A Lifted Linear Programming Branch-and-Bound Algorithm for Mixed Integer Conic Quadratic Programs

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Outline

Mixed Integer Non-Linear Programming (MINLP) Problems

$$
\begin{aligned}
z_{\text{MINLP}} := \max_{x,y} \quad & cx + dy \\
s.t. \quad & \\
& (x, y) \in \mathcal{C} \subset \mathbb{R}^{n+p} \\
& x \in \mathbb{Z}^n
\end{aligned}
\qquad \text{(MINLP)}
$$

- $\mathcal{C}$ is a convex compact set.
- Assume for simplicity that MINLP is feasible.
- Also let NLP be the nonlinear continuous relaxation obtained by eliminating $x \in \mathbb{Z}^n$.

Two Algorithm Approaches for MINLP

- Non-linear programming (NLP) based branch-and-bound algorithms (Borchers and Mitchell, 1994; Gupta and Ravindran, 1985, Leyffer 2001 and Stubbs and Mehrotra, 1999):
  - Analog of LP branch-and-bound for MILP.
  - Implementations: CPLEX 9.0 and 10.0 (ILOG, 2005) and I-BB solver in Bonmin (Bonami et al., 2005)
- Polyhedral relaxation based algorithms:
  - Outer approximation (Duran and Grossmann, 1986; Fletcher and Leyffer, 1994)
  - Generalized Benders decomposition (Geoffrion, 1972).
  - LP/NLP-based branch-and-bound (Quesada and Grossmann, 1992).
  - Extended cutting plane method (Westerlund and Pettersson, 1995;Westerlund et al., 1994).
  - Implementations: I-OA, I-QG and I-Hyb solvers in Bonmin, MINLP solver FilMINT (Abhishek et al., 2006).

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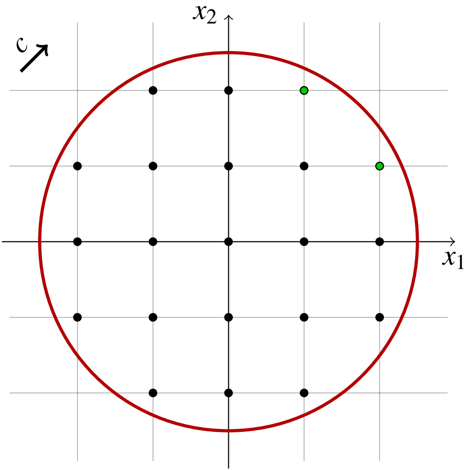
Branch-and-Bound Methods

- A branch-and-bound node is defined by $(l^k, u^k) \in \mathbb{Z}^{2n}$.
- The problem solved in a branch-and-bound node (l^k, u^k) is obtained by adding $l^k \leq x \leq u^k$ to some continuous relaxation of MINLP.
- Example:

$$
\begin{aligned}
z_{\text{NLP}(l^k,u^k)} := \max_{x,y} \quad & cx + dy \\
s.t. \quad & \\
& (x,y) \in \mathcal{C} \subset \mathbb{R}^{n+p} \qquad \text{(NLP}(l^k,u^k)) \\
& x \geq l^k \\
& x \leq u^k
\end{aligned}
$$

NLP Based Branch-and-Bound Algorithms



$$\max_x \quad x_1 + x_2$$
$$x \in \mathcal{B}^2(2.5) \quad \text{(MINLP)}$$
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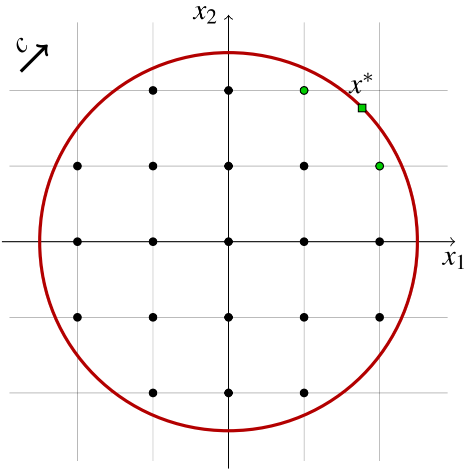

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NLP$((-\infty, -\infty)^\top, (\infty, \infty)^\top)$:

- $x_1^* = x_2^* \approx 1.77 \notin \mathbb{Z}$.
- Branch: $x_1 \leq 1 \vee x_1 \geq 2$.

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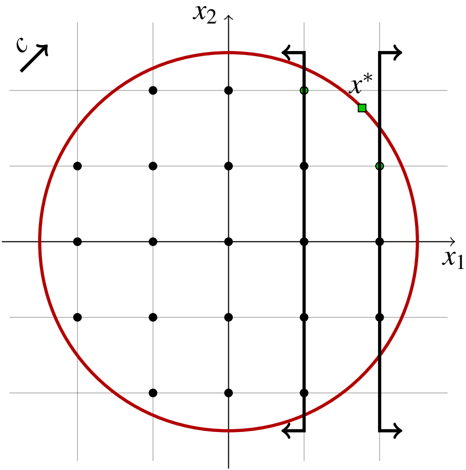

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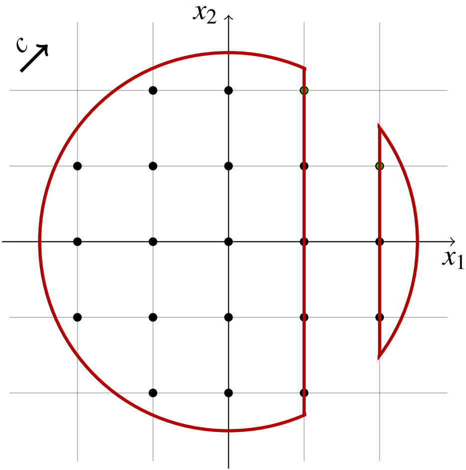

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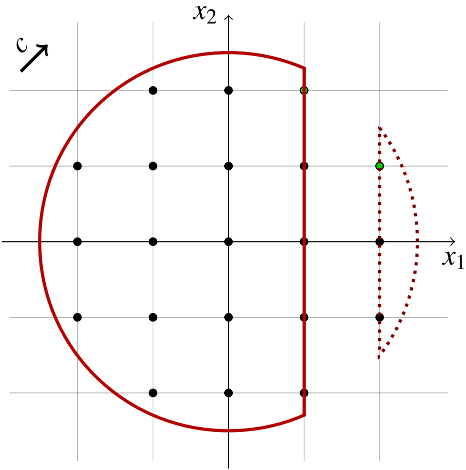

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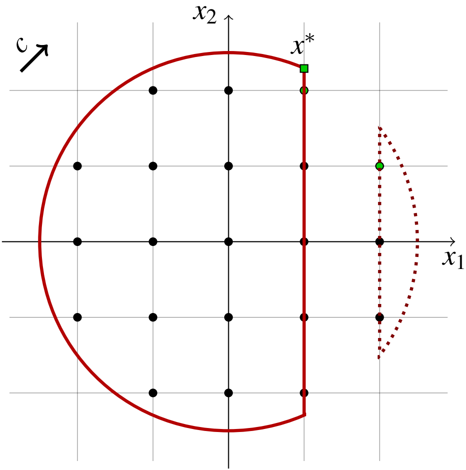

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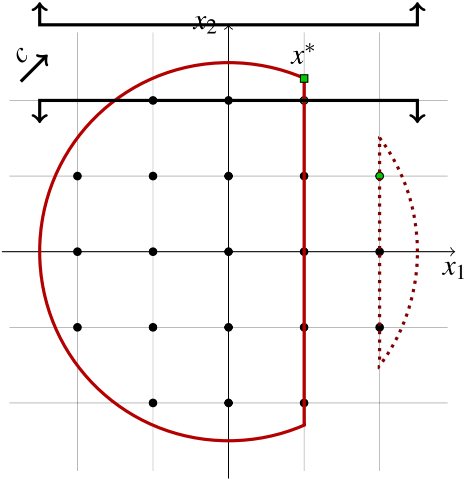

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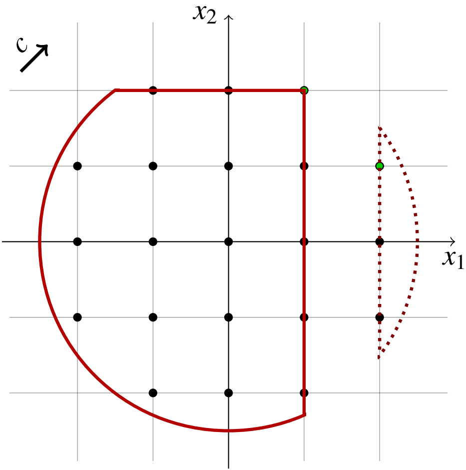

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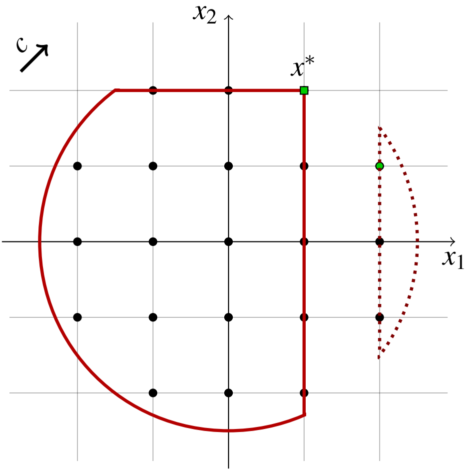

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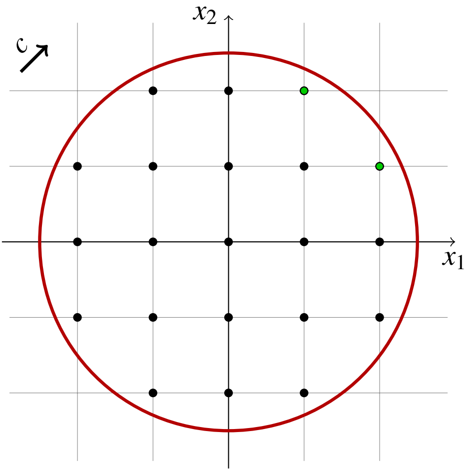

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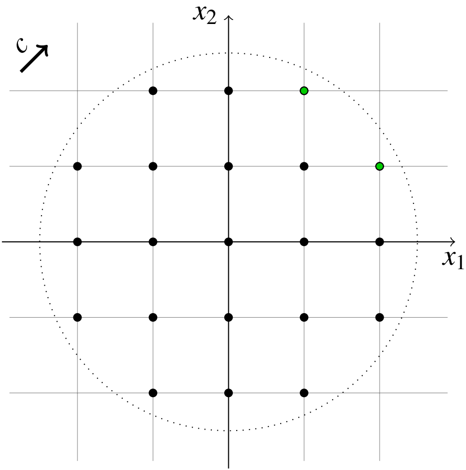

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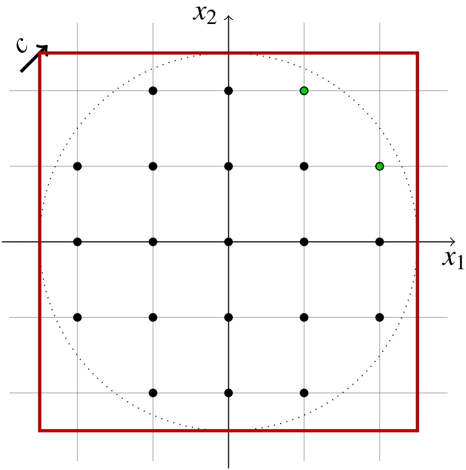

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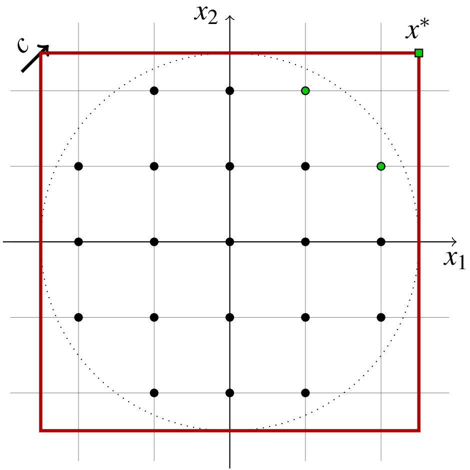

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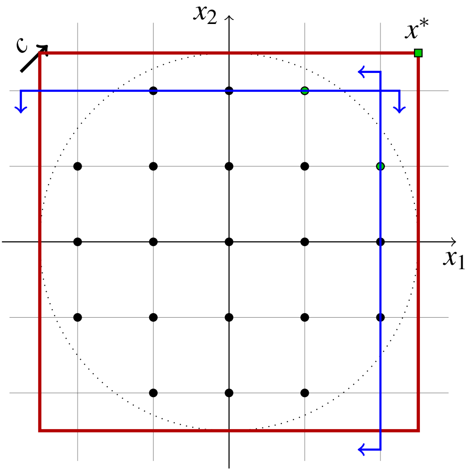

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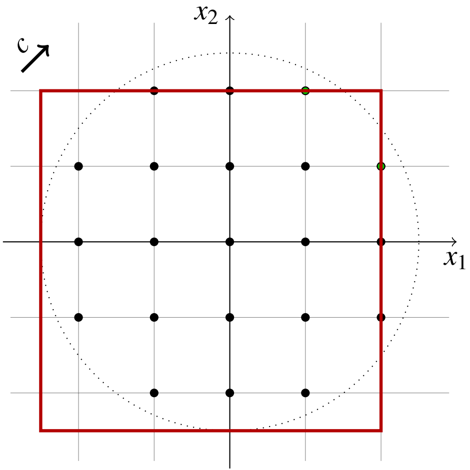

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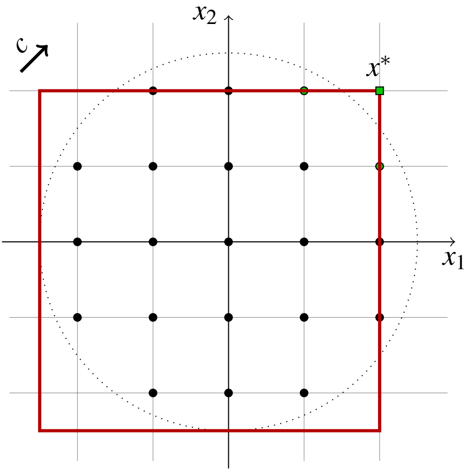

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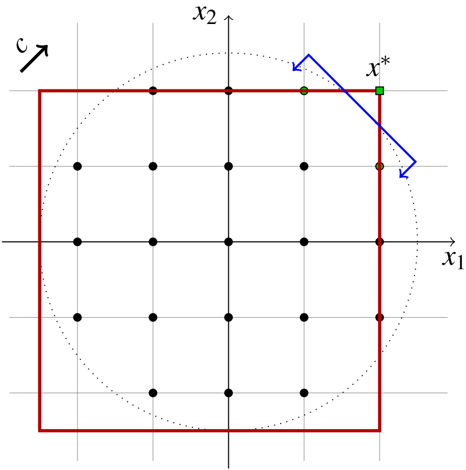

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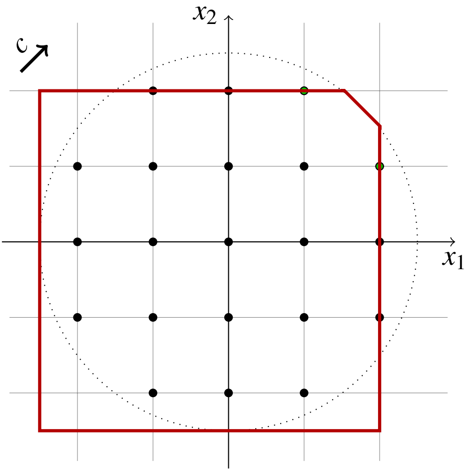

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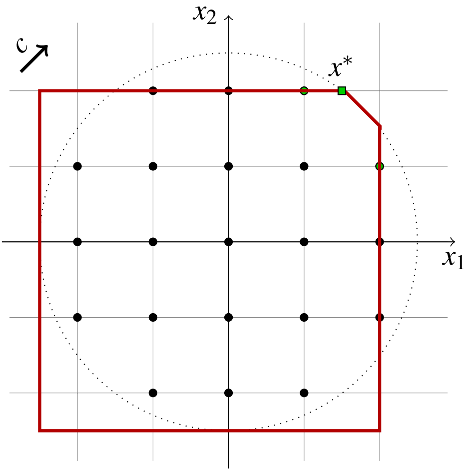

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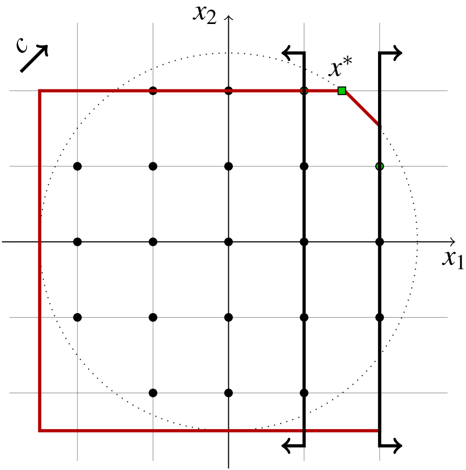

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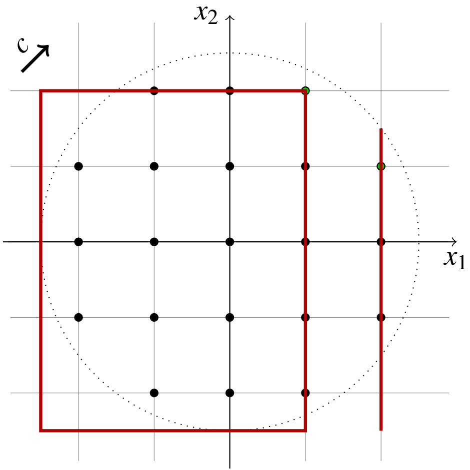

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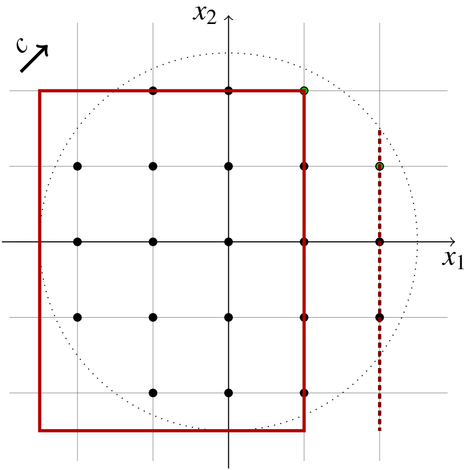

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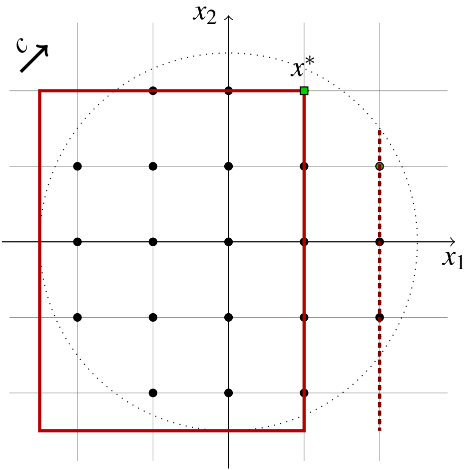

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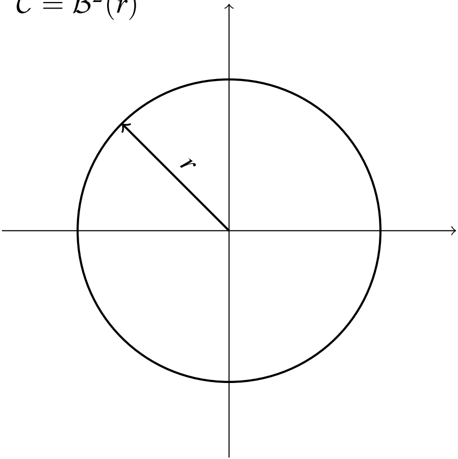
Polyheral Relaxation of Convex Sets

$\mathcal{C} = \mathcal{B}^2(r)$



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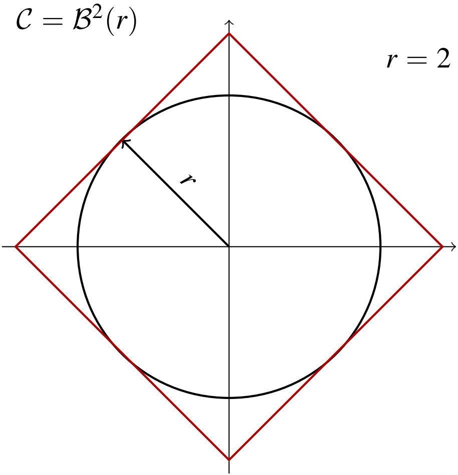

Lets relax to a regular 2^k sided polygon $\mathcal{P}_k$.

- $\mathcal{B}^2(r) \subset \mathcal{P}_k \subset (1+\varepsilon)\mathcal{B}^2(r)$ for $\varepsilon = \cos(\pi/2^k)^{-1} - 1$.
- Good news: ϵ decreases fast with k.
- Bad news: $\mathcal{P}_k$ has exponential in k number of inequalities.

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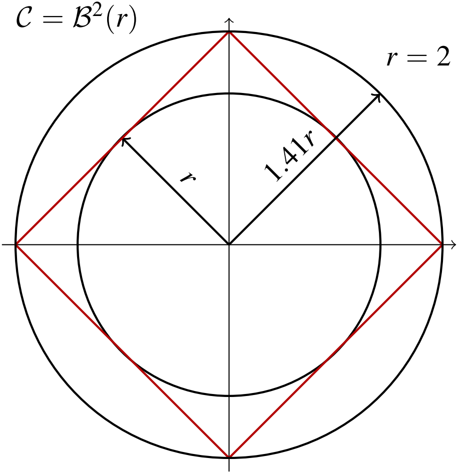

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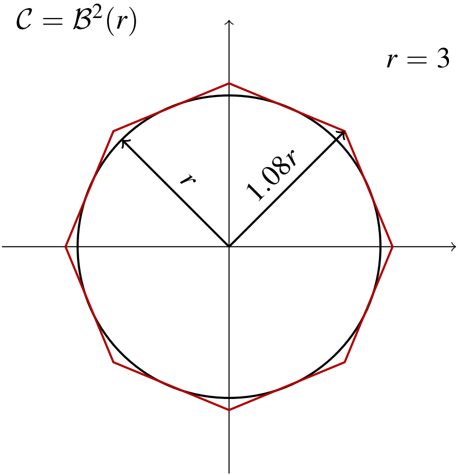

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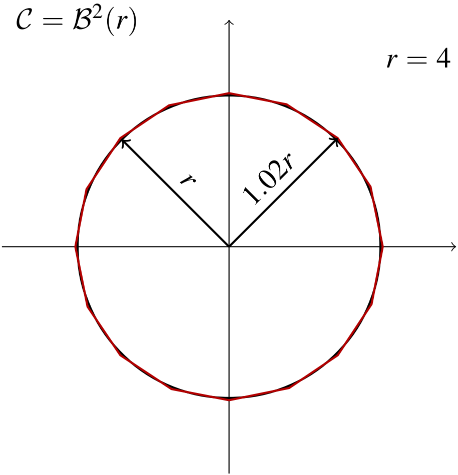

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- Really bad news: Any $\mathcal{P} \subset \mathbb{R}^d$ such that

$$\mathcal{B}^d(1) \subset \mathcal{P} \subset (1+\varepsilon)\mathcal{B}^d(1)$$

  has at least $\exp(d/(2(1+\varepsilon))^2)$ facets.
- Possible solution: Projection of $\mathcal{P} \subset \mathbb{R}^{d+q}$ to $\mathbb{R}^d$ can have an exponential (w/r to facets and variables of $\mathcal{P}$) number of facets.
- Exploiting this Ben-Tal and Nemirovski (Ben-Tal and Nemirovski, 2001) gave a relaxation of $\mathcal{B}^d(1)$ with $O(d\log(1/\varepsilon))$ facets and extra variables (Construct 2^k sided polygon using projection and an extra trick).
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- Really bad news: Any $\mathcal{P} \subset \mathbb{R}^d$ such that

$$\mathcal{B}^d(1) \subset \mathcal{P} \subset (1+\varepsilon)\mathcal{B}^d(1)$$

  has at least $\exp(d/(2(1+\varepsilon))^2)$ facets.
- Possible solution: Projection of $\mathcal{P} \subset \mathbb{R}^{d+q}$ to $\mathbb{R}^d$ can have an exponential (w/r to facets and variables of $\mathcal{P}$) number of facets.
- Exploiting this Ben-Tal and Nemirovski (Ben-Tal and Nemirovski, 2001) gave a relaxation of $\mathcal{B}^d(1)$ with $O(d\log(1/\varepsilon))$ facets and extra variables (Construct 2^k sided polygon using projection and an extra trick).
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- Higher dimensional or lifted polyhedral relaxation of convex sets.

Lifted Linear Programming Relaxation of MINLP

$$
\begin{aligned}
z_{\text{MINLP}} := \max_{x,y} \quad & cx + dy \\
s.t. \quad & \\
& (x, y) \in \mathcal{C} \subset \mathbb{R}^{n+p} \\
& x \in \mathbb{Z}^n
\end{aligned} \qquad \text{(MINLP)}
$$

- Polyhedron $\mathcal{P} \subset \mathbb{R}^{n+p+q}$ such that:

$$\mathcal{C} \subset \{(x, y) \in \mathbb{R}^{n+p} : \exists v \in \mathbb{R}^q \text{ s.t. } (x, y, v) \in \mathcal{P}\}.$$

- We get the relaxation of MINLP (and NLP):

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- We get the relaxation of MINLP (and NLP):

$$z_{\text{LP}} := \max_{x,y,v} \quad cx + dy$$
$$s.t.$$
$$(x,y,v) \in \mathcal{P} \qquad \text{(LP)}$$

Branch-and-Bound Main Loop

Set global lower bound $\mathrm{LB} := -\infty$.
Set $l_i^0 := -\infty$, $u_i^0 := +\infty$ for all $i \in \{1, \dots, n\}$.
Set node list $\mathcal{H} := \{(l^0, u^0)\}$.
while $\mathcal{H} \neq \emptyset$ **do**
    Select and remove a node $(l^k, u^k) \in \mathcal{H}$.
    `ProcessNode`(l^k, u^k).
end

ProcessNode(l^k, u^k) Version 1

Solve LP(l^k, u^k) (Let (x^*, y^*) be the optimal solution).
if LP(l^k, u^k) *is feasible* **and** $z_{\mathrm{LP}(l^k,u^k)} > \mathrm{LB}$ **then**
    if $x^* \in \mathbb{Z}^n$ **then**
        Update LB to $z_{\mathrm{LP}(l^k,u^k)}$.
    else
        Branch on x^* and add nodes to $\mathcal{H}$.
    end
end

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Solve LP(l^k, u^k) (Let (x^*, y^*) be the optimal solution).
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        Update LB to $z_{\text{LP}(l^k,u^k)}$.
    else
        Branch on x^* and add nodes to $\mathcal{H}$.
    end
end

Auxiliary Problem: Correct solution x^*

- For $x^* \in \mathbb{Z}^n$:

$$
\begin{aligned}
z_{\text{NLP}(x^*)} := \max_{y} \quad & cx^* + dy \\
s.t. \quad & \\
& (x^*, y) \in \mathcal{C} \subset \mathbb{R}^{n+p}. \qquad (\text{NLP}(x^*))
\end{aligned}
$$

$(\mathrm{LB}, \mathcal{H}) :=$ `ProcessNode`$(l^k, u^k, \mathrm{LB}, \mathcal{H})$ Version 2

Solve LP(l^k, u^k) (Let (x^*, y^*) be the optimal solution).

if LP(l^k, u^k) *is feasible* **and** $z_{\text{LP}(l^k,u^k)} > \mathrm{LB}$ **then**

    if $x^* \in \mathbb{Z}^n$ **then**

        Solve NLP(x^*).

        if NLP(x^*) *is feasible* **and** $z_{\text{NLP}(x^*)} > \mathrm{LB}$ **then**

            Update LB to $z_{\text{NLP}(x^*)}$.

        end

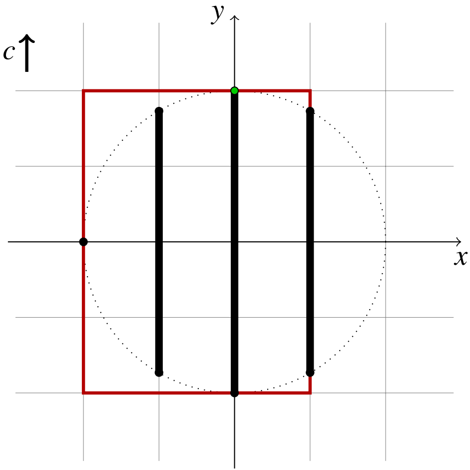
    else

        Branch on x^* and add nodes to $\mathcal{H}$.

    end

end

Correcting Integer Feasible Solutions is Not Enough



$$\max_{x,y} \quad y$$
$$(x, y) \in \mathcal{B}^2(2) \quad \text{(MINLP)}$$
$$x \in \mathbb{Z}$$

$$\max_{x,y} \quad y$$
$$(x, y) \in [-2, 2]^2 \quad \text{(LP)}$$

LP$(-\infty, 1)$:

- $x^* = 1, y^* = 2$, $(x, y) \notin \mathcal{B}^2(2)$.
- NLP$(x^*) \rightarrow (x^{cor}, y^{cor})$.
- If we fathom we loose optimum $(0, 2)$!

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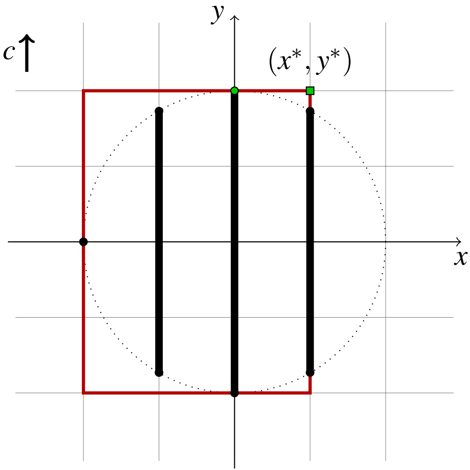

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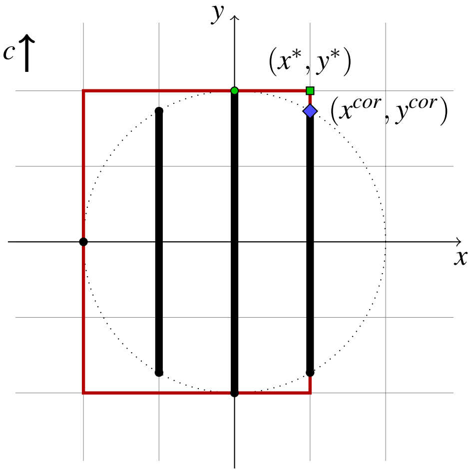

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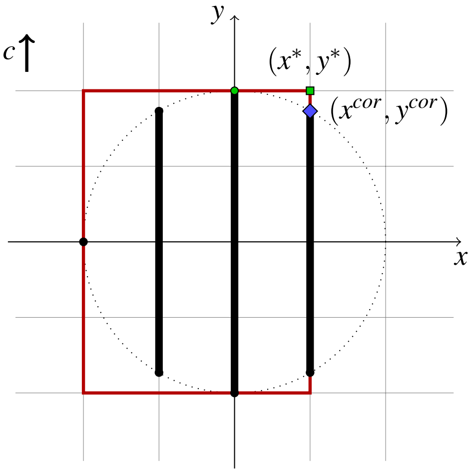

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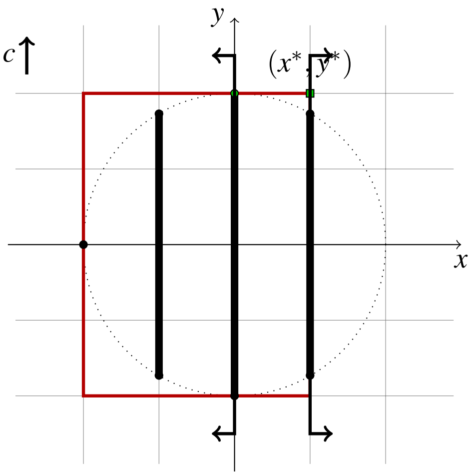

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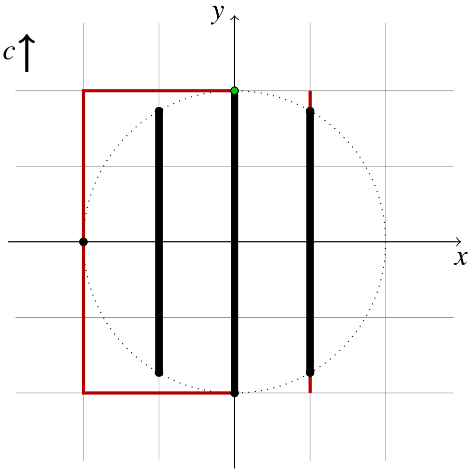
- Branch: $x \leq 0 \vee x \geq 1$.
- Solve LP$(-\infty, 0)$.
- We get optimum $(0, 2)$.

Correcting Integer Feasible Solutions is Not Enough



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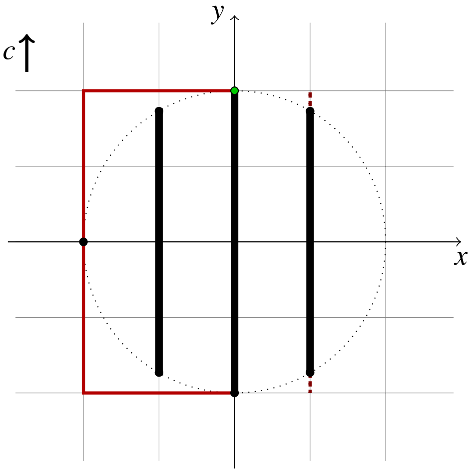

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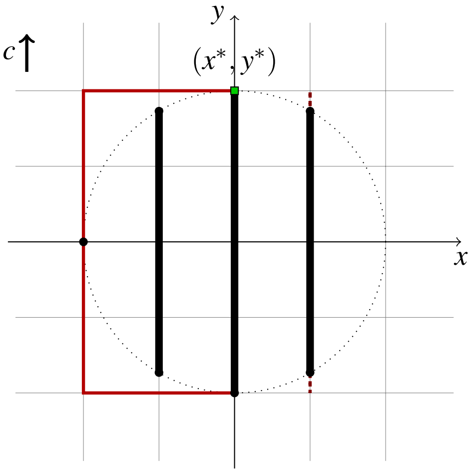

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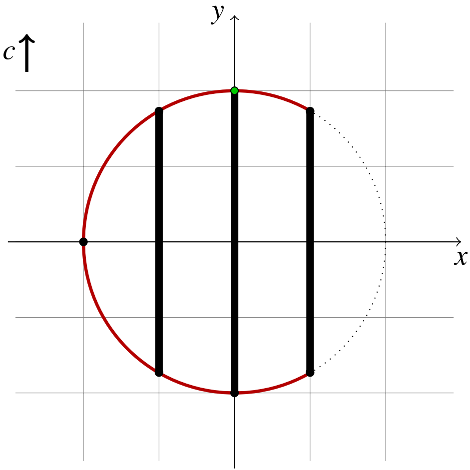

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Solution 2:

- Solve NLP$(-\infty, 1)$.
- We get optimum $(0, 2)$.

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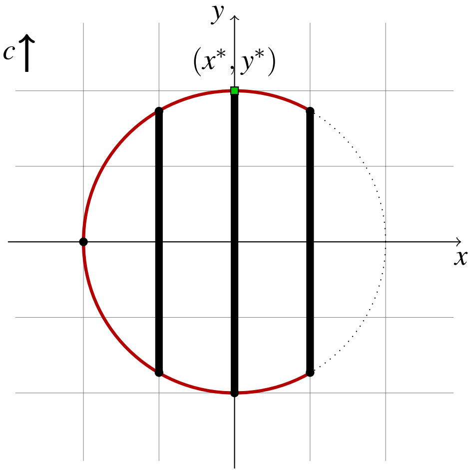

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$(\mathrm{LB}, \mathcal{H}) :=$ `ProcessNode`$(l^k, u^k, \mathrm{LB}, \mathcal{H})$ Final Version

Solve LP(l^k, u^k) (Let (x^*, y^*) be the optimal solution).
if LP(l^k, u^k) *is feasible* **and** $z_{\text{LP}(l^k,u^k)} > \mathrm{LB}$ **then**
    if $x^* \in \mathbb{Z}^n$ **then**
        Solve NLP(x^*).
        if NLP(x^*) *is feasible* **and** $z_{\text{NLP}(x^*)} > \mathrm{LB}$ **then**
            Update LB to $z_{\text{NLP}(x^*)}$.
        end
        Extra Steps
    else
        Branch on x^* and add nodes to $\mathcal{H}$.
    end
end

$(\mathrm{LB}, \mathcal{H}) := \texttt{ProcessNode}(l^k, u^k, \mathrm{LB}, \mathcal{H})$ Final Version

if $l^k \neq u^k$ **then**

  Solve NLP(l^k, u^k) (Let $(\tilde{x}, \tilde{y})$ be the optimal solution).

  if NLP(l^k, u^k) *is feasible* **and** $z_{\text{NLP}(l^k,u^k)} > \mathrm{LB}$ **then**

    if $\tilde{x} \in \mathbb{Z}^n$ **then**

      Update LB to $z_{\text{NLP}(l^k,u^k)}$.

    else

      Branch on $\tilde{x}$ and add nodes to $\mathcal{H}$.

    end

  end

end

Mixed Integer Conic Quadratic Programming Problems

$$
\begin{aligned}
z_{\text{MICP}} := \max_{x,y} \quad & cx + dy \\
s.t. \quad & \\
& Dx + Ey \leq f \\
& (x, y) \in \mathcal{CC}_i \qquad i \in \mathcal{I} \\
& (x, y) \in \mathbb{R}^{n+p} \\
& x \in \mathbb{Z}^n
\end{aligned} \qquad \text{(MICP)}
$$

- $\mathcal{CC}_i$ is a conic quadratic constraint of the form

$$\mathcal{CC} := \{(x, y) \in \mathbb{R}^{n+p} \ : \ ||Ax + By + \delta||_2 \leq ax + by + \delta_0\}$$

- Also let CP be the continuous relaxation obtained by eliminating $x \in \mathbb{Z}^n$.

Ben-Tal Nemirovski Polyhedral Relaxation of CP

$$
\begin{aligned}
z_{\text{LP}(\varepsilon)} := \max_{x,y,v} \quad & cx + dy \\
s.t. \quad & \\
& Dx + Ey \leq f \\
& (x,y,v) \in \mathcal{P}(\mathcal{CC}_i, \varepsilon) \quad i \in \mathcal{I} \\
& (x,y,v) \in \mathbb{R}^{n+p+q},
\end{aligned}
\qquad (\text{LP}(\varepsilon))
$$

- $\mathcal{P}(\mathcal{CC}_i, \varepsilon)$ polyhedron with $O((n+p)\log(1/\varepsilon))$ variables and constraints.

Implementation of Lifted LP Branch-and-Bound Algorithm: LP(ε)-BB

- Using a version of the Ben-Tal Nemirovski relaxation introduced by Glineur.
- Implemented by modifying CPLEX 10's MILP solver.
- C++, Ilog Concert Technology. Branch, incumbent and heuristic callbacks.
- $\varepsilon = 0.01$ was selected after calibration experiments.

Computational Experiments

- Dual 2.4GHz Xeon workstation with 2GB of RAM running Linux Kernel 2.4.
- LP(ε) -BB v/s CPLEX 10's MIQCP solver and Bonmin's I-BB, I-QG and I-Hyb.
- Test set: Portfolio optimization problems with cardinality constraints (Ceria and Stubbs, 2006; Lobo et al., 1998, 2007).

Problem 1: Classical

$$
\begin{aligned}
\max_{x,y} \quad & \bar{a}y \\
s.t. \quad & \\
& ||Q^{1/2}y||_2 \leq \sigma \\
& \sum_{j=1}^{n} y_j = 1 \\
& y_j \leq x_j \qquad \forall j \in \{1,\ldots,n\} \\
& \sum_{j=1}^{n} x_j \leq K \\
& x \in \{0,1\}^n \\
& y \in \mathbb{R}^n_+
\end{aligned}
$$

- y fraction of the portfolio invested in each of n assets.
- $\bar{a}$ expected returns of assets.
- $Q^{1/2}$ positive semidefinite square root of the covariance matrix Q of returns.
- K maximum number of assets to hold.

Problem 2 : Shortfall

$$
\begin{aligned}
\max_{x,y} \quad & \bar{a}y \\
s.t. \quad & \\
& ||Q^{1/2}y||_2 \leq \sigma \\
& \sum_{j=1}^{n} y_j = 1 \\
& y_j \leq x_j \qquad \forall j \in \{1, \ldots, n\} \\
& \sum_{j=1}^{n} x_j \leq K \\
& x \in \{0, 1\}^n \\
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\end{aligned}
$$

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- $\bar{a}$ expected returns of assets.
- $Q^{1/2}$ positive semidefinite square root of the covariance matrix Q of returns.
- K maximum number of assets to hold.

Problem 2 : Shortfall

$$\max_{x,y} \quad \bar{a}y$$

$$s.t.$$

$$||Q^{1/2}y||_2 \leq \frac{\bar{a}y - W_i^{low}}{\Phi^{-1}(\eta_i)} \qquad i \in \{1,2\}$$

$$\sum_{j=1}^{n} y_j = 1$$

$$y_j \leq x_j \qquad \forall j \in \{1,\ldots,n\}$$

$$\sum_{j=1}^{n} x_j \leq K$$

$$x \in \{0,1\}^n$$

$$y \in \mathbb{R}_+^n$$

- y fraction of the portfolio invested in each of n assets.
- $\bar{a}$ expected returns of assets.
- $Q^{1/2}$ positive semidefinite square root of the covariance matrix Q of returns.
- K maximum number of assets to hold.
- Approximation of $\text{Prob}(\bar{a}y \geq W_i^{low}) \geq \eta_i$

Problem 3 : Robust

$$\max_{x,y,r} \quad r$$

$$s.t.$$

$$||Q^{1/2}y||_2 \leq \sigma$$

$$\alpha||R^{1/2}y||_2 \leq \bar{a}y - r$$

$$\sum_{j=1}^{n} y_j = 1$$

$$y_j \leq x_j \qquad \forall j \in \{1, \ldots, n\}$$

$$\sum_{j=1}^{n} x_j \leq K$$

$$x \in \{0,1\}^n$$

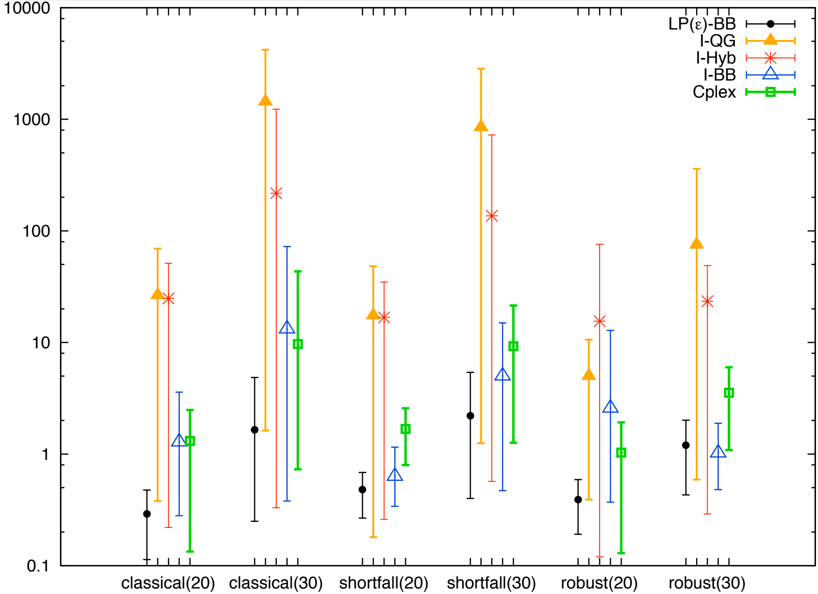
$$y \in \mathbb{R}^n_+$$

- y fraction of the portfolio invested in each of n assets.
- $\bar{a}$ expected returns of assets.
- $Q^{1/2}$ positive semidefinite square root of the covariance matrix Q of returns.
- K maximum number of assets to hold.
- Robust version from uncertainty in $\bar{a}$.

Instance Data

- Maximum number of stocks $K = 10$.
- Maximum risk $\sigma = 0.2$.
- Shortfall constraints: $\eta_1 = 80\%$, $W_1^{low} = 0.9$, $\eta_2 = 97\%$, $W_2^{low} = 0.7$ (Lobo et al., 1998, 2007).
- Data generation for Classical and Shortfall from S&P 500 data following Lobo et al. (1998), (2007).
- Data generation for Robust from S&P 500 data following Ceria and Stubbs (2006).
- Riskless asset included for Shortfall.
- Random selection of n stocks out of 462.
- 100 instances for $n \in \{20, 30, 40, 50\}$, 10 for $n \in \{100, 200\}$.

Average Solve Times [s] for $n \in \{20, 30\}$



Performance Profiles

	Solve Time				Ratio to Best		
Solver	A	B	C	Best	A	B	C
Instance 1	2	8	16	2	1	4	8

Performance Profiles

	Solve Time				Ratio to Best		
Solver	A	B	C	Best	A	B	C
Instance 1	2	8	16	2	1	4	8
Instance 2	10	5	20	5	2	1	4
Instance 3	100	300	-	100	1	3	∞

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	Solve Time				Ratio to Best		
Solver	A	B	C	Best	A	B	C
Instance 1	2	8	16	2	1	4	8
Instance 2	10	5	20	5	2	1	4
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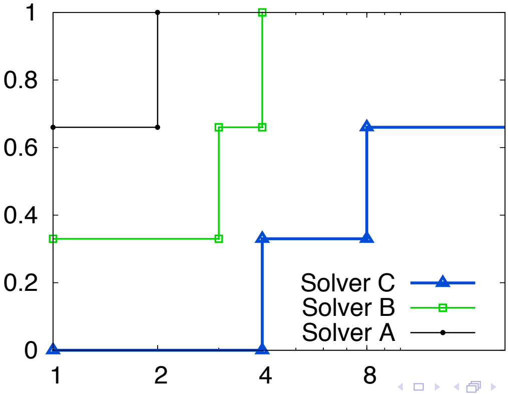
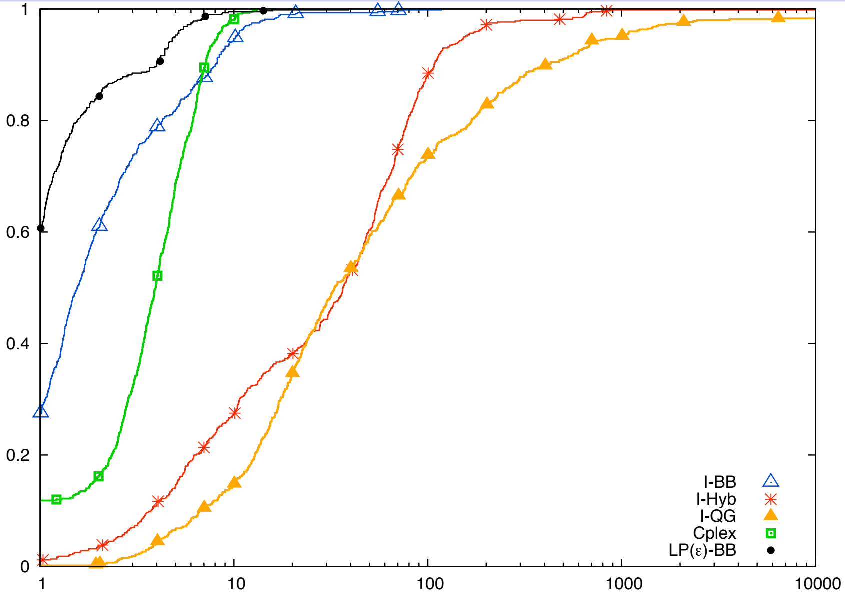
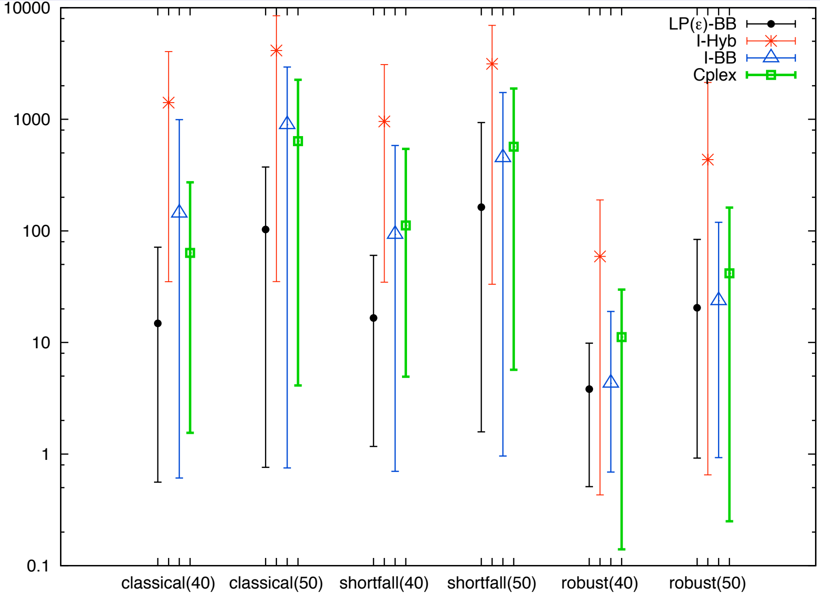


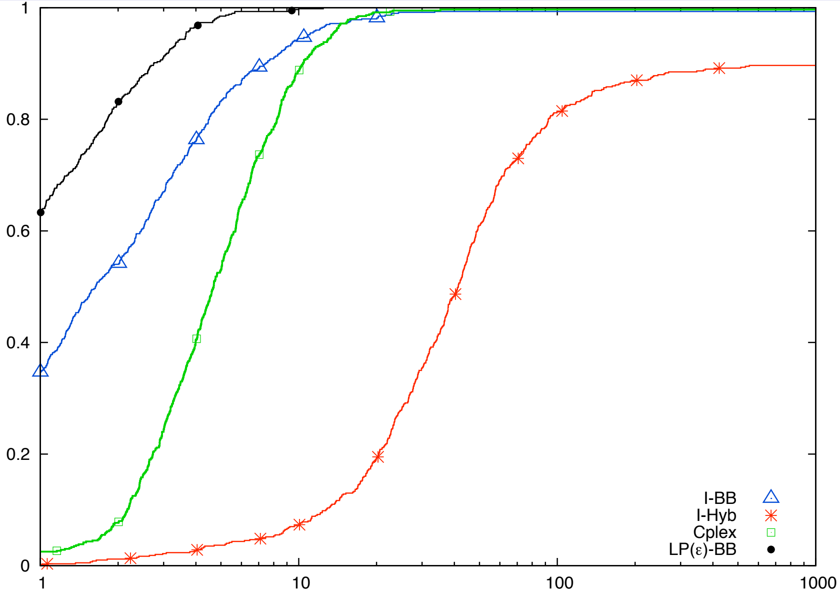
Performance Profile for $n \in \{20, 30\}$



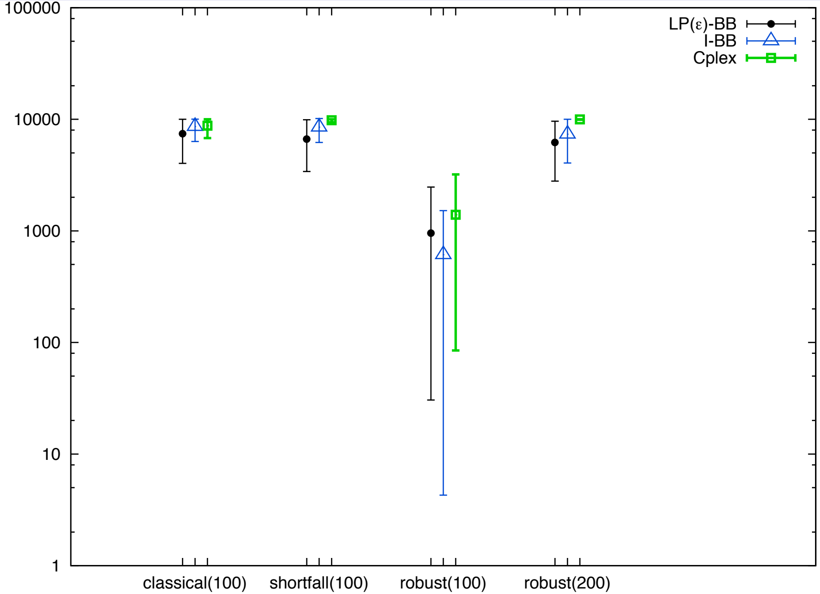
Average Solve Times [s] for $n \in \{40, 50\}$



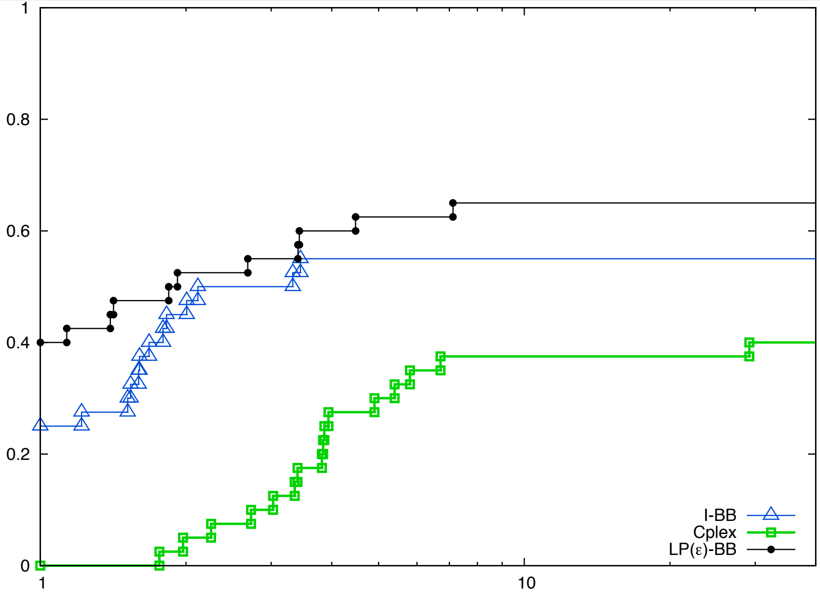
Performance Profile for $n \in \{40, 50\}$



Average Solve Times [s] for $n \in \{100, 200\}$



Performance Profile for $n \in \{100, 200\}$



Total Number of Nodes and Calls to Relaxations for Small Instances

B-and-b nodes I-QG	3580051
B-and-b nodes I-Hyb	328316
B-and-b nodes I-BB	68915
B-and-b nodes CPLEX	85957
B-and-b nodes LP(ε)-BB	57933
LP(ε)-BB calls to CP(l^k, u^k)	2305
LP(ε)-BB calls to CP(x^*)	7810

Final Remarks

- Polyhedral relaxation algorithm for MINLP:
  - Based on a lifted polyhedral relaxation.
  - Branches on integer feasible solutions.
  - “Does not update the relaxation“.
- Algorithm for the conic quadratic case:
  - Based on a lifted polyhedral relaxation by Ben-Tal and Nemirovski.
  - Implemented by modifying CPLEX MILP solver.
  - Significantly outperforms other methods for portfolio optimization problems.