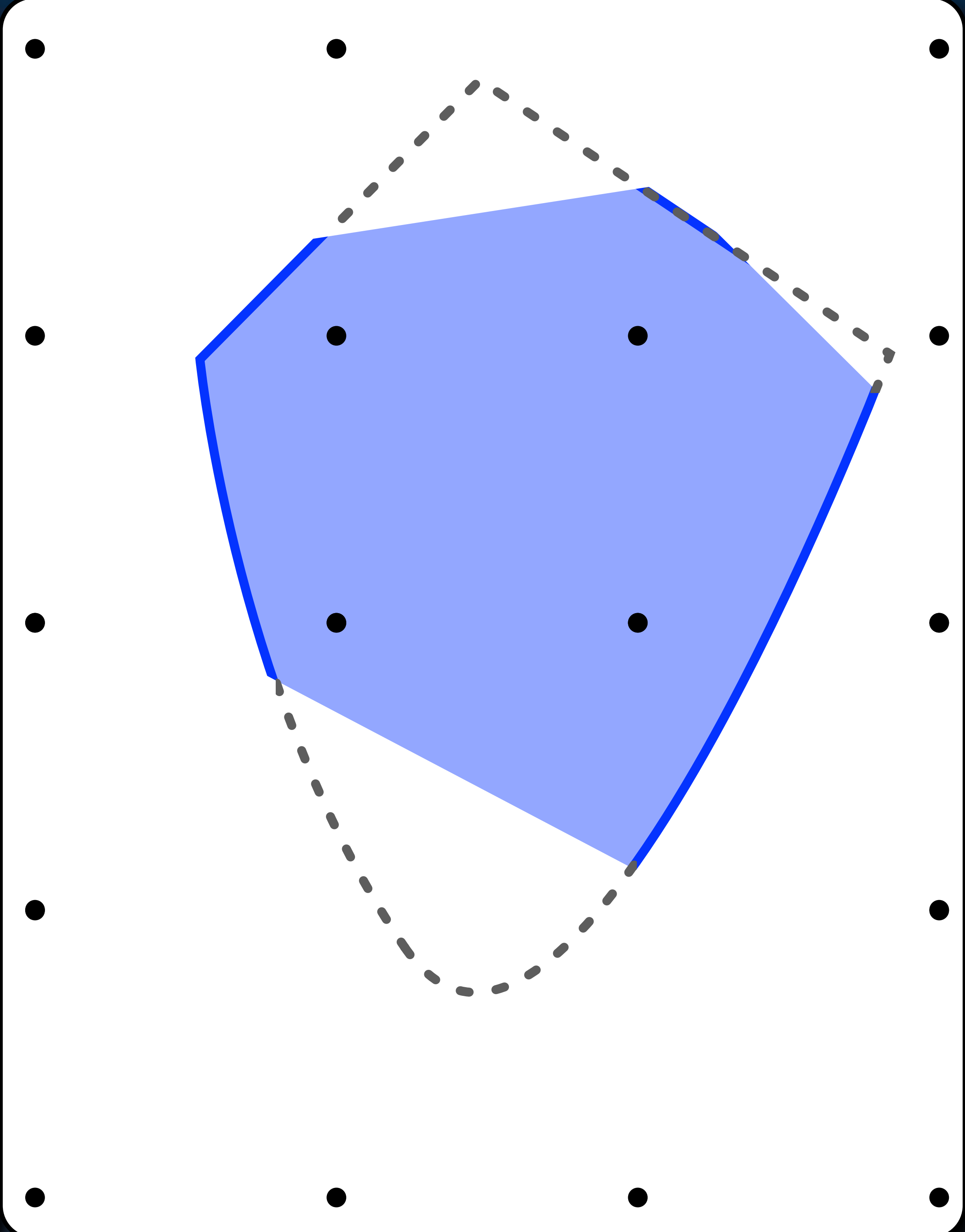

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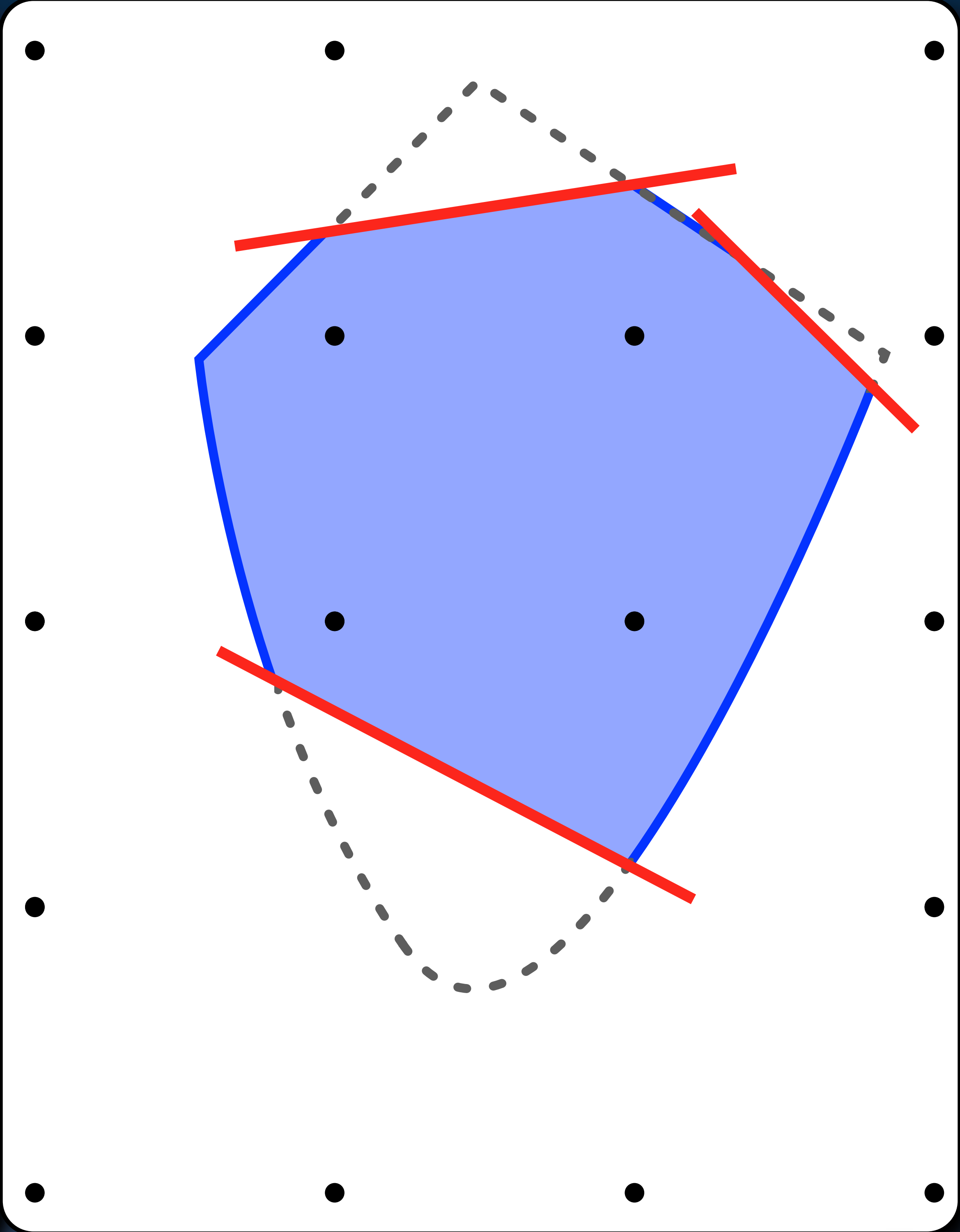
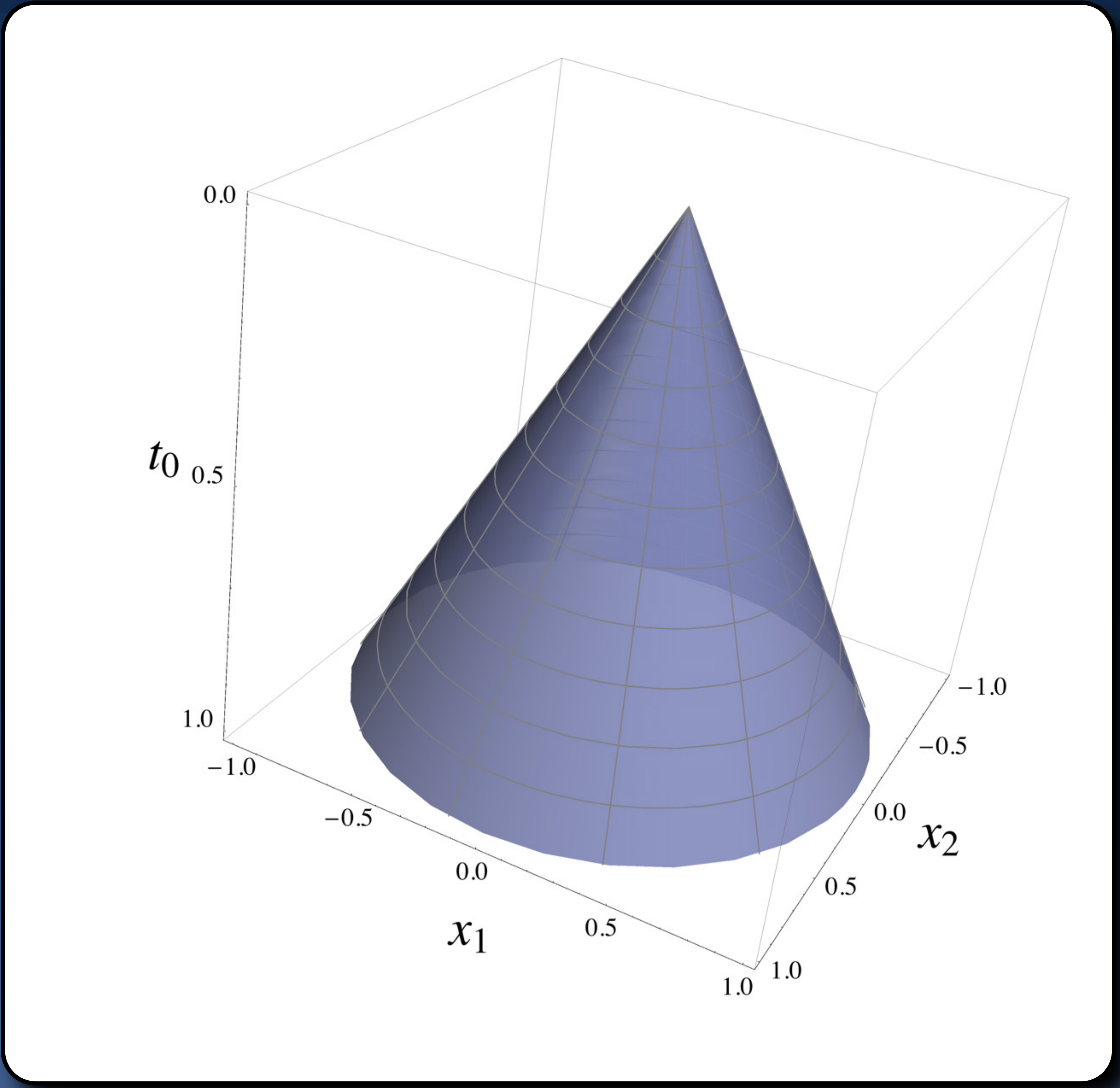
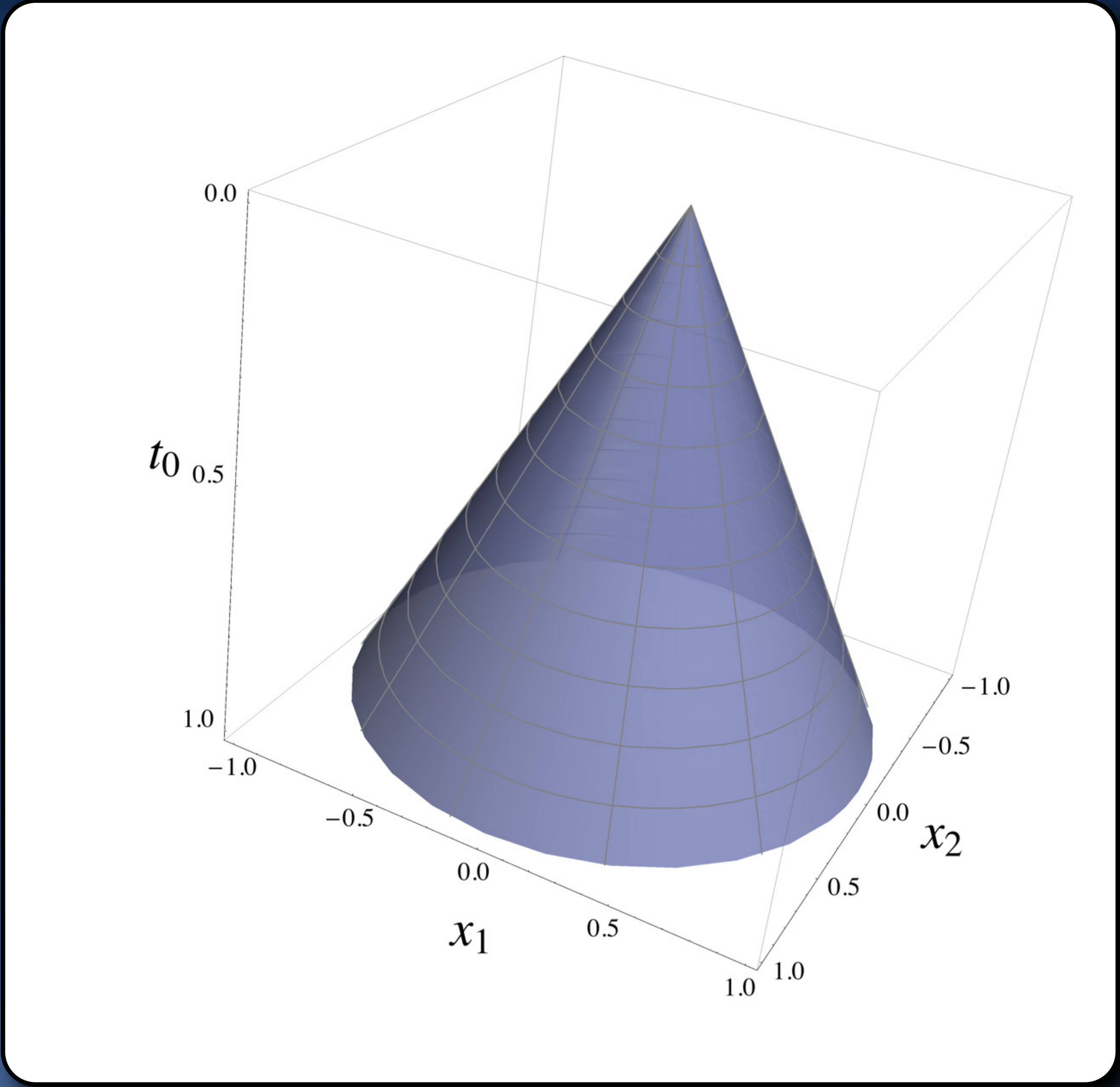


Cuts Can be Non-linear



Cuts Can be Non-linear

- Split Cuts for Cones



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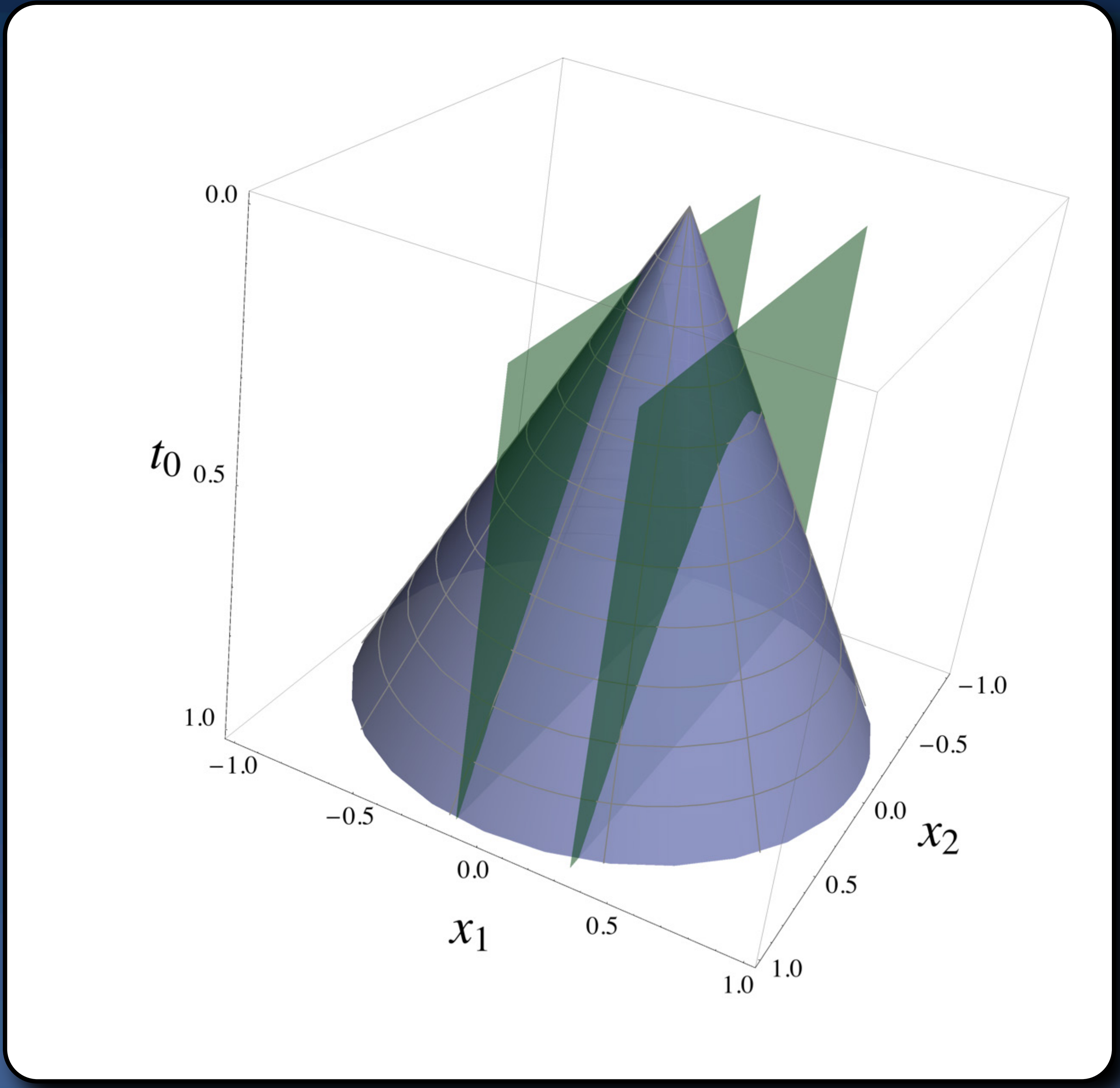

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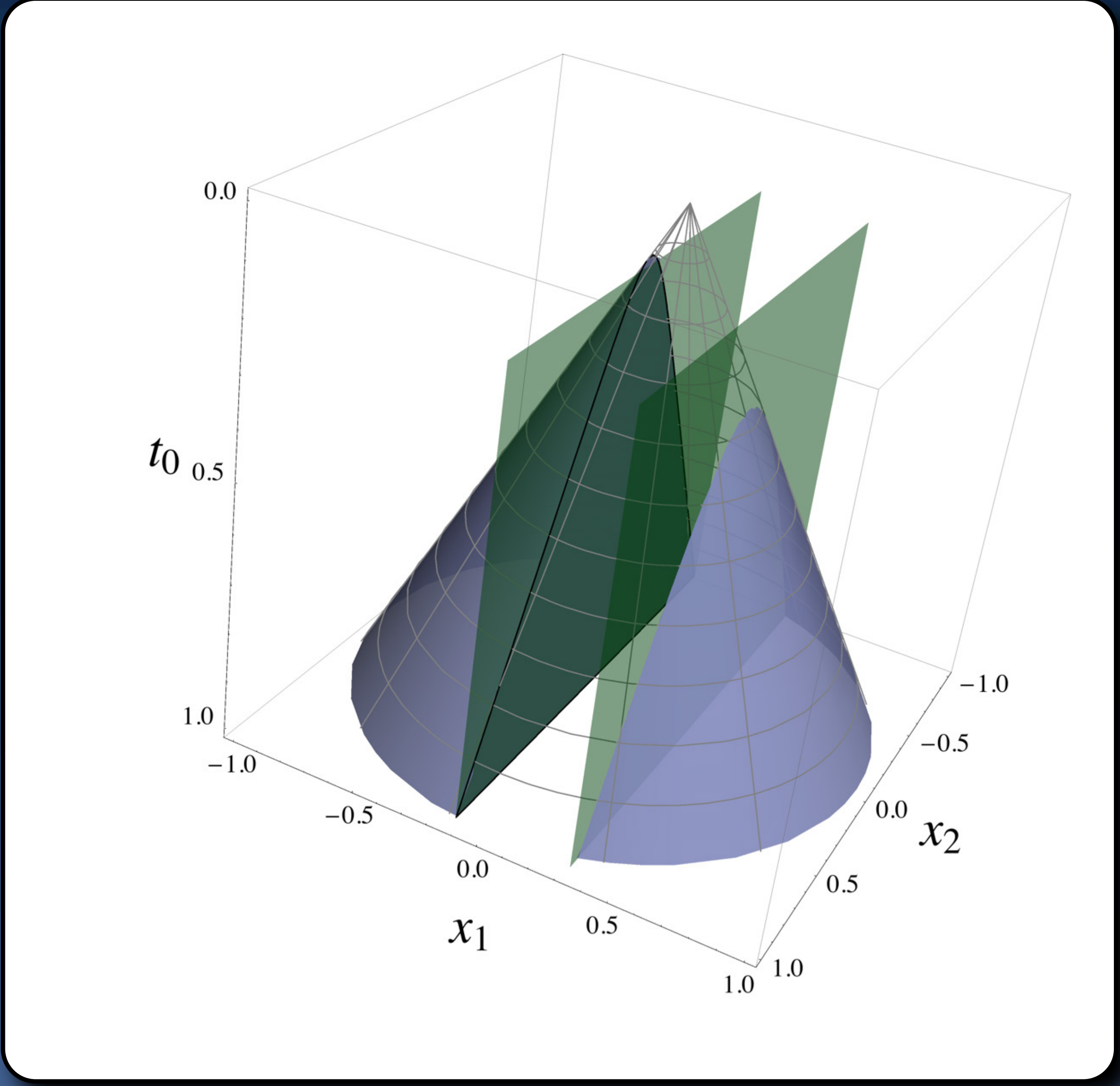

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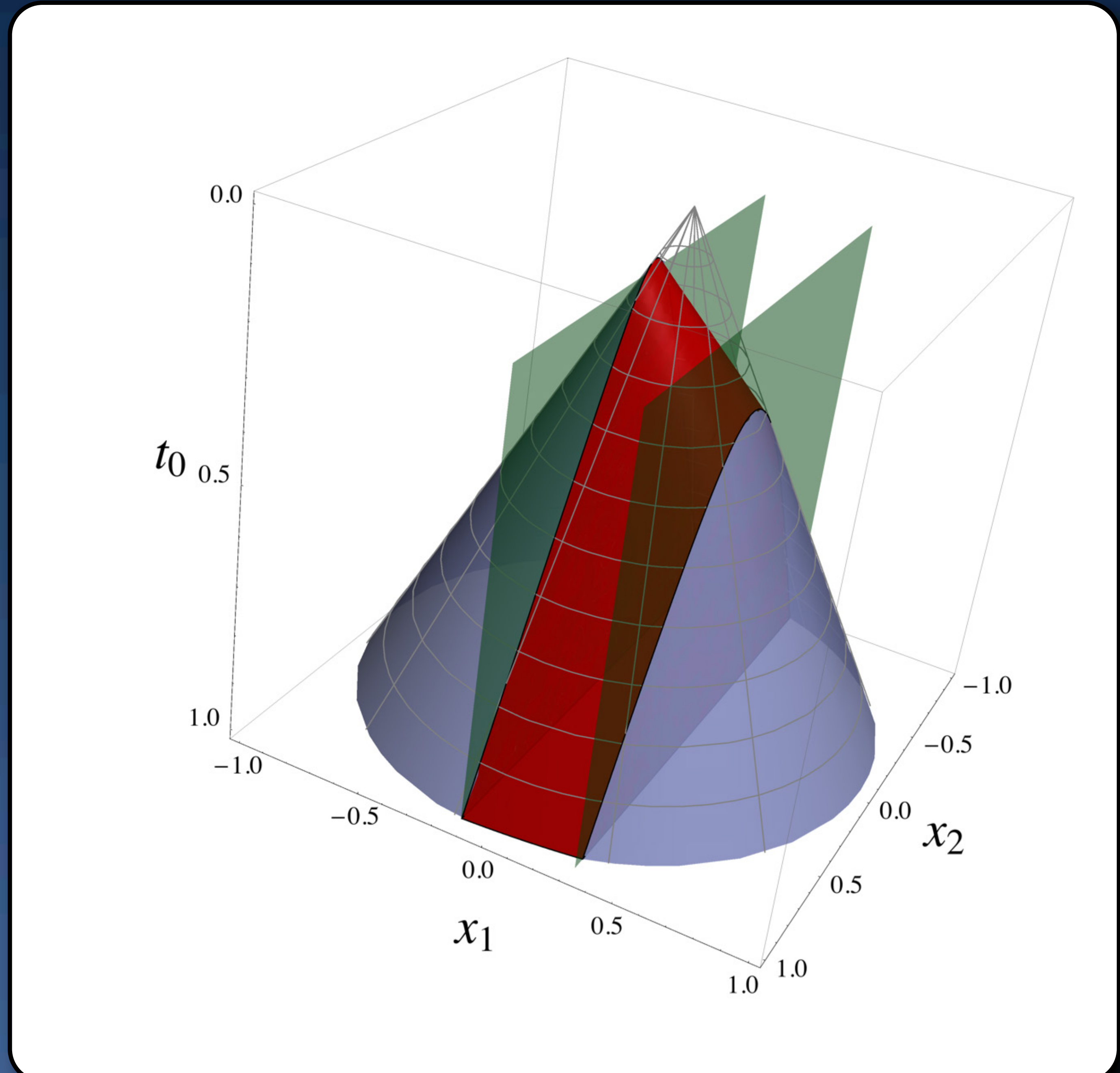

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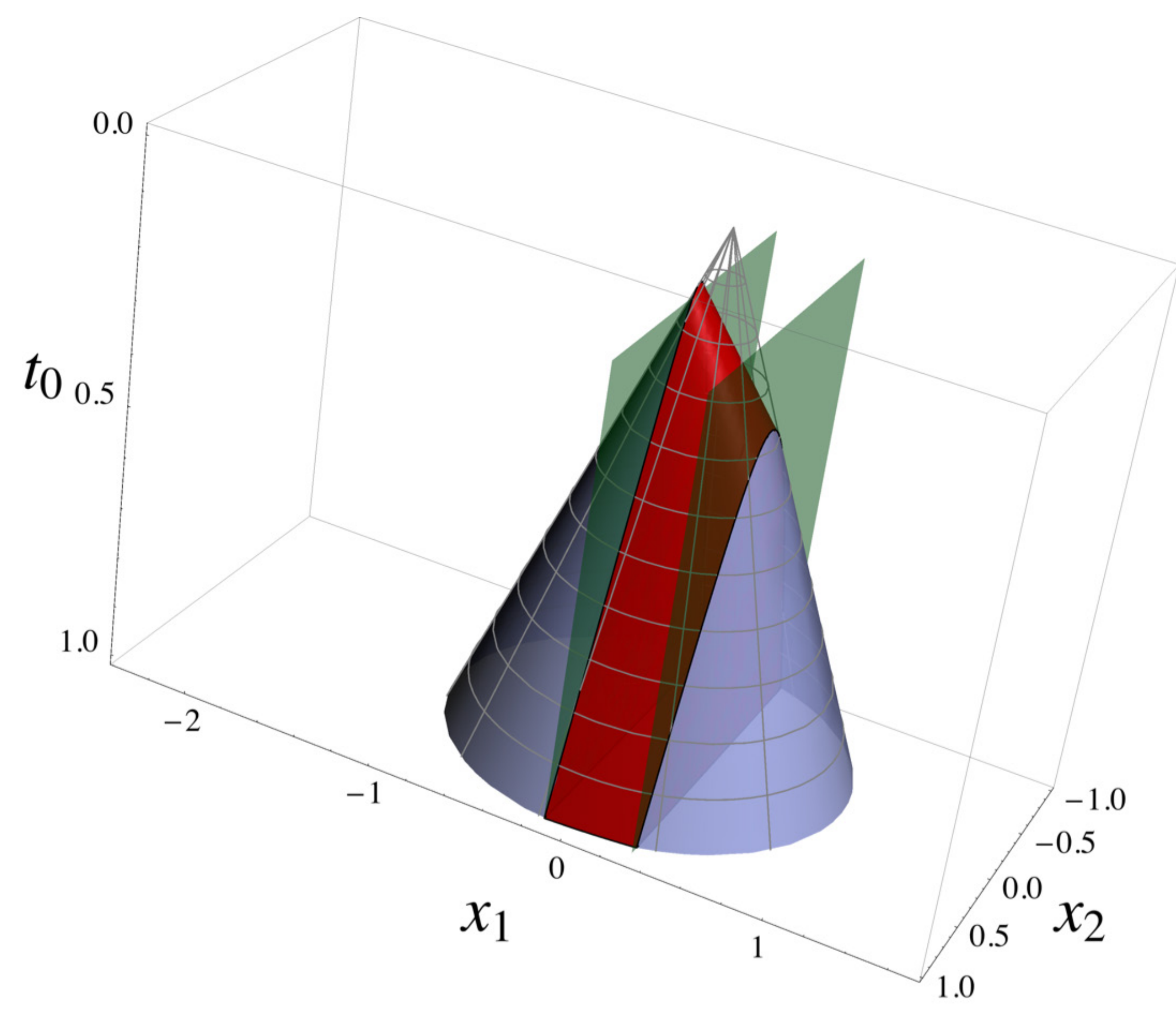

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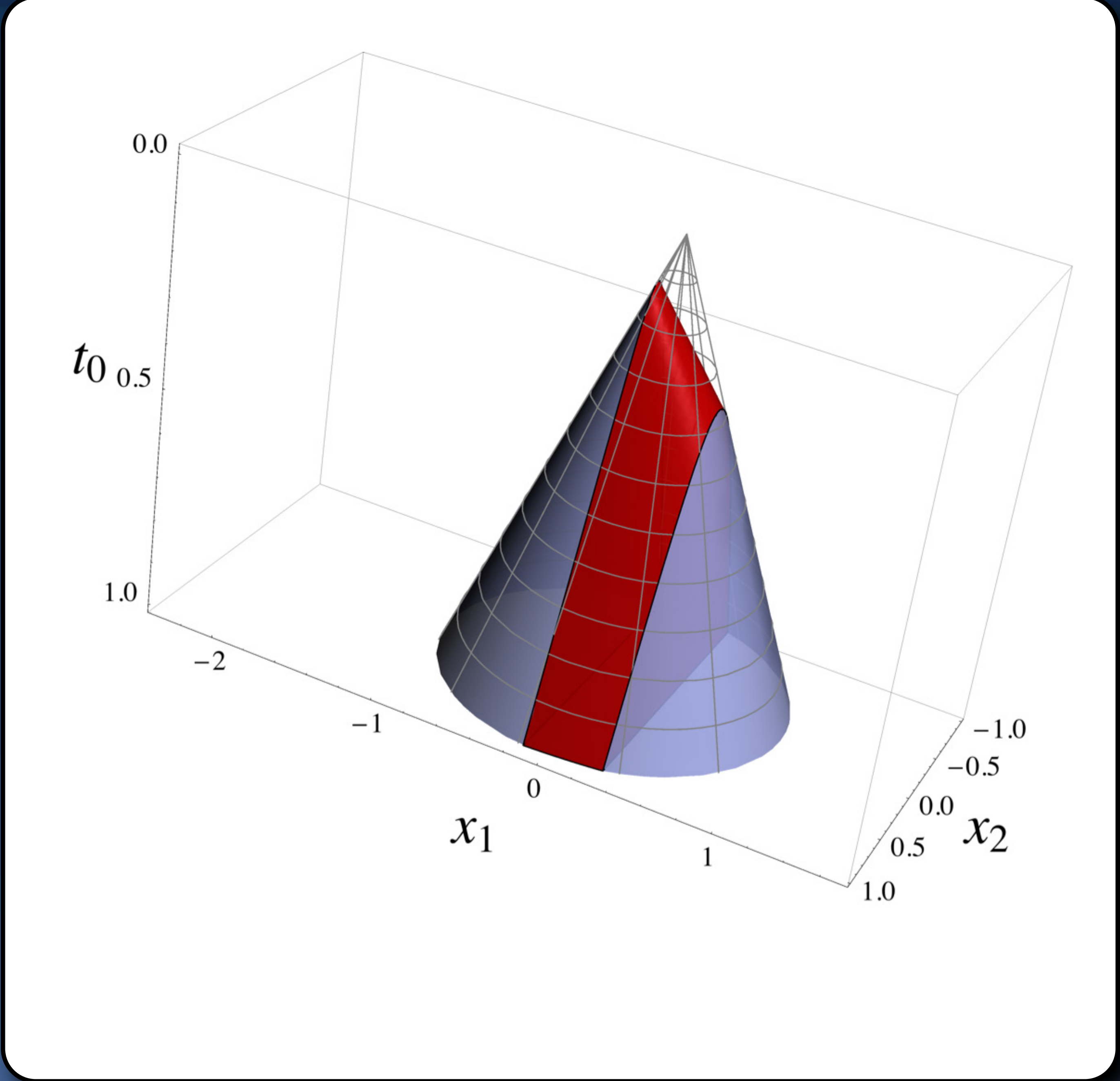

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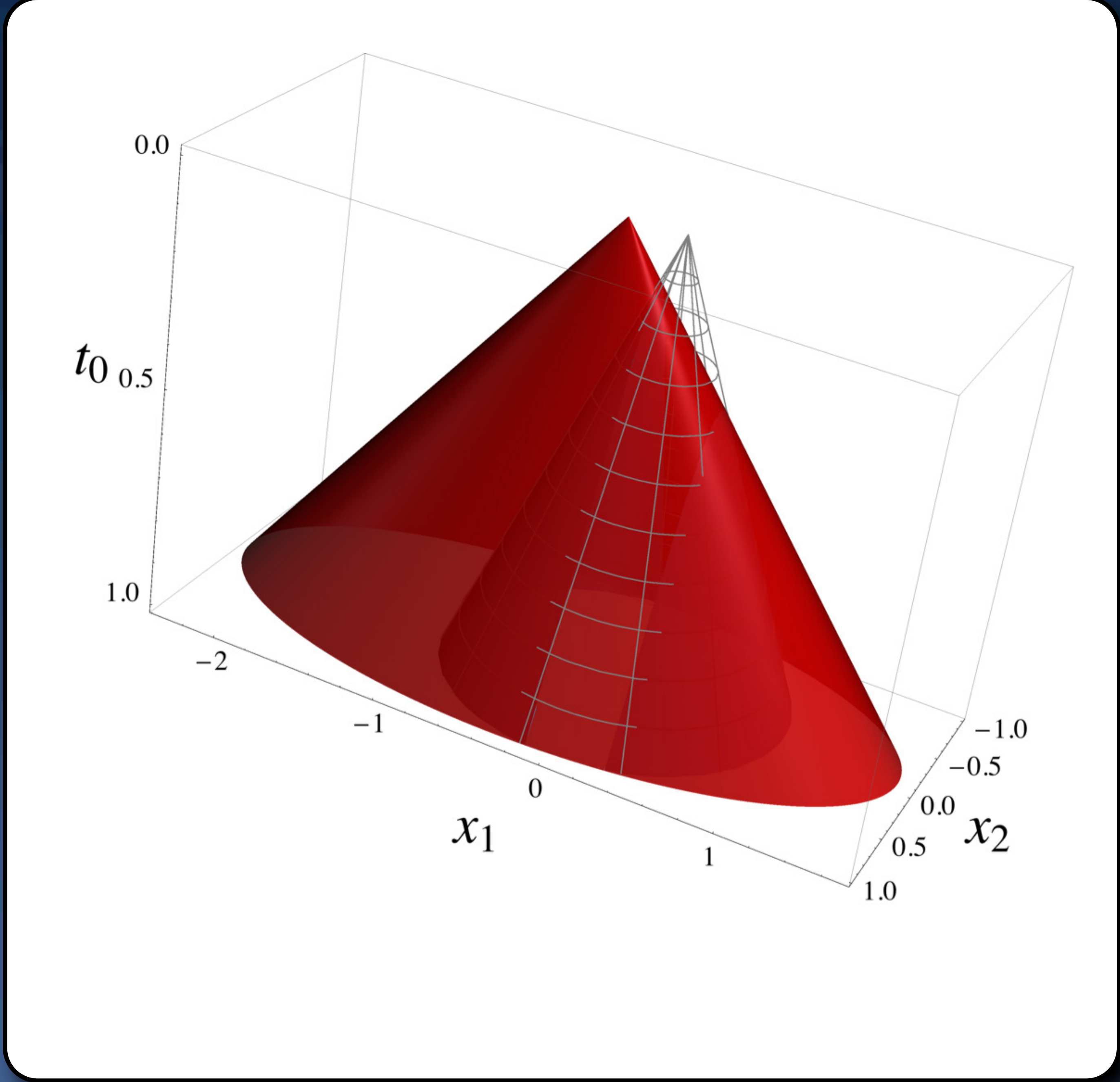

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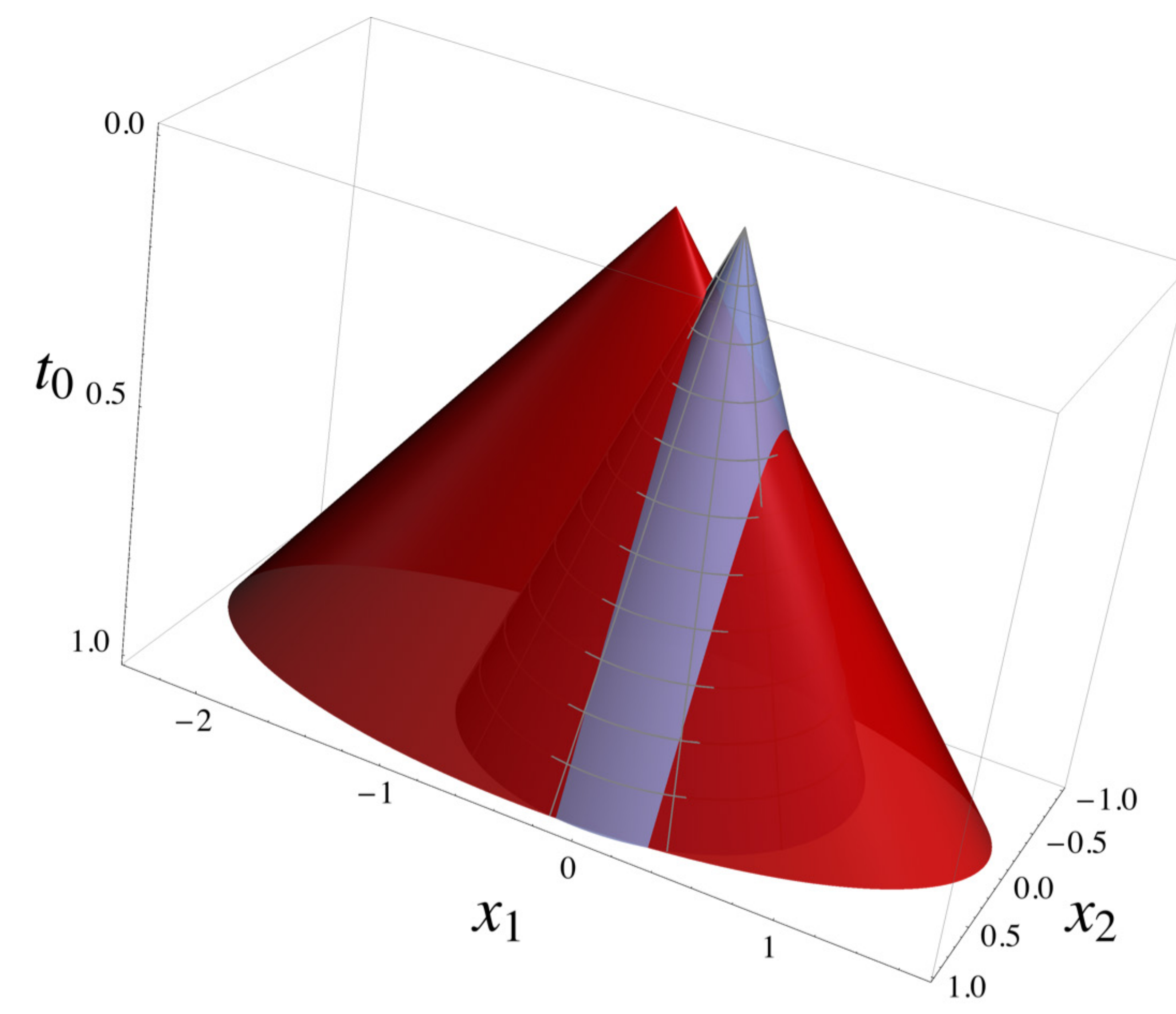

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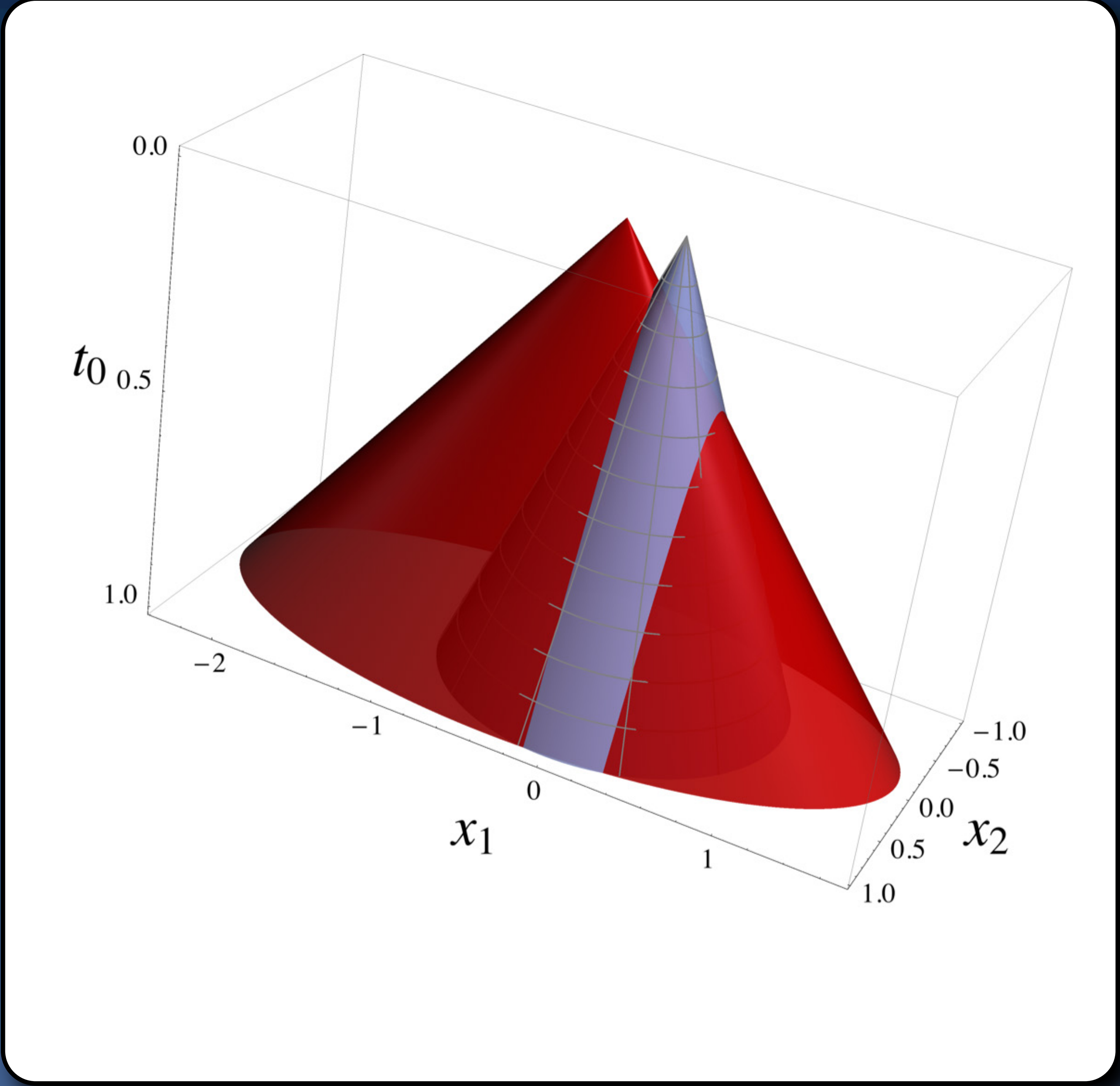

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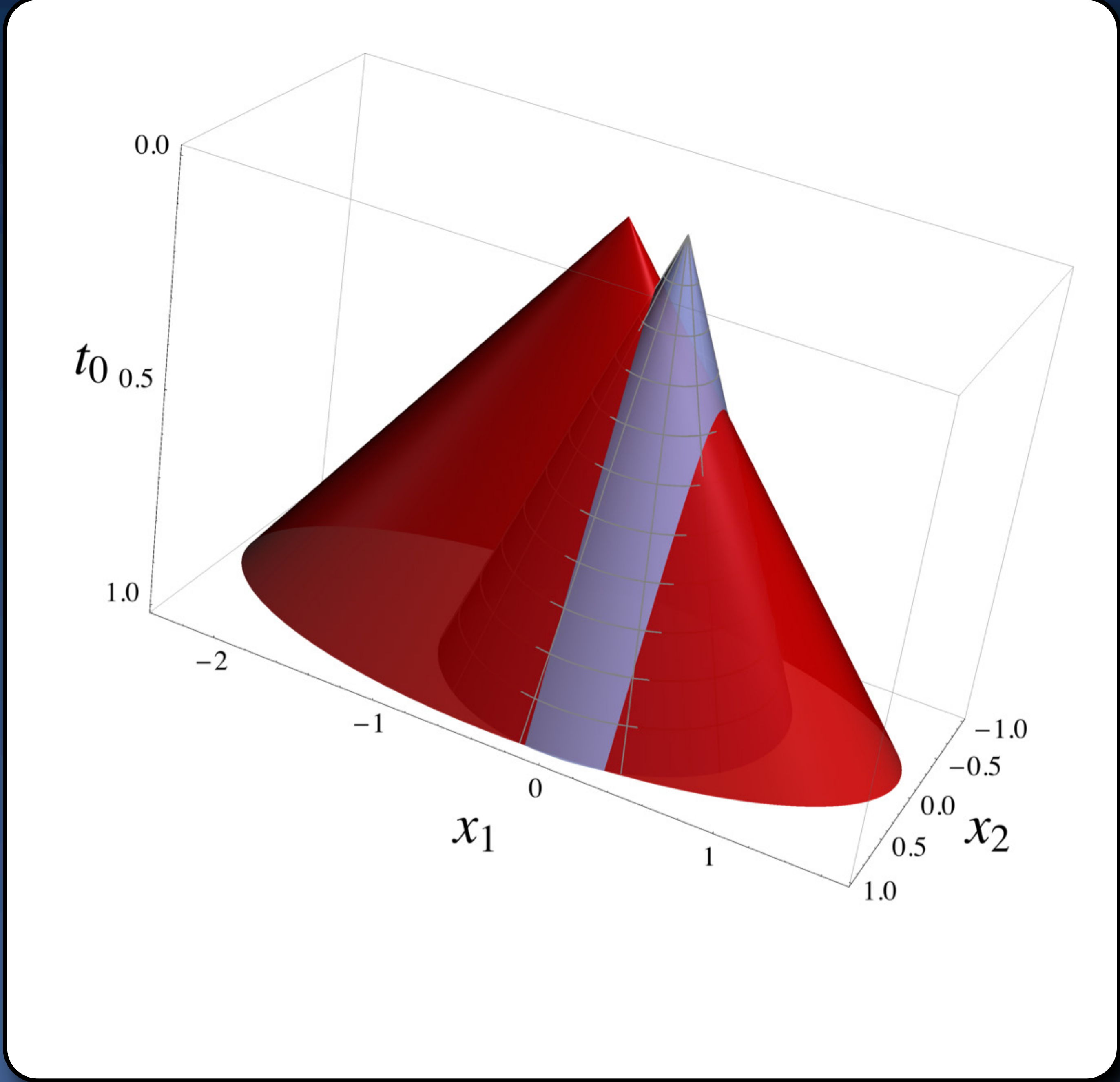
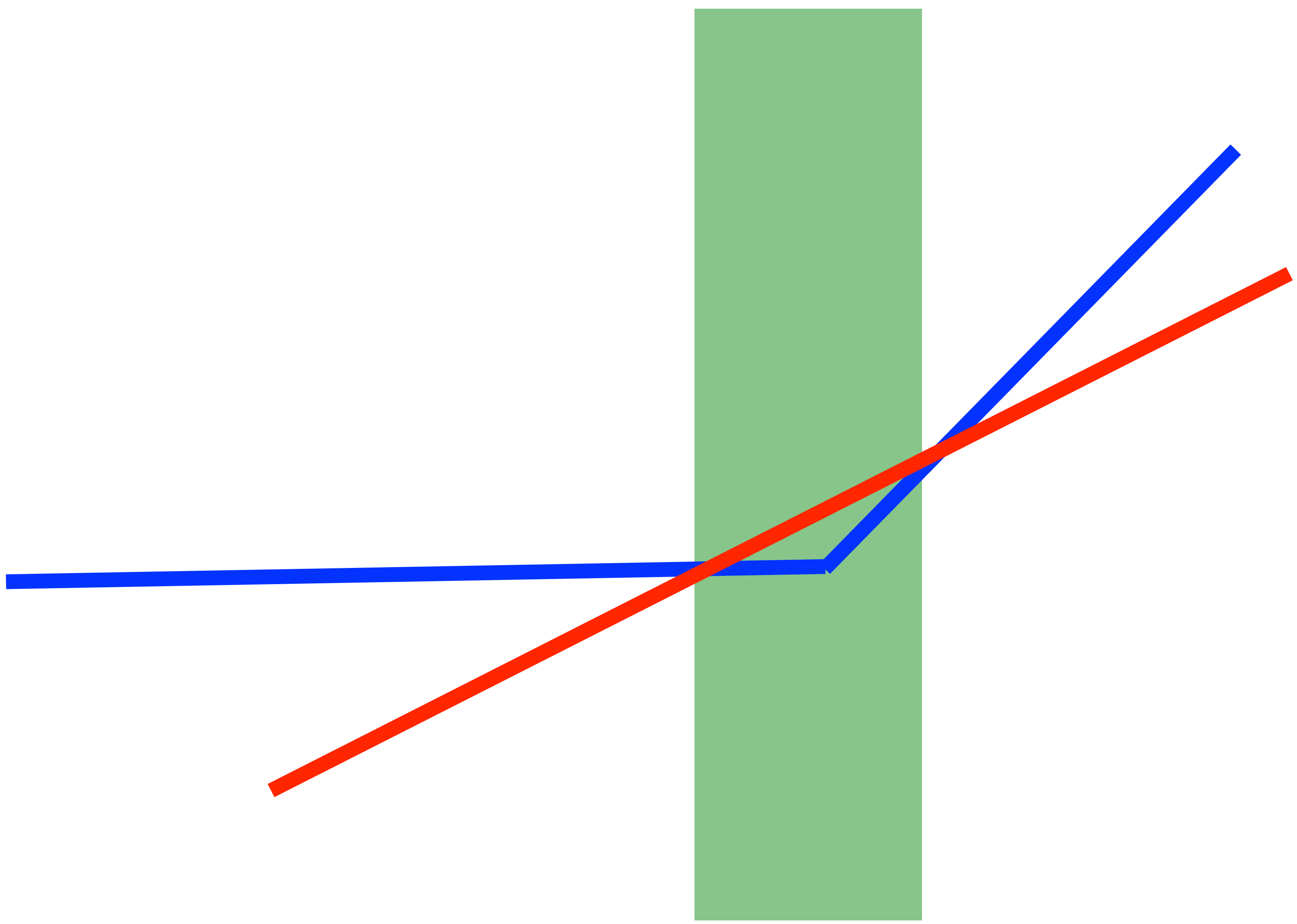
- General Convex Hulls
  - Blekherman, et al. 2013
  - Horst and Tuy. 2003



Follow the MIR way!



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- Imply general cuts for quadratic sets.
- Techniques:
    - (1) Interpolation, (2) Aggregation.

Interpolation

Interpolation for Separable Functions

- Modaresi, Kılınç, V. 2013:

$$C := \{(x, t_0) \in \mathbb{R} \times \mathbb{R} : f(x_1) \leq t_0\}$$

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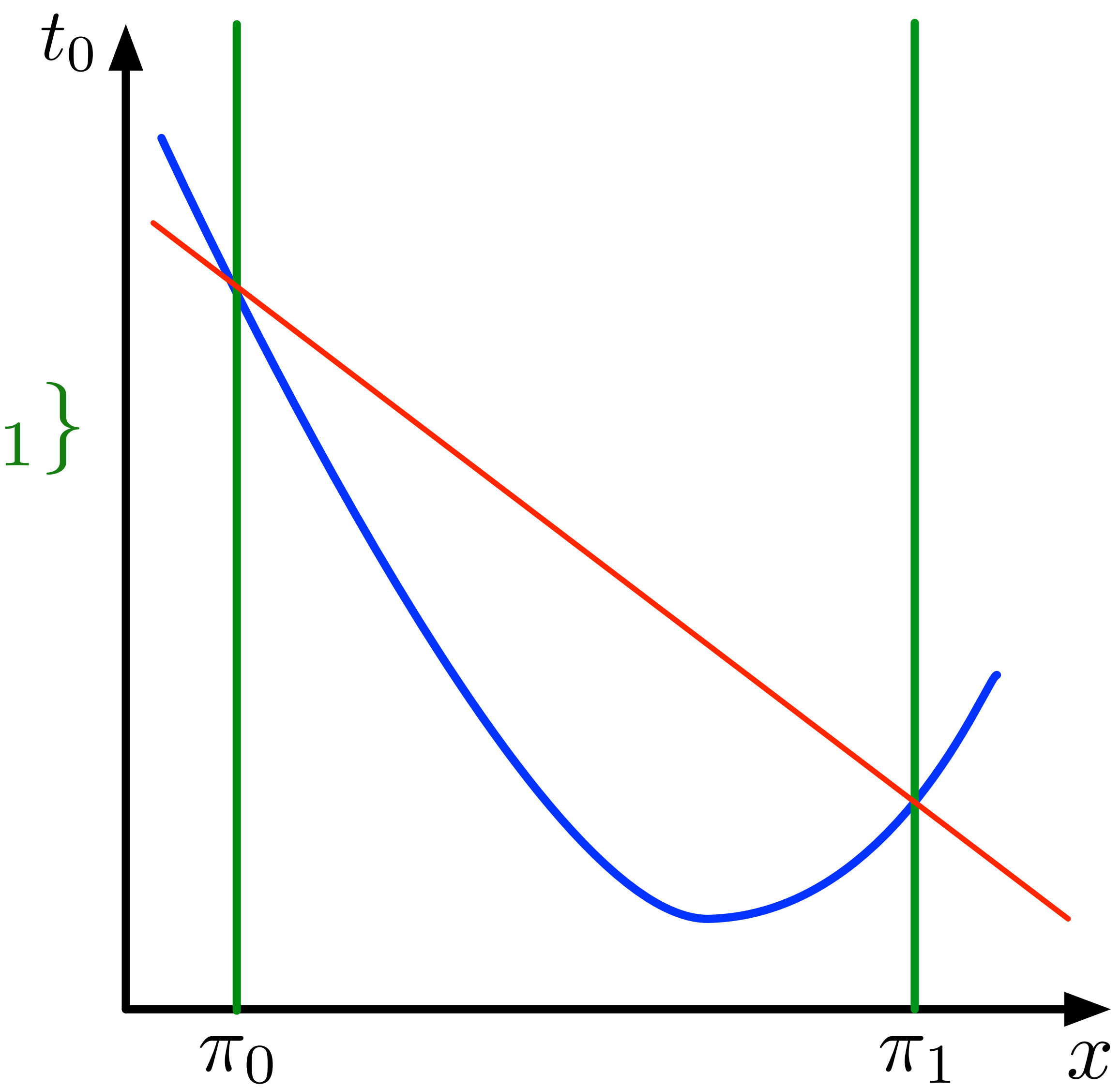

"Simple" Split = Ignores t_0 : $\pi^T x \leq \pi_0 \;\vee\; \pi^T x \geq \pi_1$

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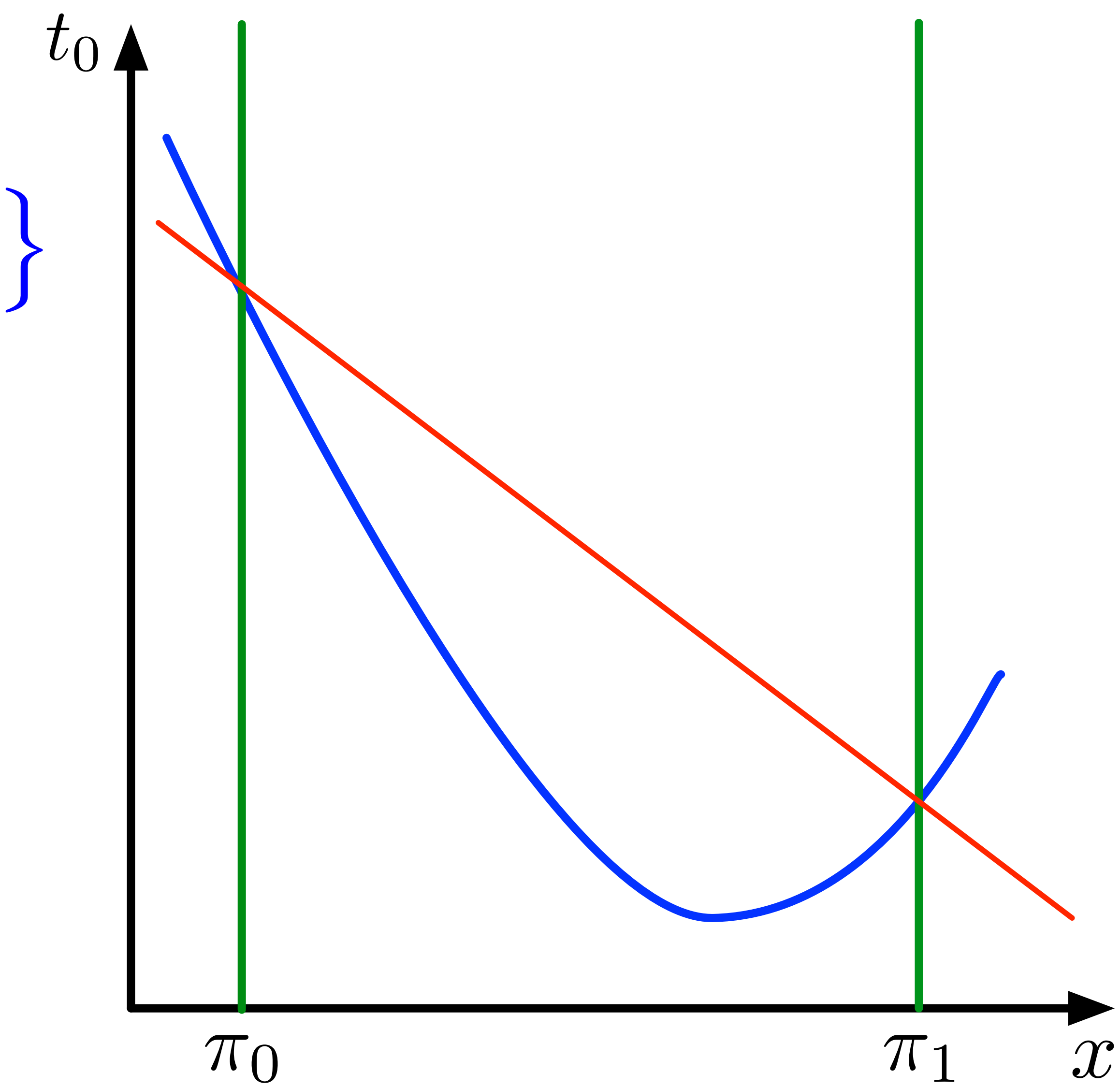
$$C := \left\{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : f\left(\pi^T x\right) + g\left(P_\pi^\perp x\right) \le t_0\right\}$$

$$P_\pi^\perp := I - \left(1/\left\|\pi\right\|_2^2\right)\pi\pi^T$$



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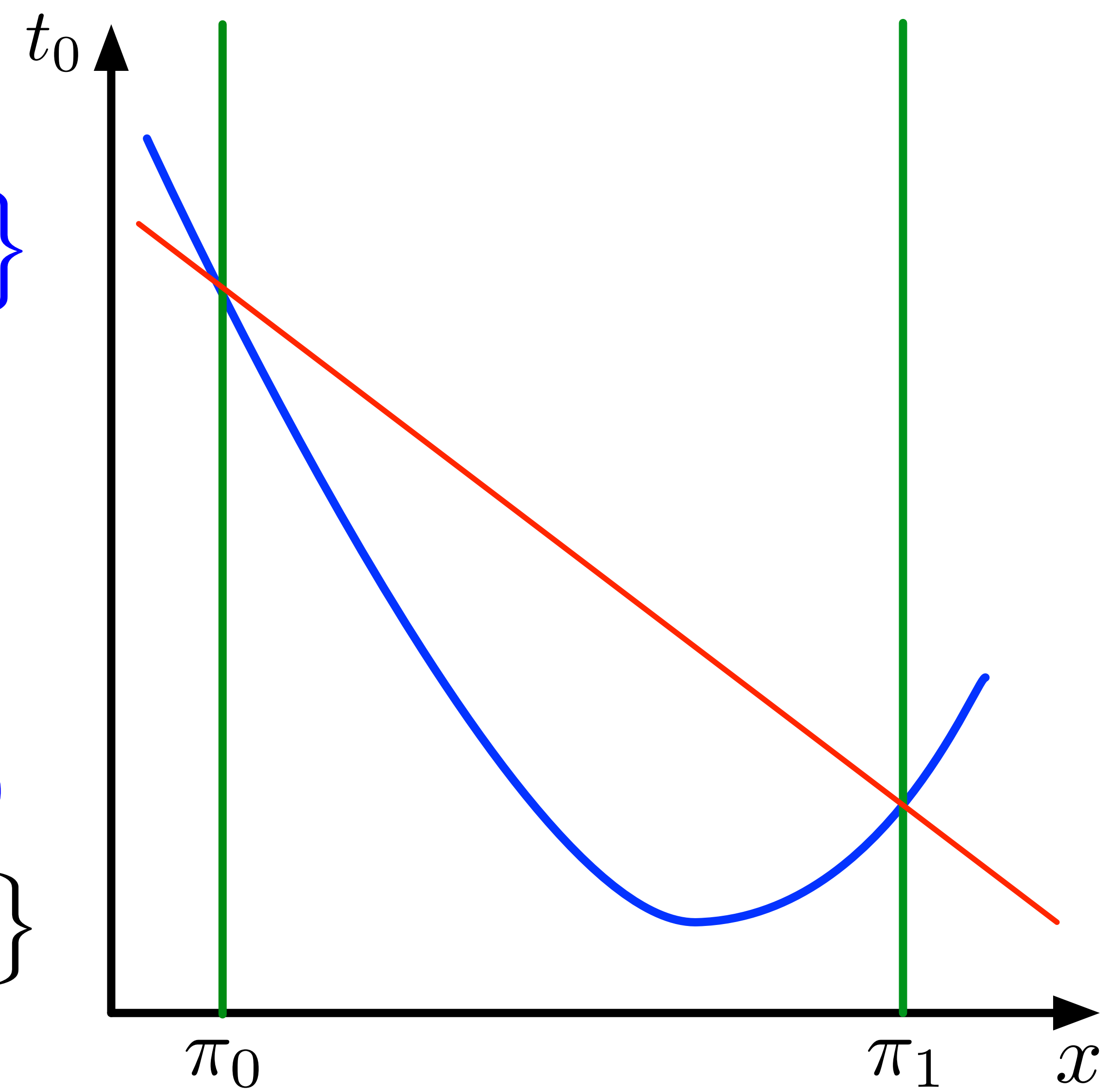

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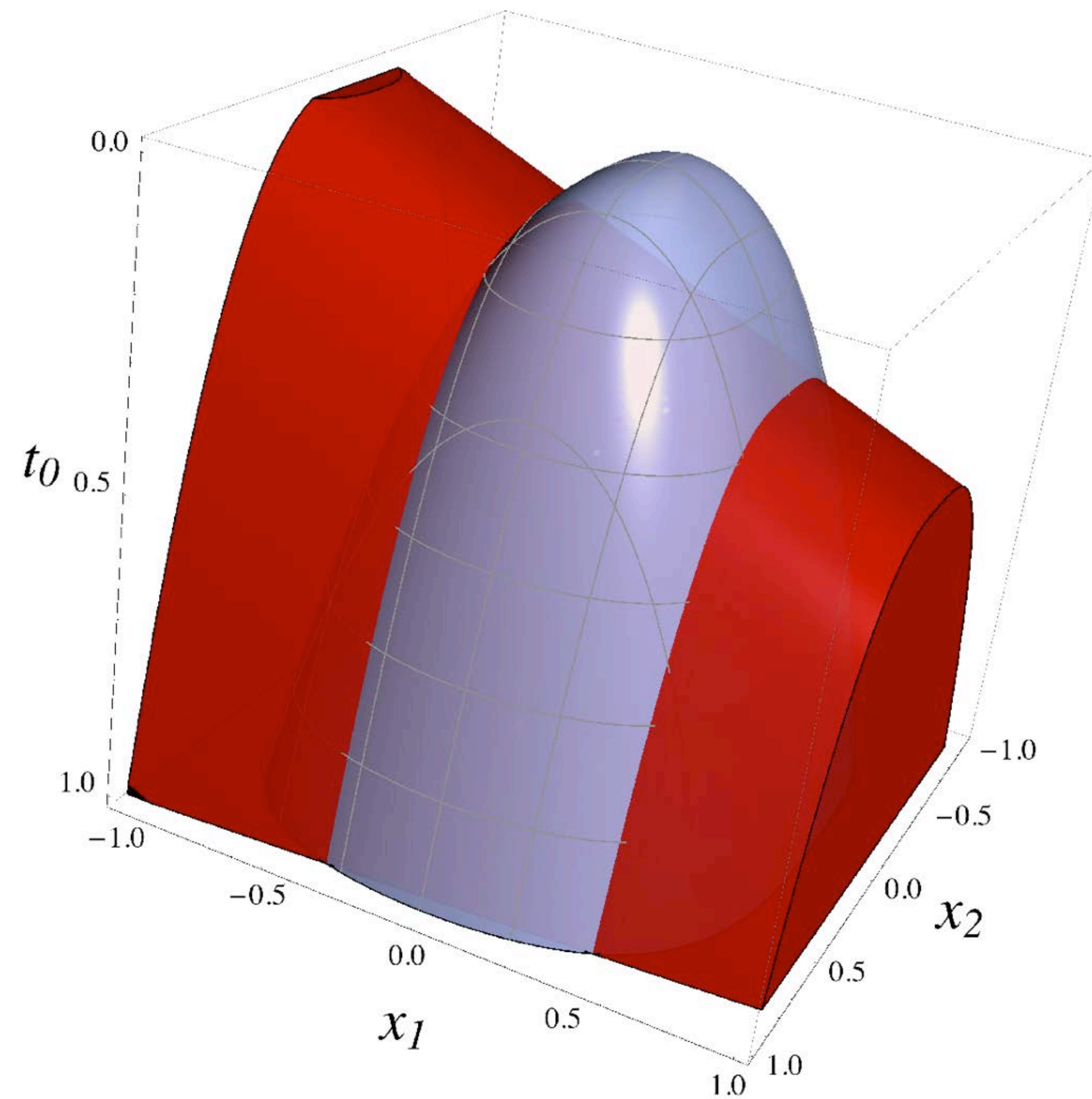
Cuts for Some General Functions

$$C := \{(x, t_0) \in \mathbb{R}^2 \times \mathbb{R} : \exp(x_1^2) - \exp(0) + |x_2|^3 \leq t_0\}$$

$$C_{e^1,\pi_0} = \{(x, t_0) \in \mathbb{R}^2 \times \mathbb{R} : \exp(x_1^2) - \exp(0) + |x_2|^3 \leq t_0, \quad ax_1 + b + |x_2|^3 \leq t_0\}$$



Cuts for All Paraboloids (Simple Split)

- Modaresi, Kılınç, V. 2012:

$$C := \left\{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} \; : \; \|B(x-c)\|_2^2 \leq t_0\right\}$$

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- $P_\pi := \left(1/\|\pi\|_2^2\right) \pi\pi^T$, $P_\pi^\perp := I - P_\pi$
- $\|x\|_2^2 = \|P_\pi x\|_2^2 + \left\|P_\pi^\perp x\right\|_2^2$
- $\|P_\pi x\|_2^2 = (1/\|\pi\|)^2 \left(\pi^T x\right)^2$

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$$\|x\|_2^2 \le t_0 \iff f\left(\pi^T x\right) + g\left(P_\pi^\perp x\right) \le t_0$$

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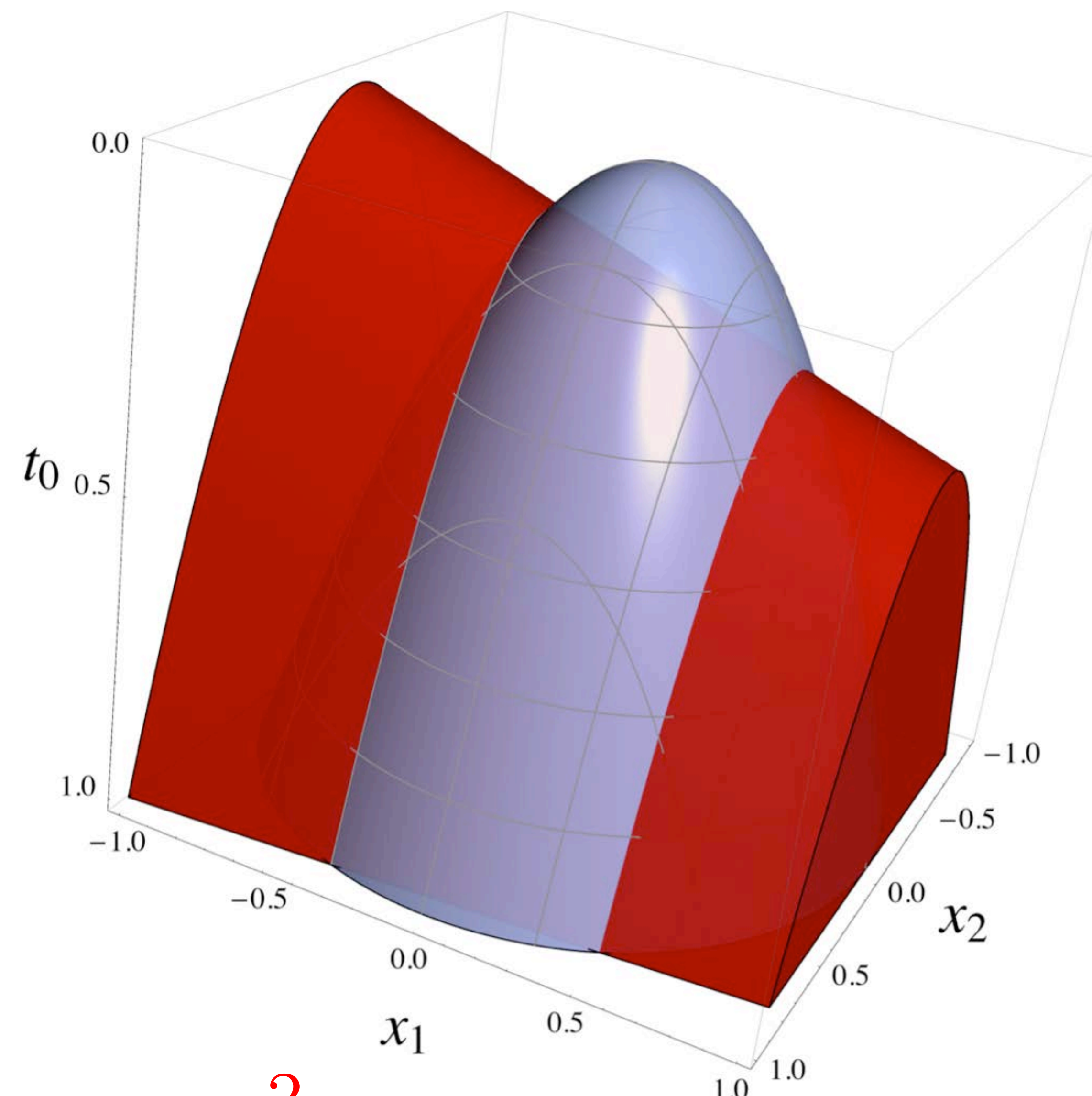

$$C_{\pi,\pi_0} = \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|B(x-c)\|_2^2 \leq t_0$$

$$a\pi^T x + b + \left\|P^{\perp}_{B^{-T}\pi} B(x-c)\right\|_2^2 \leq t_0\}$$

Interpolation for Conic Functions

- Modaresi, Kılınç, V. 2013:

- g is positively homogeneous in addition to convex.

$$C := \left\{ (x, t_0) \in \mathbb{R}^n \times \mathbb{R} \, : \, \left(|\pi^T x|^p + g\left(P_\pi^\perp x\right)^p \right)^{1/p} \leq t_0 \right\}$$

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$$\left(|a\pi^T x + b|^p + g\left(P_\pi^\perp x\right)^p \right)^{1/p} \leq t_0 \}$$

- See also Atamturk and Narayanan 2010.

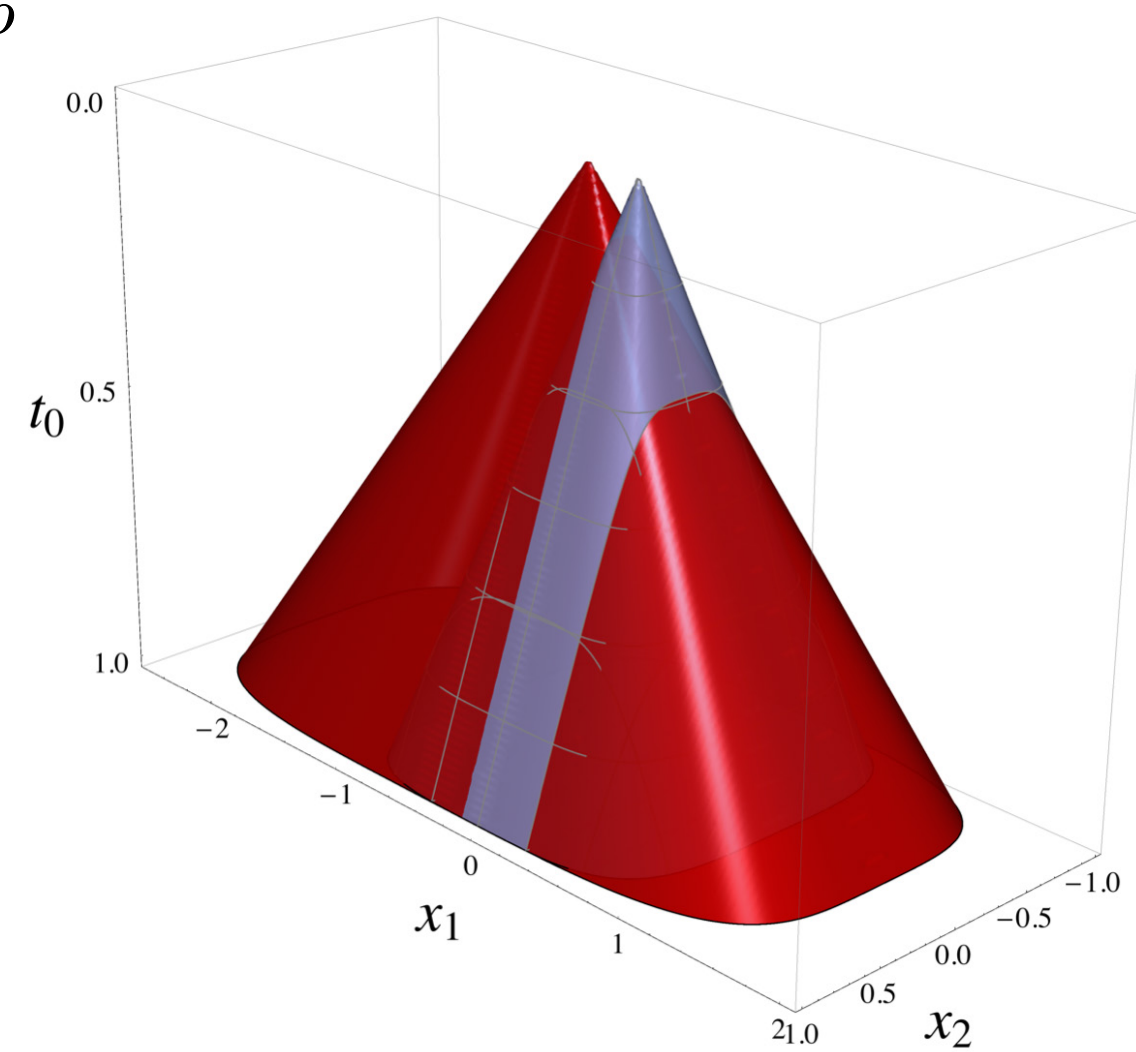
Cuts for Some General Functions

- Modaresi, Kılınç, V. 2012

$$\|x-c\|_p := \left(\sum_{i=0}^{n} (x_i - c_i)^p\right)^{1/p}$$

$$C := \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|x-c\|_p \leq t_0\}$$

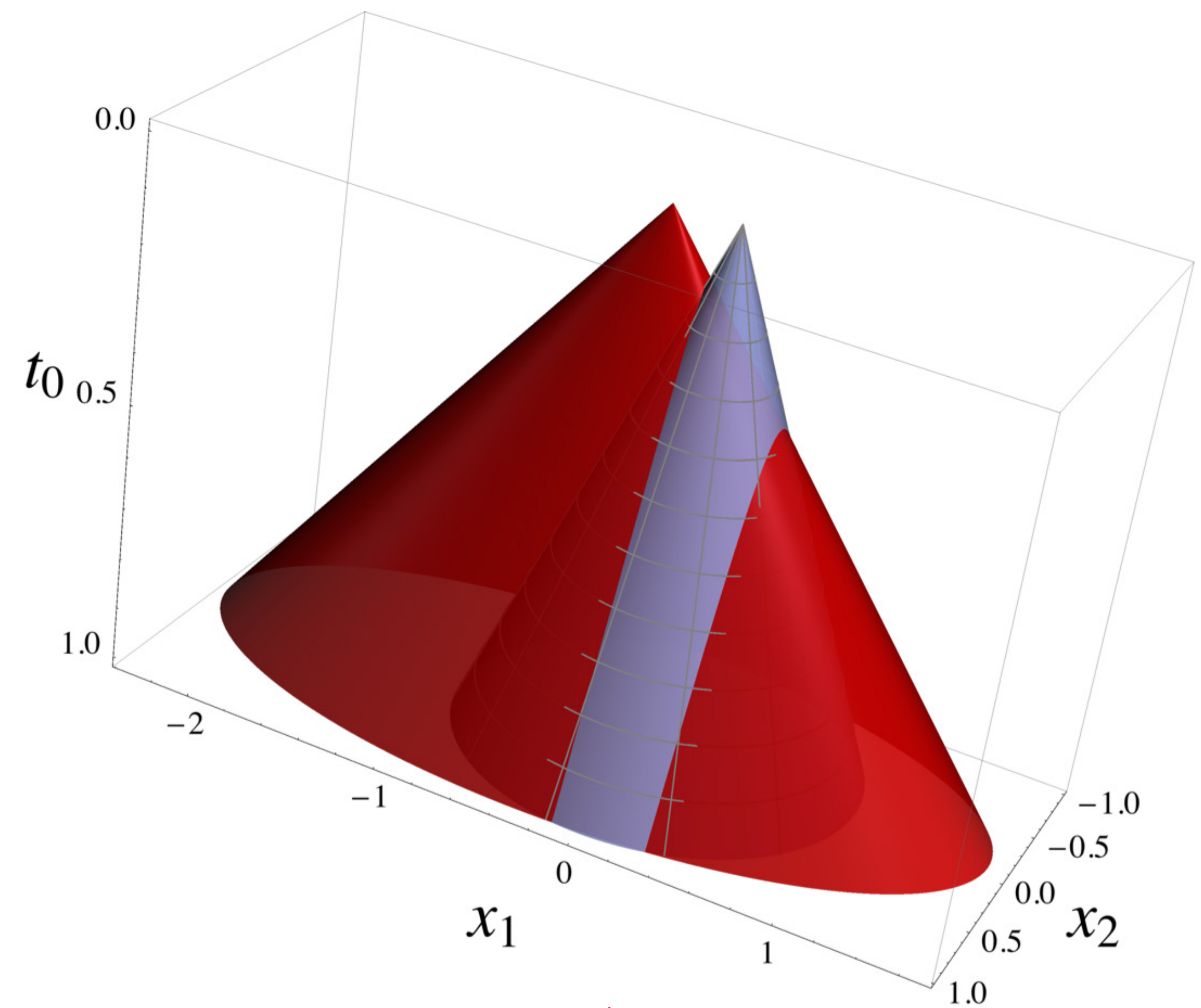
$$C_{e^1,\pi_0} := \Big\{(x,t_0) \in \mathbb{R}^n \times \mathbb{R} : \|x-c\|_p \leq t_0, \left((\alpha(x_1 - d_1) + \beta)^p + \sum_{i=2}^{n} (x_i - d_i)^p\right)^{1/p} \leq t_0\Big\}$$



Cuts for All Quadratic Cones (Simple)

- Modaresi, Kılınç, V. 2012

$$C := \left\{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|B(x - c)\|_2 \le t_0\right\}$$



$$C_{\pi,\pi_0} = \left\{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} : \|B(x - c)\|_2 \le t_0 \right.$$

$$\left. \left(\left(a\pi^T x + b\right)^2 + \left\|P^{\perp}_{B^{-T}\pi} B(x - c)\right\|_2^2\right)^{1/2} \le t_0\right\}$$

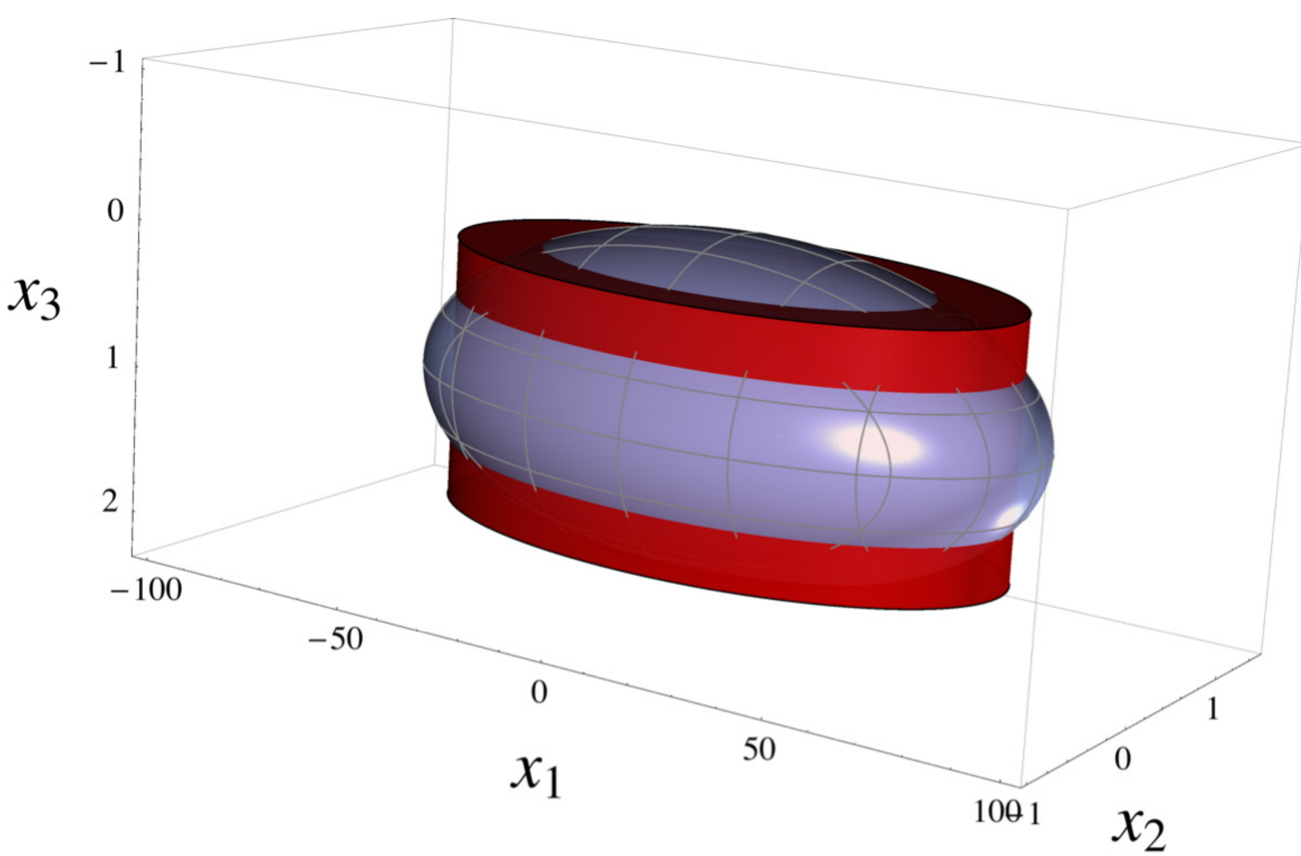
(also Atamturk and Narayanan 2010, Andersen and Jensen 2013)

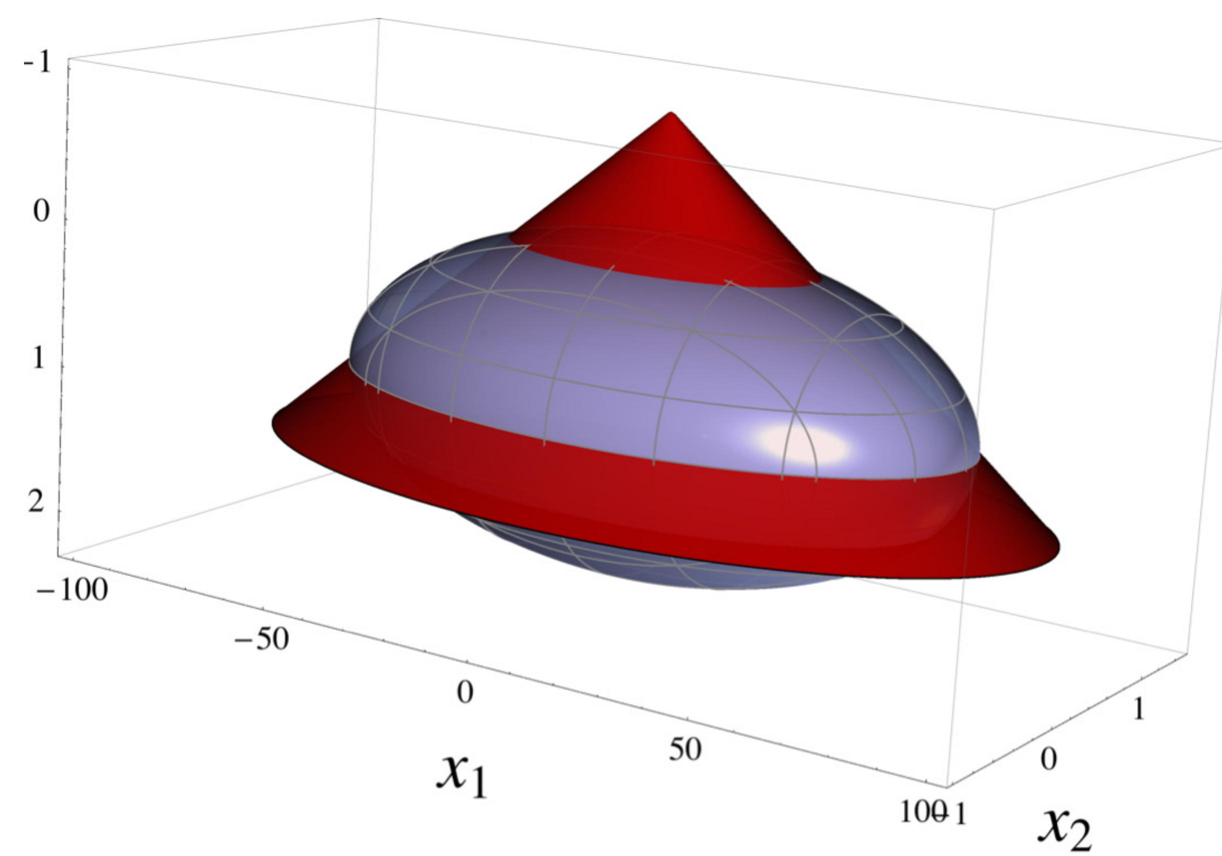
Similarly: Cuts for All Ellipsoids

- Dadush, Dey and V. 2011

  - g is positively homogeneous in addition to convex.

$$C := \left\{x \in \mathbb{R}^n \,:\, f\left(\pi^T x\right) + g\left(P_\pi^\perp x\right) \leq 0\right\}$$

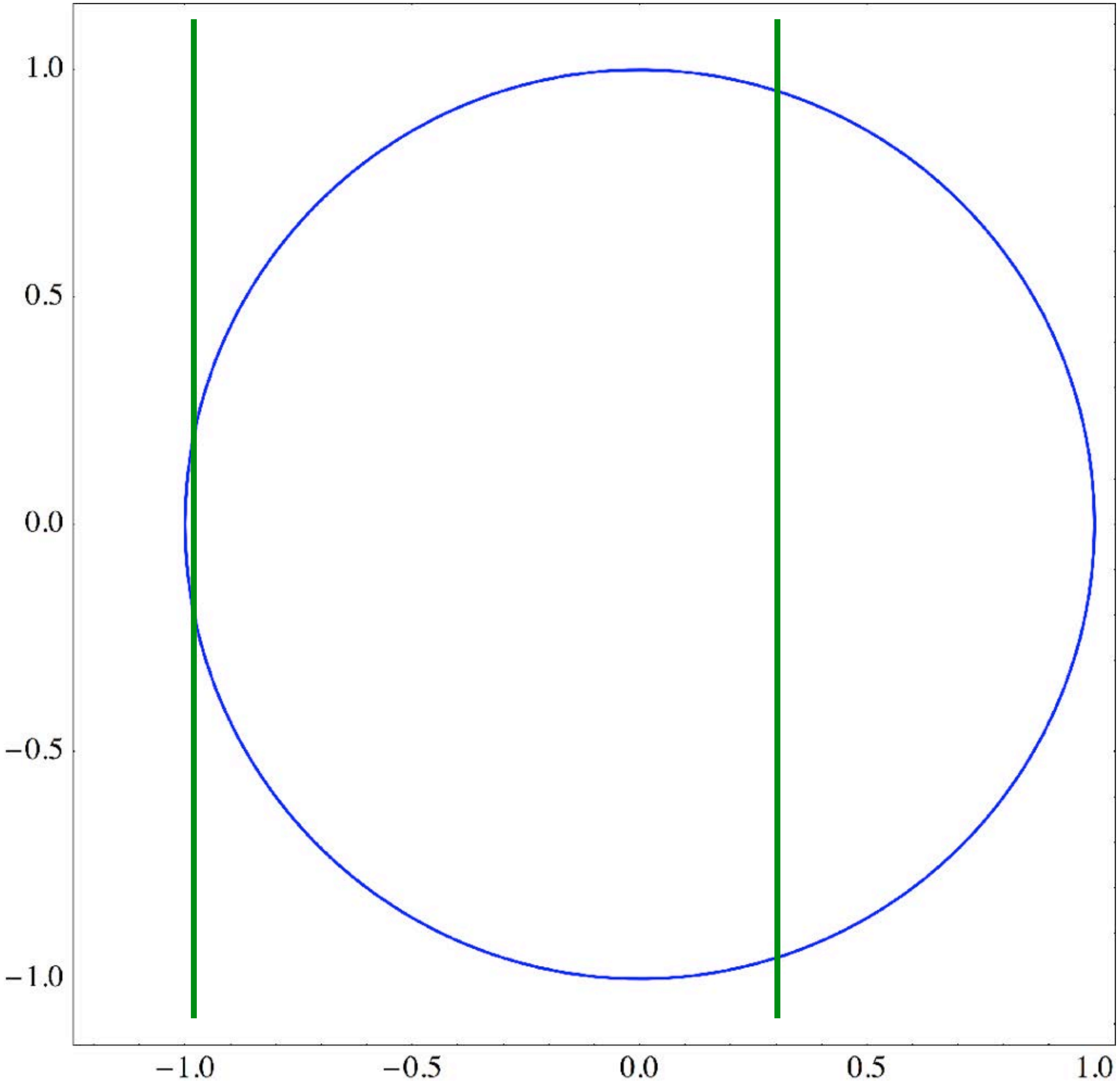




(also see Belotti, Góez, Polik, Ralphs, Terlaky 2011)

Interpolation “Form” is Crucial

$$\sqrt{x_1^2 + x_2^2} \leq t_0, \quad t_0 = 1$$



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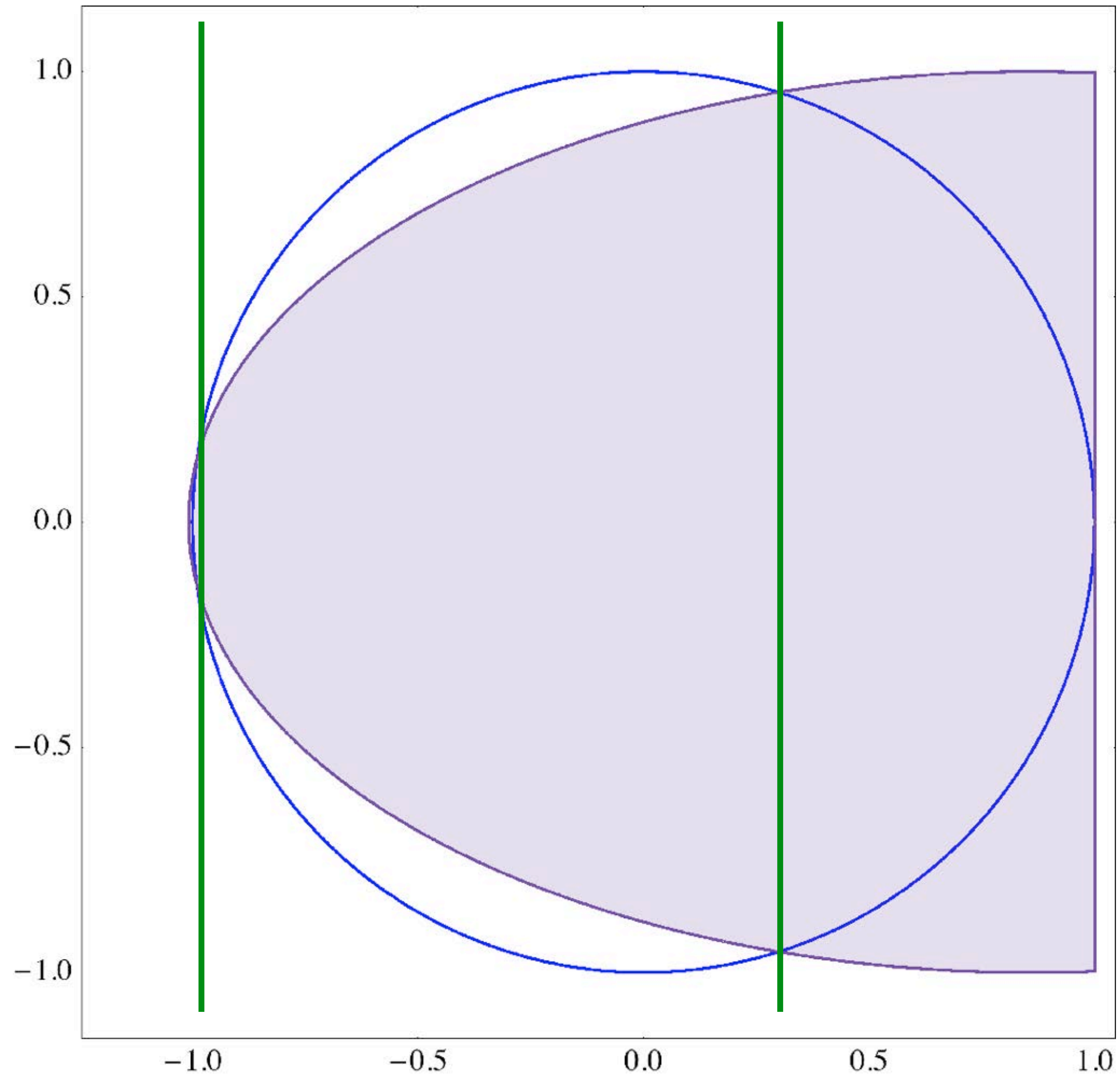
$$\sqrt{x_1^2 + x_2^2} \leq t_0, \quad t_0 = 1$$

Cone

$$\sqrt{(ax_1 + b)^2 + x_2^2} \leq 1$$



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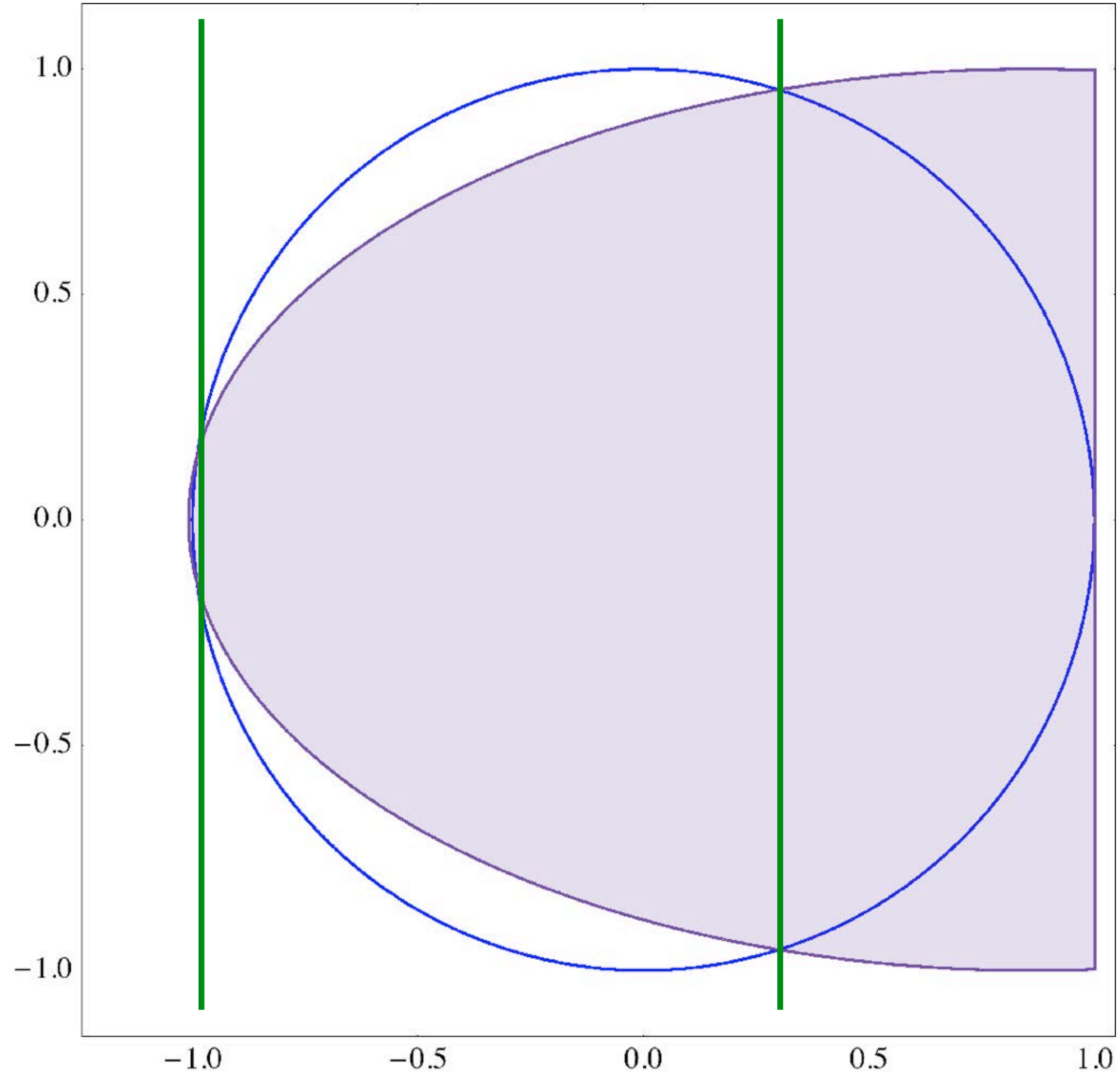

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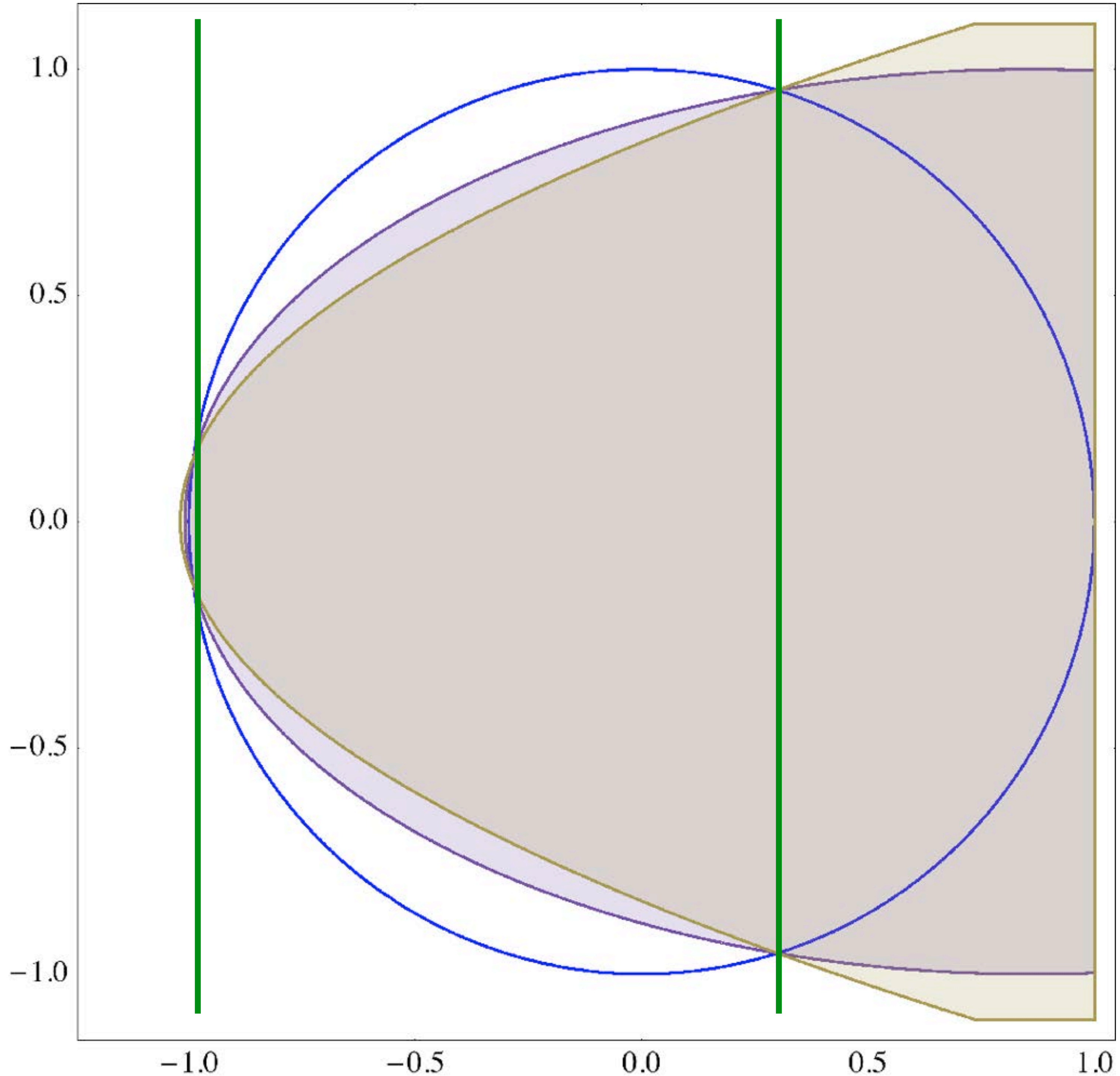

Interpolation "Form" is Crucial

$$-\sqrt{1-x_1^2}+|x_2| \leq t_0$$

Cone $\qquad t_0 = 0$

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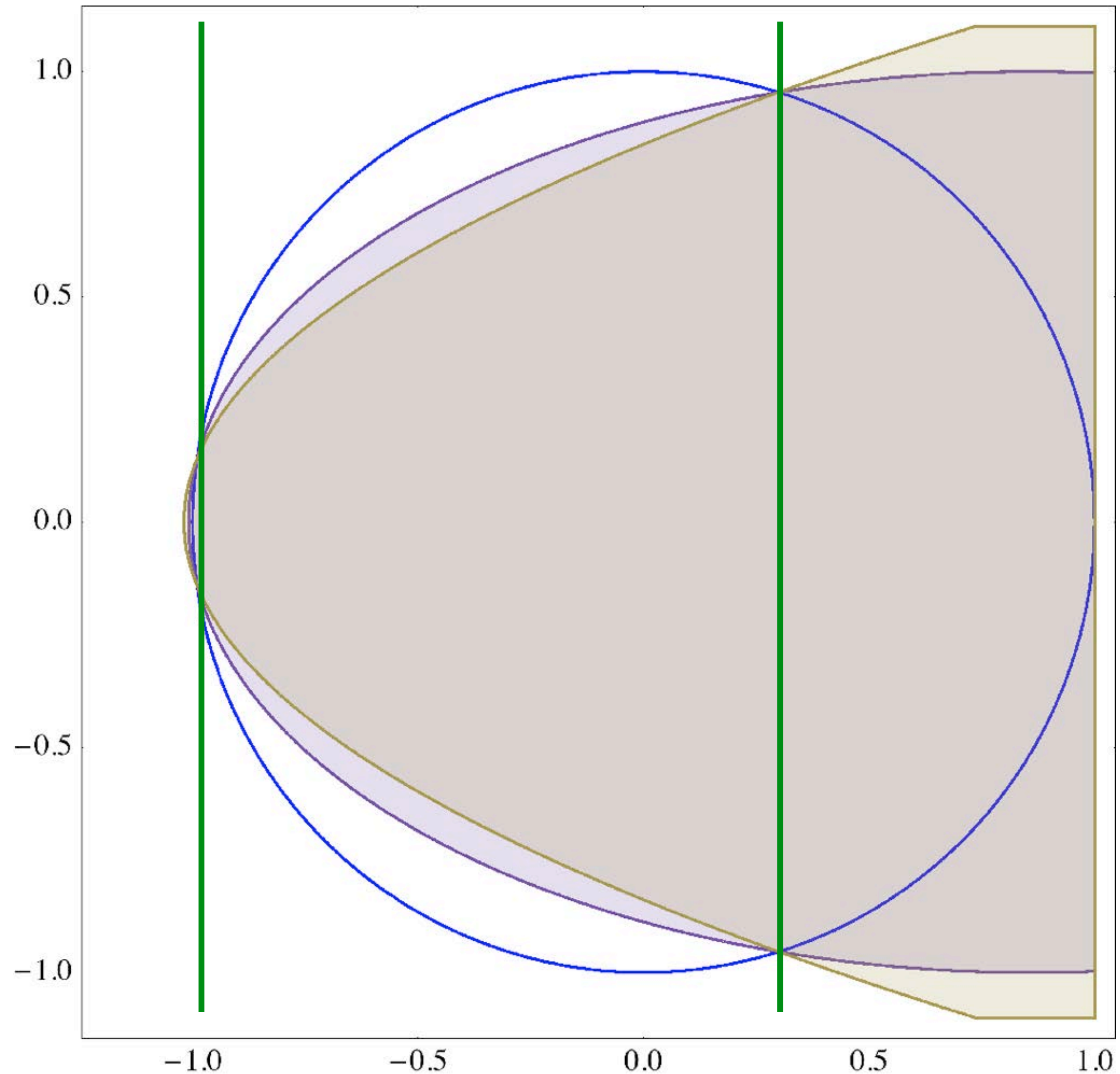
Paraboloid

$$ax_1+b+x_2^2 \leq 1$$



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Cone $\quad t_0 = 0$

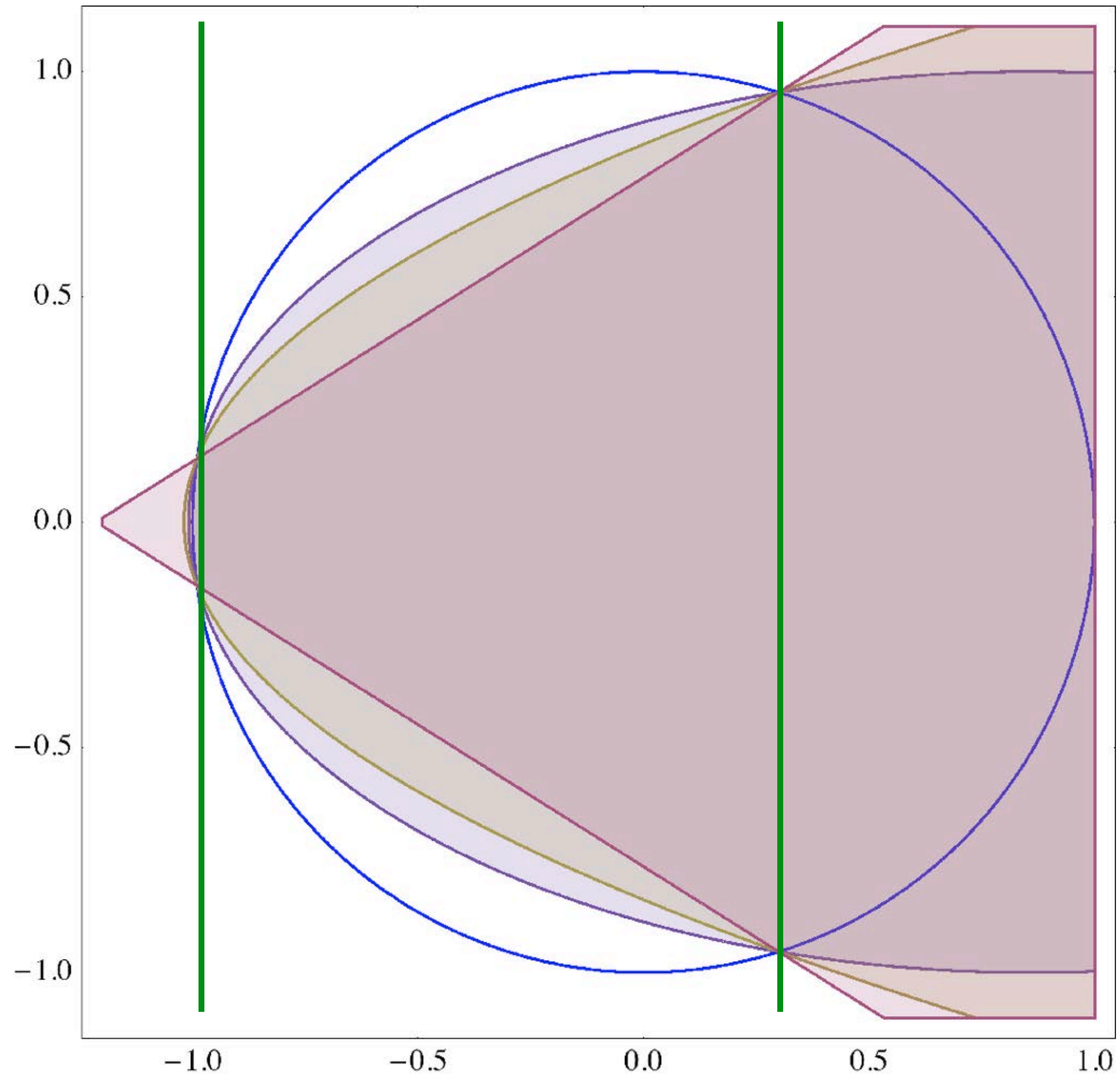
$$\sqrt{(ax_1+b)^2+x_2^2} \le 1$$

Paraboloid

$$ax_1+b+x_2^2 \le 1$$

Ellipsoid

$$ax_1+b+|x_2| \le 0$$



Extensions 1: Other QP Split Cuts

- General Split Cuts: $\pi^T x + \hat{\pi} t_0 \leq \pi_0 \vee \pi^T x + \hat{\pi} t_0 \geq \pi_1$
  - Paraboloid $\|x\|_2^2 \leq t_0$ and Cone $\|x\|_2 \leq t_0$

$$\left\| P_\pi^\perp x + \left(a\pi^T x + b \right) \pi \right\|_2 \leq c\pi^T x + dt_0 + e$$

- Other “Simple” Split Cuts: $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_1$
  - Hyperboloids: $\sqrt{\|x\|_2^2 + l^2} \leq t_0$

$$\left\| P_\pi^\perp x + \left(a\pi^T x + b \right) \pi \right\|_2 \leq t_0$$

Extensions 2: Parabolic n-split cuts

$$C := \left\{(x, t_0) \,:\, \|B(x-c)\|_2^2 \le t_0\right\}$$

$$B^{-T}\pi_i \perp B^{-T}\pi_j$$

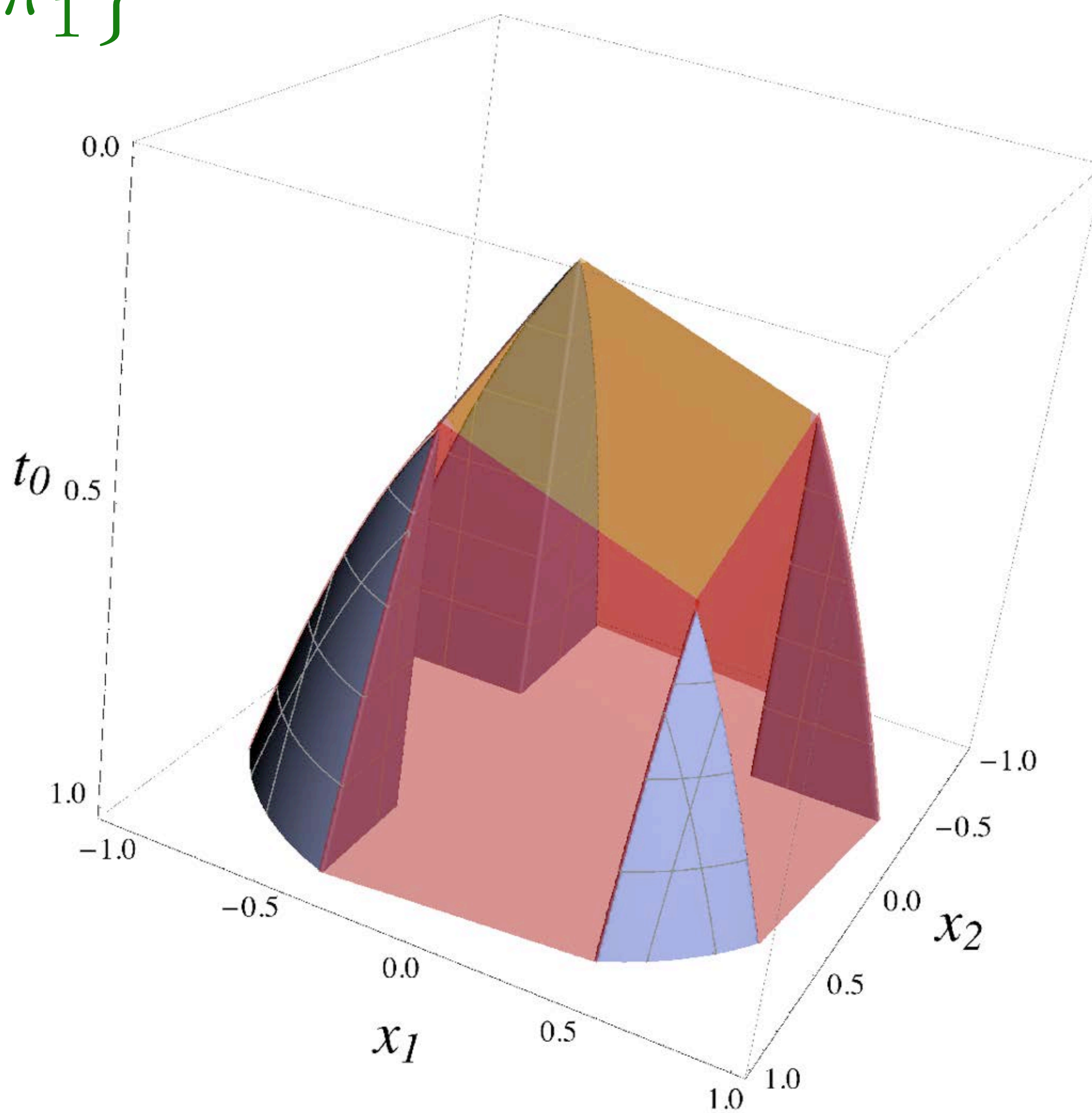
$$S := \bigcup_{i=1}^{n} \left\{(x, t_0) \,:\, \pi_0^i \le \pi_i^T x \le \pi_1^i\right\}$$

$$\operatorname{conv}(C \setminus S) =$$

$$\Big\{(x, t_0) \,:\, \|B(x-c)\|_2^2 \le t_0,$$

$$\sum_{j\in J}\left(a_j \pi_j^T x + b_j\right) + \sum_{j\notin J} d_j \left(\pi_j^T (x-c)\right)^2 \le t_0$$

$$J \subseteq \{1, \ldots, n\}\Big\}$$



Extended QCP: Parabolic n-split cuts

$C := \{(x, t_0) \, : \, \|B(x - c)\|_2^2 \leq t_0\}$ $B = I, \quad c_i = 1/2, \quad \pi_i = e^i$

$S := \bigcup_{i=1}^{n} \{(x, t_0) \, : \, \pi_0^i \leq \pi_i^T x \leq \pi_1^i\}$

$\operatorname{conv}(C \setminus S) =$

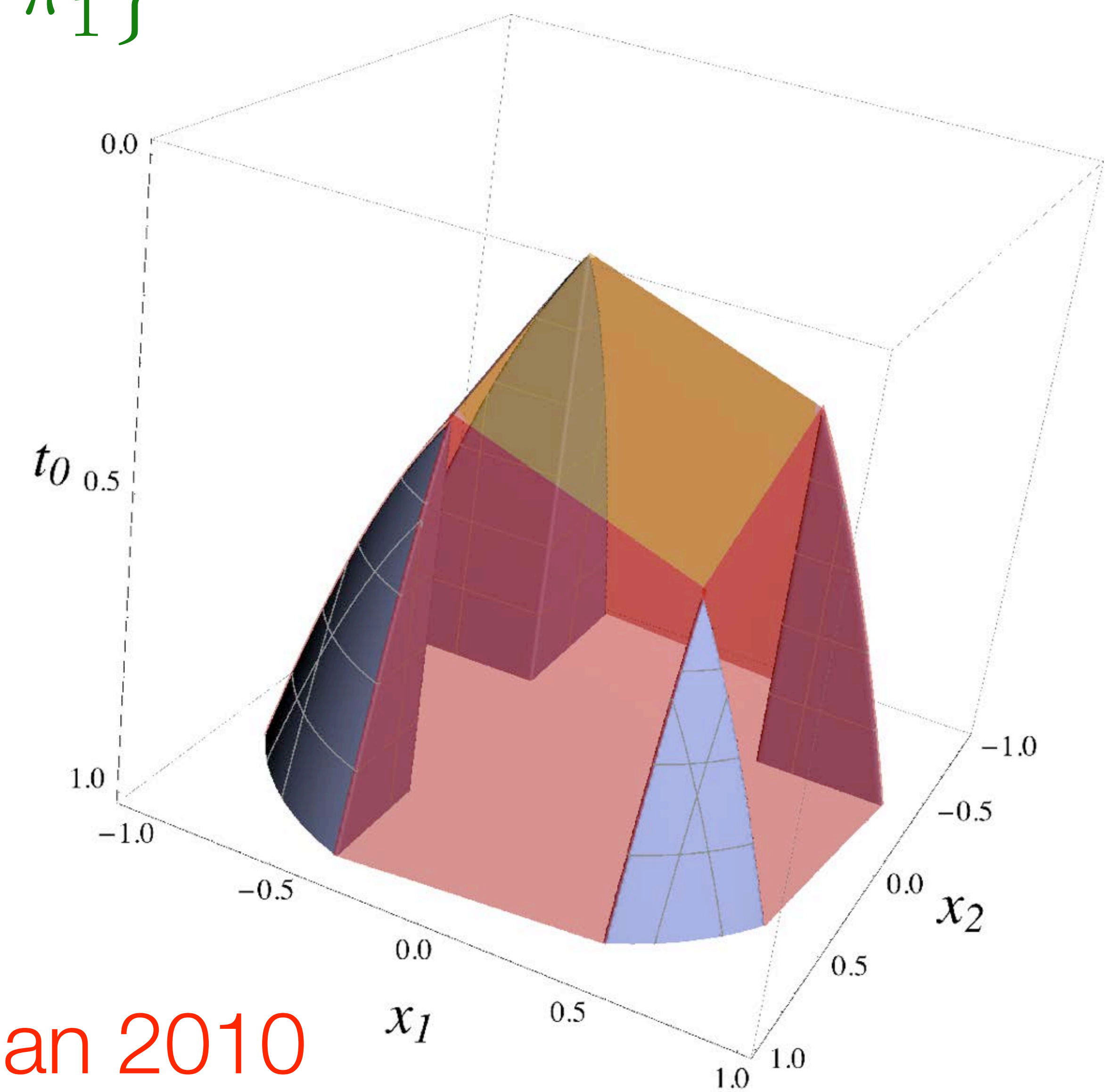
$$\{(x, t, t_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \, :$$

$$\|t\|_2^2 \leq t_0,$$

$$|x_i - c_i| \leq t_i, \; \forall i$$

$$(1 - 2F(c_i))(x_i - \lfloor c_i \rfloor)$$

$$+F(c_i) \leq t_i, \; \forall i\}$$



Conic MIR of Atamturk and Narayanan 2010

Aggregation

Aggregation ≈ SDP Approach

$$C := \left\{(x, t_0) \in \mathbb{R}^{n+1} \ : \ f(x) \leq t_0\right\}$$

$$S := \left\{(x, t_0) \in \mathbb{R}^{n+1} \ : \ g(x) > t_0\right\}$$

f convex, g concave.

$$\operatorname{conv}(C \setminus S) =$$

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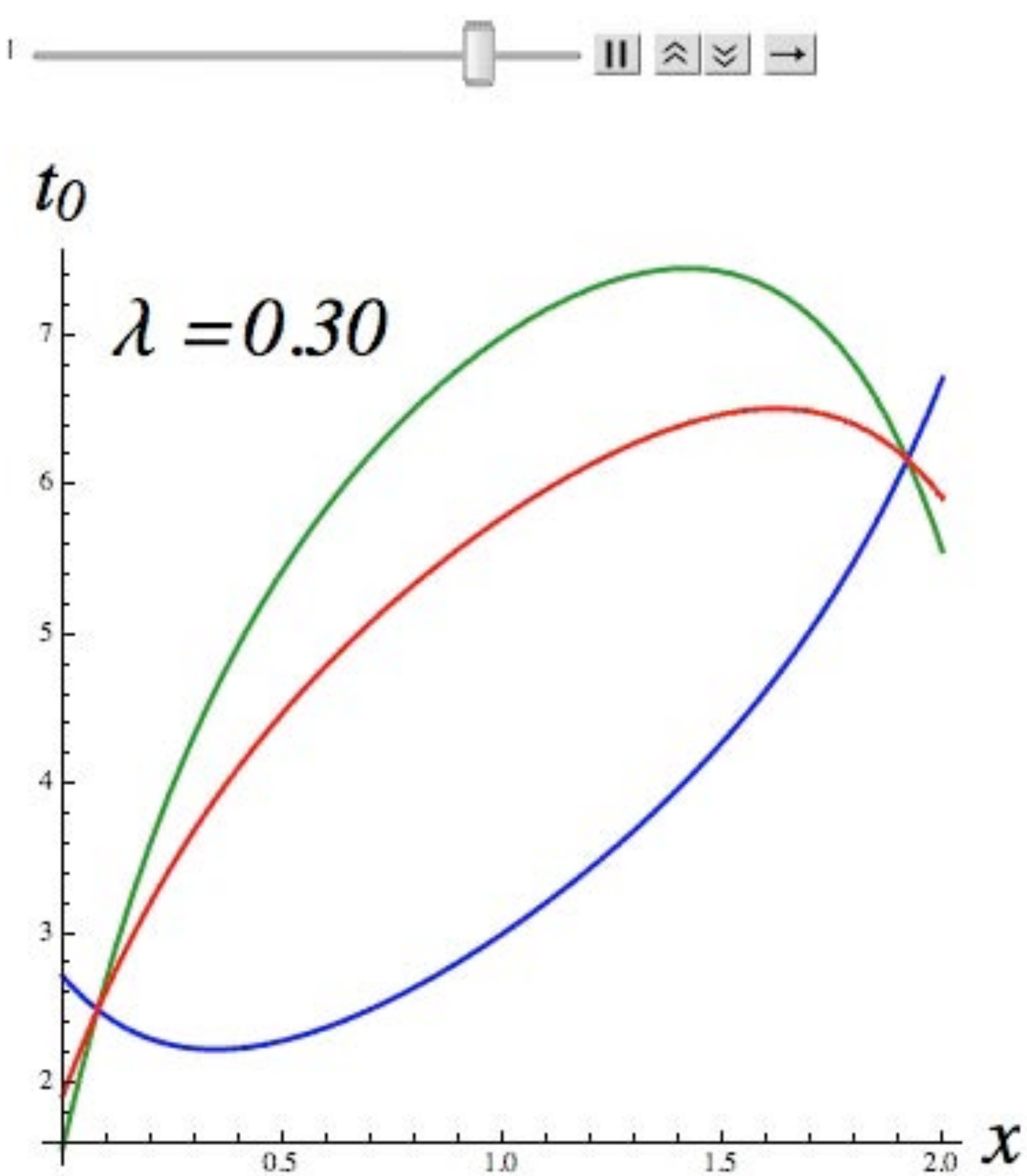
$$h_\lambda(x) := \lambda f(x) + (1-\lambda) g(x)$$



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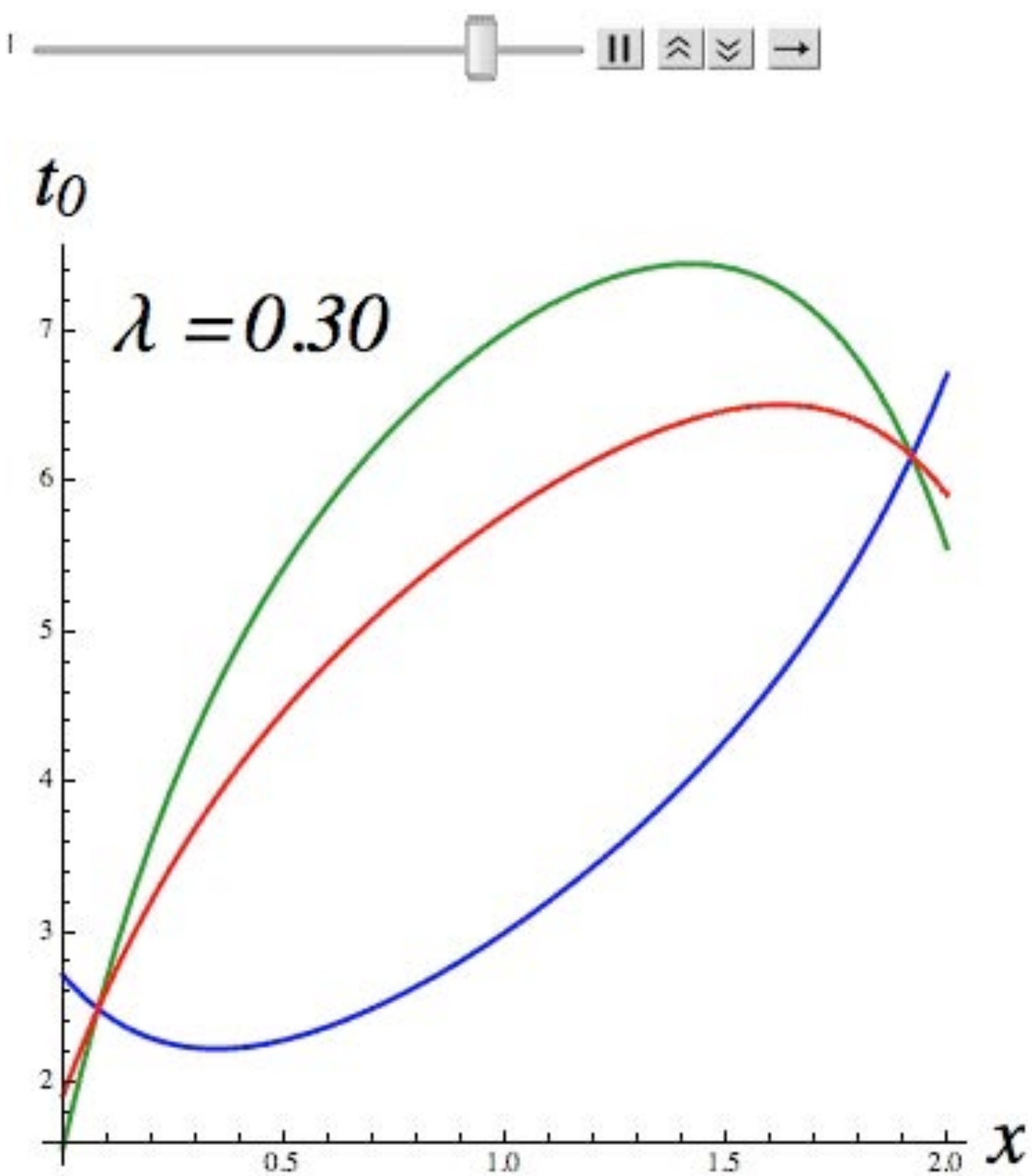
$$\lambda^* := \min\{\lambda \in [0, 1] \,:\, h_\lambda \text{ convex}\}$$



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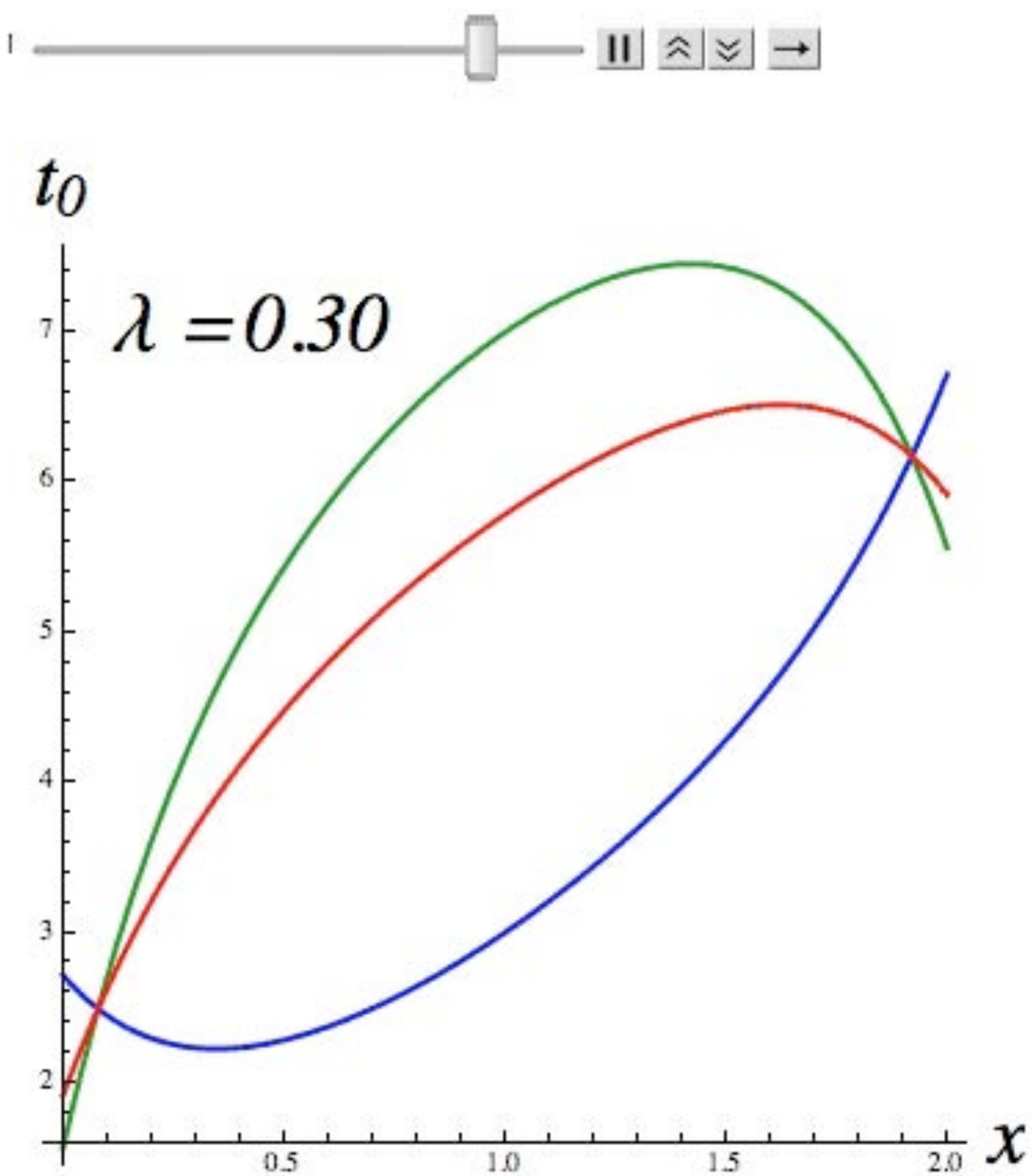

Aggregation ≈ SDP Approach

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f convex, g concave.

$$\operatorname{conv}(C \setminus S) = \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R} \; : \; f(x) \leq t_0, \; h_{\lambda^*}(x) \leq t_0\}$$

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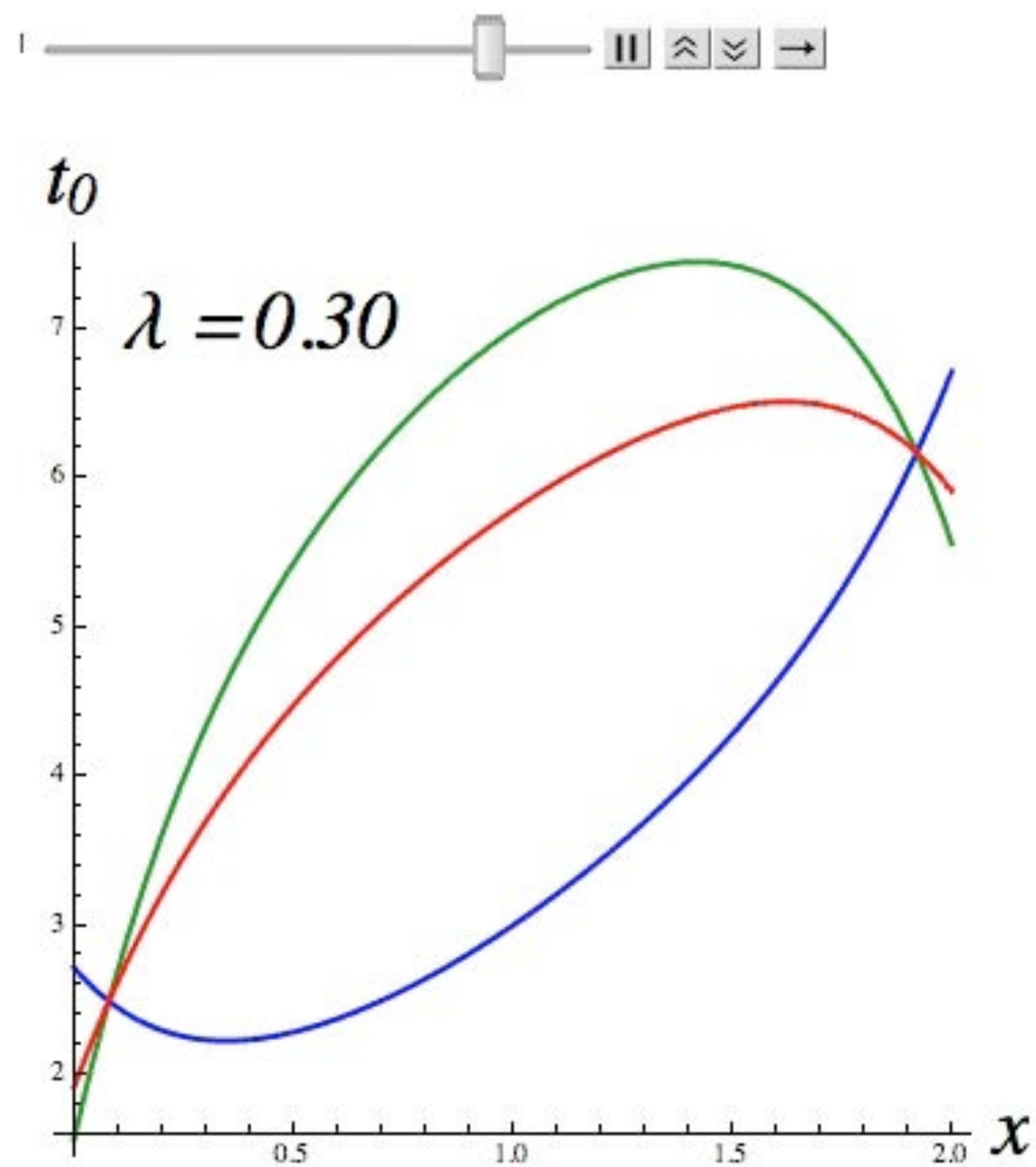


A Case Where Aggregation Works

- Modaresi, Kılınç, V. 2013:

$$C := \left\{(x, t_0) \in \mathbb{R}^{n+1} \ : \ f(x) \leq t_0\right\}$$

$$S := \left\{(x, t_0) \in \mathbb{R}^{n+1} \ : \ g(x) > \gamma t_0\right\}$$

$$f(x) = \sum_{i=1}^{n} w_i \left(a_i^T x\right) + m^T x + r$$

$$g(x) = -\sum_{i=1}^{n} \alpha_i w_i \left(a_i^T x\right) - l^T x - q$$

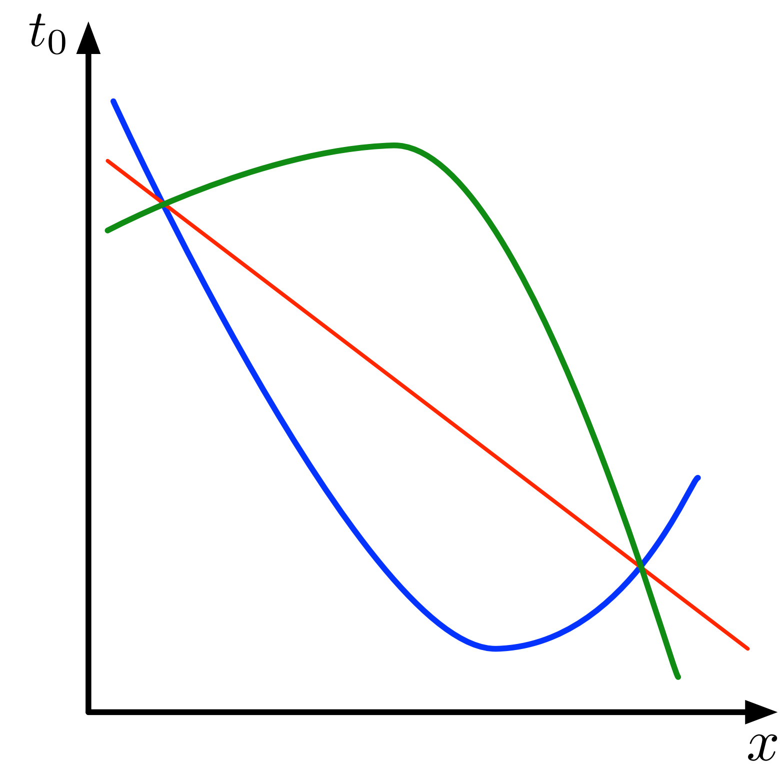
$$h_{\lambda^*}(x) := \frac{f(x) + (1/\alpha_n) g(x)}{1 + \gamma/\alpha_n}$$

$a_i \perp a_j, \quad i \neq j$

$0 \neq \alpha_n \geq \alpha_i$

w_i are convex

w_n is "coercive"



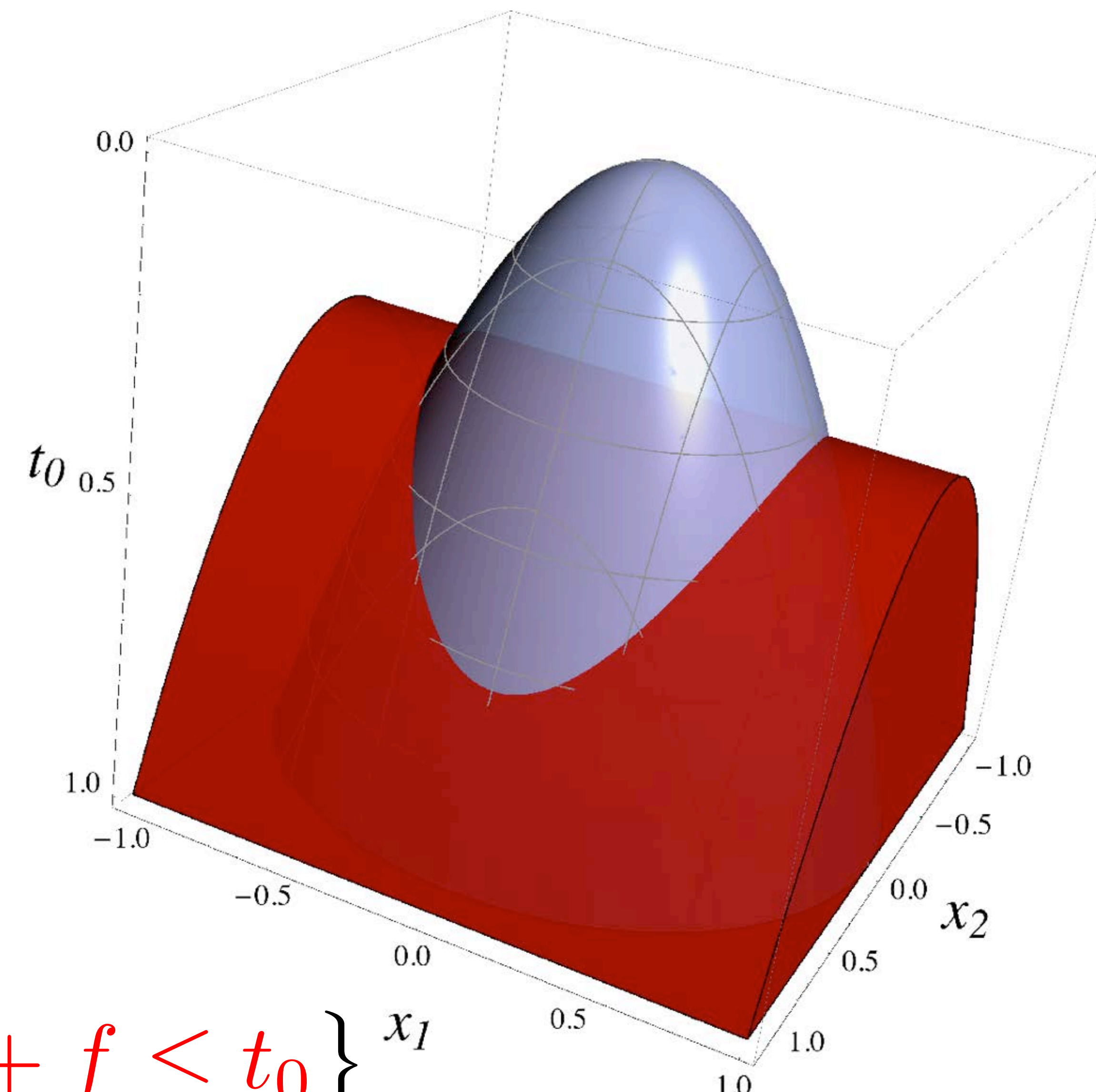
Case Gives Cuts for All Paraboloids

- Modaresi, Kılınç, V. 2013:

$$C := \left\{ (x, t_0) \in \mathbb{R}^n \times \mathbb{R} \ : \ \|B(x - c)\|_2^2 \leq t_0 \right\}$$

$$S := \left\{ (x, t_0) \in \mathbb{R}^n \times \mathbb{R} \ : \ \|A(x - d)\|_2^2 \leq 1 \right\}$$

$$\operatorname{conv}(C \setminus S) = \left\{ (x, t_0) \in \mathbb{R}^n \times \mathbb{R} \ : \ f(x) \leq t_0, \ x^T E x + a^T x + f \leq t_0 \right\}$$



- Also Bienstock and Michalka 2011 and 2013.

Summary

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  - General geometry for quadratic sets.
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- Cut Name? Intersection/Concavity Cuts?