Mixed Integer Gomory Cuts for Quadratic Programming: The Power of Extended Formulations?

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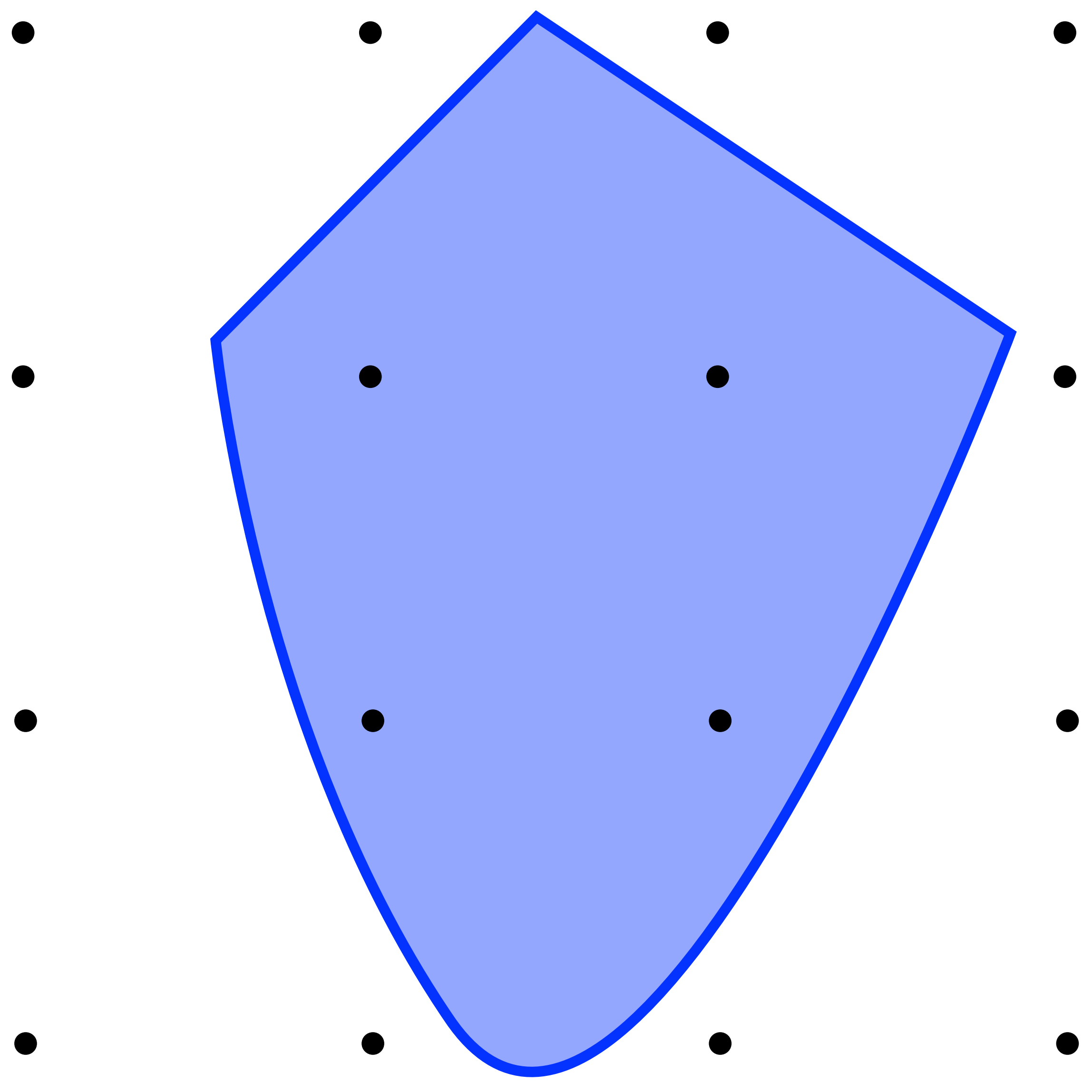
Outline

- Introduction: Split Cuts
- Nonlinear Split Cuts
- Conic MIR: Extended Formulations
- Strength Comparison
- Summary

Split Disjunctions and Split Cuts



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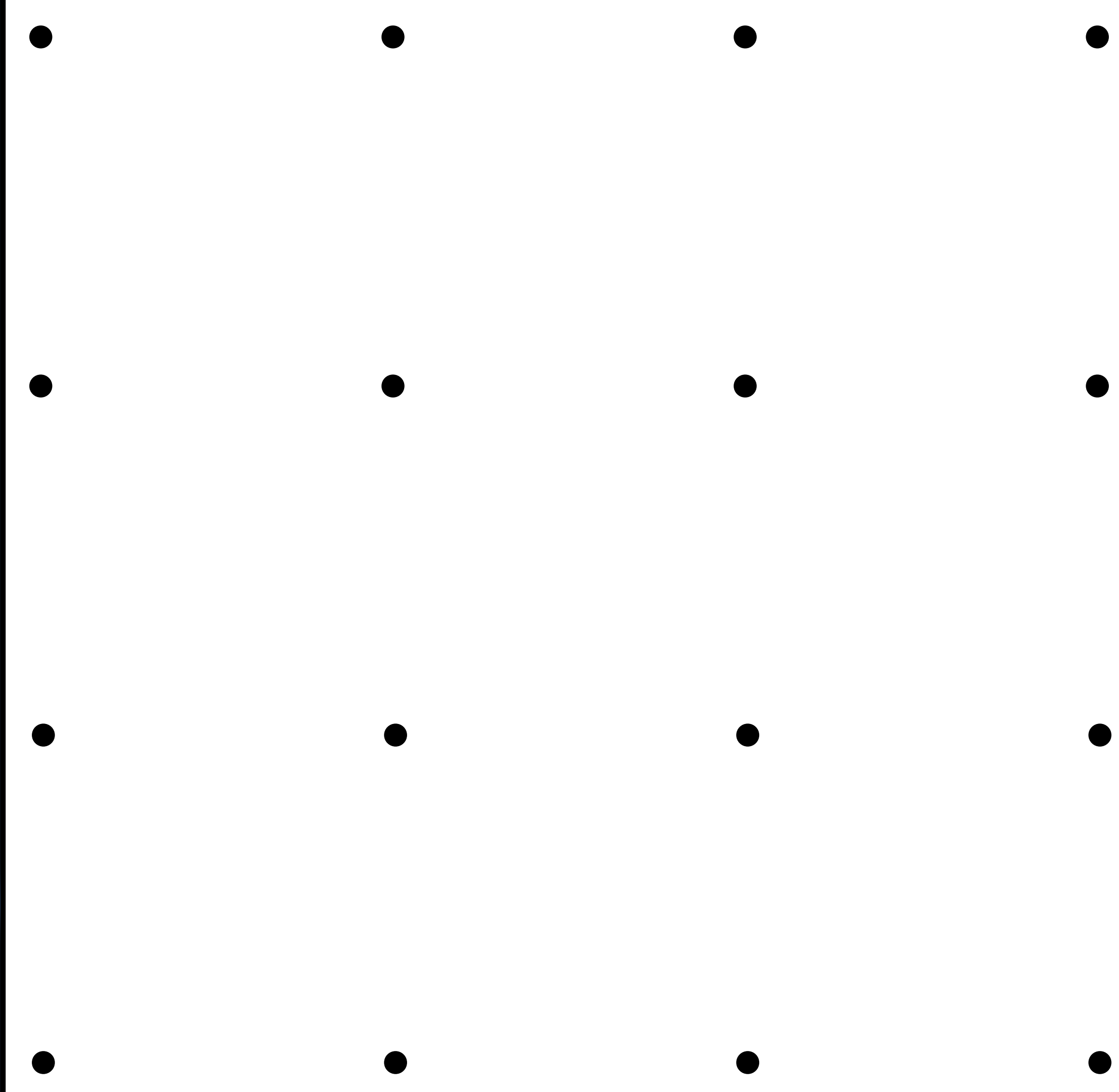

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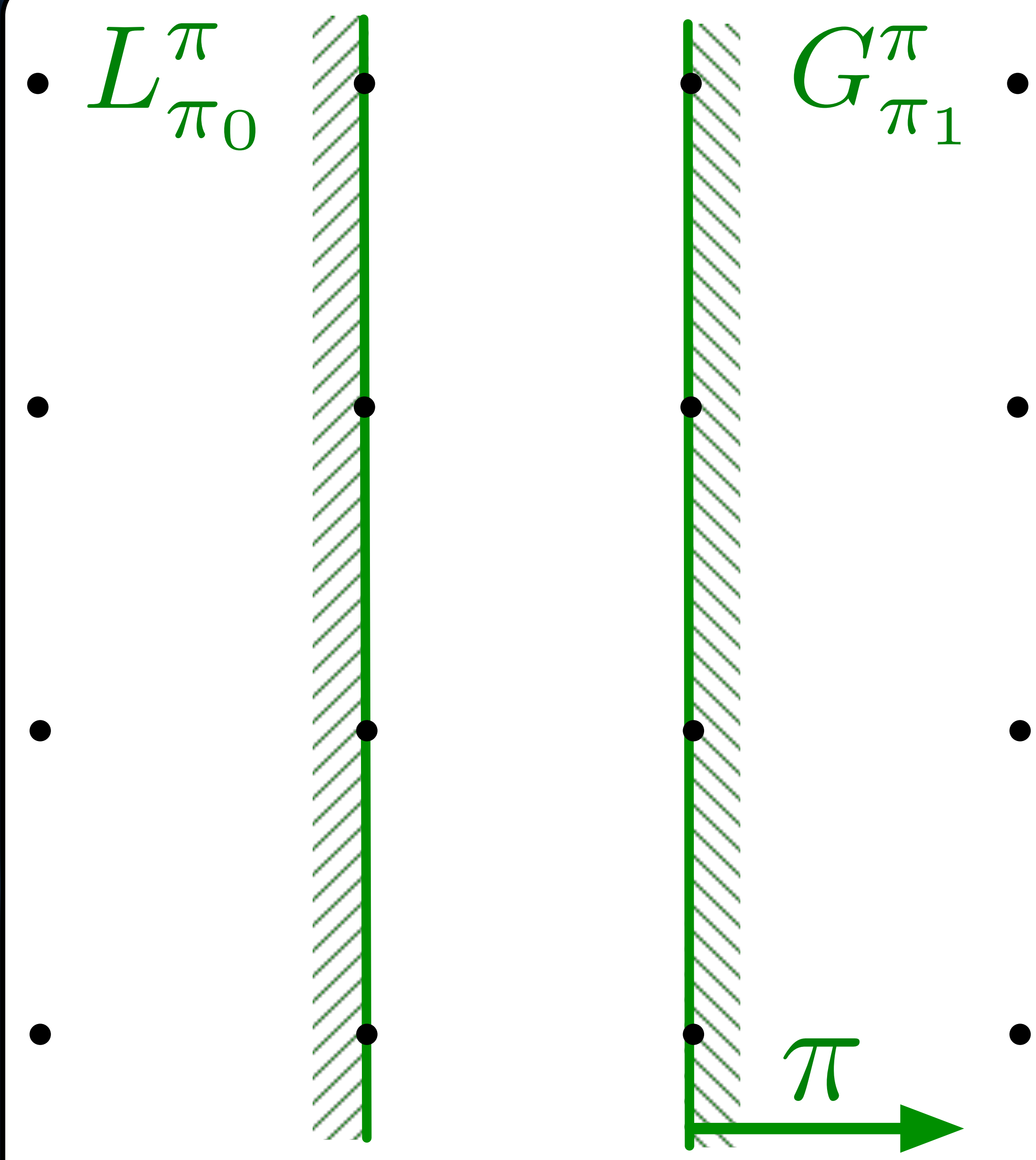

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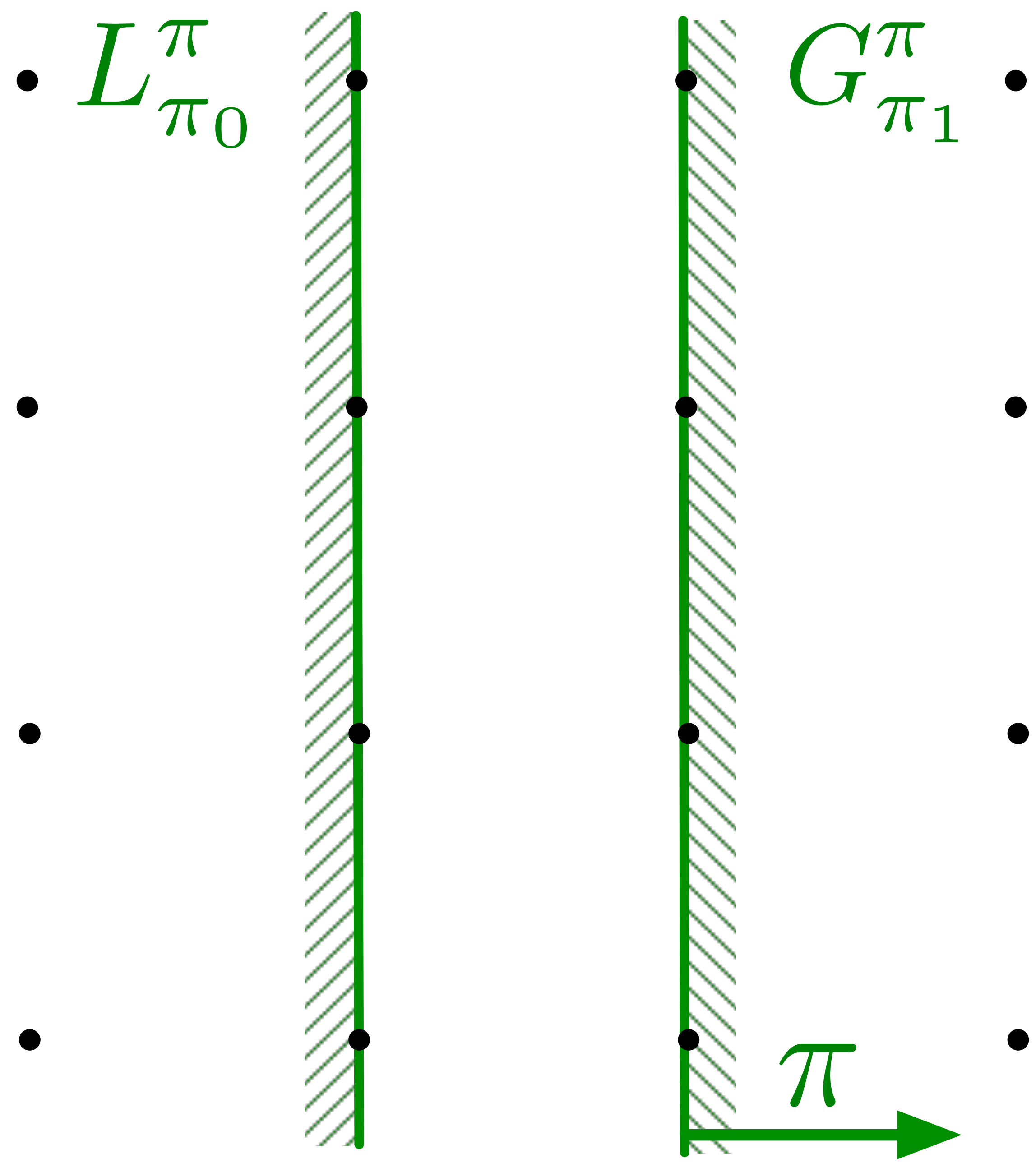
$$\pi \in \mathbb{Z}^n, \quad \pi_1 = \pi_0 + 1 \in \mathbb{Z}$$

$$\Downarrow$$

$$\mathbb{Z}^n \subseteq L^{\pi}_{\pi_0} \cup G^{\pi}_{\pi_1}$$



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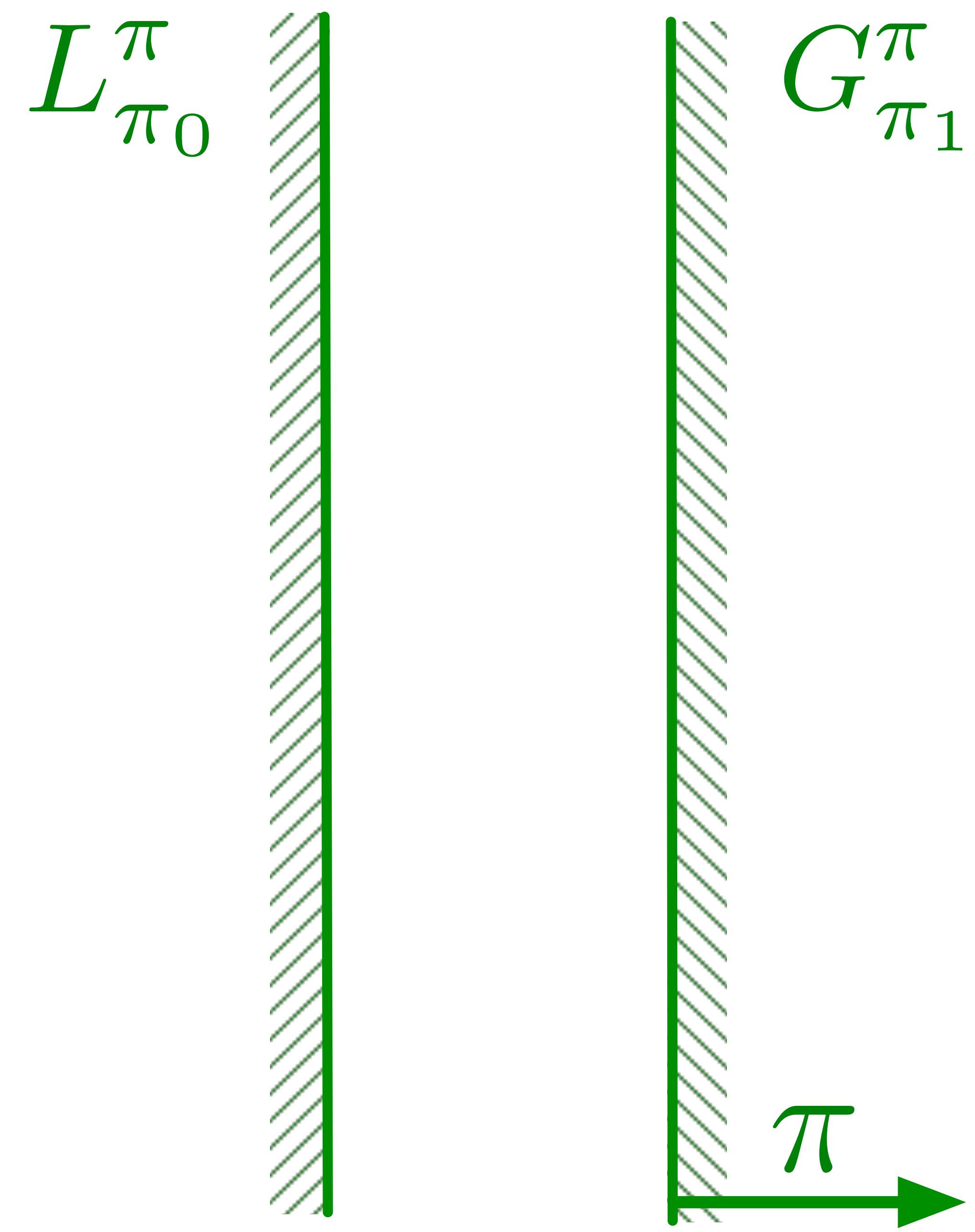

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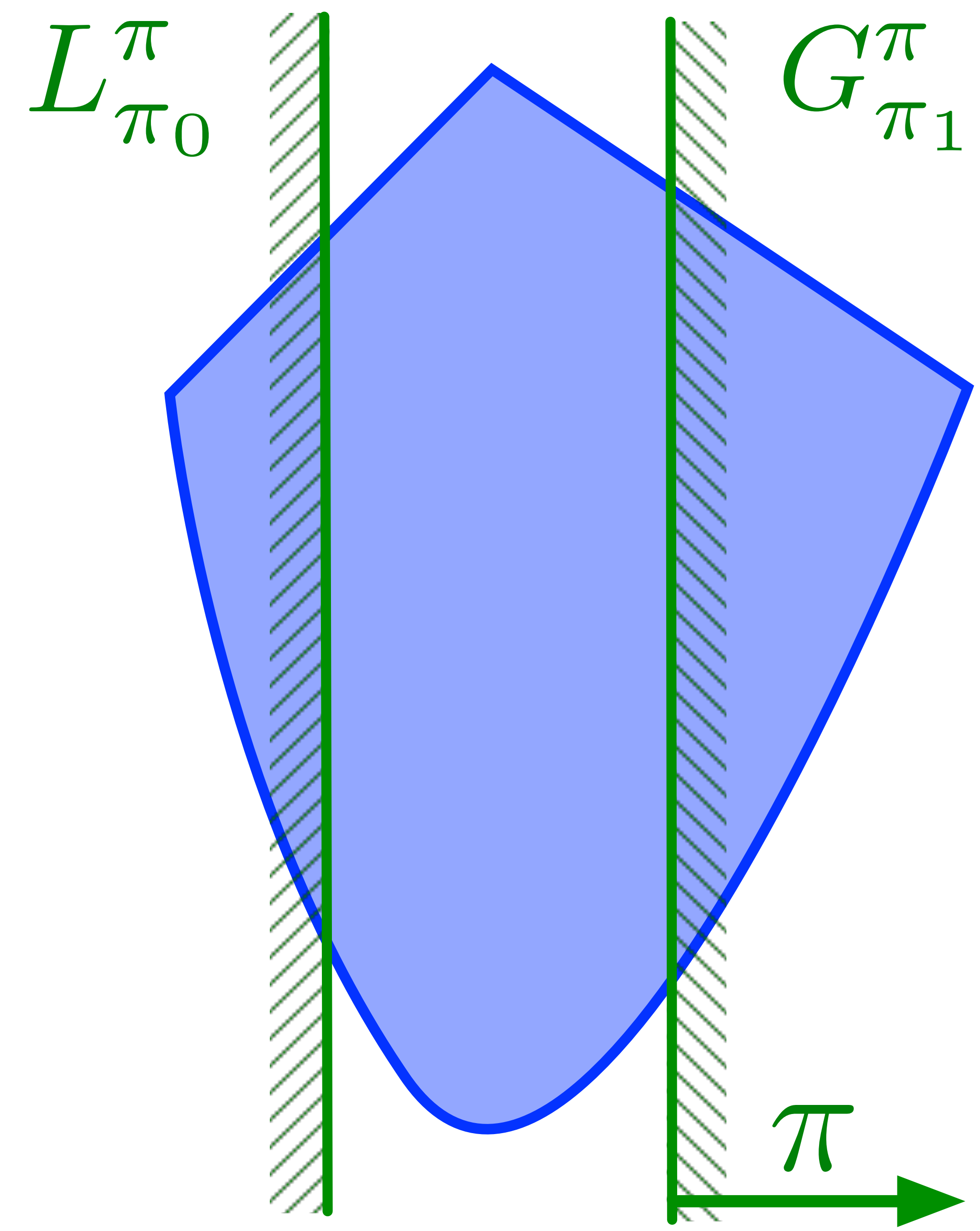

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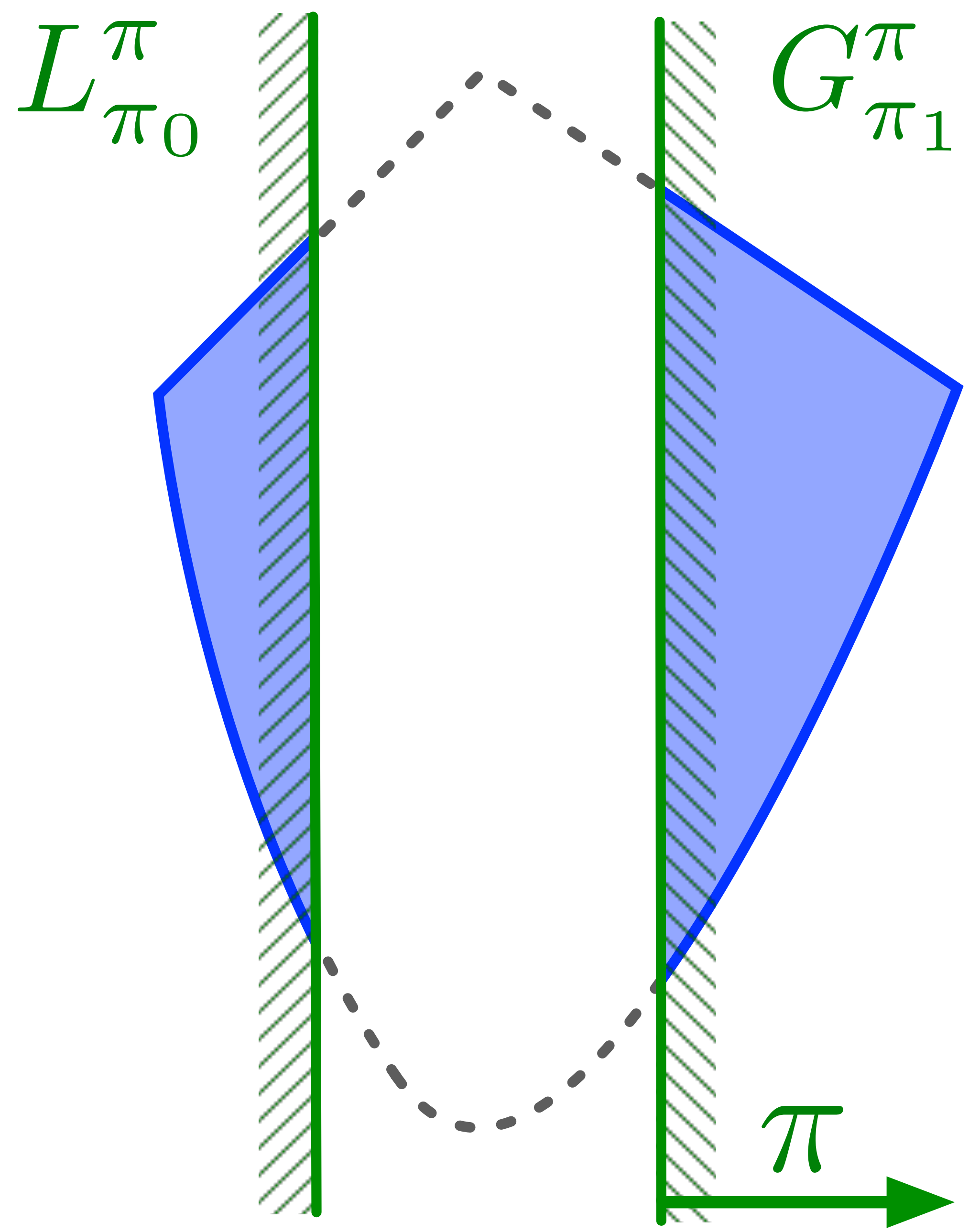

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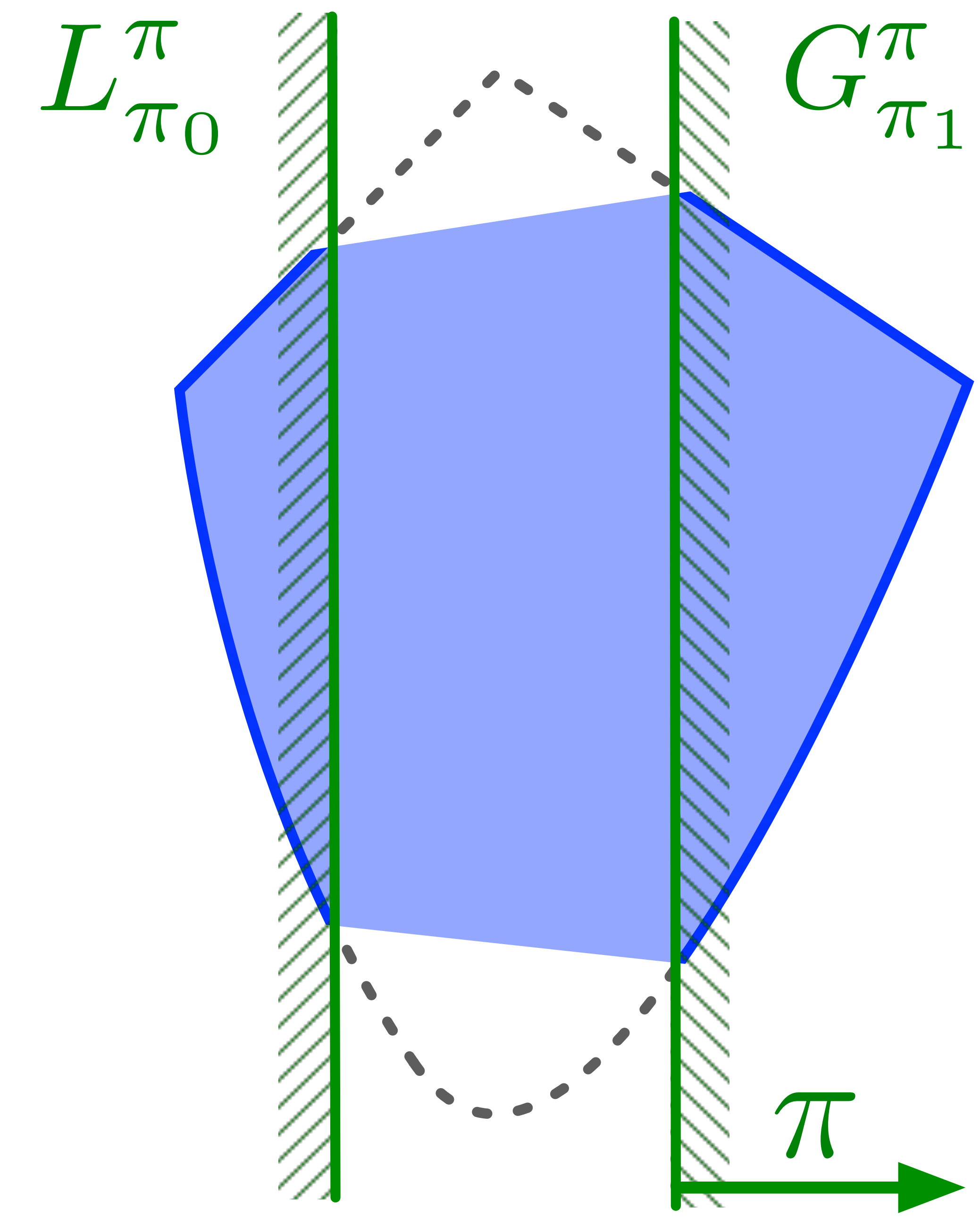

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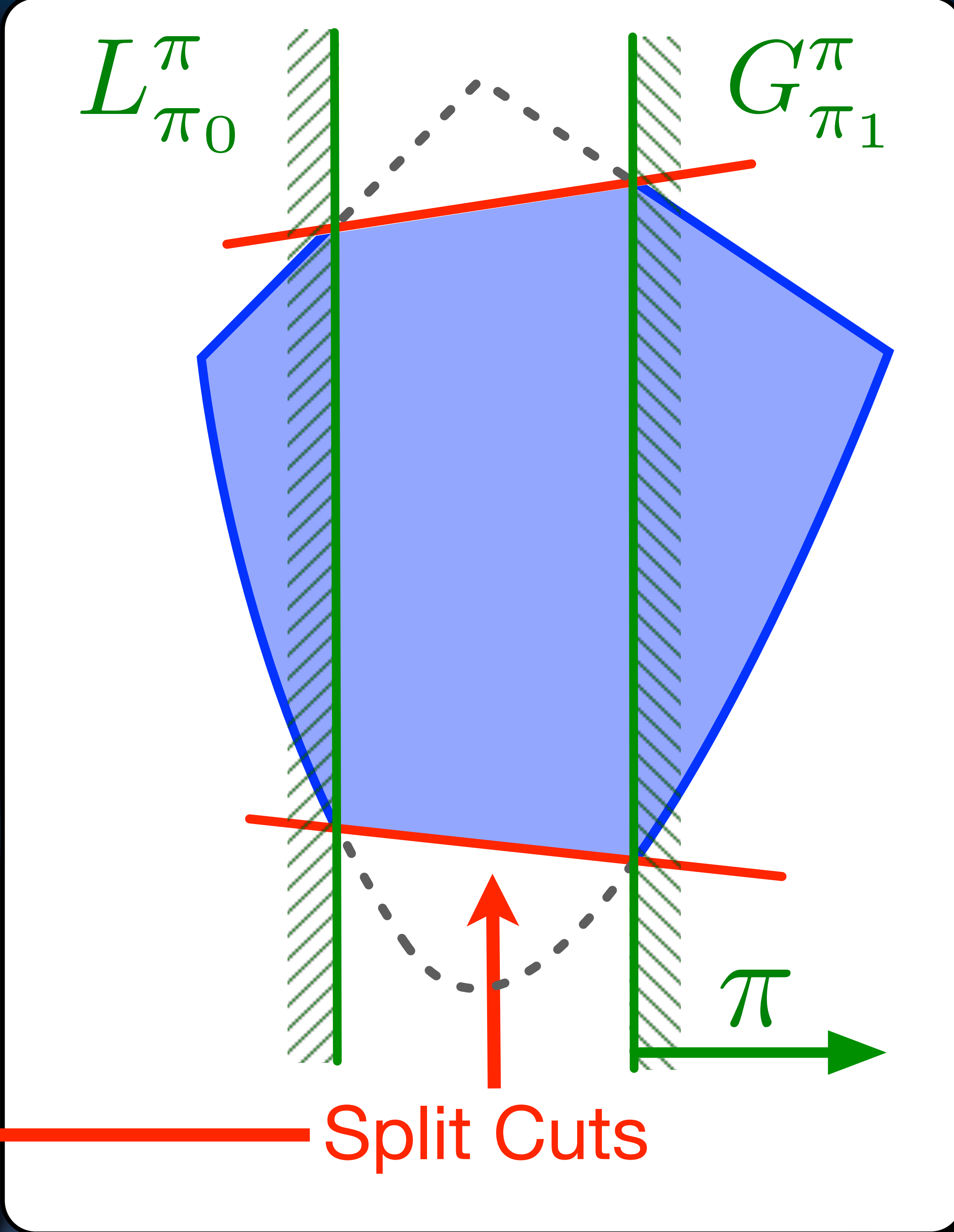

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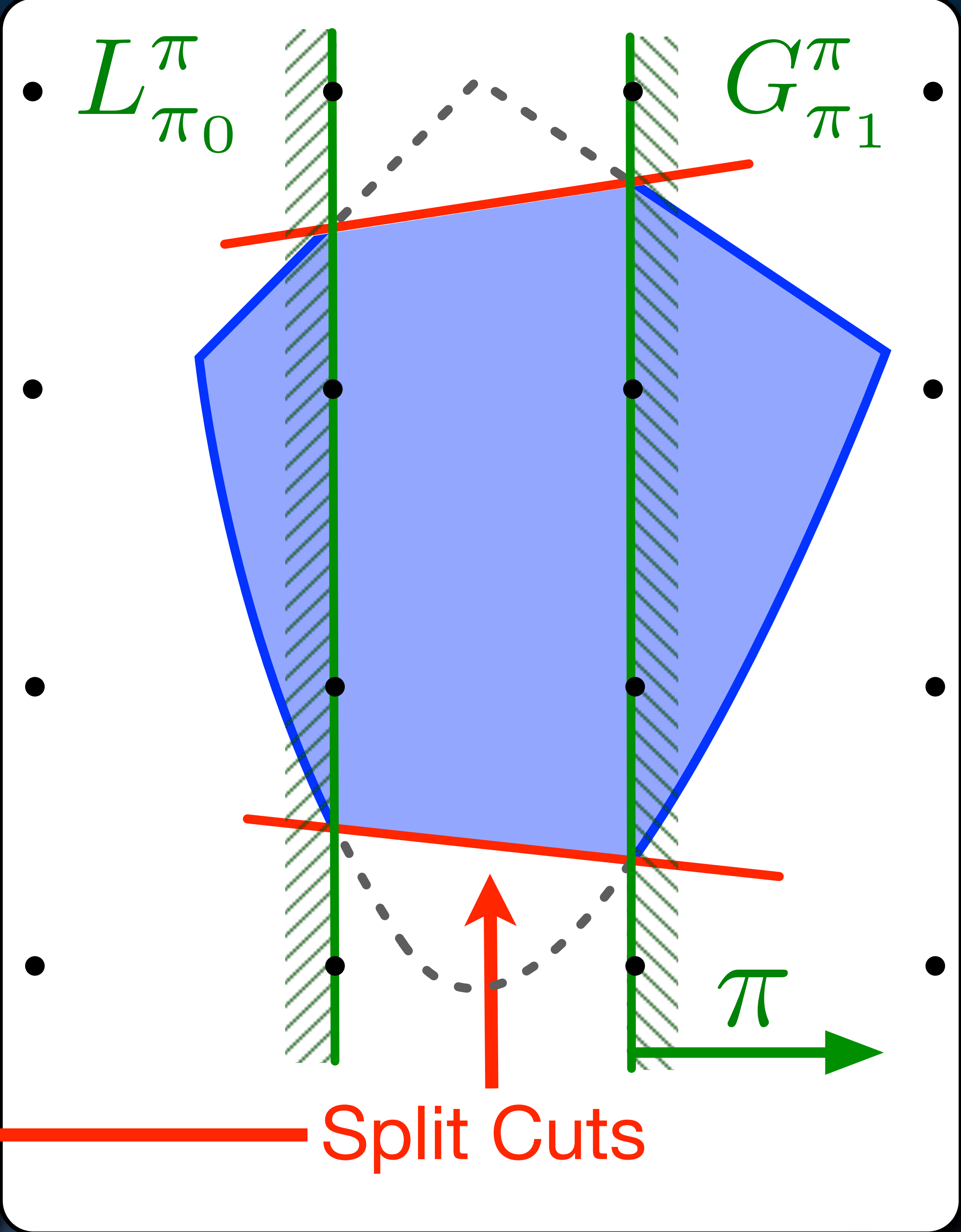

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Elementary Splits:

$$\pi = e^i$$

$$x_i \leq \pi_0 \quad \vee \quad x_i \geq \pi_0 + 1$$

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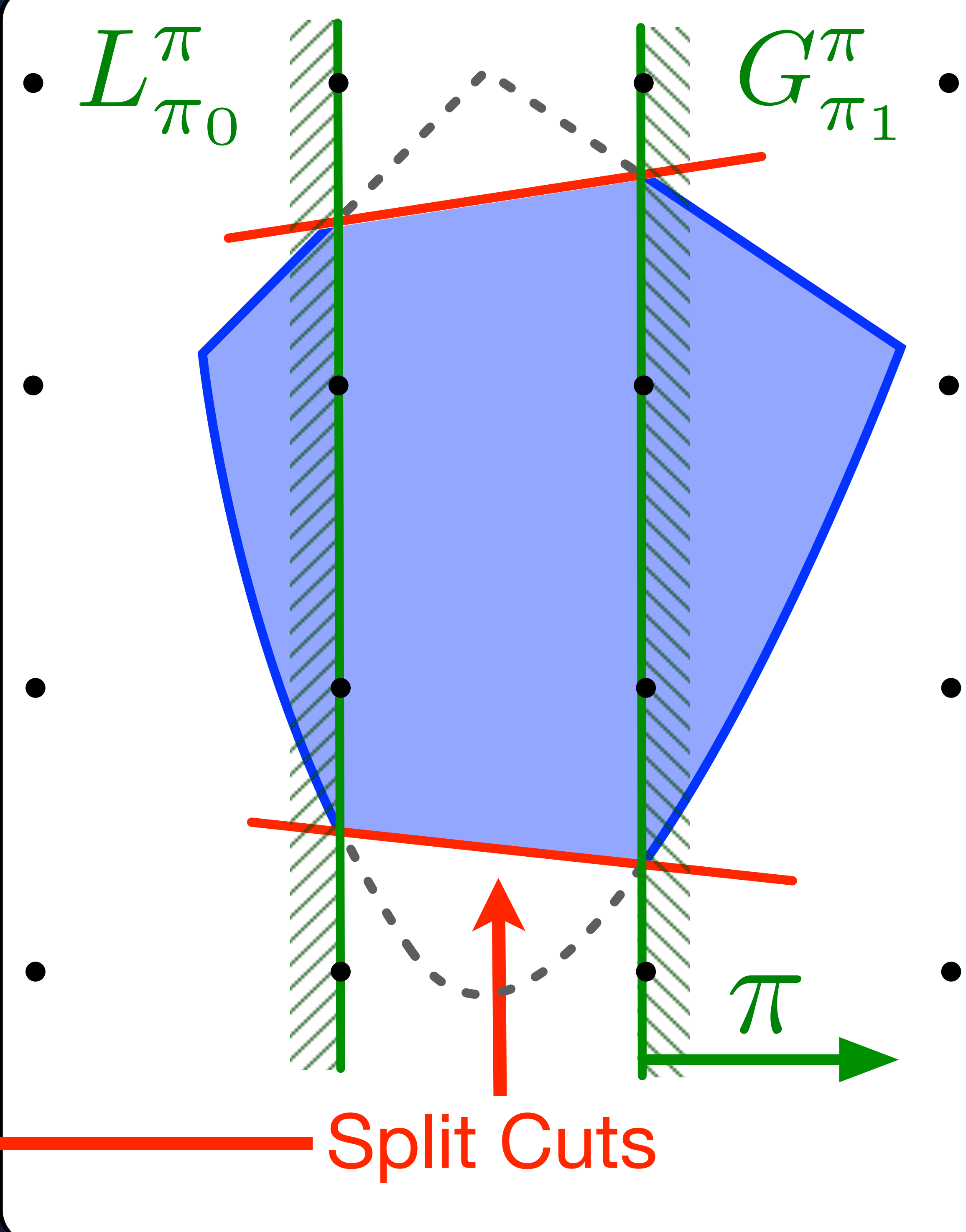


Split Cuts for Simplicial Cones

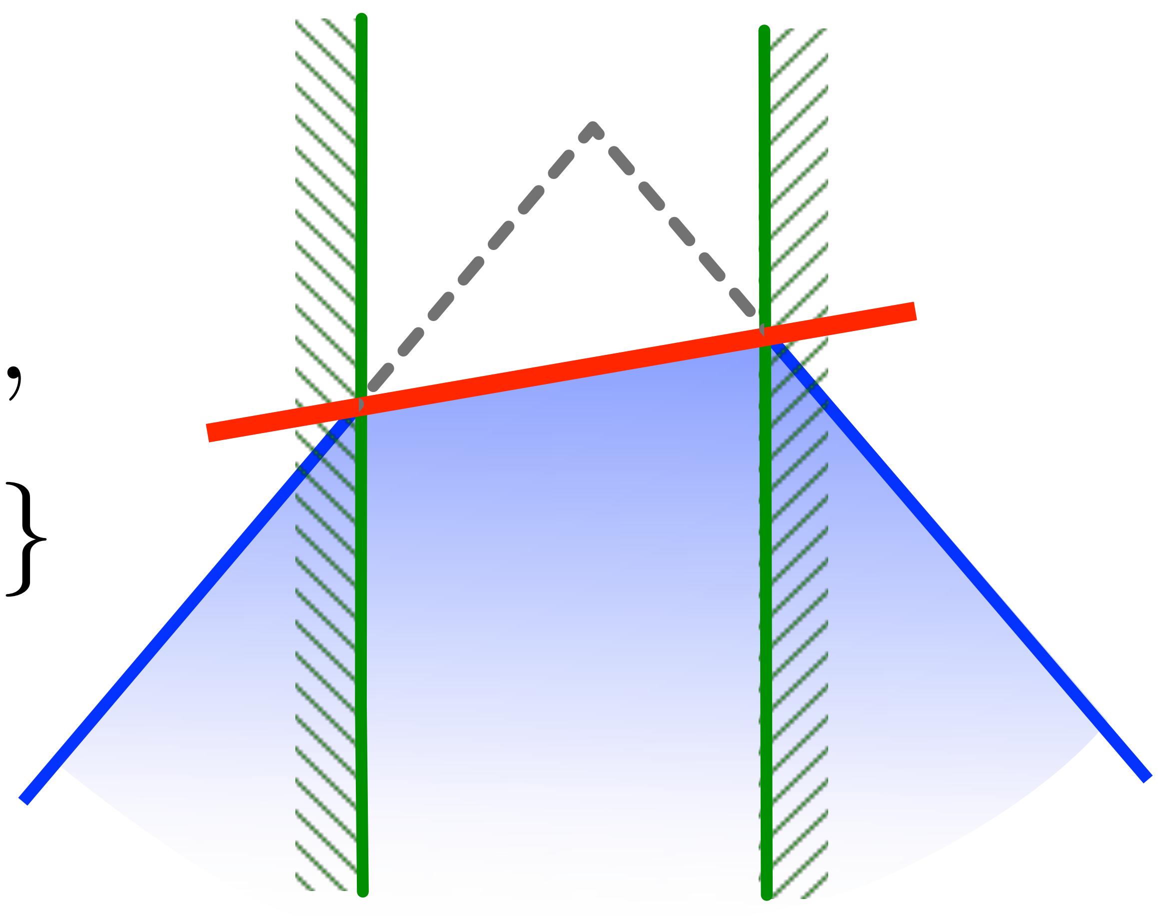
- Formulas: (MIG: Gomory 1960 and MIR: Nemhauser and Wolsey 1988)

$P := \{x \in \mathbb{R}^n : Ax \leq b\},$

$\det(A) \neq 0$

$\pi_0 < \pi^T A^{-1} b < \pi_1$

$P_{\pi,\pi_0} := \{x \in \mathbb{R}^n : \quad Ax \leq b,$
$\quad a^T x \leq b\}$



Split Cuts for Quadratic Cones

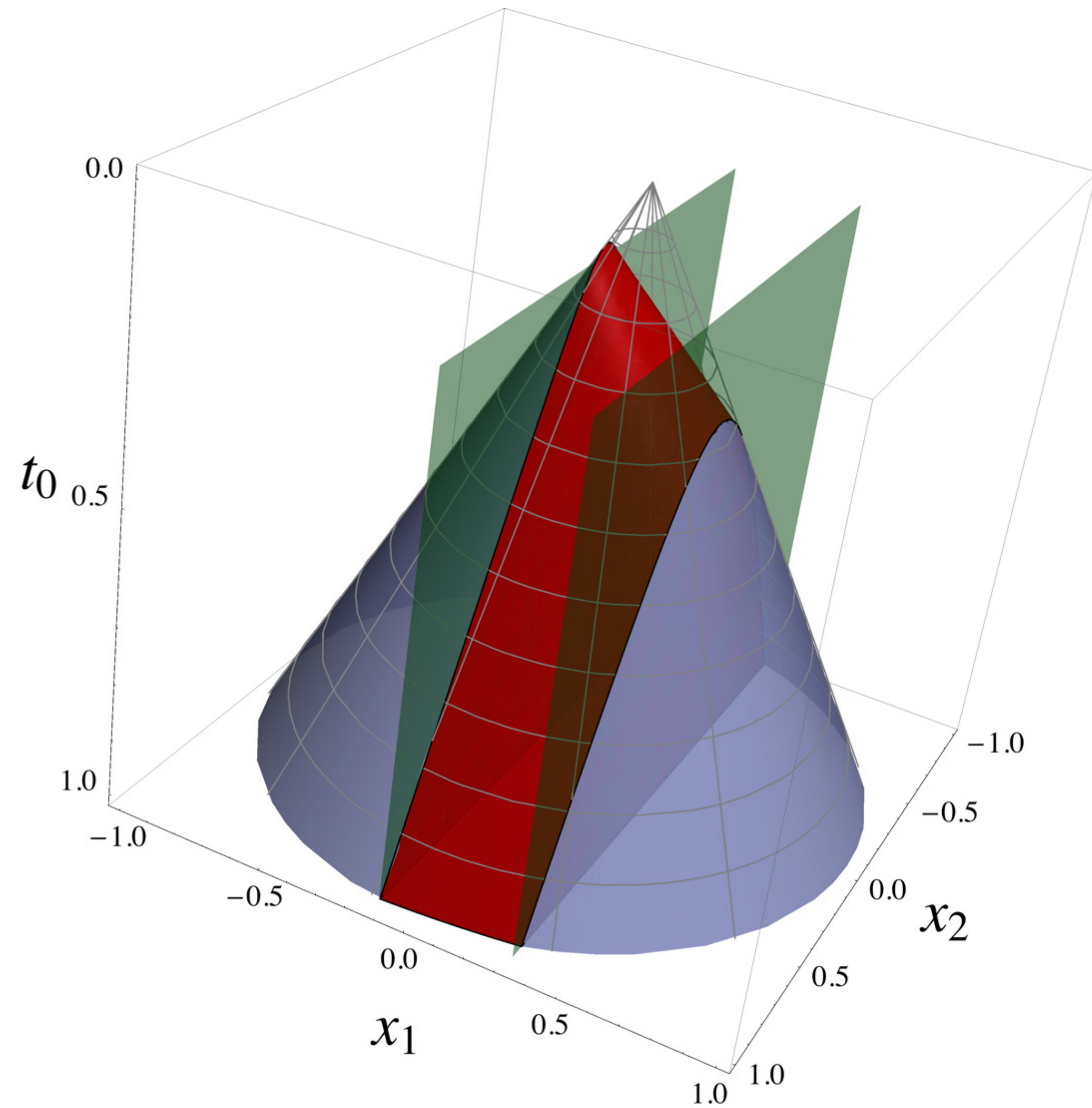
- Formulas: (Modaresi, Kılınç, V. 2011)

$$C(B,c) := \{(x,t_0) \in \mathbb{R}^n \times \mathbb{R} : \|B(x-c)\|_2 \le t_0\}$$

$$\{(x,t_0) \in \mathbb{R}^{n+1} : \langle \pi, x\rangle \le \pi_0\}$$

$$\{(x,t_0) \in \mathbb{R}^{n+1} : \langle \pi, x\rangle \ge \pi_1\}$$

$$C(B,c)_{\pi,\pi_0} = \{(x,t_0) \in \mathbb{R}^n \times \mathbb{R} : \|B(x-c)\|_2 \le t_0, \|Dx-d\|_2 \le t_0\}$$



(also Atamturk and Narayanan 2010, Andersen and Jensen 2013)

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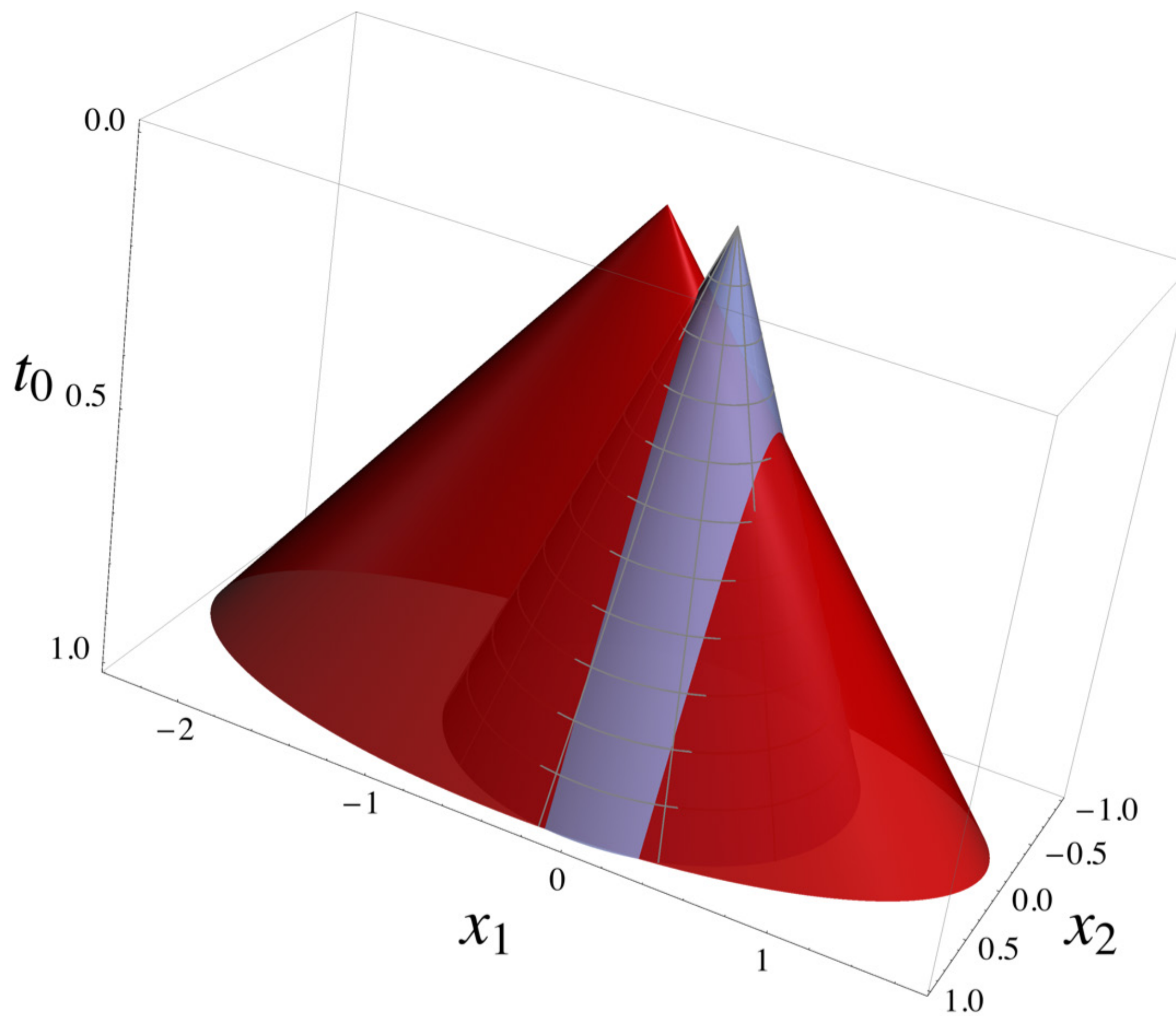

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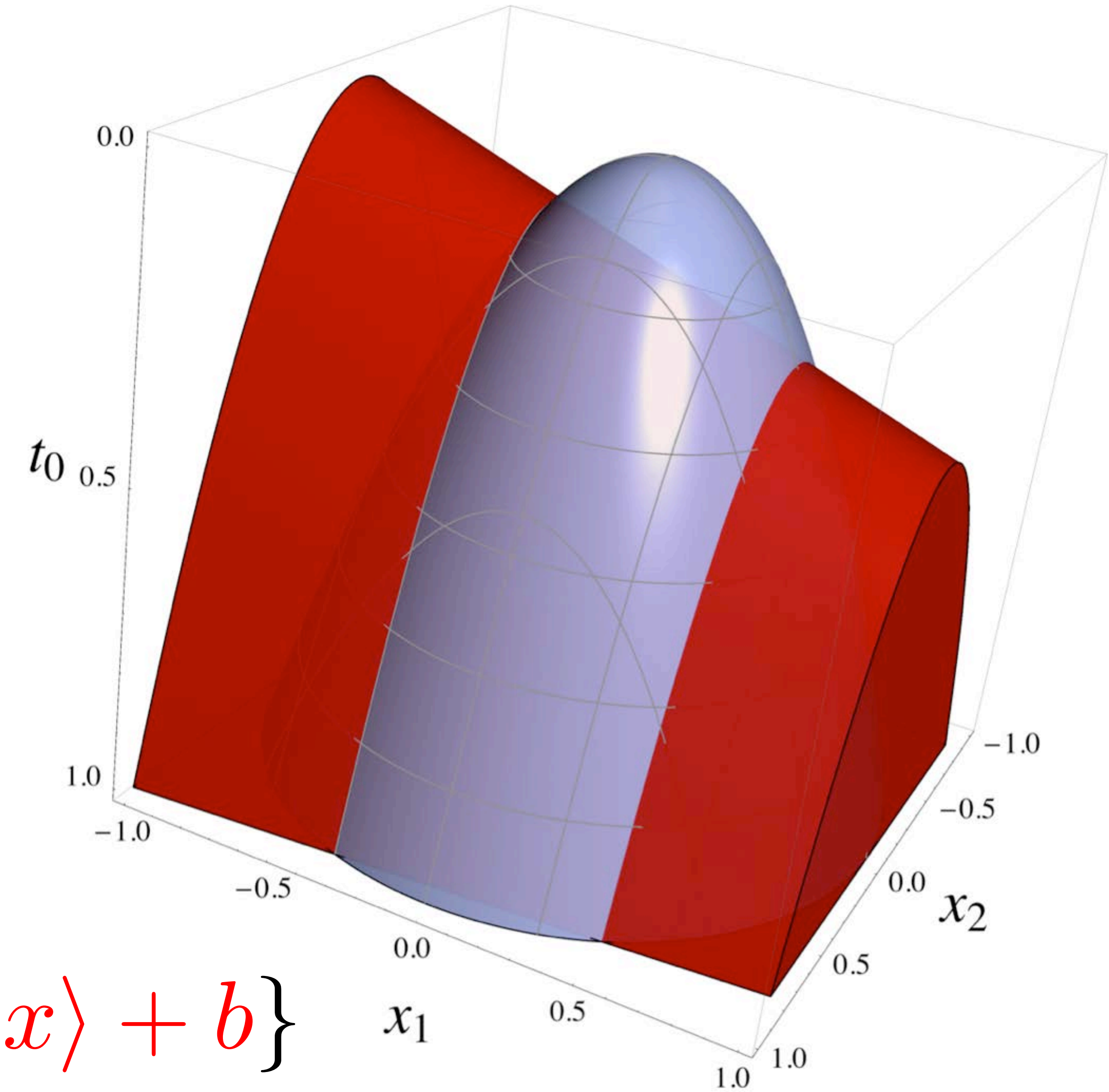
Split Cuts for Paraboloids

- Formulas: (Modaresi, Kılınç, V. 2012)

$$Q(B,c) := \{(x, s_0) \in \mathbb{R}^n \times \mathbb{R} : \|B(x-c)\|_2^2 \leq s_0\}$$

$$Q := Q(B,c)$$

$$Q_{\pi,\pi_0} = \{(x, s_0) \in \mathbb{R}^n \times \mathbb{R} : \|B(x-c)\|_2^2 \leq s_0, \|D(x-d)\|_2^2 \leq s_0 + \langle a, x\rangle + b\}$$



- Stronger than "conic" cut for Shortest Vector.

Conic MIR and Extended Formulation

$$Q(B, c) := \left\{(x, s_0) \in \mathbb{R}^n \times \mathbb{R} \,:\, \|B(x-c)\|_2^2 \leq s_0\right\}$$

Extended Formulation: $(x, t, s_0) \in \mathbb{Z}^n \times \mathbb{R}^n \times \mathbb{R}_+$

$|B(x-c)| \leq t$ ← Linear Part

$\|t\|_2^2 \leq s_0$ ← Nonlinear Part

$$|v| \leq t \;\Updownarrow\; |v_i| \leq t_i \,\forall i$$

- Atamturk and Narayanan 2007, Modaresi, Kılınç, V. 2011

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Conic MIR = Split cut for linear part

$$f := \pi^T c - \pi_0 \in (0,1)$$

$$L := \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R}^n \;:\; |B(x-c)| \leq t\right\}$$

$$L_{\pi,\pi_0} := \left\{(x,t) \;:\; \begin{array}{c} |B(x-c)| \leq t \\ (1-2f)\left(\pi^T x - \pi_0\right) + f \leq \left|B^{-T}\pi\right|^T t \end{array}\right\}$$

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Non-linear Effect of Linear Cut

$$MIR^2_{\pi,\pi_0} := \left\{ (x,t,s_0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \; : \; \begin{array}{c} (x,t) \in L_{\pi,\pi_0} \\ \|t\|_2^2 \leq s_0 \end{array} \right\}$$

$$Q_{\pi,\pi_0} \subseteq \mathrm{Proj}_{(x,s_0)} \left(MIR^2_{\pi,\pi_0} \right)$$

Equality for $B = I$ and $\pi = e^i$.

Containment can be strict.

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Containment can be strict.

For $B = I$

$$\mathrm{Proj}_{(x,s_0)}\left(\bigcap_{i=1}^n MIR^2_{e^i,\lfloor c_i \rfloor}\right) \subseteq \bigcap_{i=1}^n Q_{e^i,\lfloor c_i \rfloor}$$

Strict containment for $n = 2$ and $c = (1/2, 1/2)$.

MIR CVP Bound > Split CVP Bound

For $B = I$ and $c_i = 1/2$

$$n/4 = \min_{x} \left\{ \|x - c\|_2^2 \ : \ x \in \mathbb{Z}^n \right\}$$
$$= \min_{x, s_0} \left\{ s_0 \ : \ (x, s_0) \in \mathrm{Proj}_{(x, s_0)} \left(\bigcap_{i=1}^{n} MIR^2_{e^i, \lfloor c_i \rfloor} \right) \right\}$$

$$1/4 = \min_{x, s_0} \left\{ s_0 \ : \ (x, s_0) \in \bigcap_{(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}} Q_{\pi, \pi_0} \right\}$$

CMIR Bound

$$n/4 = \min_{x,s_0} \left\{ s_0 \;:\; (x,s_0) \in \mathrm{Proj}_{(x,s_0)} \left(\bigcap_{i=1}^{n} MIR^2_{e^i,\lfloor c_i \rfloor} \right) \right\}$$

$$\bigcap_{i=1}^{n} MIR^2 = \left\{ (x,t,s_0) \;:\; \begin{array}{rl} |x_i - 1/2| \leq t_i & \forall\; i, \\ (1-2f)\left(x_i - \lfloor 1/2 \rfloor\right) + f \leq t_i & \forall\; i, \\ \|t\|_2^2 \leq s_0 & \end{array} \right\}$$

$$= \left\{ (x,t,s_0) \;:\; \begin{array}{rl} |x_i - 1/2| \leq t_i & \forall\; i, \\ 1/2 \leq t_i & \forall\; i, \\ \|t\|_2^2 \leq s_0 & \end{array} \right\}.$$

$f = \pi^T c - \pi_0 = 1/2$

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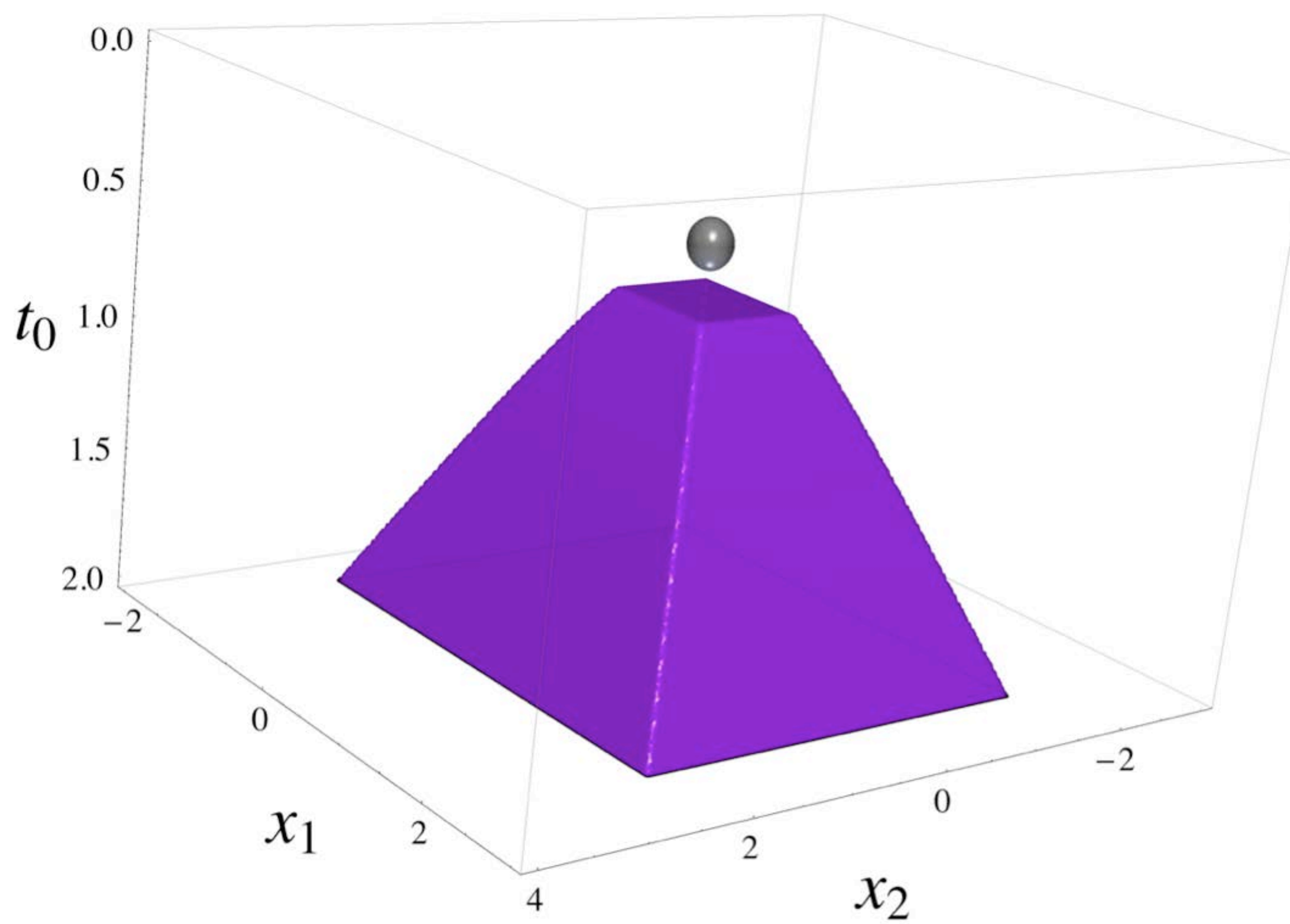

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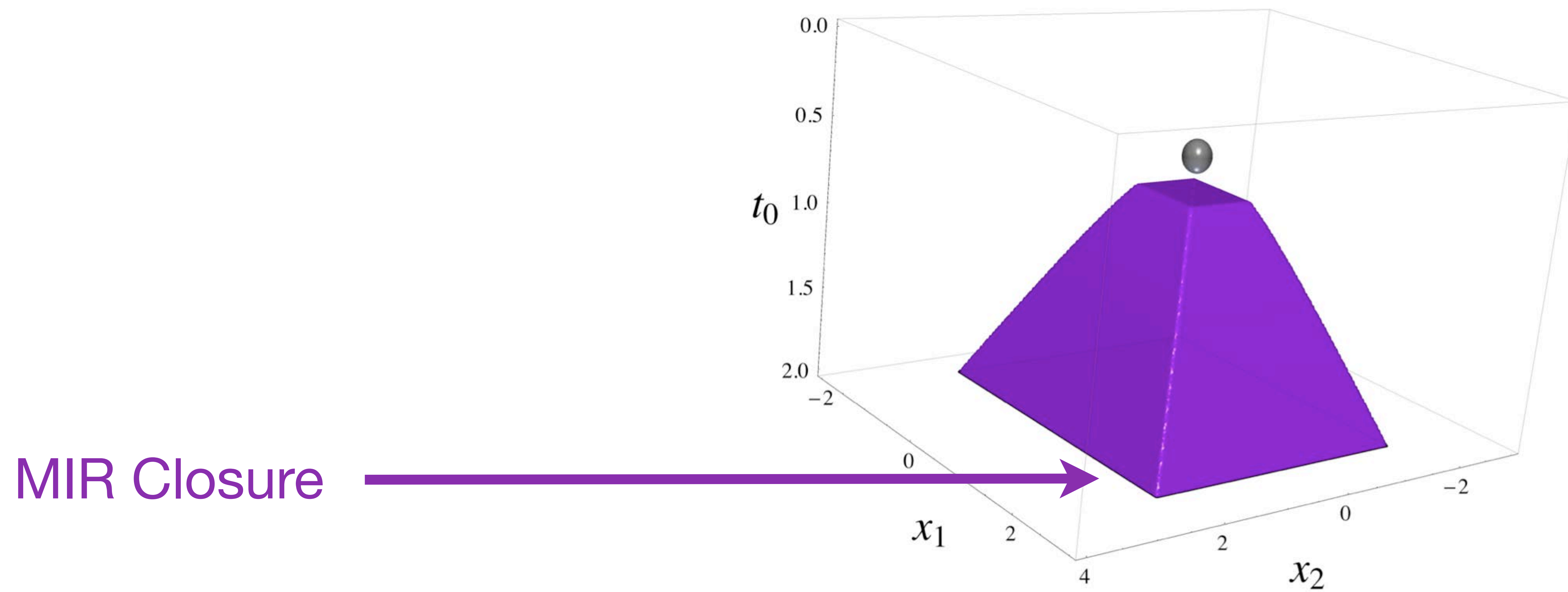

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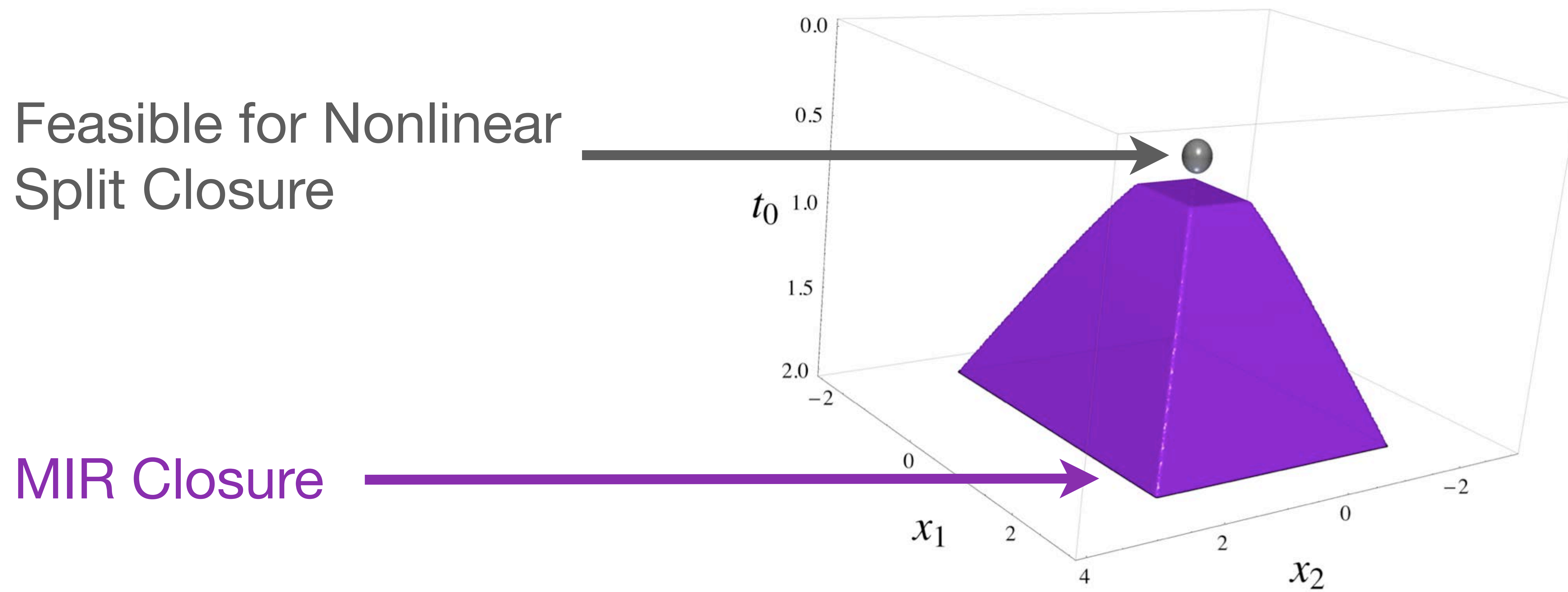

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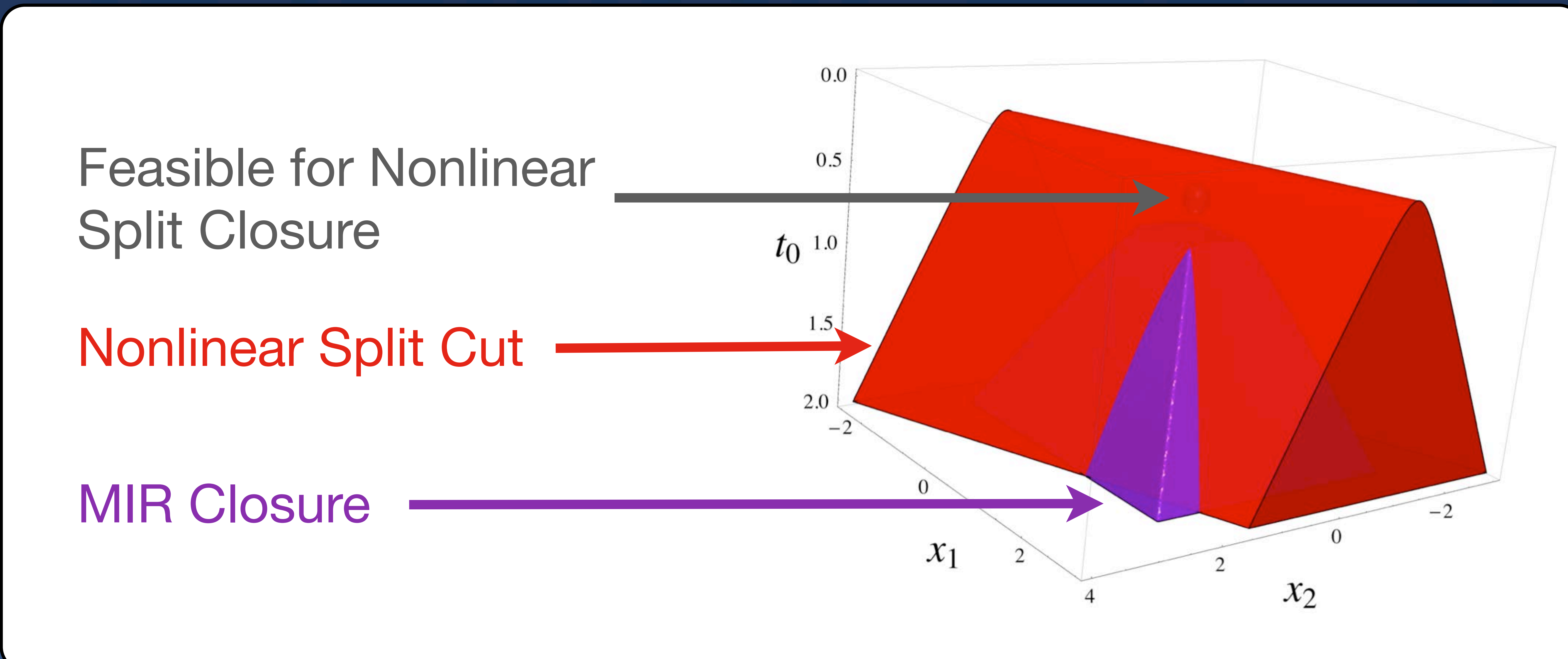
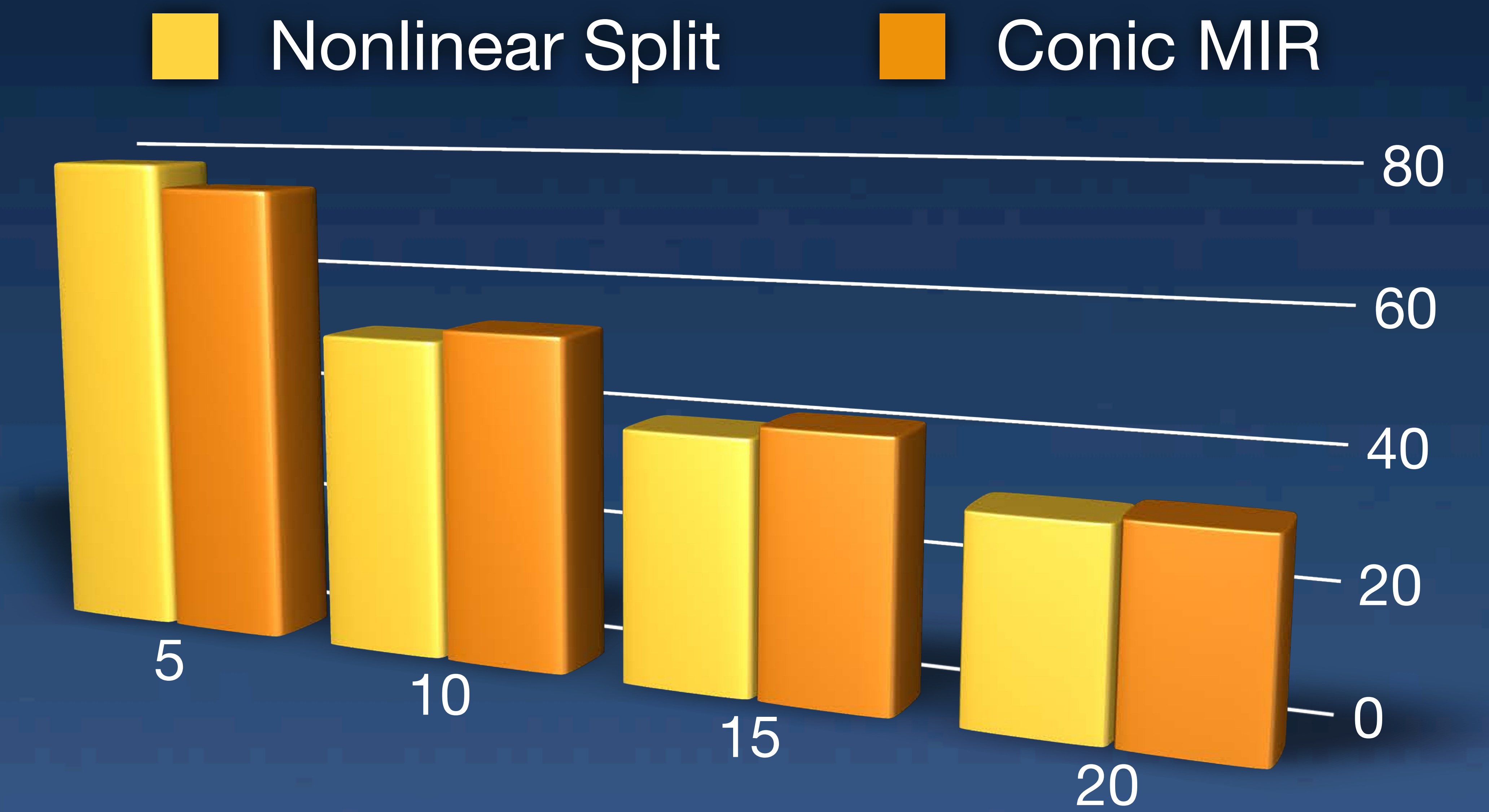


GAP Closed [%] ≈ 2 n Cuts



$$\text{GAP Cosed} = 100 \times \left(1 - \frac{\text{Cut Bound}}{\text{Optimal CVP}}\right)$$

Summary and Open Questions

- Non-linear Split Cuts
  - Too Expensive.
  - Solution: Non-linear extended formulation.
- Computationally:
  - Non-linear extended formulation helps by itself.
  - Neither cut seems to help.
  - Do you have difficult quadratic MIP problems?