The Chvátal-Gomory Closure of a Compact Convex Set is a Rational Polytope

Daniel Dadush and Santanu S. Dey
Georgia Institute of Technology

Juan Pablo Vielma
University of Pittsburgh

SIAM Conference on Optimization
May, 2011 – Darmstadt, Germany

Outline

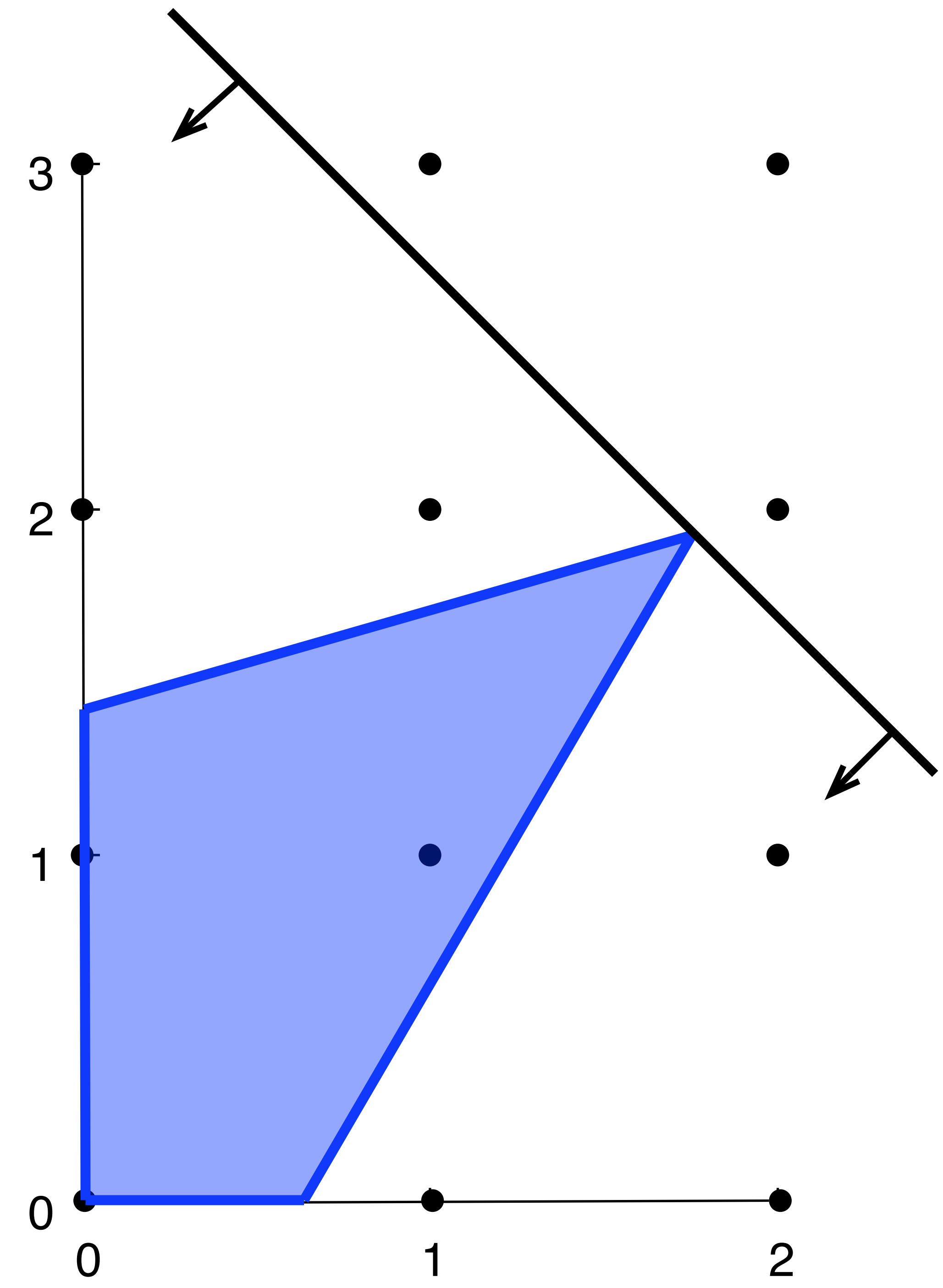
- Introduction
- Proof of Main Result:
  - Step 1
  - Step 2
- Conclusions and Current Work

CG Cuts for Rational Polyhedra

$$P := \left\{ x \in \mathbb{R}^2 \;:\; \begin{aligned} x_1 + x_2 &\leq 3, \\ 5x_1 - 3x_2 &\leq 3 \end{aligned} \right\}$$

$$\subseteq$$

$$H := \{ x \in \mathbb{R}^2 \;:\; 4x_1 + 3x_2 \leq 10.5 \}$$

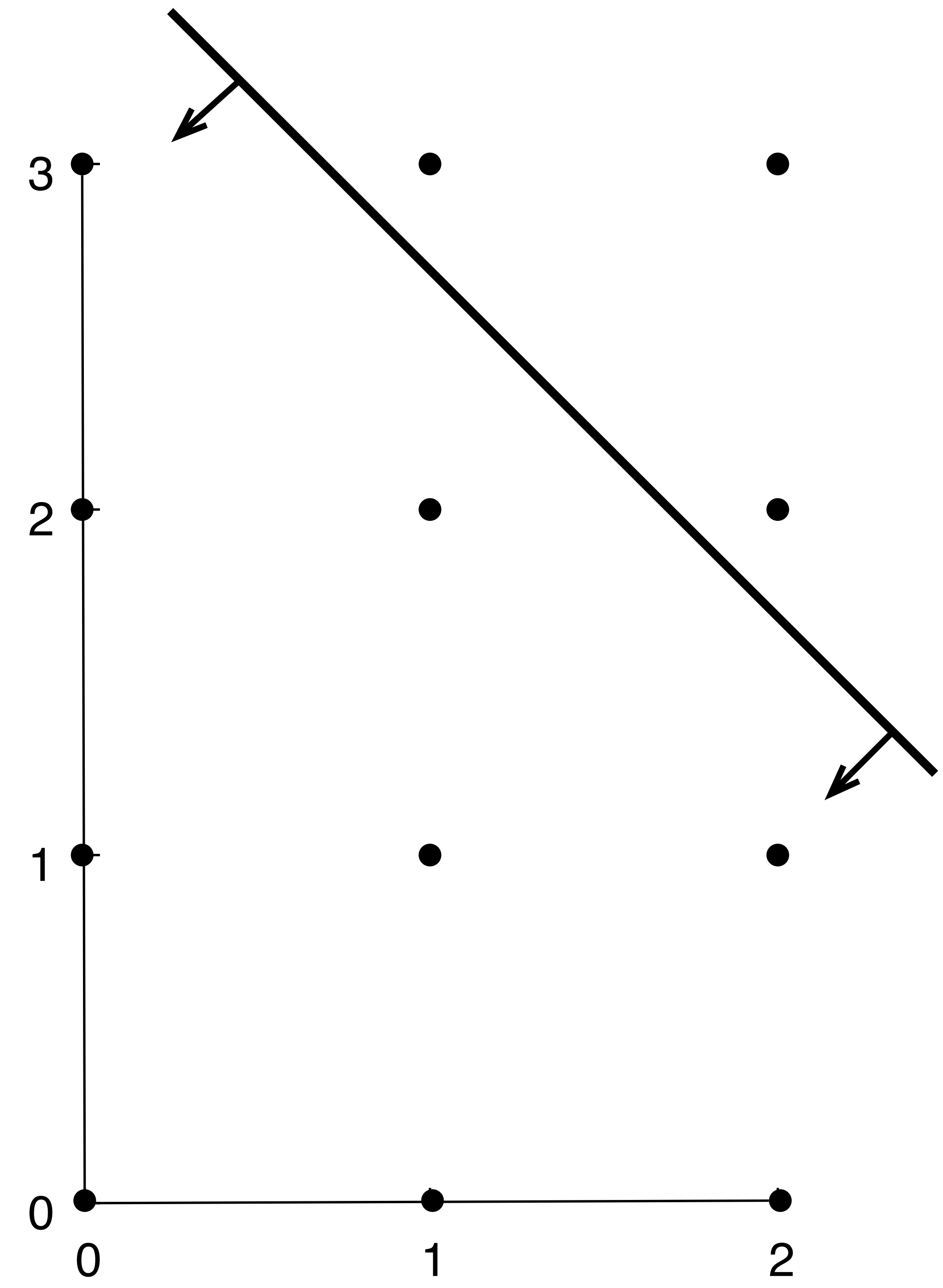


CG Cuts for Rational Polyhedra

$$P := \left\{ x \in \mathbb{R}^2 \;:\; \begin{array}{r} x_1 + x_2 \leq 3, \\ 5x_1 - 3x_2 \leq 3 \end{array} \right\}$$

$$\rotatebox{90}{$\subseteq$}$$

$$H := \{x \in \mathbb{R}^2 \;:\; 4x_1 + 3x_2 \leq 10.5\}$$



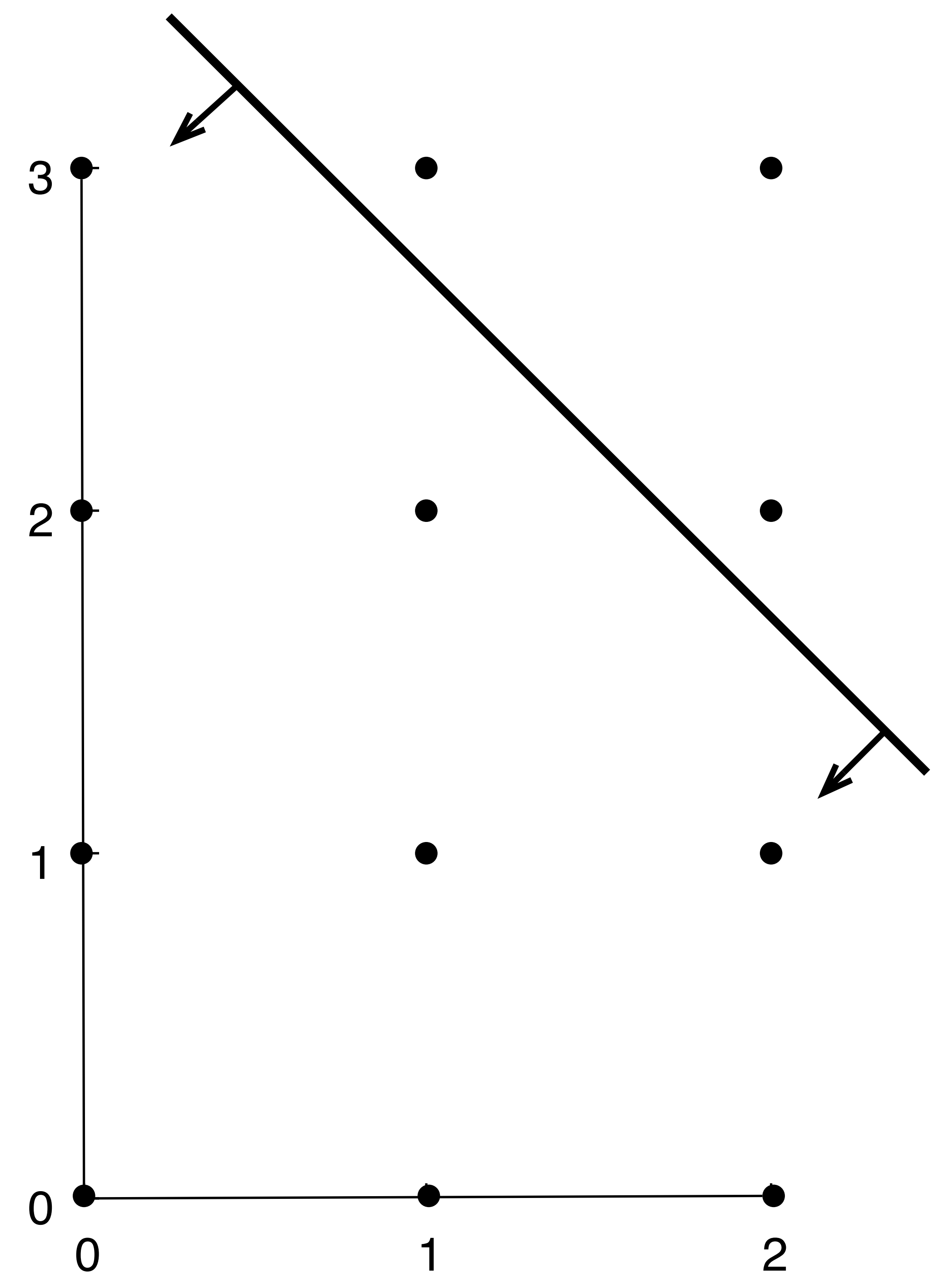
CG Cuts for Rational Polyhedra

$$P := \left\{ x \in \mathbb{R}^2 \; : \; \begin{array}{r} x_1 + x_2 \leq 3, \\ 5x_1 - 3x_2 \leq 3 \end{array} \right\}$$

$$|\cap$$

$$H := \{ x \in \mathbb{R}^2 \; : \; \underbrace{4x_1 + 3x_2}_{\in \mathbb{Z}} \leq 10.5 \}$$

$$\text{if } x \in \mathbb{Z}^2$$



CG Cuts for Rational Polyhedra

$$P := \left\{ x \in \mathbb{R}^2 : \begin{array}{r} x_1 + x_2 \le 3, \\ 5x_1 - 3x_2 \le 3 \end{array} \right\}$$

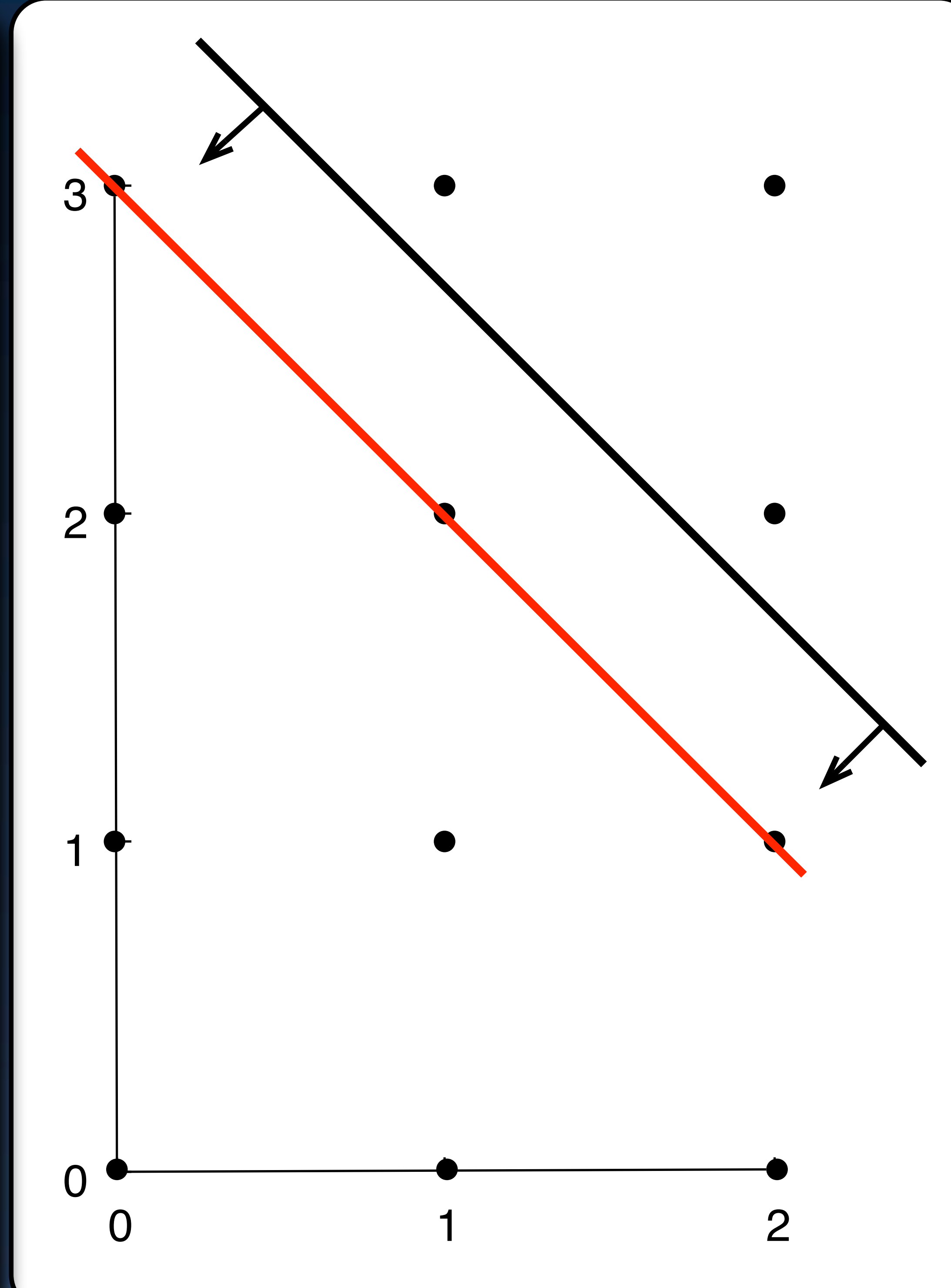
$$\subseteq$$

$$H := \{ x \in \mathbb{R}^2 : \underbrace{4x_1 + 3x_2}_{\in \mathbb{Z}} \le 10.5 \}$$

if $x \in \mathbb{Z}^2$

$$4x_1 + 3x_2 \le \lfloor 10.5 \rfloor$$

Valid for $H \cap \mathbb{Z}^2$



CG Cuts for Rational Polyhedra

$$P := \left\{ x \in \mathbb{R}^2 : \begin{array}{r} x_1 + x_2 \le 3, \\ 5x_1 - 3x_2 \le 3 \end{array} \right\}$$

$$\cap|$$

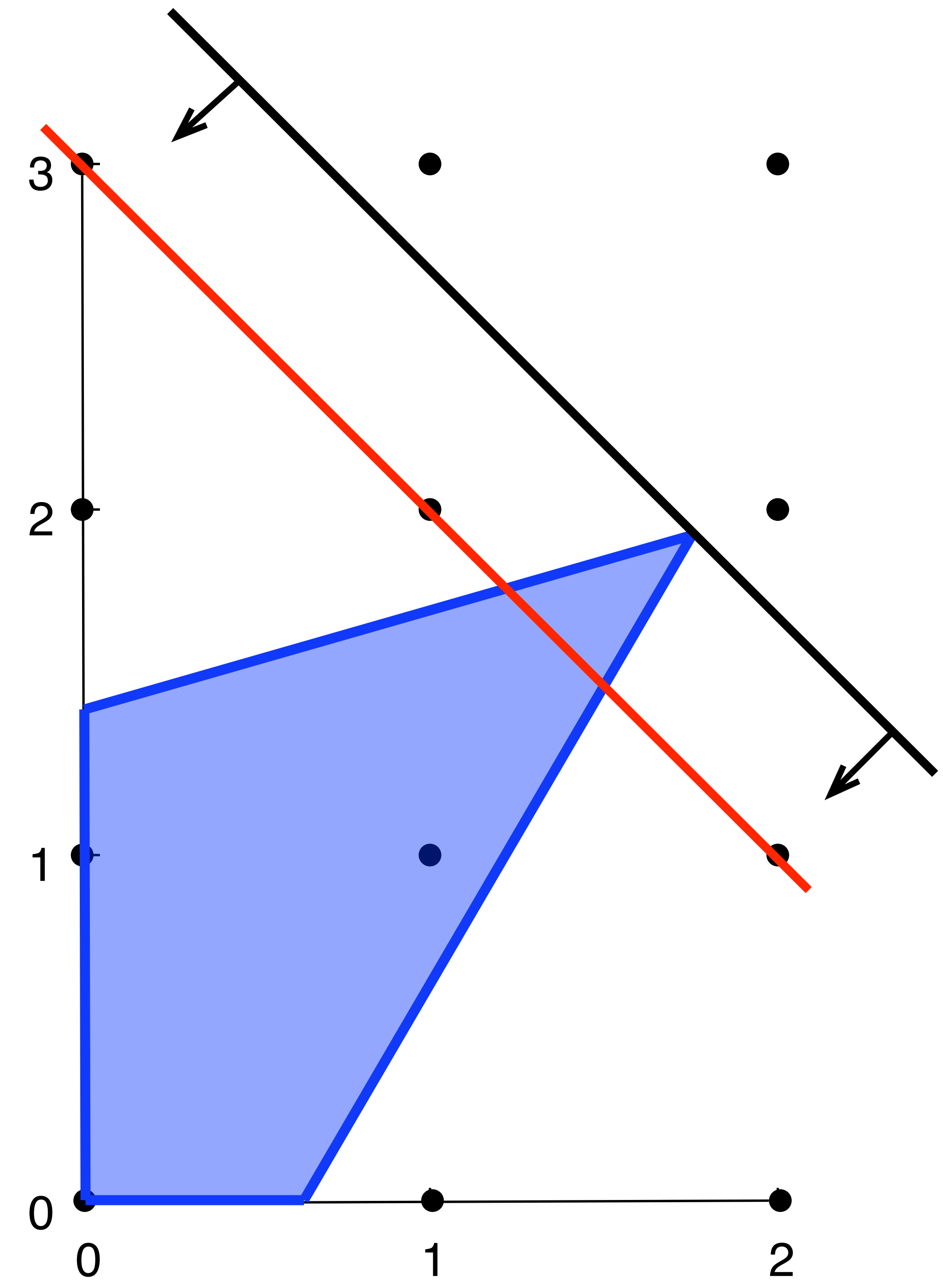
$$H := \{ x \in \mathbb{R}^2 : \underbrace{4x_1 + 3x_2}_{\in \mathbb{Z}} \le 10.5 \}$$

if $x \in \mathbb{Z}^2$

$$4x_1 + 3x_2 \le \lfloor 10.5 \rfloor$$

Valid for $H \cap \mathbb{Z}^2$

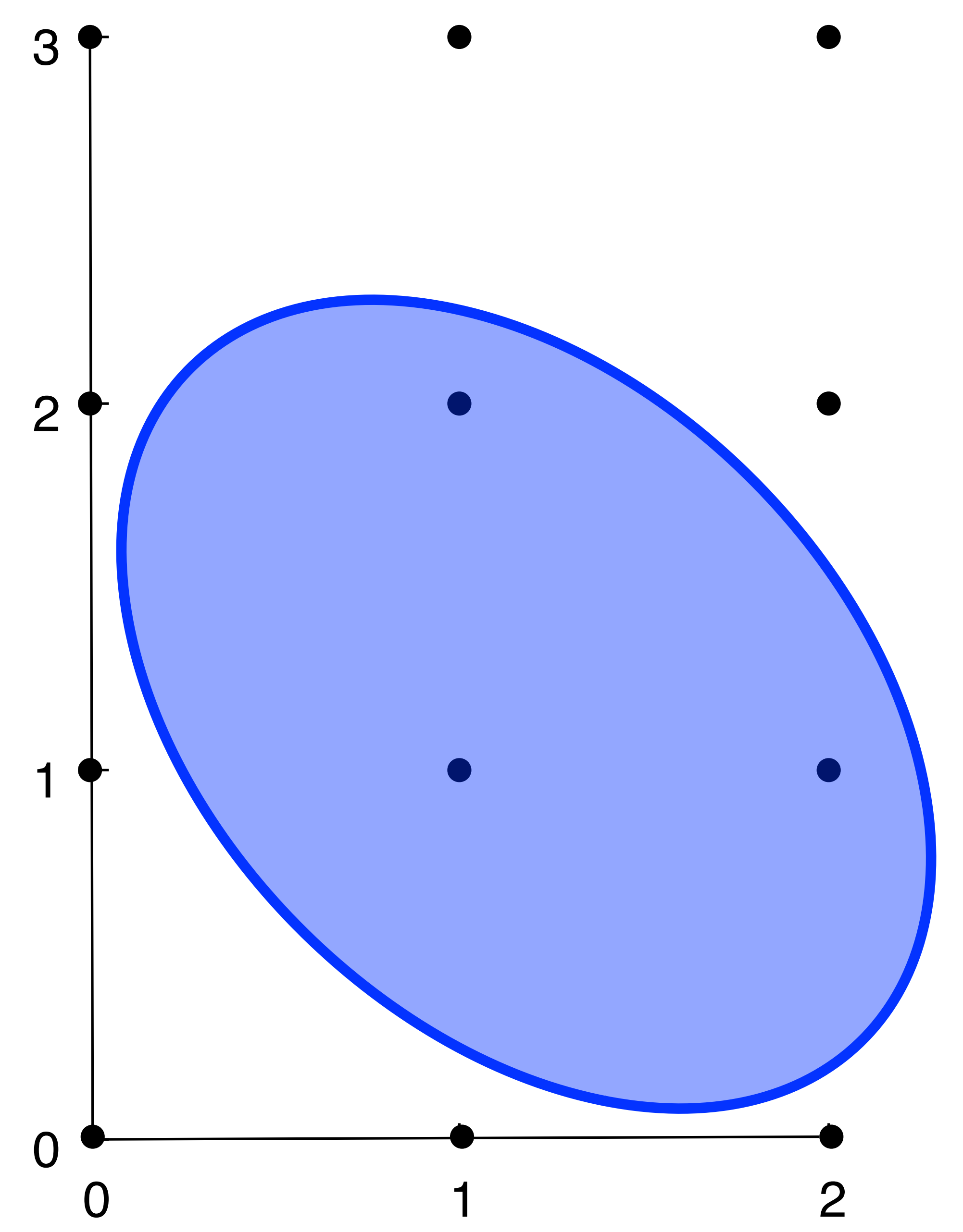
Valid for $C \cap \mathbb{Z}^2$



CG Cuts for General Convex Sets

$$\sigma_C(a) := \sup\{\langle a, x\rangle \,:\, x \in C\}$$

$$C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n \,:\, \langle a, x\rangle \le \sigma_C(a)\}$$



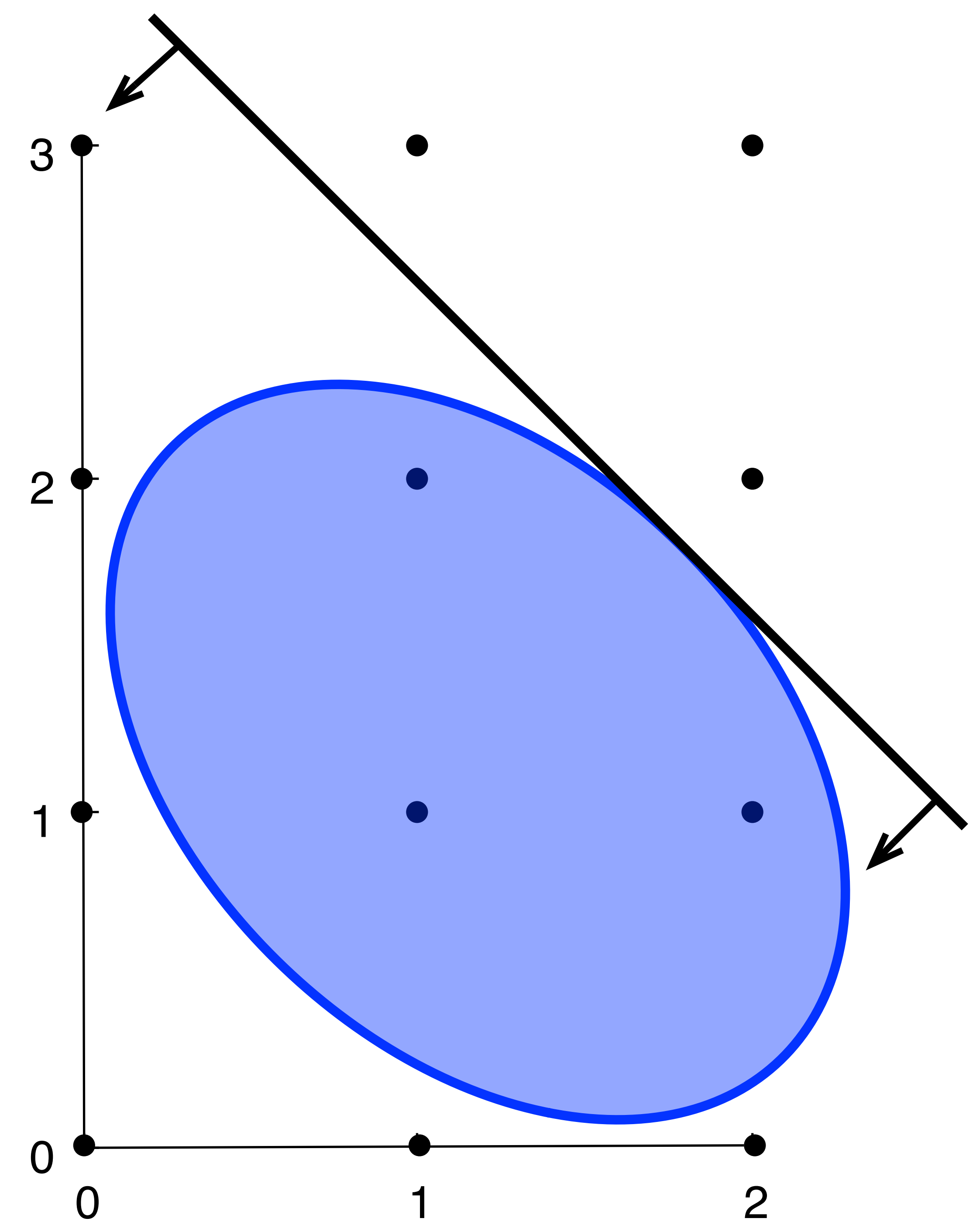
CG Cuts for General Convex Sets

$$\sigma_C(a) := \sup\{\langle a, x\rangle \ : \ x \in C\}$$

$$C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n \ : \ \langle a, x\rangle \leq \sigma_C(a)\}$$

$$\cap\!|$$

$$H := \{x \in \mathbb{R}^n \ : \ \langle a, x\rangle \leq \sigma_C(a)\}$$



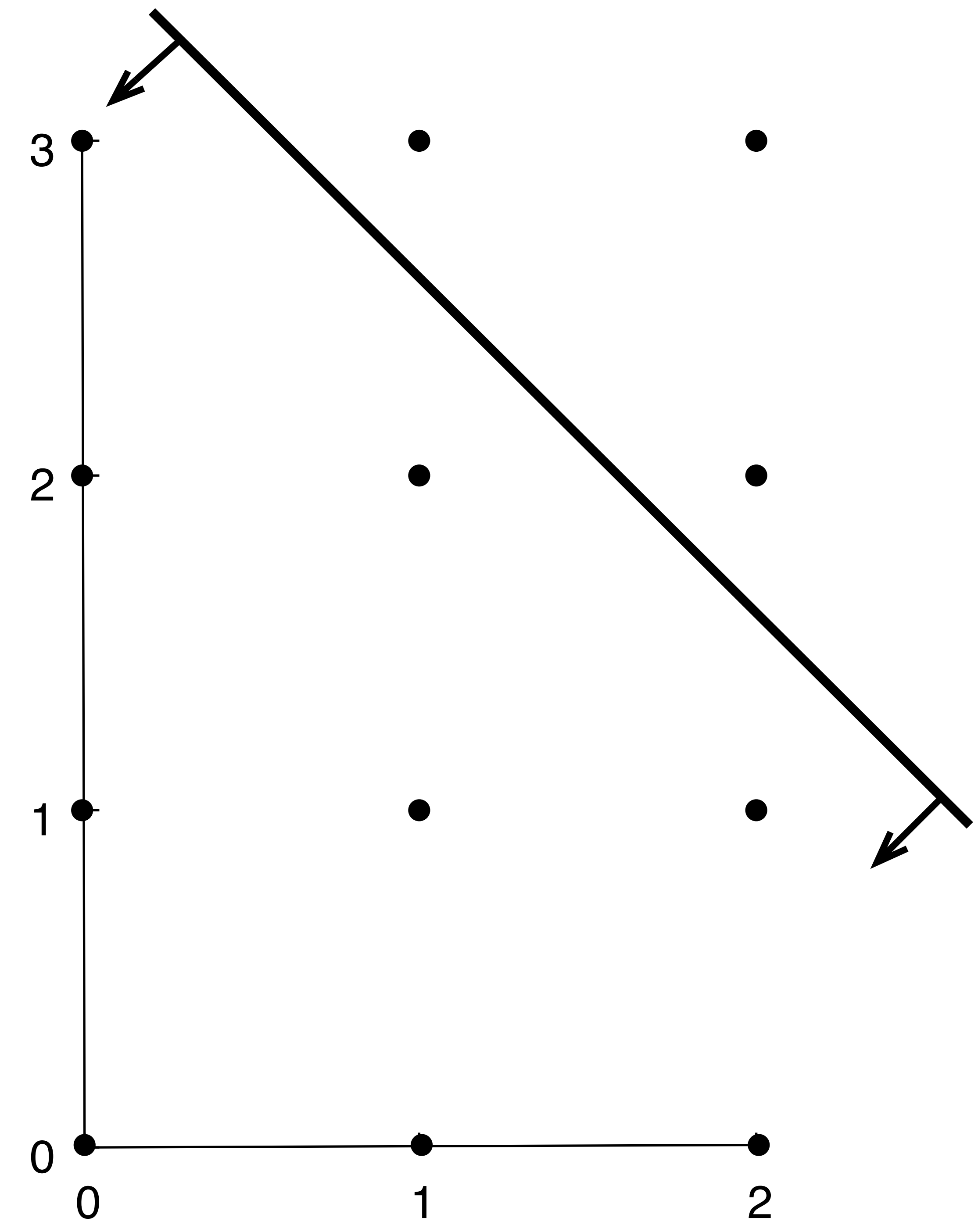
CG Cuts for General Convex Sets

$$\sigma_C(a) := \sup\{\langle a, x\rangle \,:\, x \in C\}$$

$$C = \bigcap_{a\in\mathbb{Z}^n} \{x \in \mathbb{R}^n \,:\, \langle a, x\rangle \le \sigma_C(a)\}$$

$$\cap\!\!\mid$$

$$H := \{x \in \mathbb{R}^n \,:\, \langle a, x\rangle \le \sigma_C(a)\}$$



CG Cuts for General Convex Sets

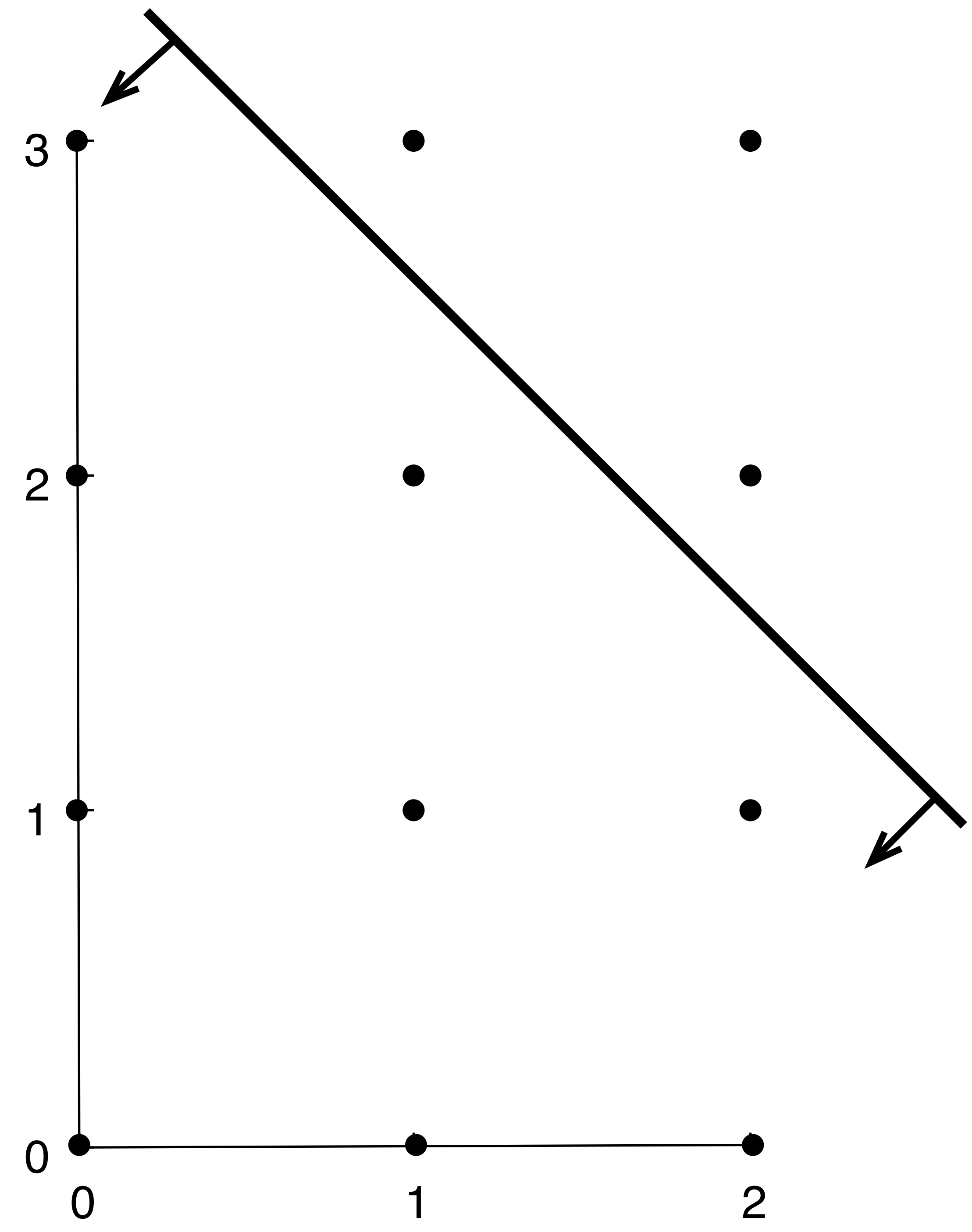
$$\sigma_C(a) := \sup\{\langle a, x\rangle \ : \ x \in C\}$$

$$C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n \ : \ \langle a, x\rangle \leq \sigma_C(a)\}$$

$$\cap\!\!|$$

$$H := \{x \in \mathbb{R}^n \ : \ \underbrace{\langle a, x\rangle}_{\in \mathbb{Z}} \leq \sigma_C(a)\}$$

if $x \in \mathbb{Z}^n$



CG Cuts for General Convex Sets

$$\sigma_C(a) := \sup\{\langle a, x\rangle \,:\, x \in C\}$$

$$C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n \,:\, \langle a, x\rangle \leq \sigma_C(a)\}$$

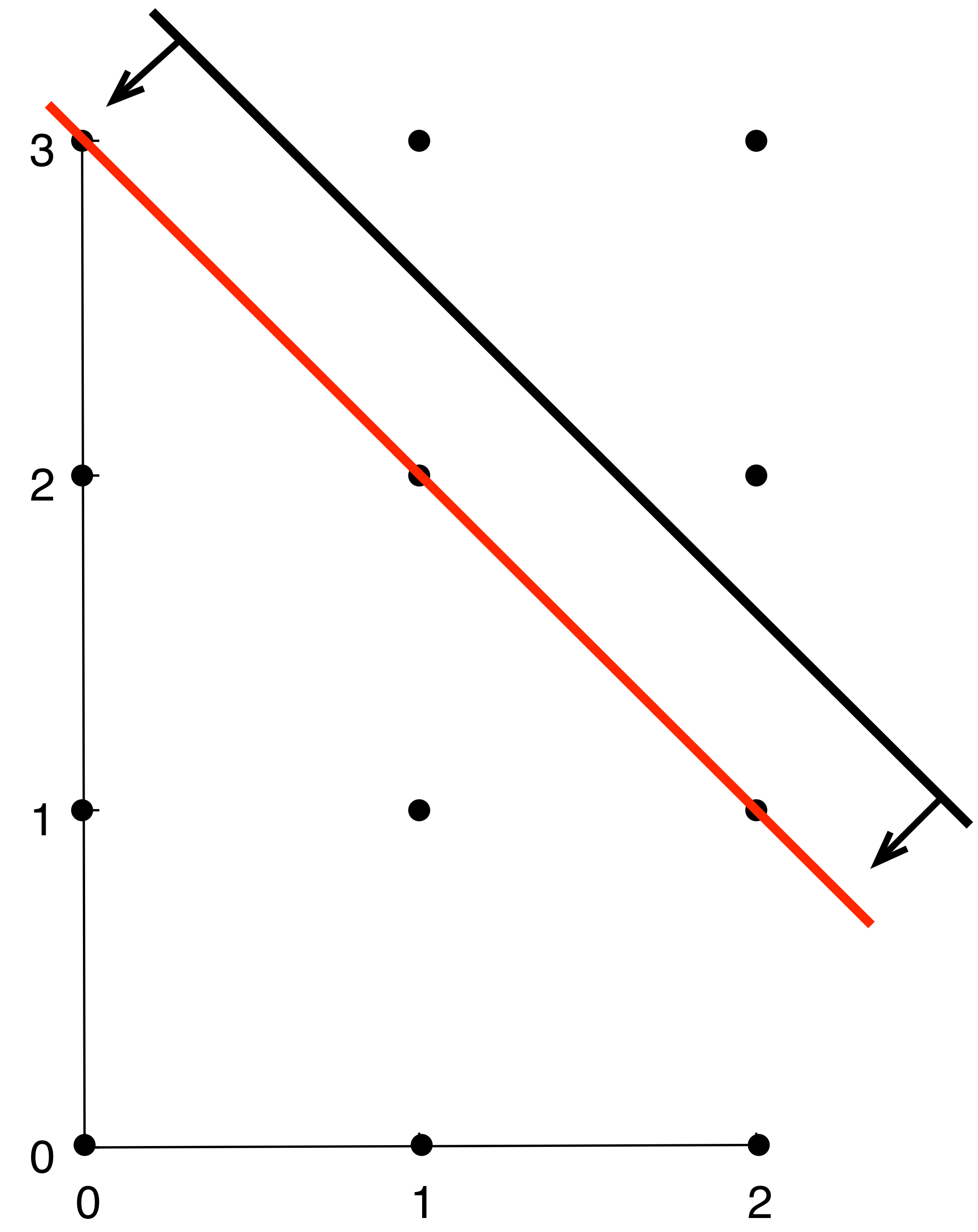
$$\cap\!\!|$$

$$H := \{x \in \mathbb{R}^n \,:\, \underbrace{\langle a, x\rangle}_{\in \mathbb{Z}} \leq \sigma_C(a)\}$$

if $x \in \mathbb{Z}^n$

$$\langle a, x\rangle \leq \lfloor \sigma_C(a) \rfloor$$

Valid for $H \cap \mathbb{Z}^n$



CG Cuts for General Convex Sets

$$\sigma_C(a) := \sup\{\langle a, x\rangle \,:\, x \in C\}$$

$$C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n \,:\, \langle a, x\rangle \leq \sigma_C(a)\}$$

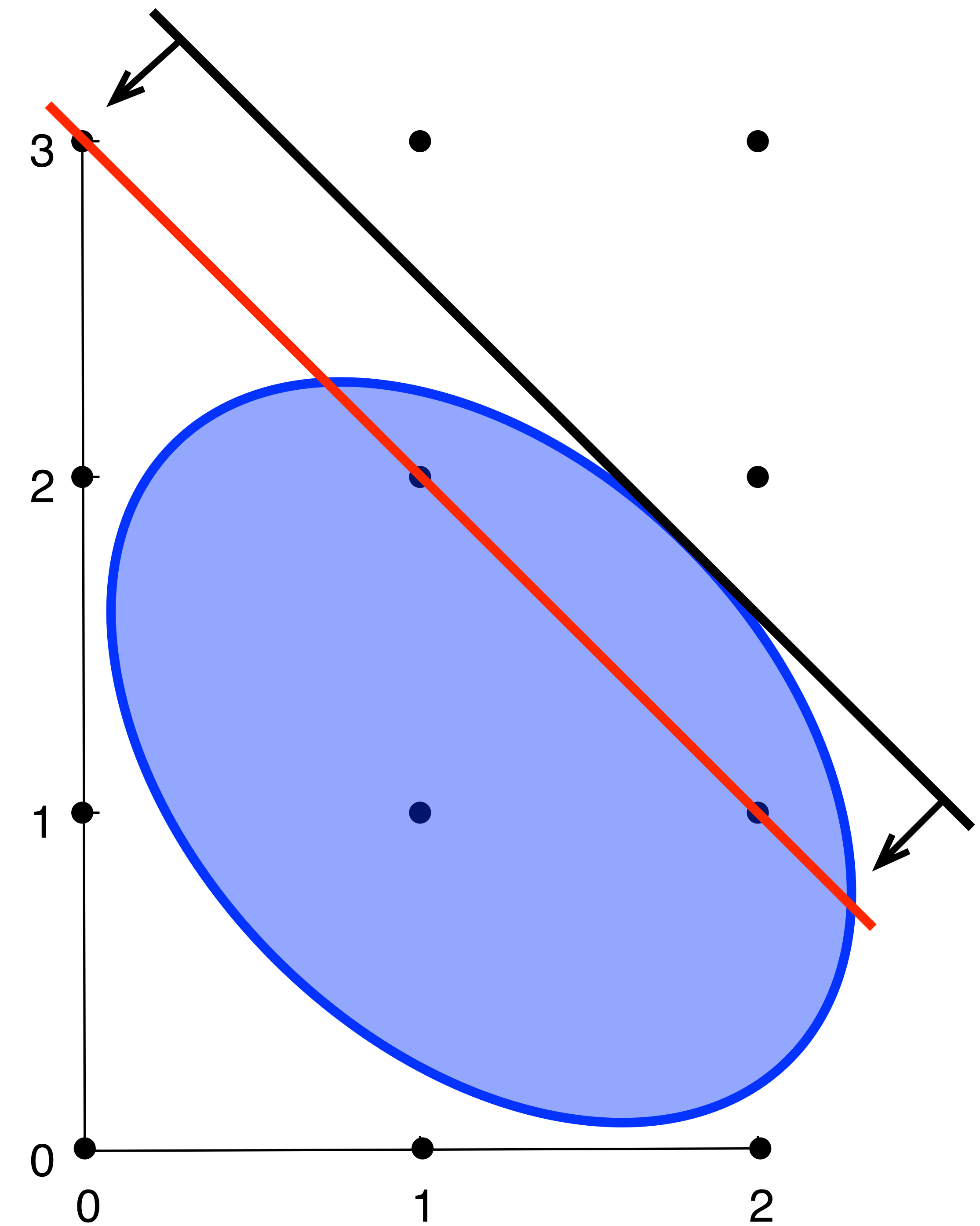
$$\cap\!\!|$$

$$H := \{x \in \mathbb{R}^n \,:\, \underbrace{\langle a, x\rangle}_{\in \mathbb{Z} \text{ if } x \in \mathbb{Z}^n} \leq \sigma_C(a)\}$$

$$\langle a, x\rangle \leq \lfloor \sigma_C(a) \rfloor$$

Valid for $H \cap \mathbb{Z}^n$

Valid for $C \cap \mathbb{Z}^n$



CG Closure = Add all CG Cuts

$$\mathtt{CC}(C) := \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}$$

- Not necessarily a polyhedron, remember:

$$C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \sigma_C(a)\}$$

CG Closure = Add all CG Cuts

$$\mathrm{cc}(C) := \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}$$

- Not necessarily a polyhedron, remember:

$$C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \sigma_C(a)\}$$

- $\mathrm{cc}(C)$ is a polyhedron if C is:
  - a rational polyhedron (Schrijver, 1980).
  - a strictly convex set (D., D. and V. 2010).

CG Closure is Finitely Generated

Theorem: There exists finite $S \subseteq \mathbb{Z}^n$ such that

$$\mathtt{cc}(C) = \underbrace{\bigcap_{a \in S} \{x \in \mathbb{R}^n \ : \ \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor \}}_{\mathtt{cc}(C,S)}$$

CG Closure is Finitely Generated

Theorem: There exists finite $S \subseteq \mathbb{Z}^n$ such that

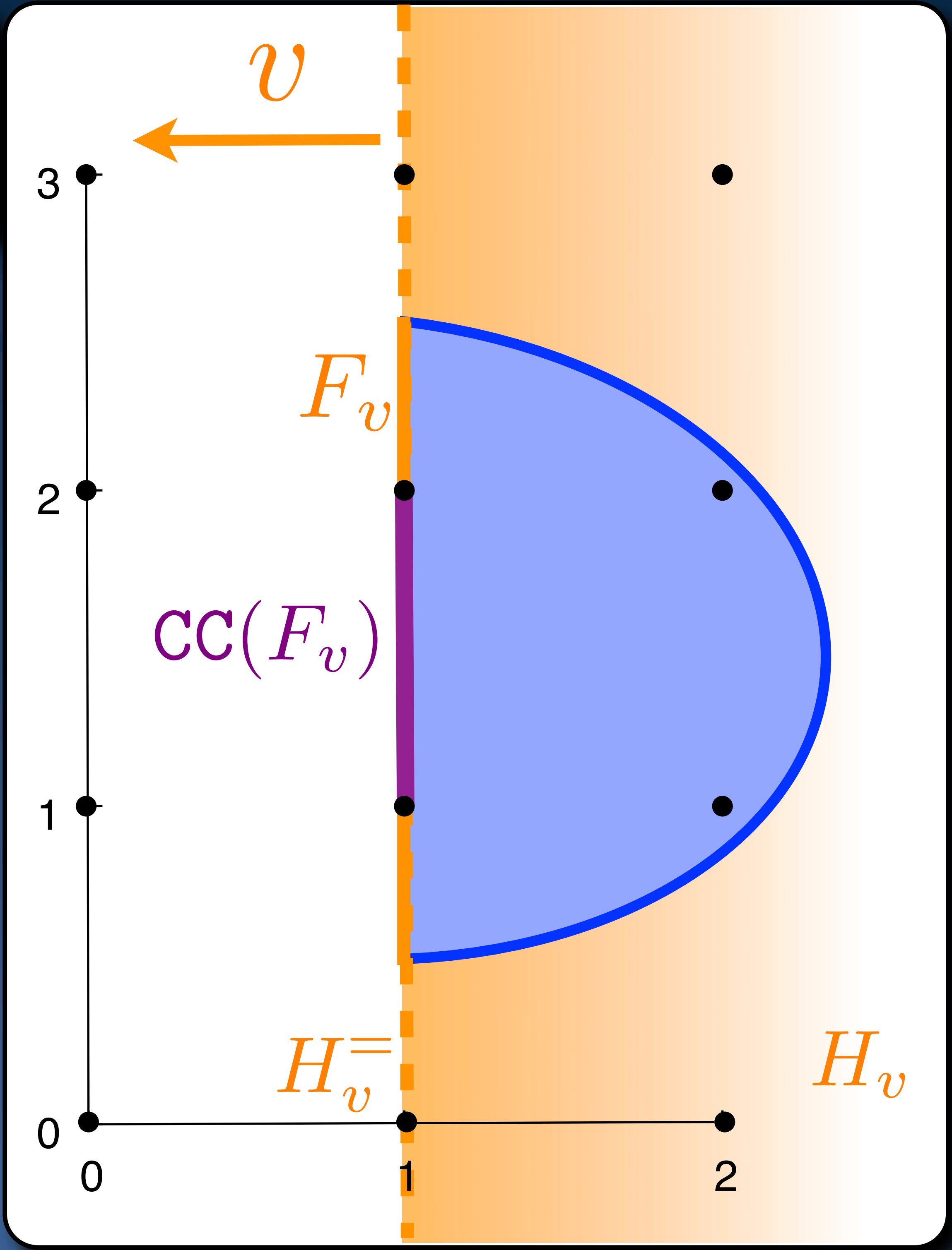
$$\mathrm{cc}(C) = \underbrace{\bigcap_{a \in S} \{x \in \mathbb{R}^n \ : \ \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}}_{\mathrm{cc}(C,S)}$$

- Proof by induction on $\dim(C)$:
  - Step 1: Create finite S_1 s.t. $\mathrm{cc}(C, S_1) \subseteq C$, etc.
  - Step 2: Show only missed finite number of cuts

Main Tool: Lifting Cuts for Faces

P polyhedron, F face of P

$\mathsf{cc}(F) = \mathsf{cc}(C) \cap F$ (Schrijver, '86)



Main Tool: Lifting Cuts for Faces

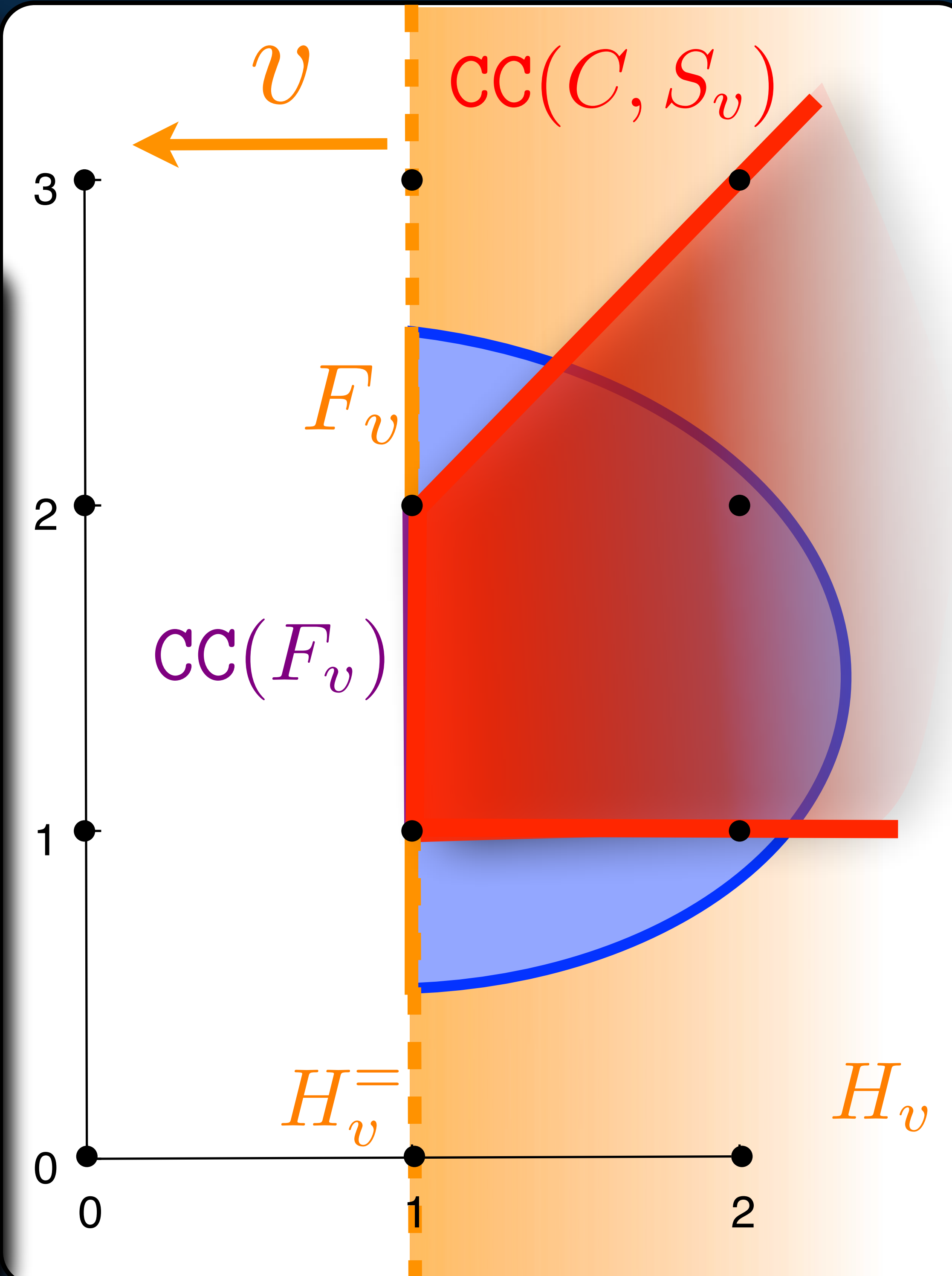
P polyhedron, F face of P

$\mathtt{CC}(F) = \mathtt{CC}(C) \cap F$ (Schrijver, '86)

If $\mathtt{CC}(F_v)$ is finitely generated then:

$\exists\, S_v$ s.t.

- $|S_v| < \infty$.
- $\mathtt{CC}(C, S_v) \cap H_v^{=} = \mathtt{CC}(F_v)$
- $\mathtt{CC}(C, S_v) \subseteq H_v$



Lifting Cuts for Faces Proof

- Part 1: Kill Irrationality: $\operatorname{aff}_I(C) := \operatorname{aff}\left(\operatorname{aff}(C) \cap \mathbb{Z}^n\right)$

$$|S_I| < \infty \text{ s.t. } \mathtt{cc}(C, S_I) \cap H_v^= \subseteq \operatorname{aff}_I\left(H_v^=\right)$$
$$\mathtt{cc}(C, S_I) \subseteq H_v$$

  - Kronecker's approximation theorem

- Part 2: Lift inside $\operatorname{aff}_I\left(H_v^=\right)$

$$|S_R| < \infty \text{ s.t. } \mathtt{cc}(C, S_R) \cap \operatorname{aff}_I\left(H_v^=\right) = \mathtt{cc}(F_v) \cap \operatorname{aff}_I\left(H_v^=\right)$$

  - Dirichlet's approximation theorem

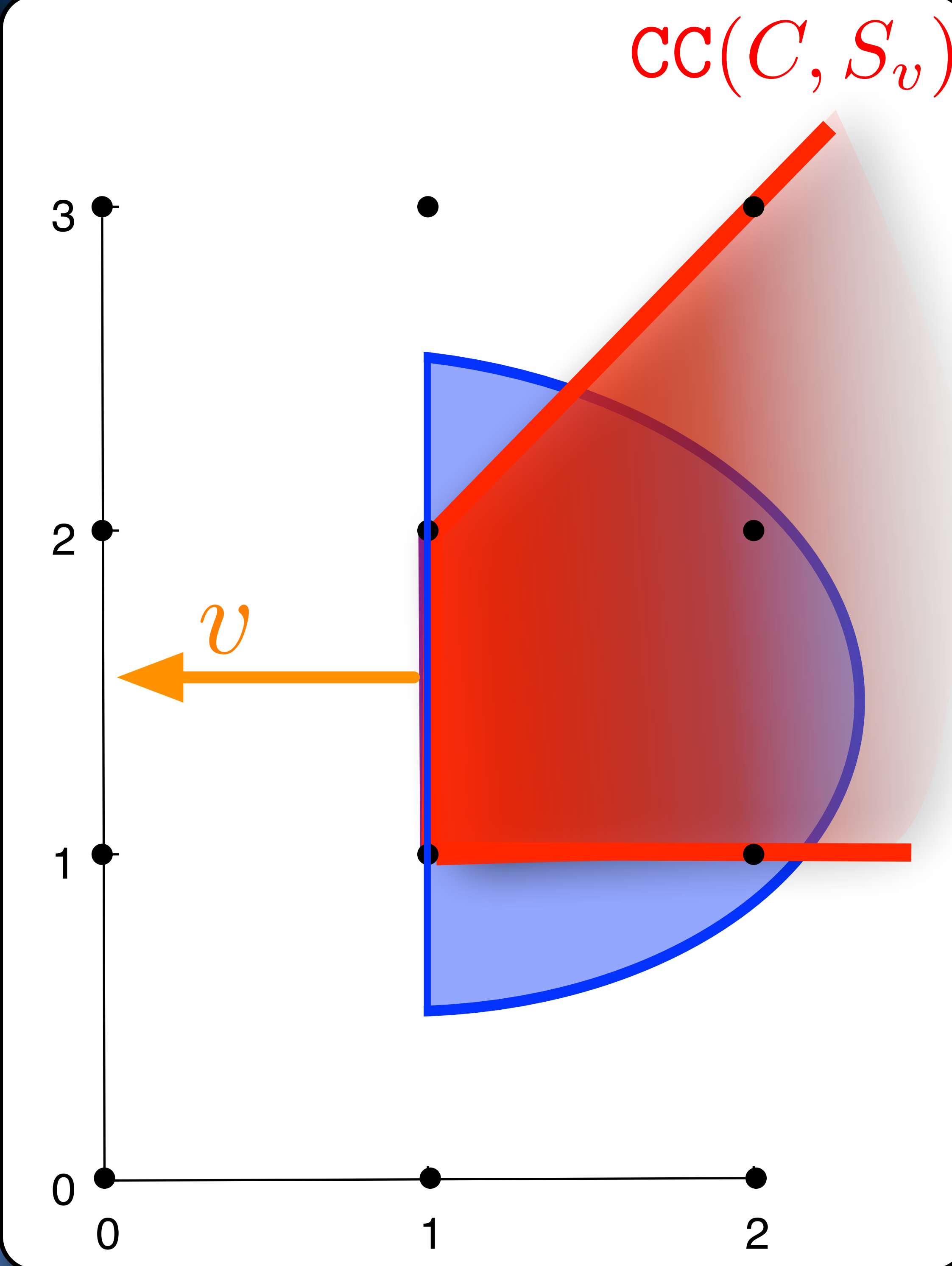
Step 1: Two Approximations

- Approximation A:
  - Finite S_A such that $\mathtt{cc}(C, S_A) \subseteq C \cap \mathrm{aff}_I(C)$
  - Proof: Compactness argument
- Approximation B:
  - Finite S_B such that

    $\mathtt{cc}(C, S_A \cup S_B) \cap \mathrm{relbd}(C) = \mathtt{cc}(C) \cap \mathrm{relbd}(C)$

Approximation B: (for Full-dim C)

- $\mathrm{cc}(C, S_v) \cap H_v^{=} = \mathrm{cc}(F_v)$
- $\mathrm{cc}(C, S_v) \subseteq H_v$

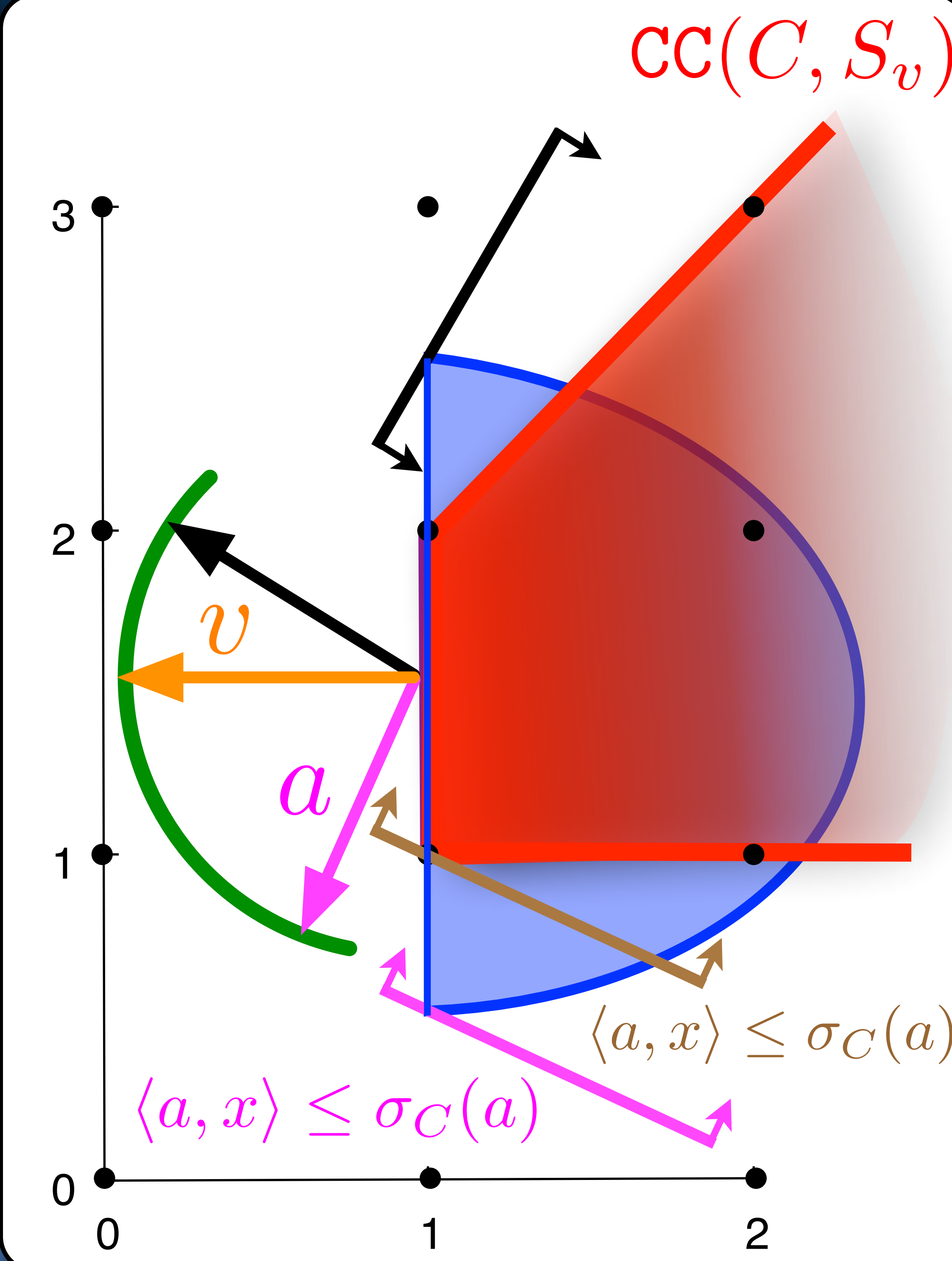


Approximation B: (for Full-dim C)

- $\mathtt{cc}(C, S_v) \cap H_v^{=} = \mathtt{cc}(F_v)$
- $\mathtt{cc}(C, S_v) \subseteq H_v$

$\exists$ neigbourhood $N_v \subseteq S^{n-1}$ s.t. $\forall a \in N_v$

$$\{x \in \mathbb{R}^n \ : \ \langle a, x\rangle \leq \sigma_C(a)\}$$
$$\cap\!|$$
$$\{x \in \mathbb{R}^n \ : \ \langle a, x\rangle \leq \sigma_{\mathtt{cc}(C,S_v)}(a)\}$$



Approximation B: (for Full-dim C)

- $\mathtt{cc}(C, S_v) \cap H_v^{=} = \mathtt{cc}(F_v)$
- $\mathtt{cc}(C, S_v) \subseteq H_v$

$\exists$ neigbourhood $N_v \subseteq S^{n-1}$ s.t. $\forall a \in N_v$

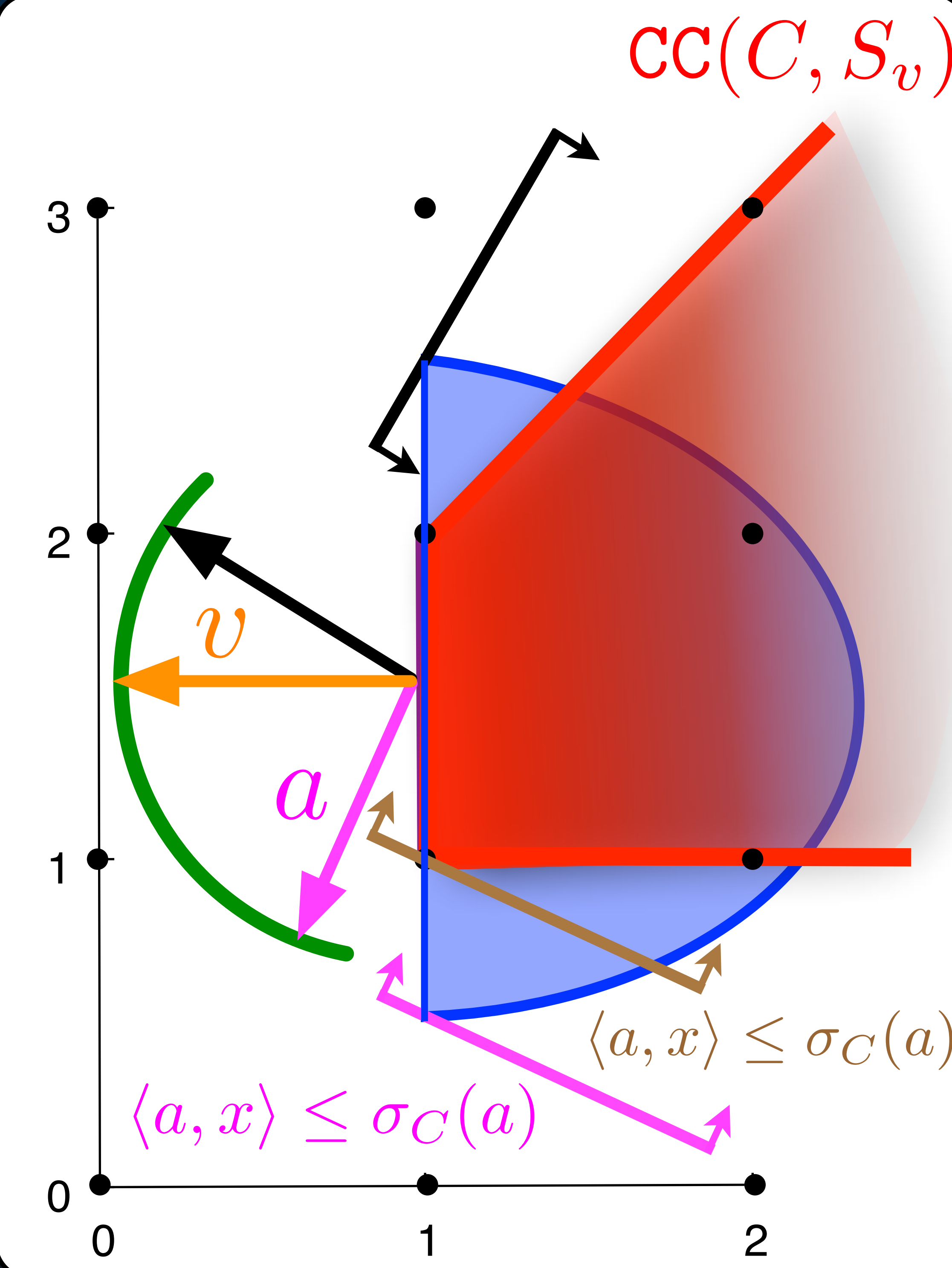
$$\{x \in \mathbb{R}^n \ : \ \langle a, x\rangle \leq \sigma_C(a)\}$$

$$\cap|$$

$$\{x \in \mathbb{R}^n \ : \ \langle a, x\rangle \leq \sigma_{\mathtt{cc}(C,S_v)}(a)\}$$

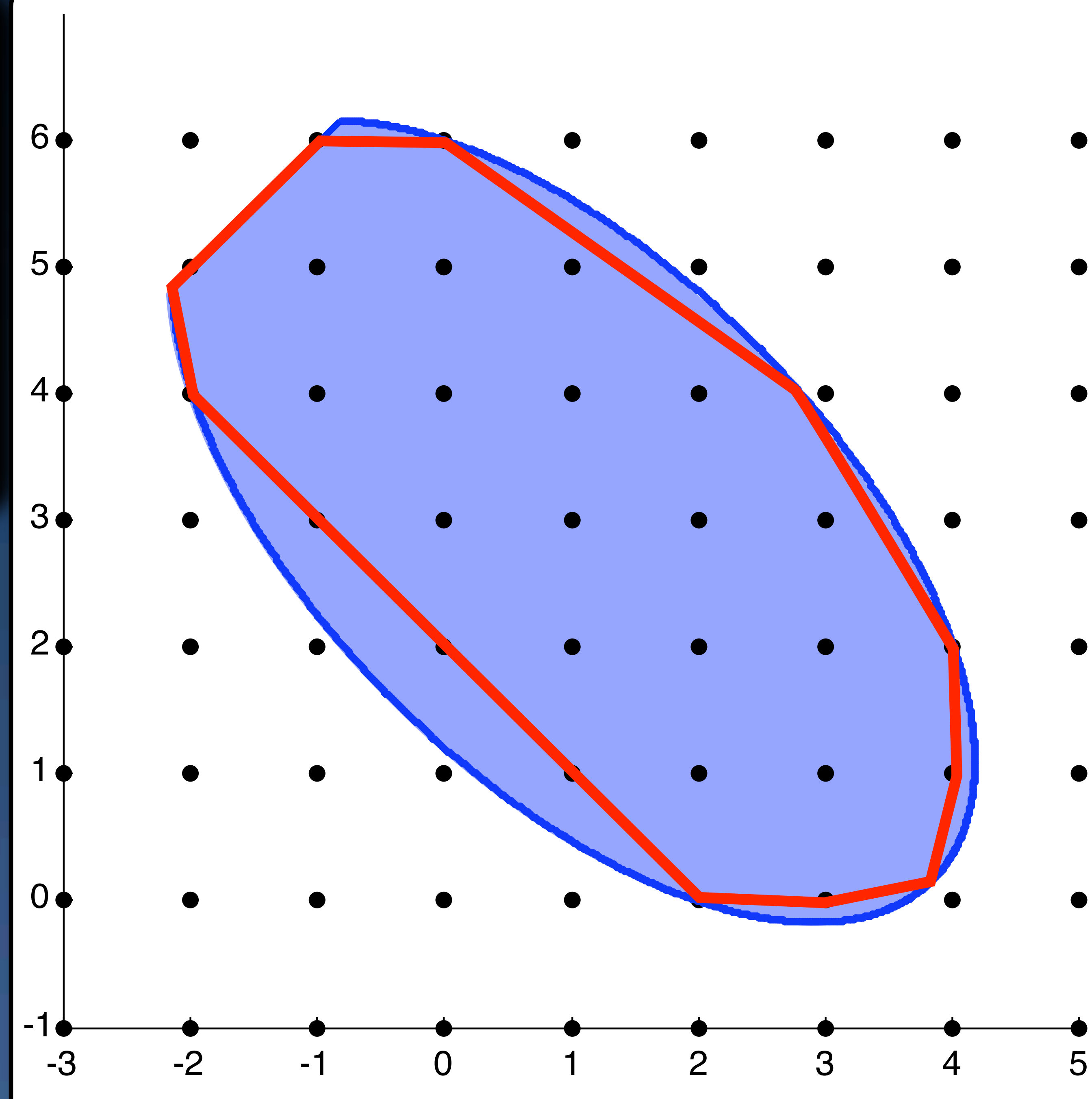
$$S^{n-1} \subseteq \bigcup_{v \in S^{n-1}} N_v \longrightarrow \bigcup_{i=1}^{k} N_{v_i}$$

compactness



Approximation B

$\texttt{cc}(C, S_A)$ is a polytope

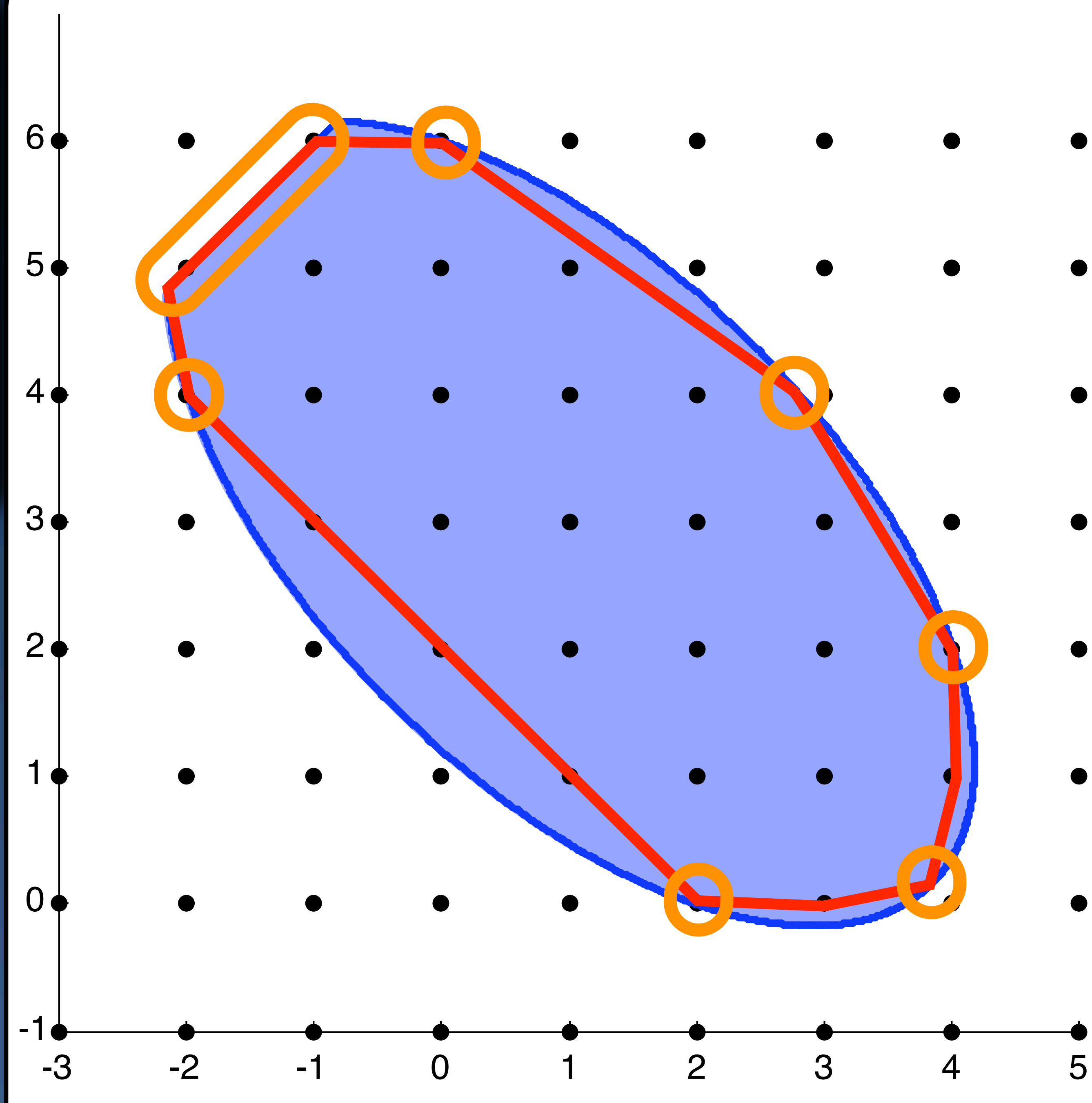


Approximation B

$\texttt{cc}(C, S_A)$ is a polytope

$$\Downarrow$$

$$\texttt{cc}(C, S_A) \cap \mathrm{relbd}(C) \subseteq \bigcup_{i=1}^{k} F_{v_i}$$



Approximation B

$\texttt{CC}(C, S_A)$ is a polytope

$$\Downarrow$$

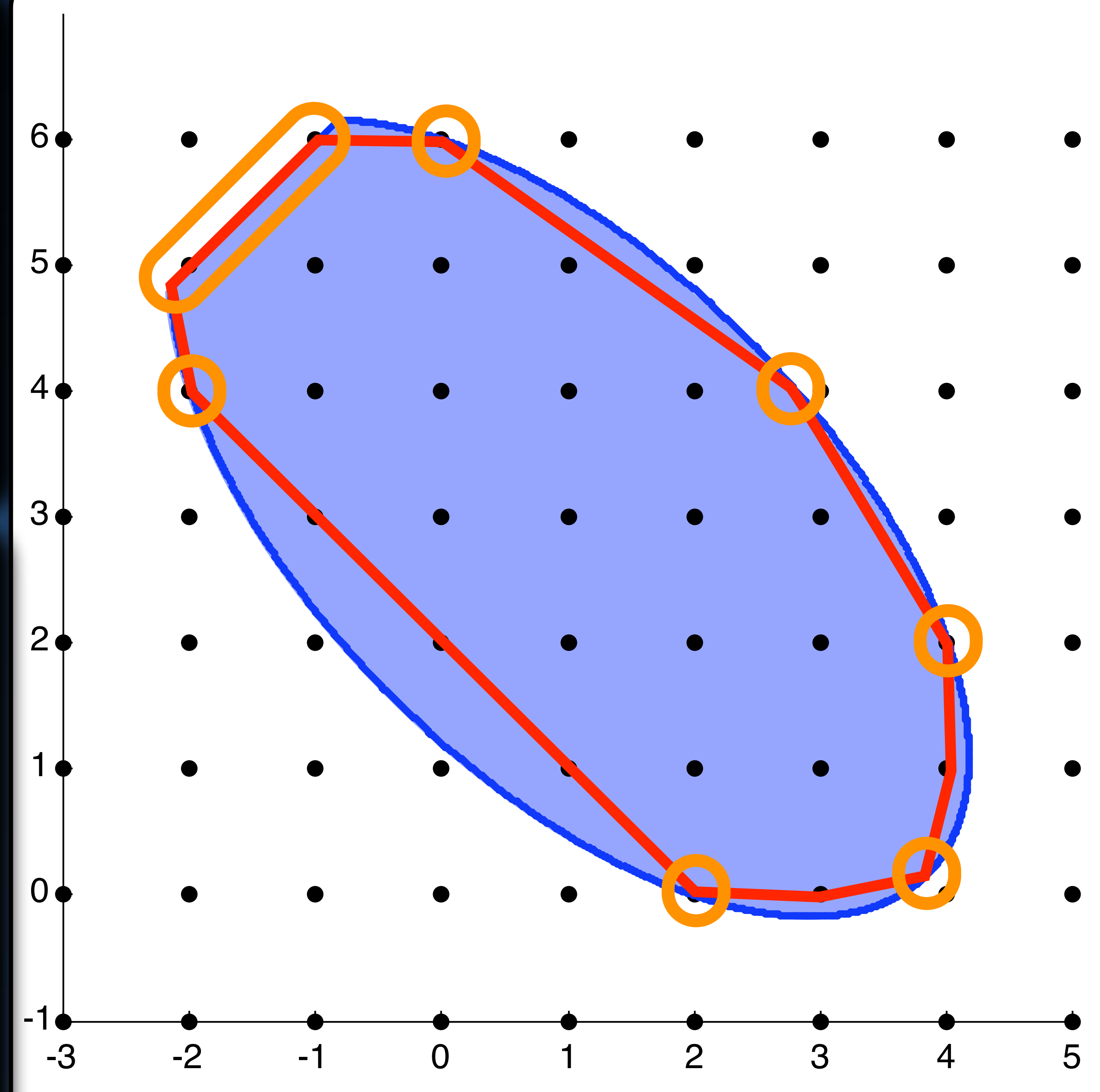
$$\texttt{CC}(C, S_A) \cap \text{relbd}(C) \subseteq \bigcup_{i=1}^{k} F_{v_i}$$

Induction Hypothesis

$\texttt{CC}(F_{v_i})$ is finitely generated:

$$S_B = \text{lifting of } \texttt{CC}(F_{v_i})$$

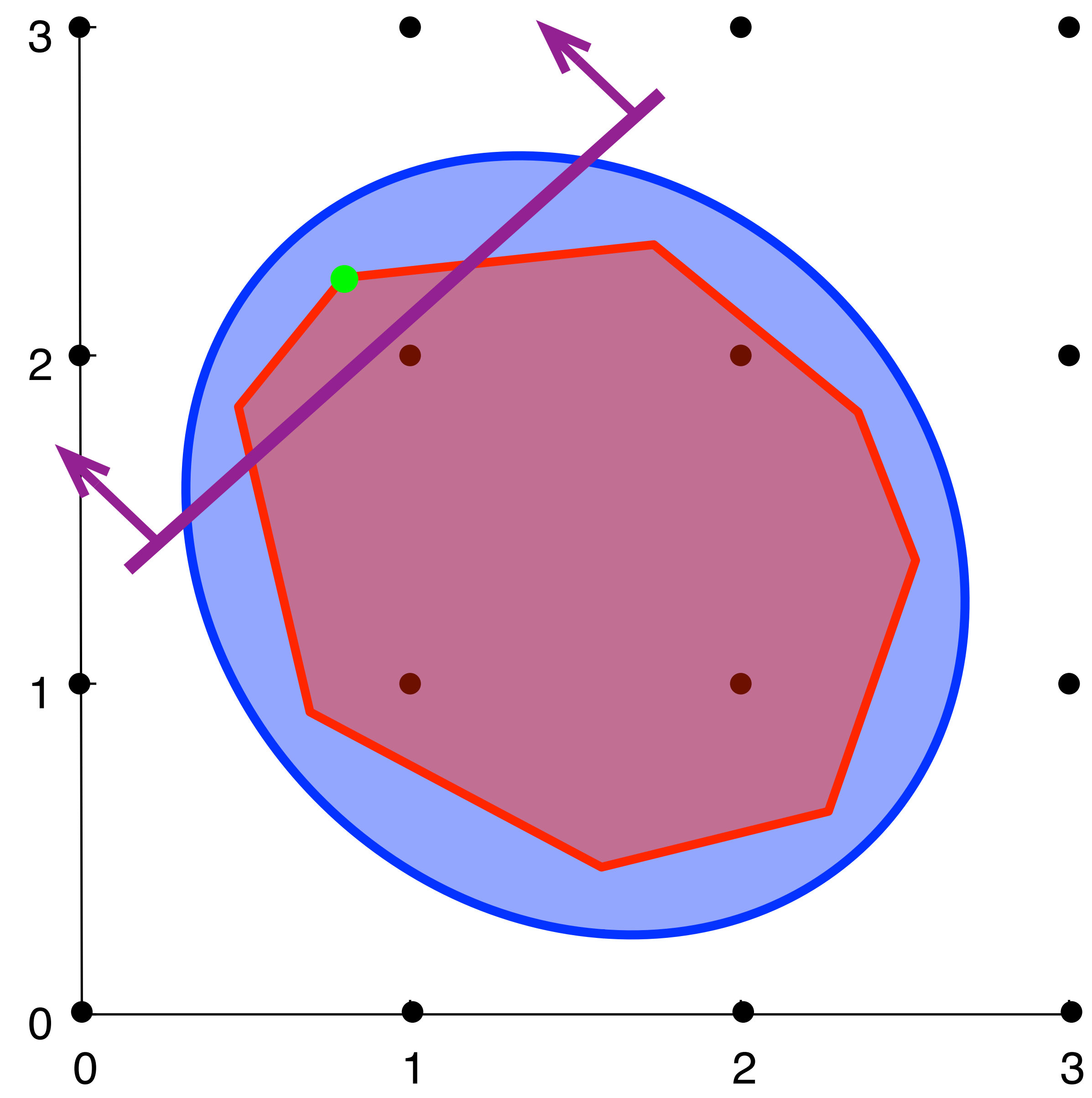
$$S_1 = S_A \cup S_B$$



Step 2 : Missed cuts (for Full-dim C)

$$V := \operatorname{ext}\left(\mathtt{CC}(S_1, C)\right) \setminus \mathbb{Z}^n$$

$$\langle a, v\rangle > \lfloor \sigma_C(a) \rfloor$$

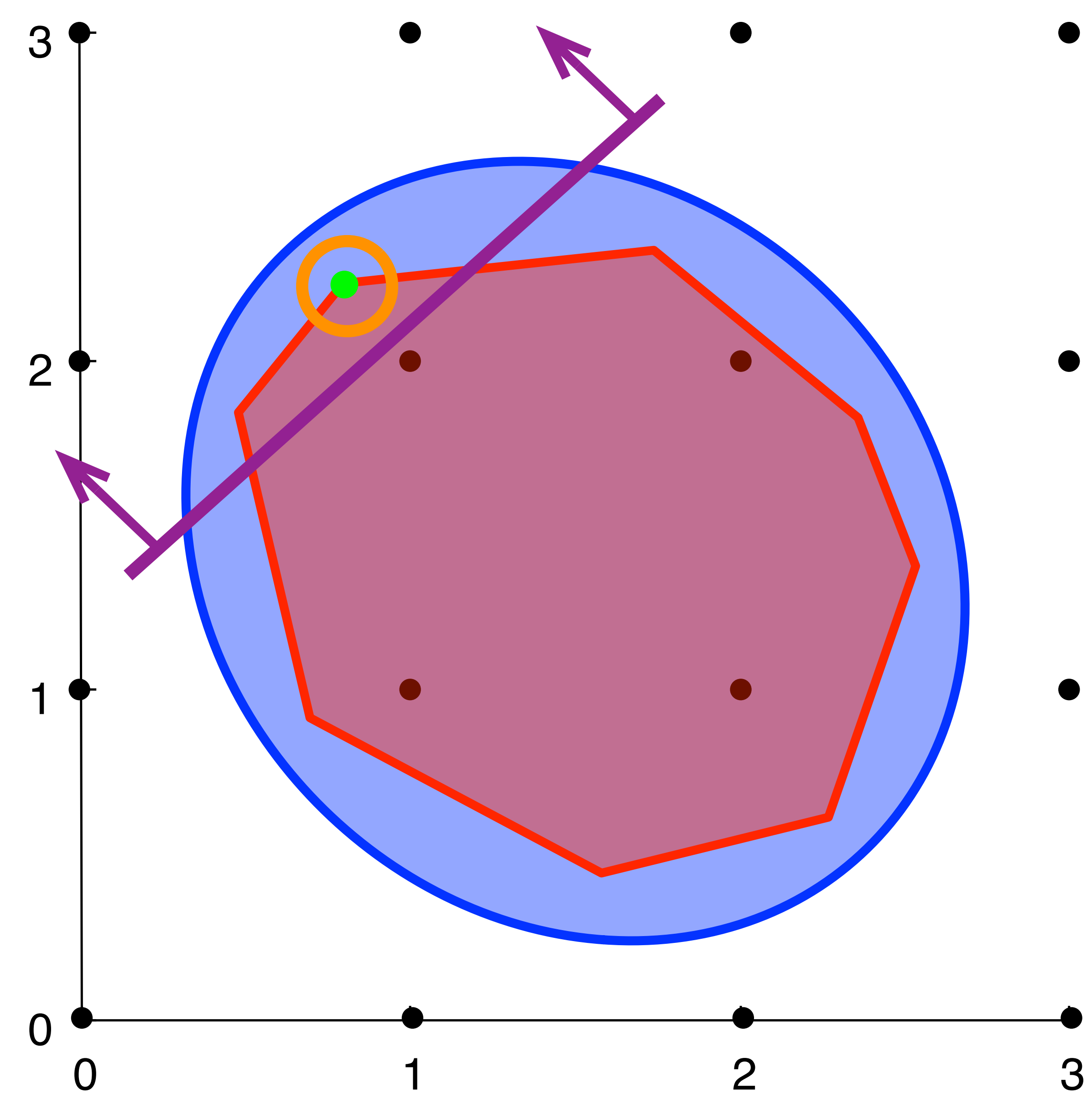


Step 2 : Missed cuts (for Full-dim C)

$V := \text{ext}\,(\text{CC}(S_1, C)) \setminus \mathbb{Z}^n$

$\langle a, v \rangle > \lfloor \sigma_C(a) \rfloor$

$\exists \varepsilon > 0 \quad \varepsilon B^n + v \subset C \quad \forall v \in V$



Step 2 : Missed cuts (for Full-dim C)

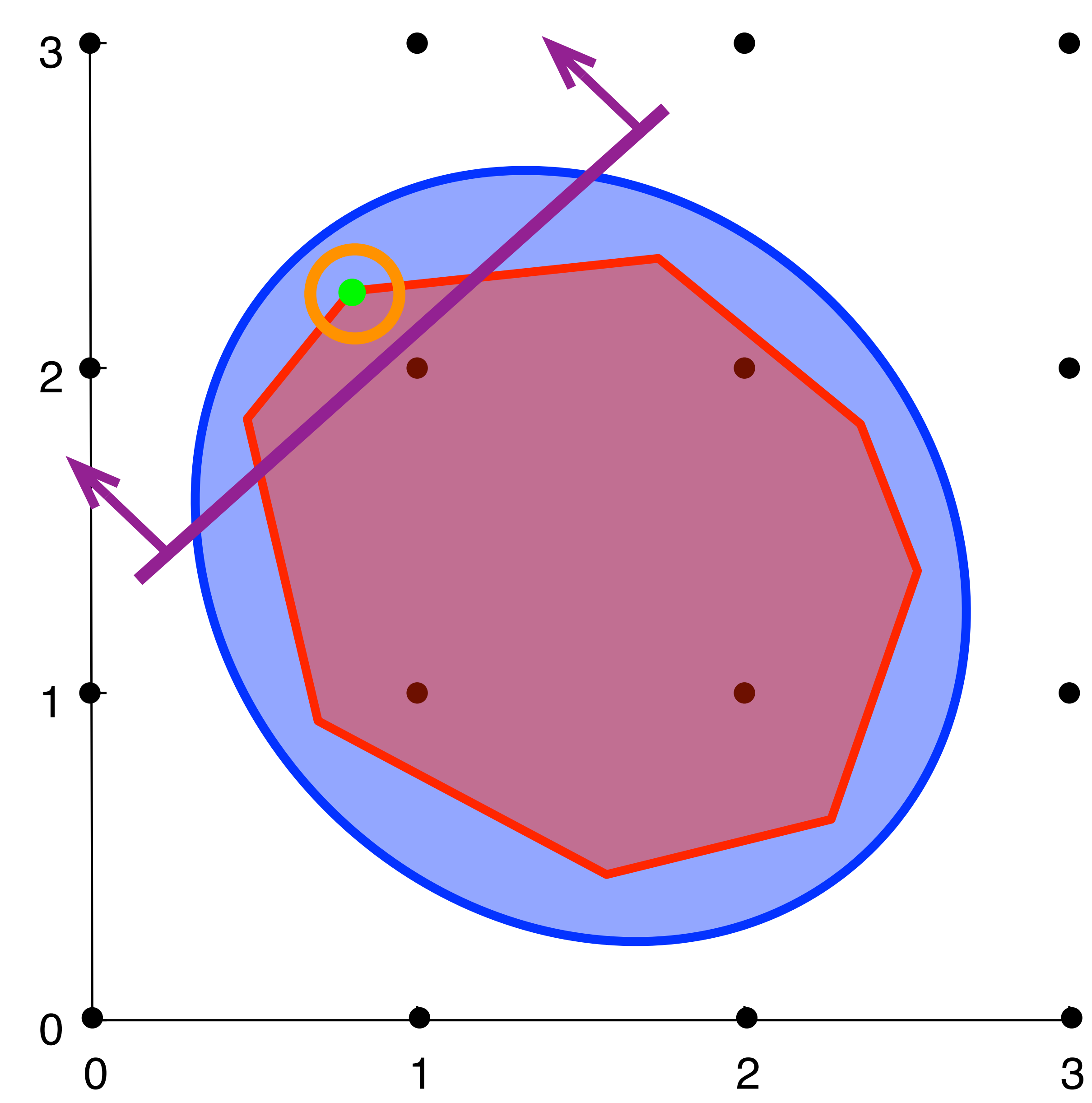
$$V := \operatorname{ext}\left(\mathtt{CC}(S_1, C)\right) \setminus \mathbb{Z}^n$$

$$\langle a, v\rangle > \lfloor \sigma_C(a) \rfloor$$

$$\exists \varepsilon > 0 \quad \varepsilon B^n + v \subset C \quad \forall v \in V$$

$$\|a\| \geq \frac{1}{\varepsilon} \Rightarrow$$

$$\begin{aligned} \lfloor \sigma_C(a) \rfloor &\geq \sigma_C(a) - 1 \\ &\geq \sigma_{v+\varepsilon B^n}(a) - 1 \\ &= \langle v, a\rangle + \varepsilon \|a\| - 1 \\ &\geq \langle v, a\rangle \end{aligned}$$



Step 2 : Missed cuts (for Full-dim C)

$$V := \operatorname{ext}\left(\mathrm{CC}(S_1, C)\right) \setminus \mathbb{Z}^n$$

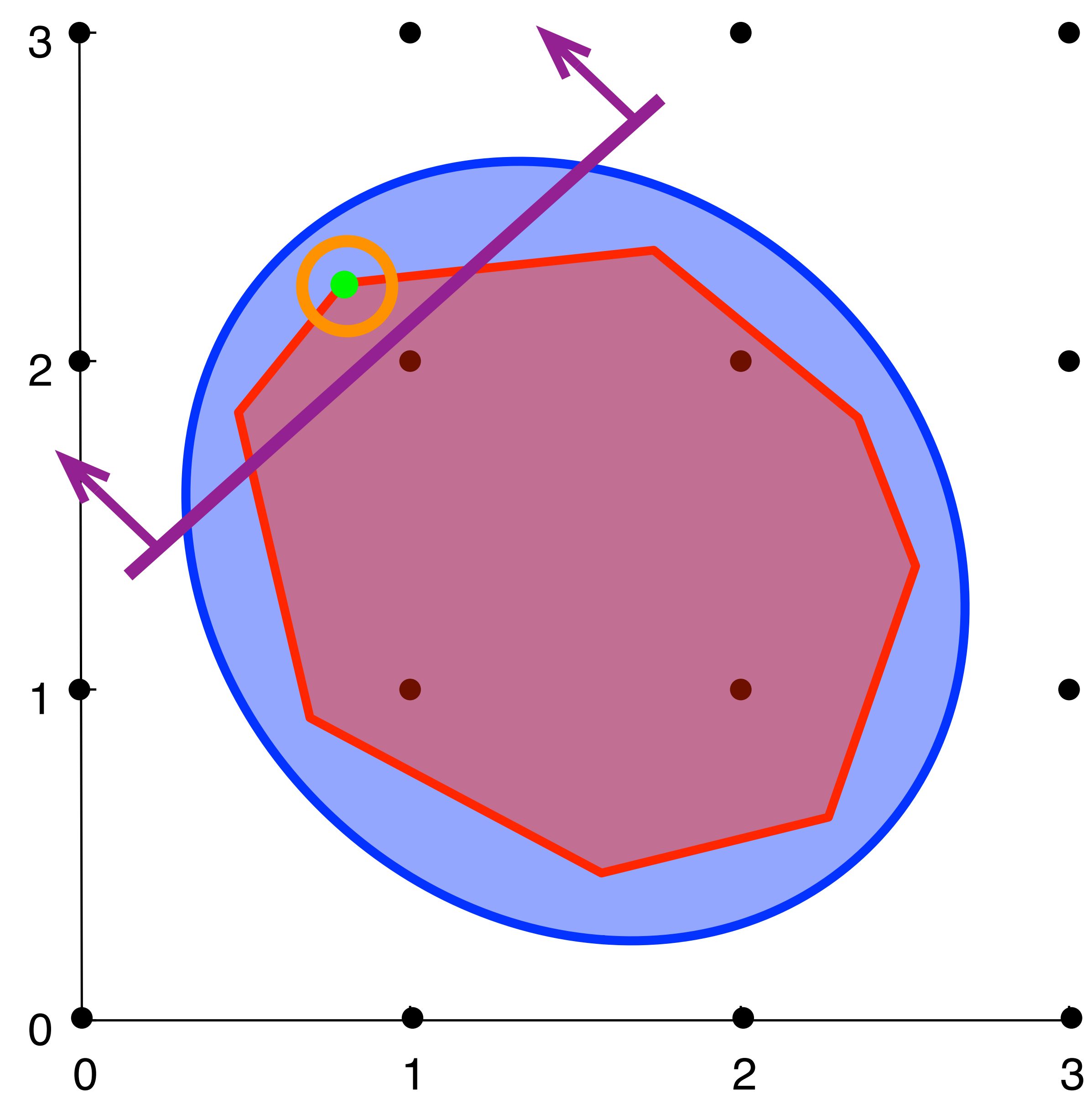
$$\langle a, v \rangle > \lfloor \sigma_C(a) \rfloor$$

$$\exists \varepsilon > 0 \quad \varepsilon B^n + v \subset C \quad \forall v \in V$$

$$\|a\| \geq \frac{1}{\varepsilon} \Rightarrow$$

$$\begin{aligned}
\lfloor \sigma_C(a) \rfloor &\geq \sigma_C(a) - 1 \\
&\geq \sigma_{v+\varepsilon B^n}(a) - 1 \\
&= \langle v, a \rangle + \varepsilon \|a\| - 1 \\
&\geq \langle v, a \rangle
\end{aligned}$$

$$S_2 = (1/\varepsilon) B \cap \mathbb{Z}^n$$



Conclusions and Current Work

- CG Closure of Compact Convex set is Polytope:
  - Answers 30 year old question by Schrijver for "irrational" polytopes (see also Dunkel and Schulz 2010)
- What about unbounded sets?
  - CG closure is polyhedron for a class of unbounded sets:
    - Class includes rational polyhedra = True generalization of Schrijver theorem.