A Lifted Linear Programming Branch-and-Bound Algorithm for Mixed Integer Conic Quadratic Programs

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Outline

"Convex" Mixed Integer Non-Linear Programming (MINLP) Problems

$$
\begin{aligned}
z_{\text{MINLP}} := \max_{x,y} \quad & cx + dy \\
s.t. \quad & (x,y) \in \mathcal{C} \subset \mathbb{R}^{n+p} \\
& x \in \mathbb{Z}^n
\end{aligned}
\qquad \text{(MINLP)}
$$

- $\mathcal{C}$ is a convex compact set.
- Advanced algorithms and Software:
  - NLP based branch-and-bound algorithms (Borchers and Mitchell, 1994, Gupta and Ravindran, 1985, Leyffer 2001 and Stubbs and Mehrotra, 1999,...)
  - Polyhedral relaxation based algorithms (Duran and Grossmann, 1986, Fletcher and Leyffer, 1994, Geoffrion, 1972,Quesada and Grossmann, 1992, Westerlund and Pettersson, 1995,Westerlund et al., 1994,...)
  - CPLEX 9.0+ (ILOG, 2005), Bonmin (Bonami et al., 2005), FilMINT (Abhishek et al., 2006), ...

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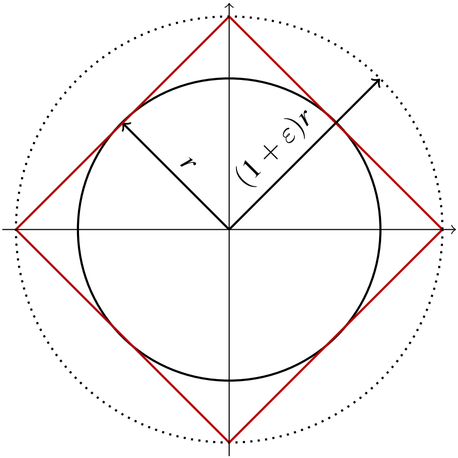
  - CPLEX 9.0+ (ILOG, 2005), Bonmin (Bonami et al., 2005), FilMINT (Abhishek et al., 2006), . . .
- Polyhedral relaxation algorithms try to exploit the technology for Mixed Integer Linear Programming

Polyheral Relaxation of Convex Sets

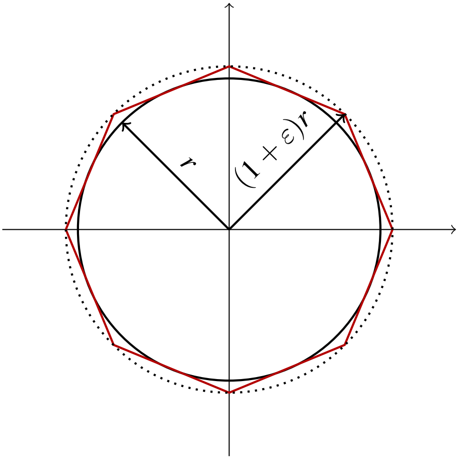
$\mathcal{C} = \mathcal{B}^d(r)$, $d = 2$, $\varepsilon = 0.41$



- It is known that at least $\exp(d/(2(1+\varepsilon))^2)$ facets are needed in the original space.
- Ben-Tal and Nemirovski (2001) approximate $\mathcal{B}^d(r)$ as the projection of a polyhedron with $O(d \log(1/\varepsilon))$ variables and constraints.
- Glineur (2000) refined the approximation and showed that it is algorithmically and computationally “impractical” for (pure continuous) conic quadratic optimization.

Polyheral Relaxation of Convex Sets

$\mathcal{C} = \mathcal{B}^d(r),\ d = 2,\ \varepsilon = 0.08$



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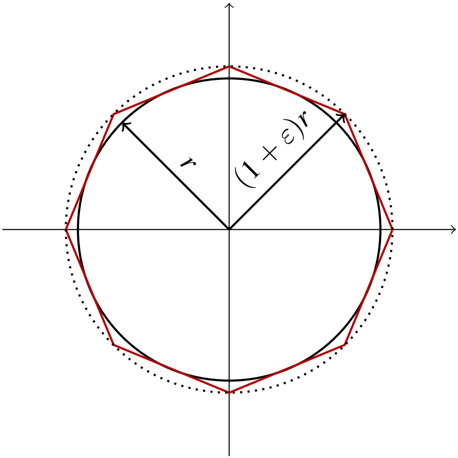

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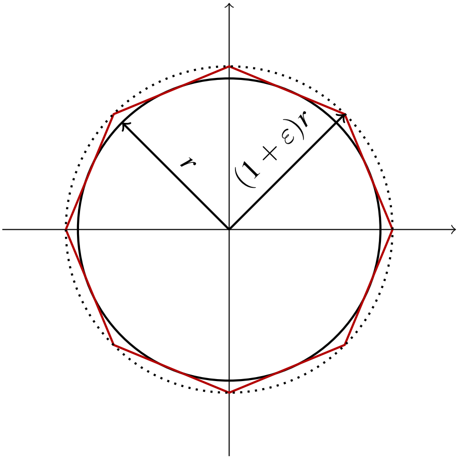

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Using Ben-Tal Nemirovski Approximation to Exploit Mixed Integer Linear Programming Solver Technology

- Lifted linear programming relaxation: Polyhedron $\mathcal{P} \subset \mathbb{R}^{n+p+q}$ such that

$$\mathcal{C} \subset \{(x,y) \in \mathbb{R}^{n+p} \ : \ \exists\, v \in \mathbb{R}^{q} \text{ s.t. } (x,y,v) \in \mathcal{P}\} \approx \mathcal{C}$$

- Use a state of the art MILP solver to solve

$$\begin{aligned} \max_{x,y,v} \quad & cx + dy \\ s.t. \quad & (x,y,v) \in \mathcal{P} \\ & x \in \mathbb{Z}^n \end{aligned} \qquad \text{(MILP)}$$

- Problem: Obtained solution might not even be feasible for MINLP
- Solution: Modify Solve of MILP

Idea: Simulate NLP Branch-and-Bound

- Problem solved in NLP B&B node $(l^k, u^k) \in \mathbb{Z}^{2n}$ is:

$$
\begin{aligned}
z_{\mathrm{NLP}(l^k,u^k)} := \max_{x,y} \quad & cx + dy \\
s.t. \quad & (x,y) \in \mathcal{C} \subset \mathbb{R}^{n+p} \\
& l^k \le x \le u^k
\end{aligned}
\qquad (\mathrm{NLP}(l^k, u^k))
$$

- Problem solved by state of the art MILP solver is:

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- Advantages of second subproblem:
  - Algorithmic Advantage: Simplex has warm starts.
  - Computational Advantage: Use MILP solver's technology.

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- Issues:
  1. Integer feasible solutions may be infeasible for $\mathcal{C}$.
  2. Need to be careful when fathoming by integrality.

First Issue: Correcting Integer Feasible Solutions

- Let $(x^*, y^*, v^*) \in \mathcal{P}$ such that $x^* \in \mathbb{Z}^n$, but $(x^*, y^*) \notin \mathcal{C}$.
- We reject (x^*, y^*, v^*) and try to correct it using:

$$z_{\text{NLP}(x^*)} := \max_{y} \quad cx^* + dy$$
$$s.t.$$
$$(x^*, y) \in \mathcal{C} \subset \mathbb{R}^{n+p}. \qquad (\text{NLP}(x^*))$$

- This can be done for solutions found by heuristics, at integer feasible nodes, etc.

Second Issue: Correct Fathoming by Integrality

- Suppose that for a node (l^k, u^k) with $l^k \neq u^k$ we have that the solution (x^*, y^*, v^*) of LP(l^k, u^k) is such that $x^* \in \mathbb{Z}^n$
- If $(x^*, y^*) \in \mathcal{C}$ then (x^*, y^*) is also the optimal for NLP(l^k, u^k) and we can fathom by integrality.
- If $(x^*, y^*) \notin \mathcal{C}$ it is not sufficient to solve NLP(x^*):
  - Problem: Corrected solution is not necessarily optimal for NLP(l^k, u^k).
  - Solution: Solve NLP(l^k, u^k) and process node according to its solution.

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Branch-and-Bound Main Loop

Set global lower bound $\mathrm{LB} := -\infty$.
Set $l_i^0 := -\infty$, $u_i^0 := +\infty$ for all $i \in \{1, \ldots, n\}$.
Set node list $\mathcal{H} := \{(l^0, u^0)\}$.
while $\mathcal{H} \neq \emptyset$ **do**
    Select and remove a node $(l^k, u^k) \in \mathcal{H}$.
    `ProcessNode`(l^k, u^k).
end

$(\mathrm{LB}, \mathcal{H}) := \texttt{ProcessNode}(l^k, u^k, \mathrm{LB}, \mathcal{H})$

```
Solve LP(l^k, u^k) (Let (x*, y*) be the optimal solution).
if LP(l^k, u^k) is feasible and z_{LP(l^k,u^k)} > LB then
    if x* ∈ Z^n then
        Solve NLP(x*).
        if NLP(x*) is feasible and z_{NLP(x*)} > LB then
            Update LB to z_{NLP(x*)}.
        end
        Extra Steps
    else
        Branch on x* and add nodes to H.
    end
end
```

$(\mathrm{LB}, \mathcal{H}) := \texttt{ProcessNode}(l^k, u^k, \mathrm{LB}, \mathcal{H})$

```
if l^k ≠ u^k then
    Solve NLP(l^k, u^k) (Let (x̃, ỹ) be the optimal solution).
    if NLP(l^k, u^k) is feasible and z_NLP(l^k,u^k) > LB then
        if x̃ ∈ Z^n then
            Update LB to z_NLP(l^k,u^k).
        else
            Branch on x̃ and add nodes to H.
        end
    end
end
```

Computational Experiments

- Implementation of Lifted LP B&B Algorithm (LP(ε)-BB):
  - Using Ben-Tal Nemirovski relaxation from Glineur (2000).
  - Implemented by modifying CPLEX 10's MILP solver using branch, incumbent and heuristic callbacks.
  - $\varepsilon = 0.01$ was selected after calibration experiments.
- Portfolio optimization problems with cardinality constraints (Ceria and Stubbs, 2006; Lobo et al., 1998, 2007)
- Computer and solvers:
  - Dual 2.4GHz Xeon Linux workstation with 2GB of RAM.
  - LP(ε)-BB v/s CPLEX 10's MIQCP solver and Bonmin's I-BB, I-QG and I-Hyb.

Problem 1: Classical

$$
\begin{aligned}
\max_{x,y} \quad & \bar{a}y \\
s.t. \quad & \\
& ||Q^{1/2}y||_2 \leq \sigma \\
& \sum_{j=1}^{n} y_j = 1 \\
& y_j \leq x_j \qquad \forall j \in \{1, \ldots, n\} \\
& \sum_{j=1}^{n} x_j \leq K \\
& x \in \{0,1\}^n \\
& y \in \mathbb{R}^n_+
\end{aligned}
$$

- y fraction of the portfolio invested in each of n assets.
- $\bar{a}$ expected returns of assets.
- $Q^{1/2}$ positive semidefinite square root of the covariance matrix Q of returns.
- K maximum number of assets to hold.

Problem 2 : Shortfall

$$
\begin{aligned}
\max_{x,y} \quad & \bar{a}y \\
s.t. \quad & \\
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Problem 2 : Shortfall

$$\max_{x,y} \quad \bar{a}y$$

$$s.t.$$

$$||Q^{1/2}y||_2 \leq \frac{\bar{a}y - W_i^{low}}{\Phi^{-1}(\eta_i)} \qquad i \in \{1,2\}$$

$$\sum_{j=1}^{n} y_j = 1$$

$$y_j \leq x_j \qquad \forall j \in \{1,\ldots,n\}$$

$$\sum_{j=1}^{n} x_j \leq K$$

$$x \in \{0,1\}^n$$

$$y \in \mathbb{R}^n_+$$

- y fraction of the portfolio invested in each of n assets.
- $\bar{a}$ expected returns of assets.
- $Q^{1/2}$ positive semidefinite square root of the covariance matrix Q of returns.
- K maximum number of assets to hold.
- Approximation of $\mathrm{Prob}(\bar{a}y \geq W_i^{low}) \geq \eta_i$

Problem 3 : Robust

$$
\begin{aligned}
\max_{x,y,r} \quad & r \\
s.t. \quad & \\
& ||Q^{1/2}y||_2 \leq \sigma \\
& \alpha||R^{1/2}y||_2 \leq \bar{a}y - r \\
& \sum_{j=1}^{n} y_j = 1 \\
& y_j \leq x_j \qquad \forall j \in \{1, \ldots, n\} \\
& \sum_{j=1}^{n} x_j \leq K \\
& x \in \{0, 1\}^n \\
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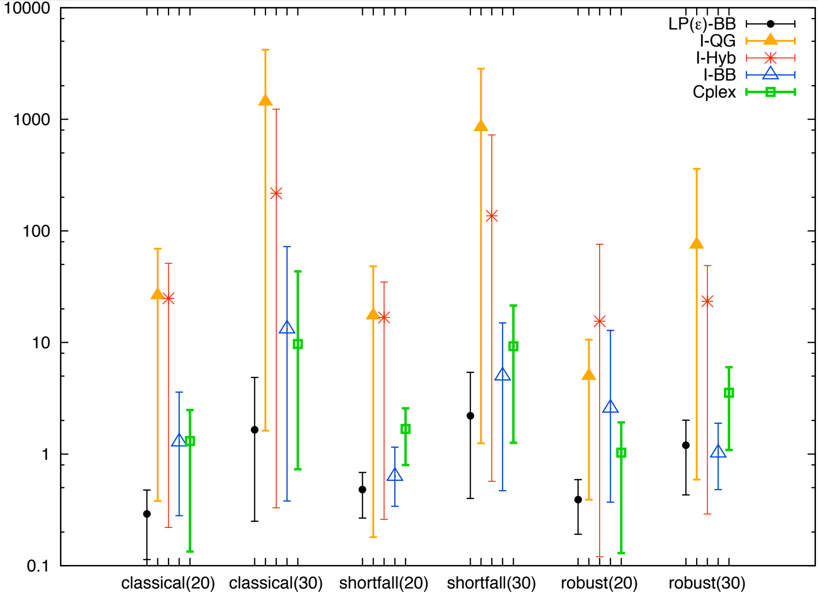
- y fraction of the portfolio invested in each of n assets.
- $\bar{a}$ expected returns of assets.
- $Q^{1/2}$ positive semidefinite square root of the covariance matrix Q of returns.
- K maximum number of assets to hold.
- Robust version from uncertainty in $\bar{a}$.

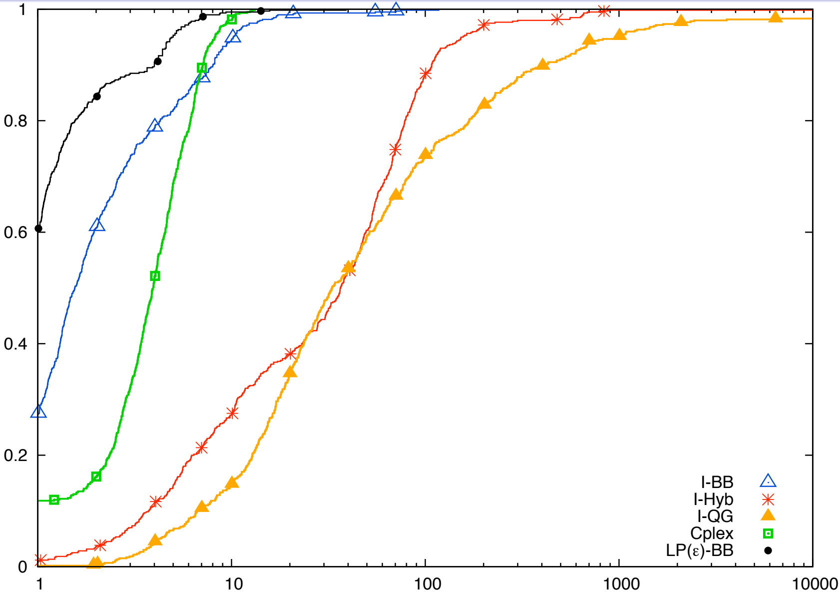
Instance Data

- Maximum number of stocks $K = 10$.
- Maximum risk $\sigma = 0.2$.
- Shortfall constraints: $\eta_1 = 80\%$, $W_1^{low} = 0.9$, $\eta_2 = 97\%$, $W_2^{low} = 0.7$ (Lobo et al., 1998, 2007).
- Data generation for Classical and Shortfall from S&P 500 data following Lobo et al. (1998), (2007).
- Data generation for Robust from S&P 500 data following Ceria and Stubbs (2006).
- Riskless asset included for Shortfall.
- Random selection of n stocks out of 462.
- 100 instances for $n \in \{20, 30, 40, 50\}$, 10 for $n \in \{100, 200\}$.

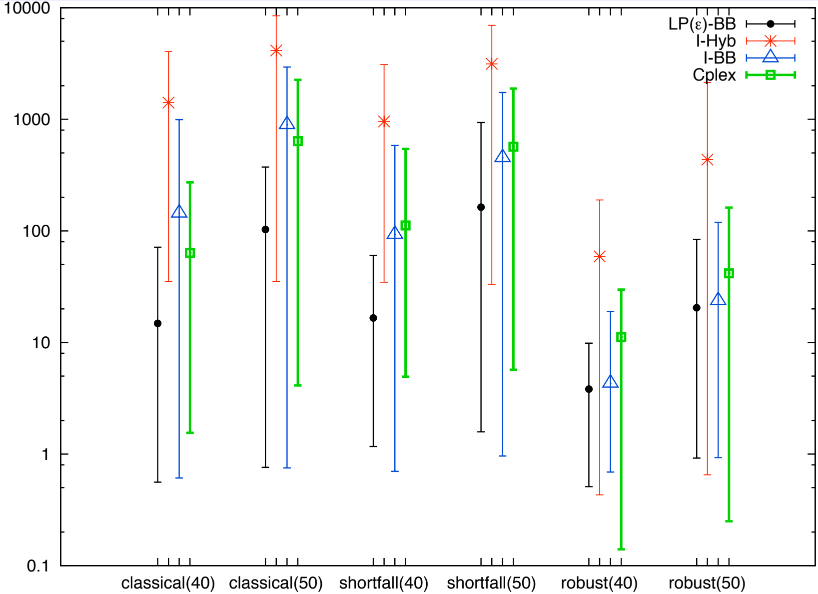
Average Solve Times [s] for $n \in \{20, 30\}$



Performance Profile for $n \in \{20, 30\}$



Average Solve Times [s] for $n \in \{40, 50\}$



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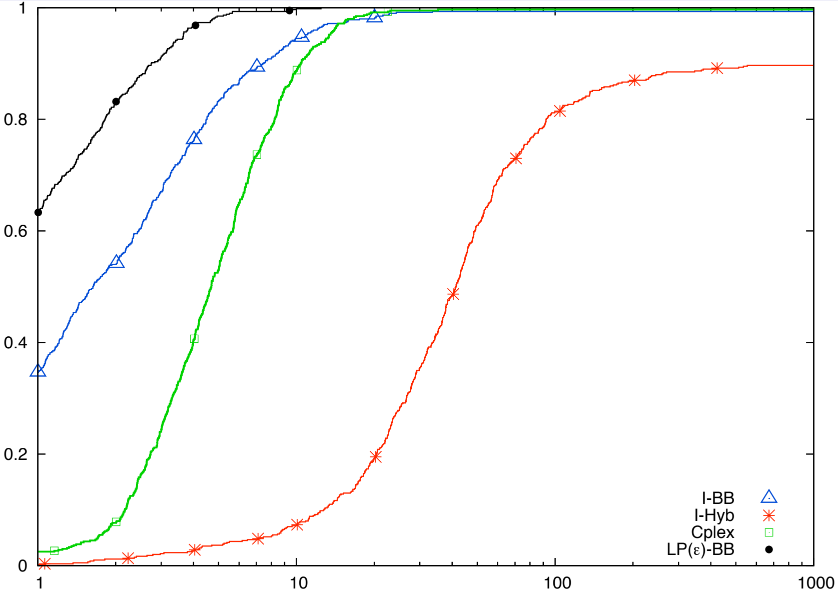
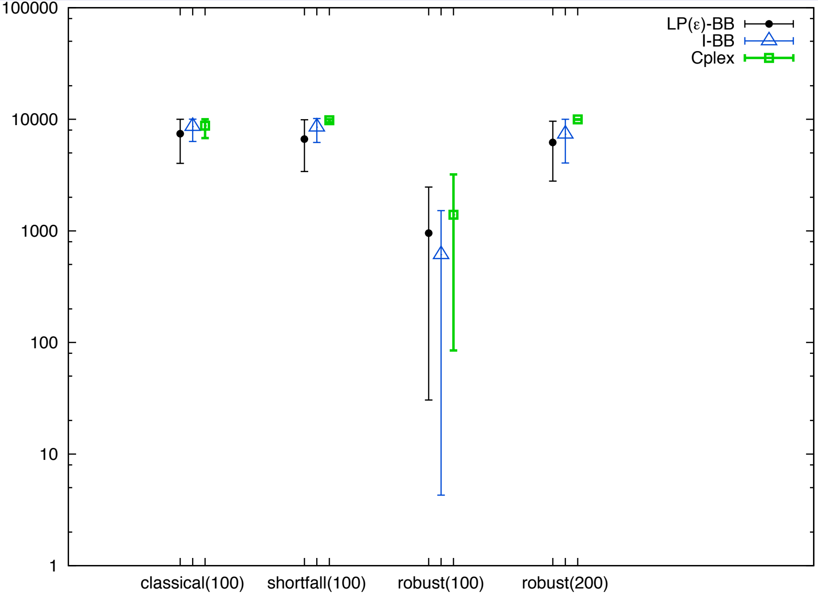


Average Solve Times [s] for $n \in \{100, 200\}$



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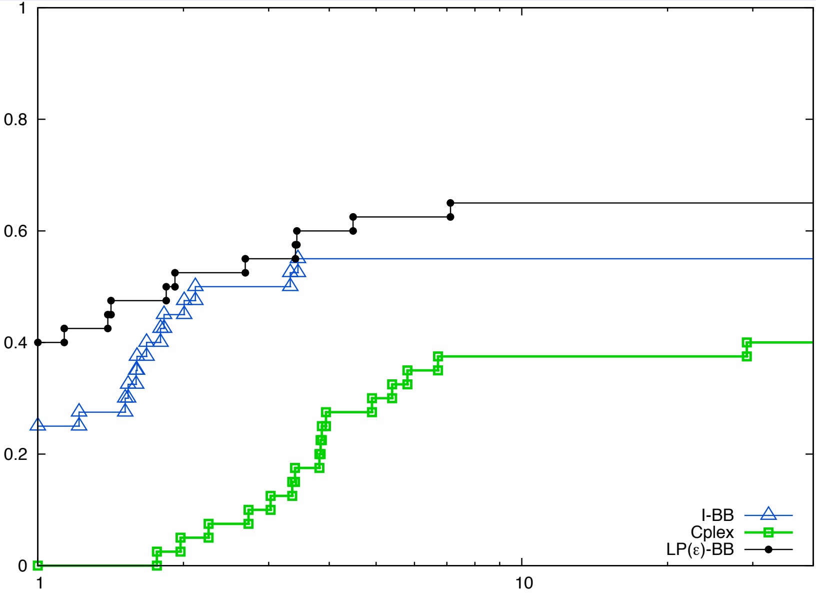

Total Number of Nodes and Calls to Relaxations for Small Instances

I-QG (B&B nodes)	3,580,051
I-Hyb (B&B nodes)	328,316
I-BB (B&B nodes)	68,915
CPLEX (B&B nodes)	85,957
LP(ε) -BB (B&B nodes)	57,933

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LP(ε) -BB (B&B nodes)	57,933
NLP(l^k, u^k) (LP(ε) -BB calls)	2,305
NLP(x^*) (LP(ε) -BB calls)	7,810

Final Remarks

- Polyhedral relaxation algorithm for "convex" MINLP:
  - Based on a lifted polyhedral relaxation.
  - "Does not update the relaxation".
- Algorithm for the conic quadratic case:
  - Characteristics:
    - Based on a lifted polyhedral relaxation by Ben-Tal and Nemirovski.
    - Implemented by modifying CPLEX MILP solver.
  - Advantages:
    - Can outperform other methods for portfolio optimization problems.
    - Shows that Ben-Tal and Nemirovski approximation can be computationally "practical".