The Chvátal-Gomory Closure of a Strictly Convex Body is a Rational Polyhedron

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Outline

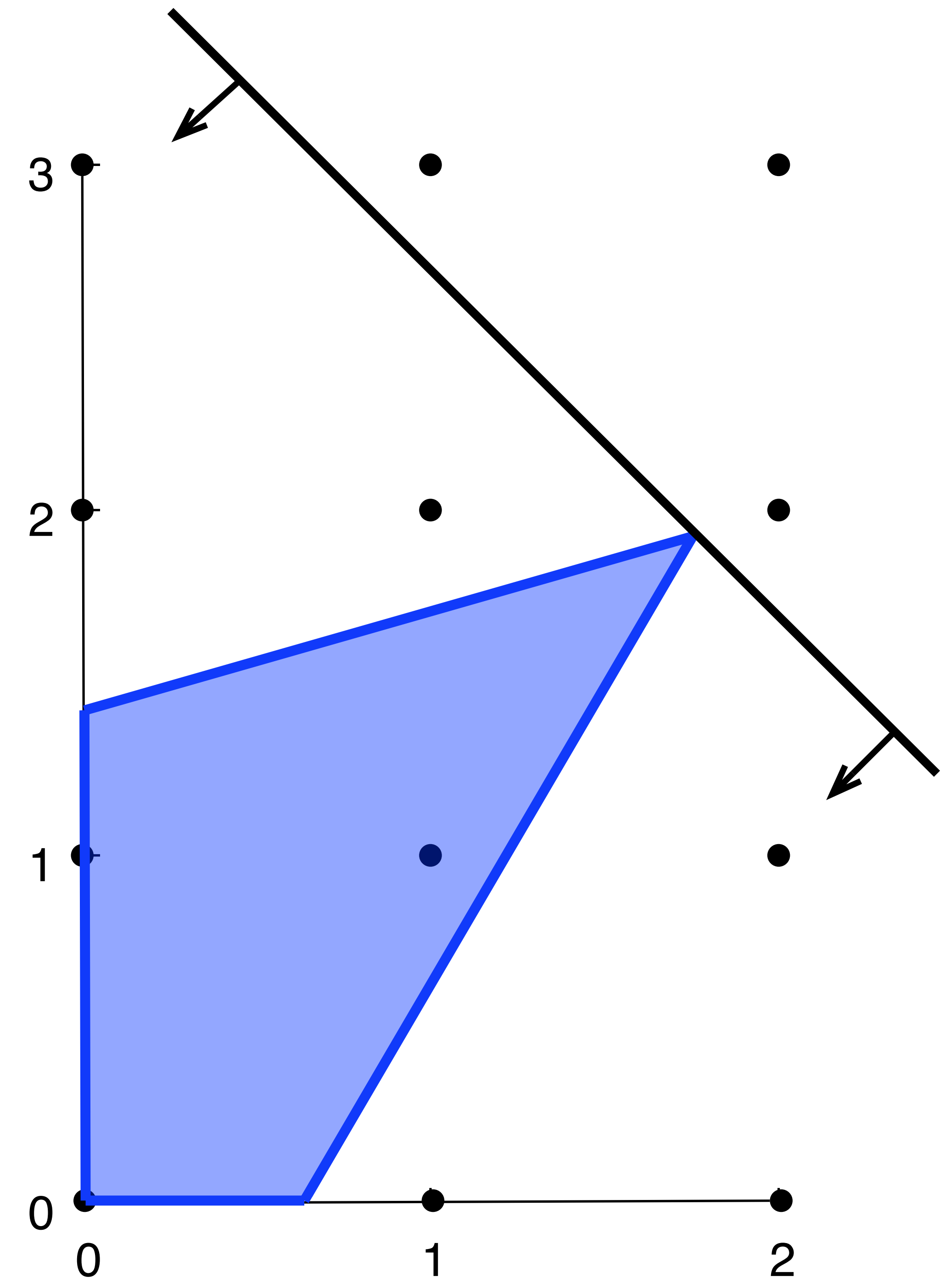
- Introduction
- Proof:
  - Step 1
  - Step 2
- Conclusions and Future Work

CG Cuts for Rational Polyhedra

$$P := \left\{ x \in \mathbb{R}^2 \ : \ \begin{aligned} x_1 + x_2 &\leq 3, \\ 5x_1 - 3x_2 &\leq 3 \end{aligned} \right\}$$

$$\cap$$

$$H := \{ x \in \mathbb{R}^2 \ : \ 4x_1 + 3x_2 \leq 10.5 \}$$



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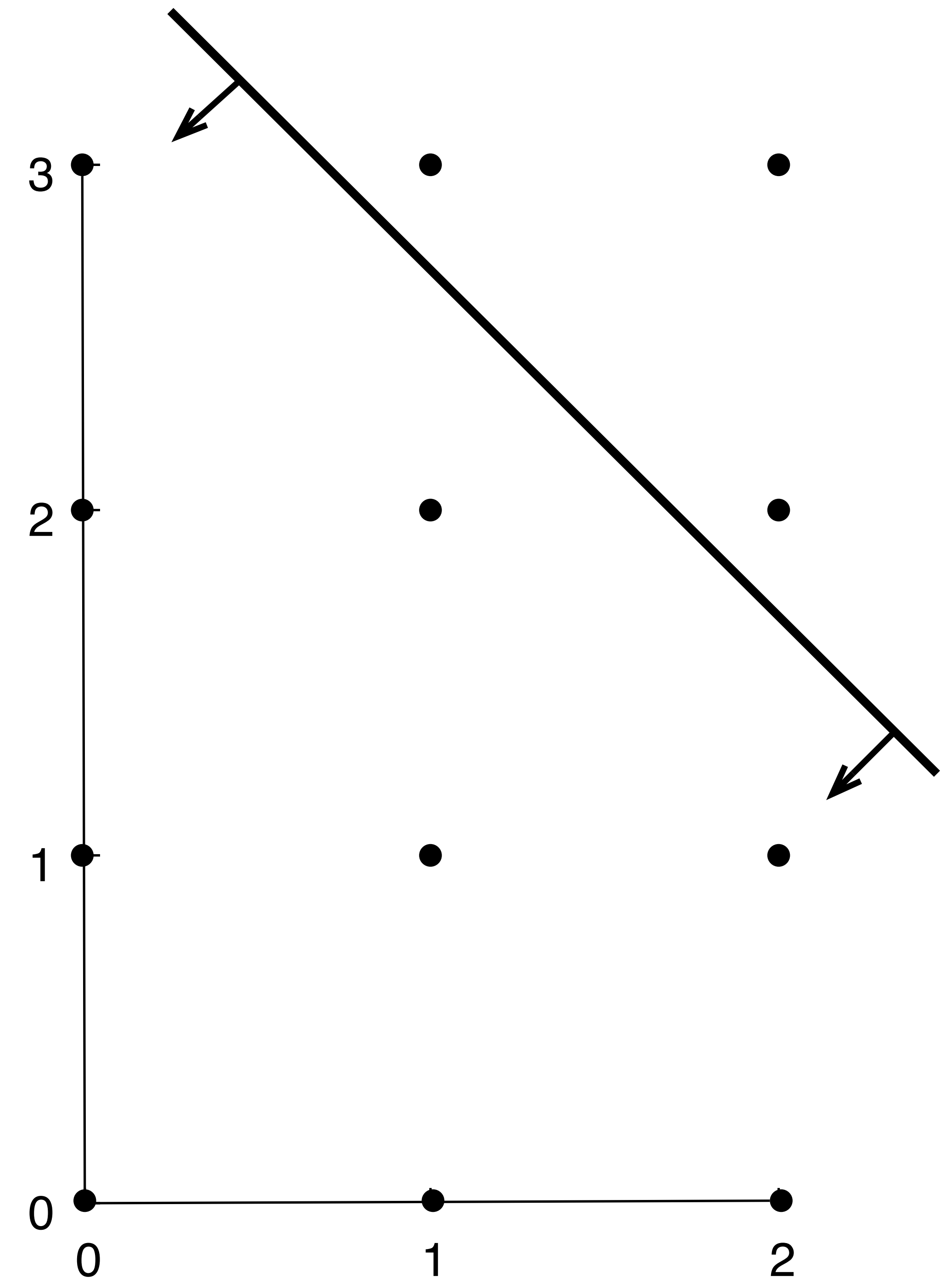

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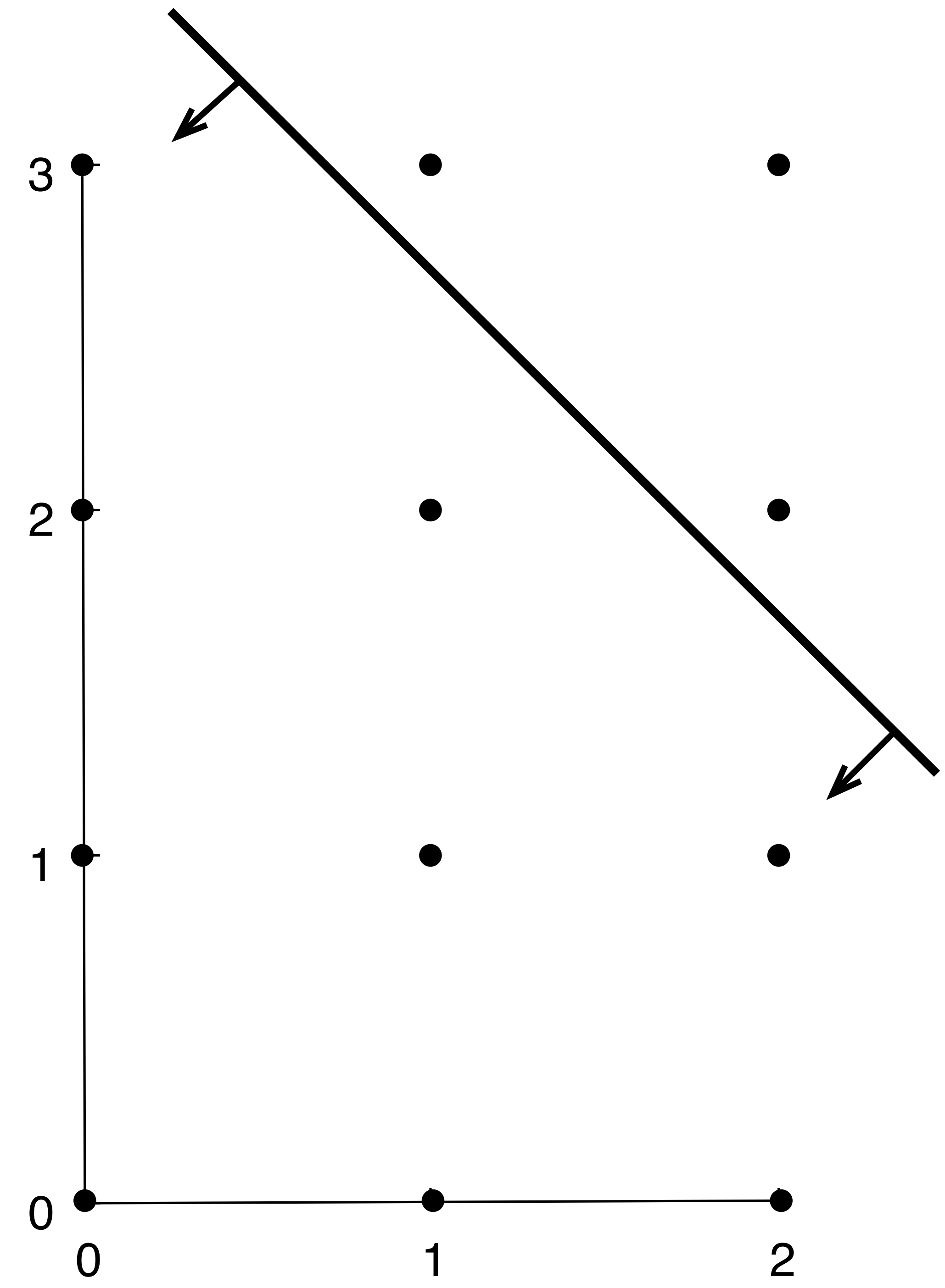
$$\text{if } x \in \mathbb{Z}^2$$



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$$\mathrel{\rotatebox{90}{$\subseteq$}}$$

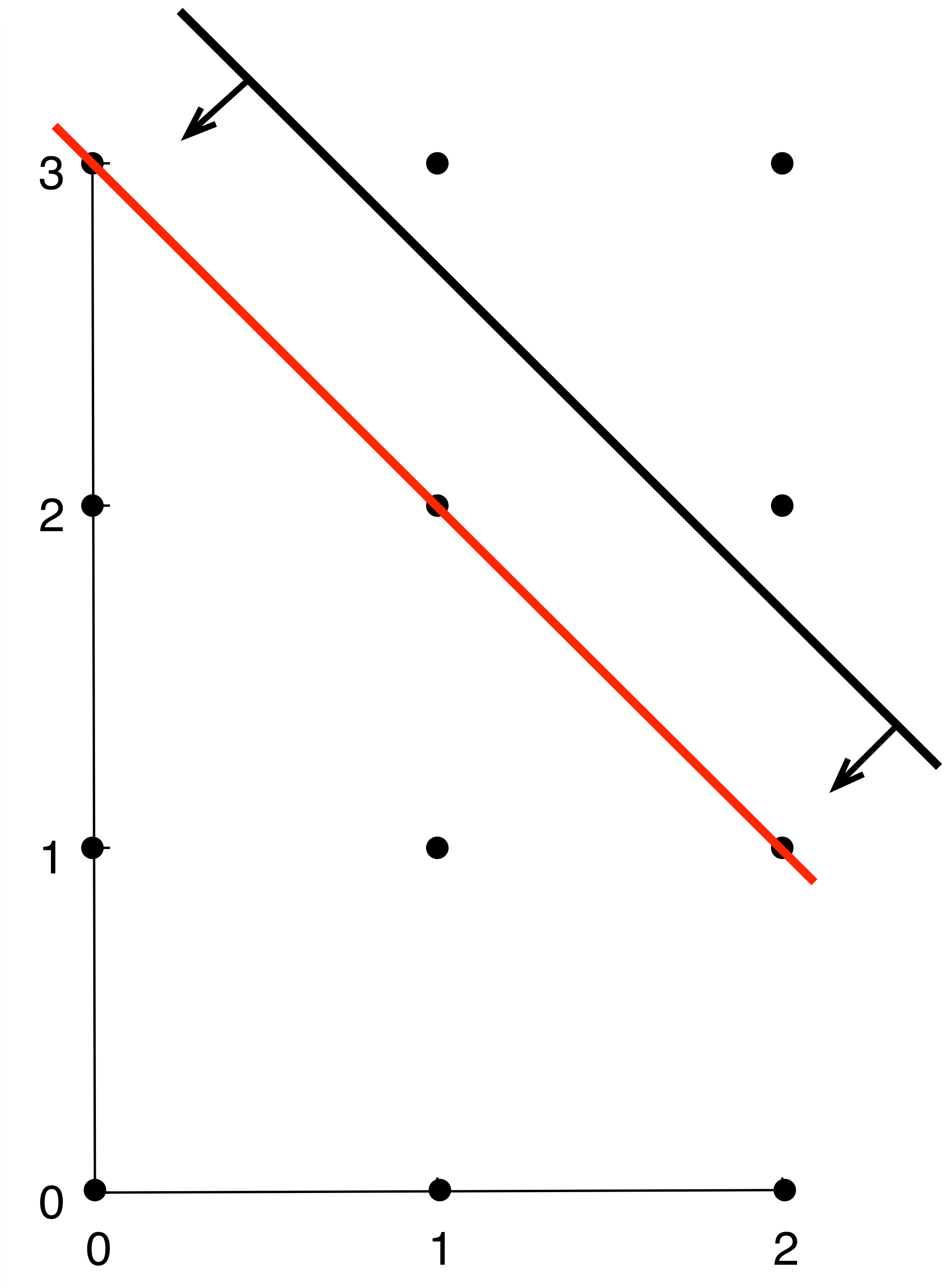
$$H := \{ x \in \mathbb{R}^2 \; : \; \underbrace{4x_1 + 3x_2}_{\in \mathbb{Z}} \leq 10.5 \}$$

if $x \in \mathbb{Z}^2$

$$\downarrow$$

$$4x_1 + 3x_2 \leq \lfloor 10.5 \rfloor$$

Valid for $H \cap \mathbb{Z}^2$



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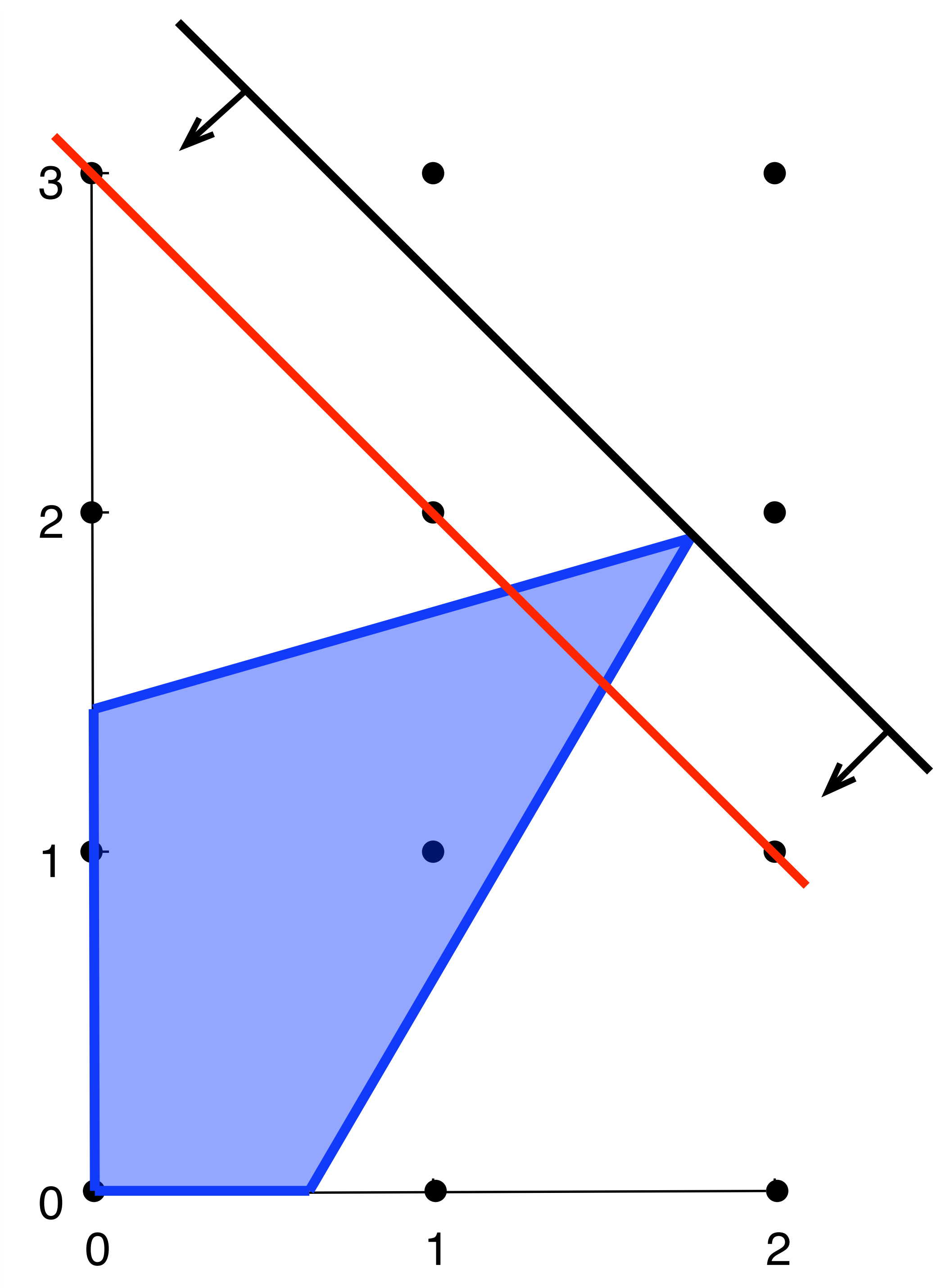


Importance of CG Cuts

- 3 main papers (Chvátal, Gomory, Schrijver)
  - +1,200 citations.
- First pure cutting plane algorithm for Integer Programming.
- CG cuts yield Matching Polytope.
- Cutting plane proofs.
- Still practical computationally.

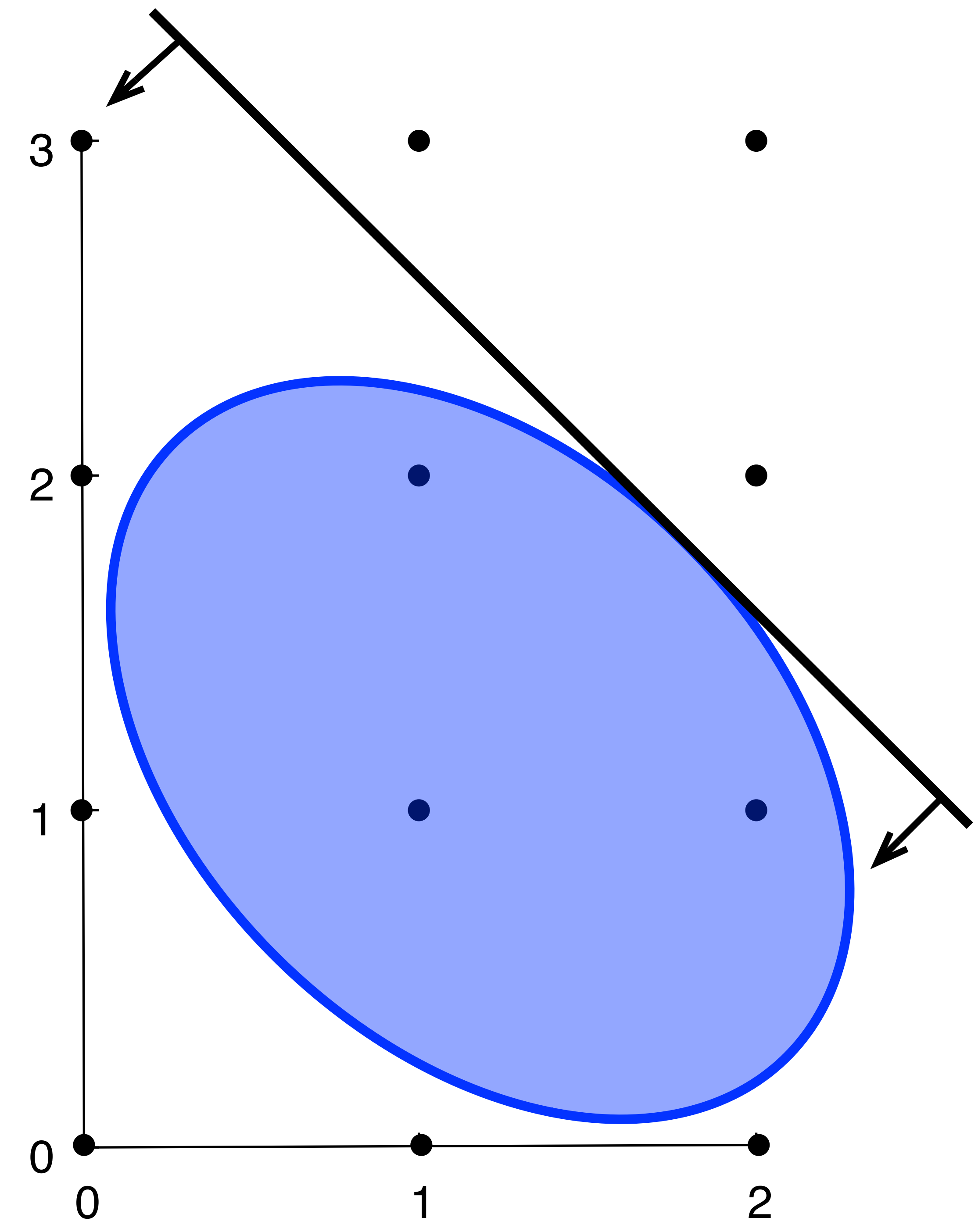
CG Cuts for General Convex Sets

$$\sigma_C(a) := \sup\{\langle a, x\rangle \;:\; x \in C\}$$

$$C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n \;:\; \langle a, x\rangle \leq \sigma_C(a)\}$$

$$|\bigcap$$

$$H := \{x \in \mathbb{R}^n \;:\; \langle a, x\rangle \leq \sigma_C(a)\}$$



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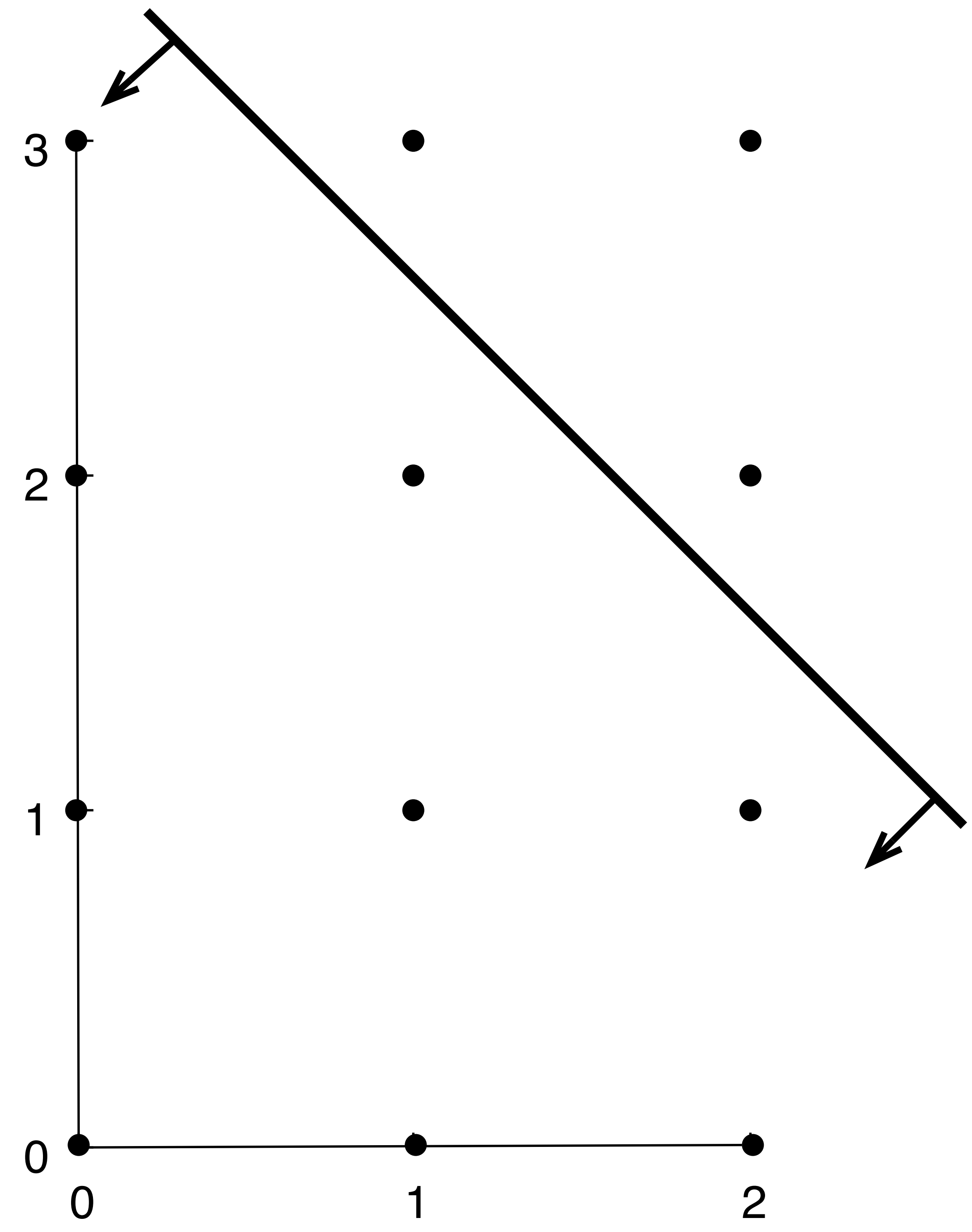

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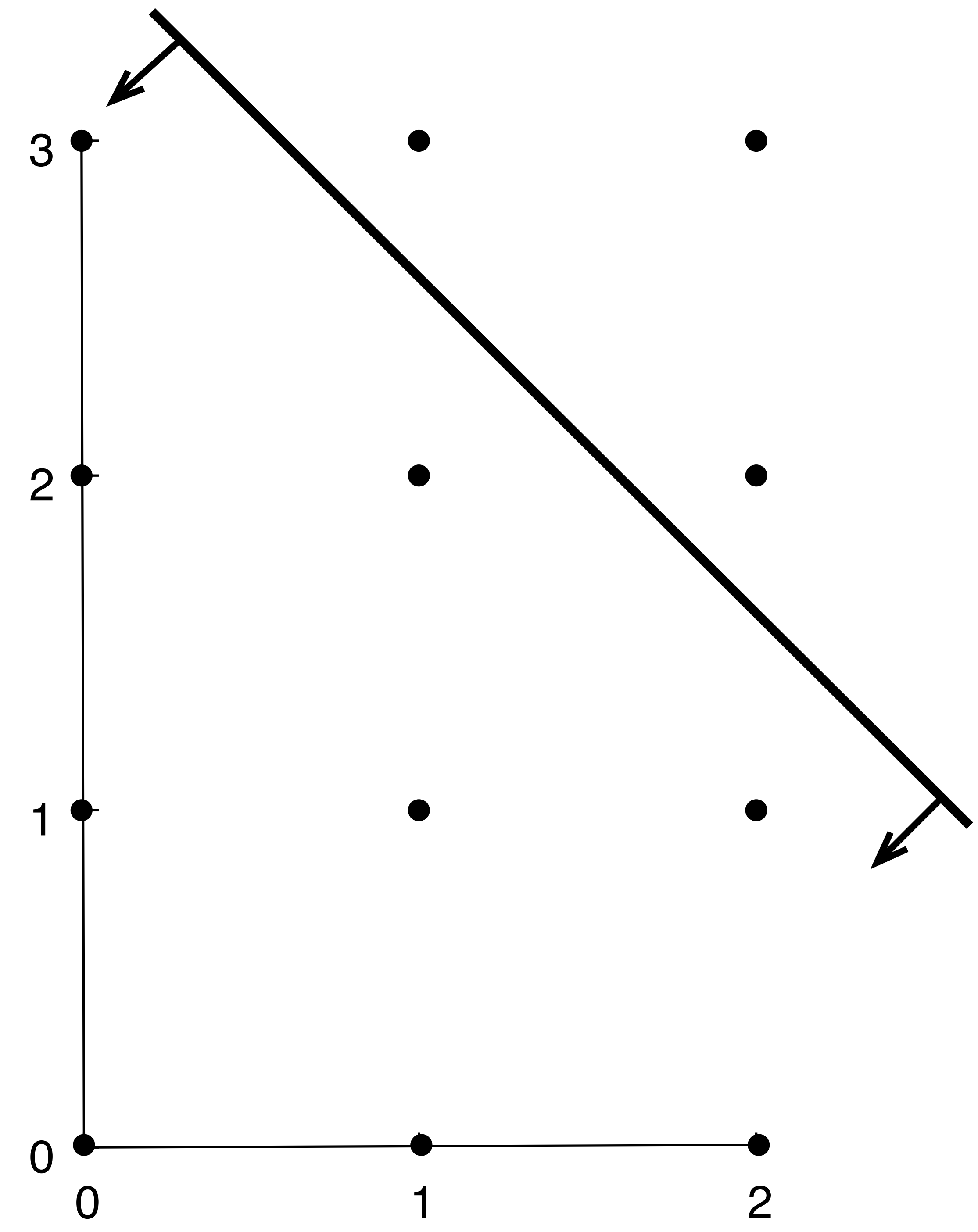

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$$\langle a, x\rangle \le \lfloor \sigma_C(a) \rfloor$$

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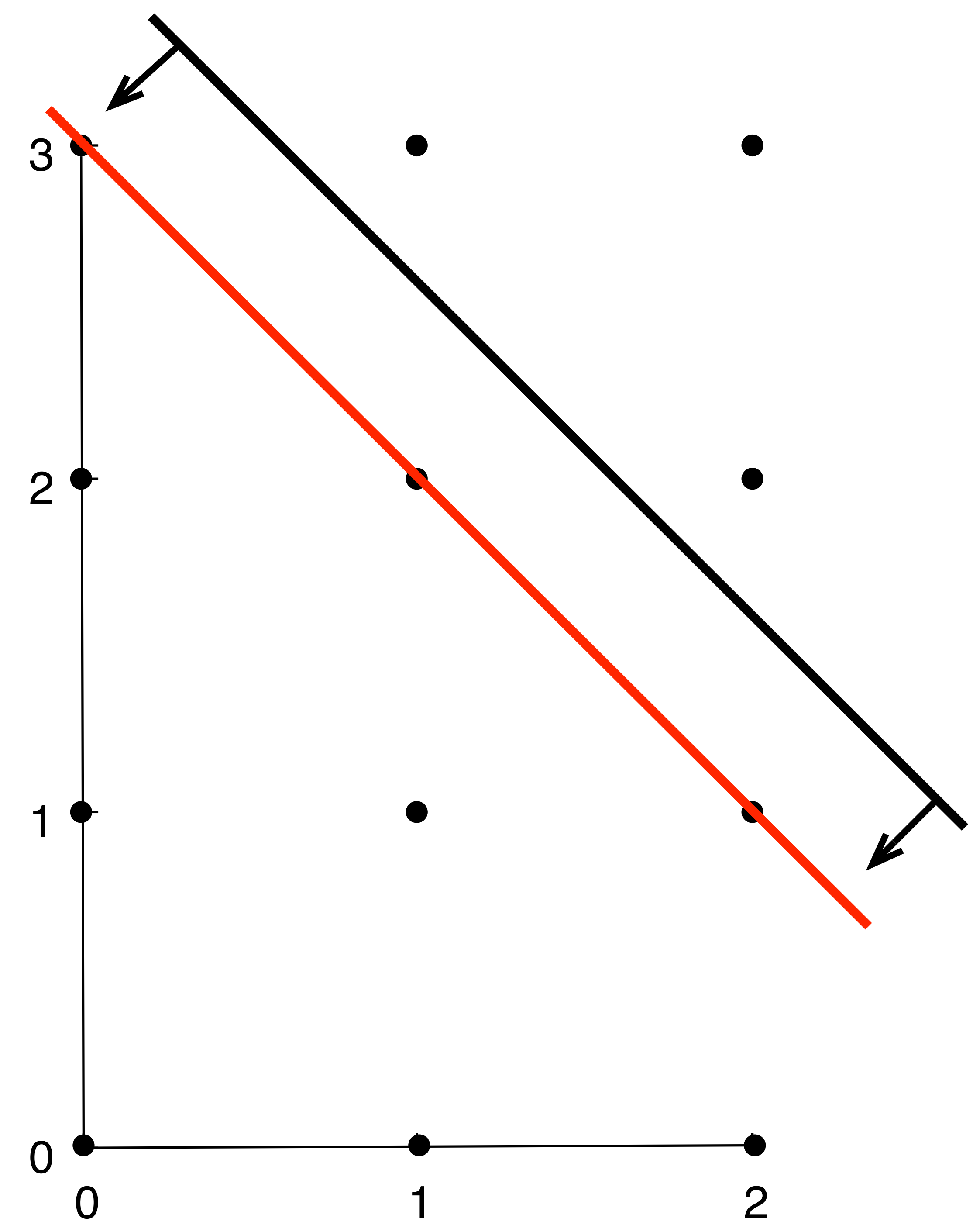

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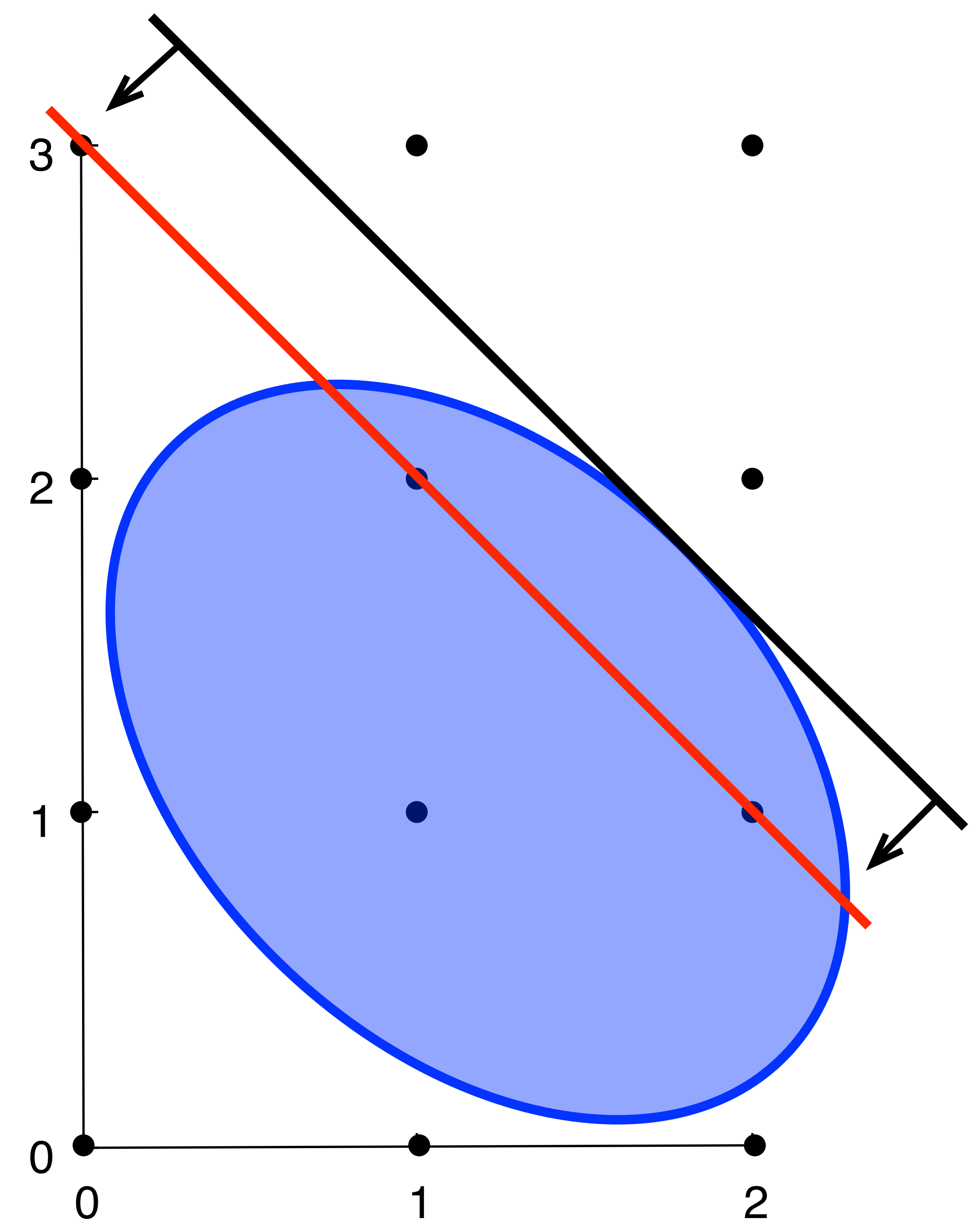

CG Closure = Add all CG Cuts

$$\mathtt{CGC}(C) := \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n \; : \; \langle a, x \rangle \leq \boxed{\lfloor \sigma_C(a) \rfloor}\}$$

- Not necessarily a polyhedron, remember:

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- $\mathtt{CGC}(C)$ is a polyhedron if C is:
  - a rational polyhedra (Schrijver, 1980).
  - a “rational” ellipsoid (Dey and V. 2010).

Finite Number of Important CG Cuts

Theorem: There exists finite $S \subseteq \mathbb{Z}^n$ such that

$$\texttt{CGC}(C) = \bigcap_{a \in S} \{x \in \mathbb{R}^n \ : \ \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}$$

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- For this talk C is intersection of:
  - A strictly convex body (full dim, compact).
  - and a rational polyhedron.

Proof Outline for $\mathrm{bd}(C) \cap \mathbb{Z}^n = \emptyset$

Theorem: There exists finite $S \subseteq \mathbb{Z}^n$ such that

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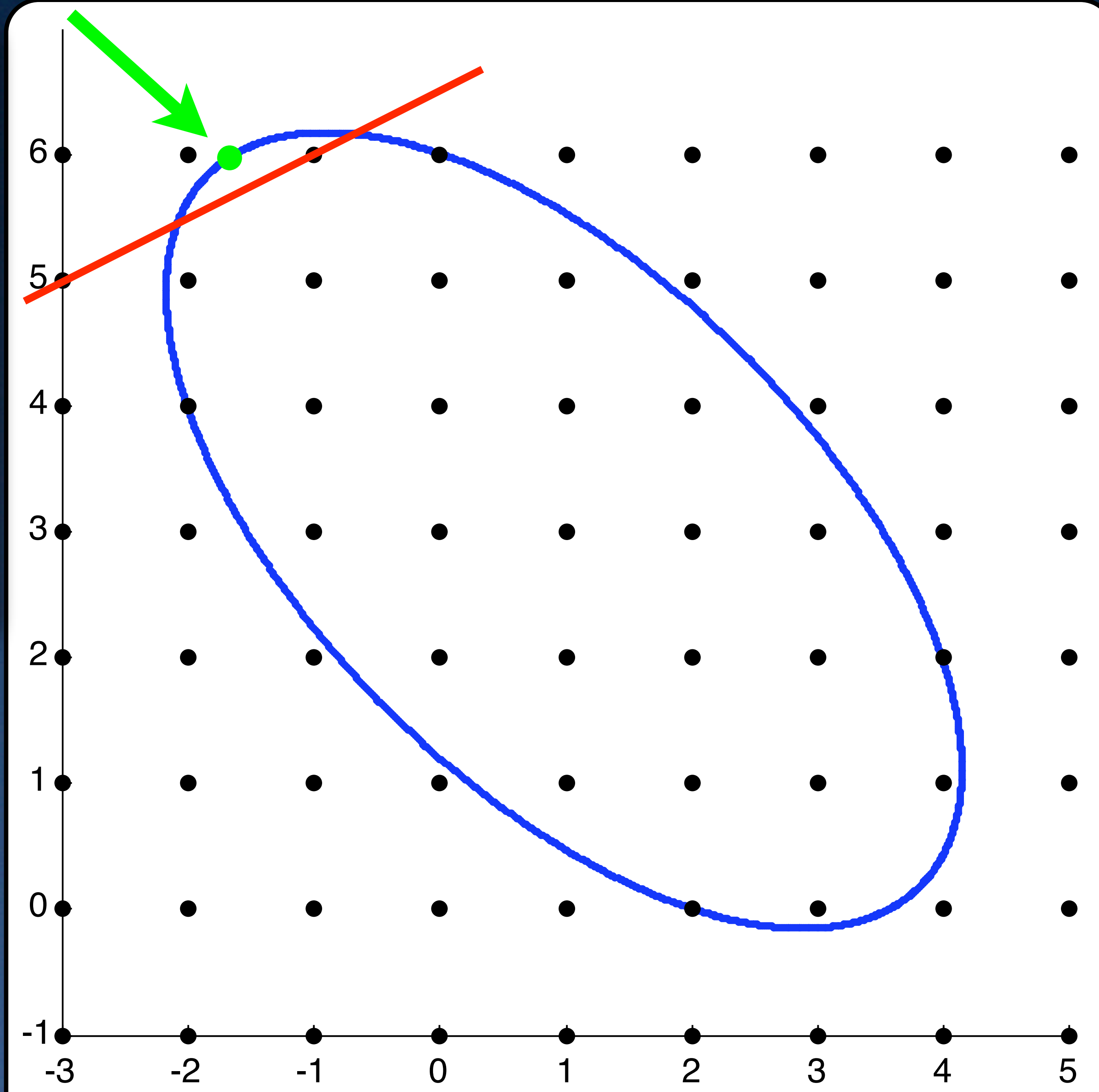
- Step 1: There exists finite $S^1 \subseteq \mathbb{Z}^n$ such that
  - $\mathtt{CGC}(S^1, C) \subseteq \mathrm{int}(C)$
- Step 2: There exists finite $S^2 \subseteq \mathbb{Z}^n$ such that
  - $\mathtt{CGC}(C) = \mathtt{CGC}(S^1, C) \cap \mathtt{CGC}(S^2, C)$

Separate points in $\mathrm{bd}(C)$

$u \in \mathrm{bd}(C) \setminus \mathbb{Z}^n$ $\quad \exists\, a^u \in \mathbb{Z}^n$

$$\langle a^u, u \rangle > \lfloor \sigma_C(a^u) \rfloor$$



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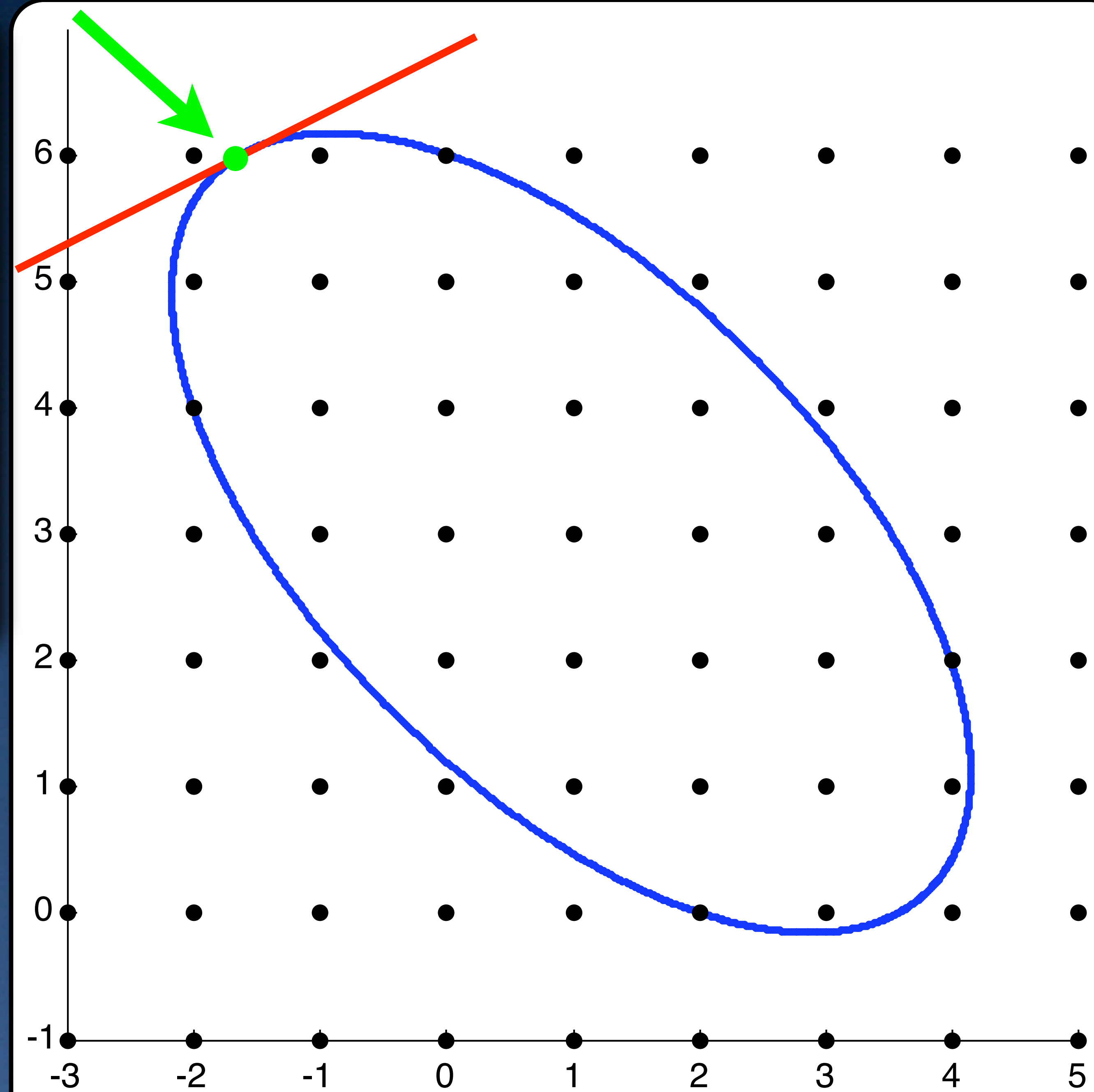
$$\langle s(u), u\rangle = \sigma_C\left(s(u)\right)$$



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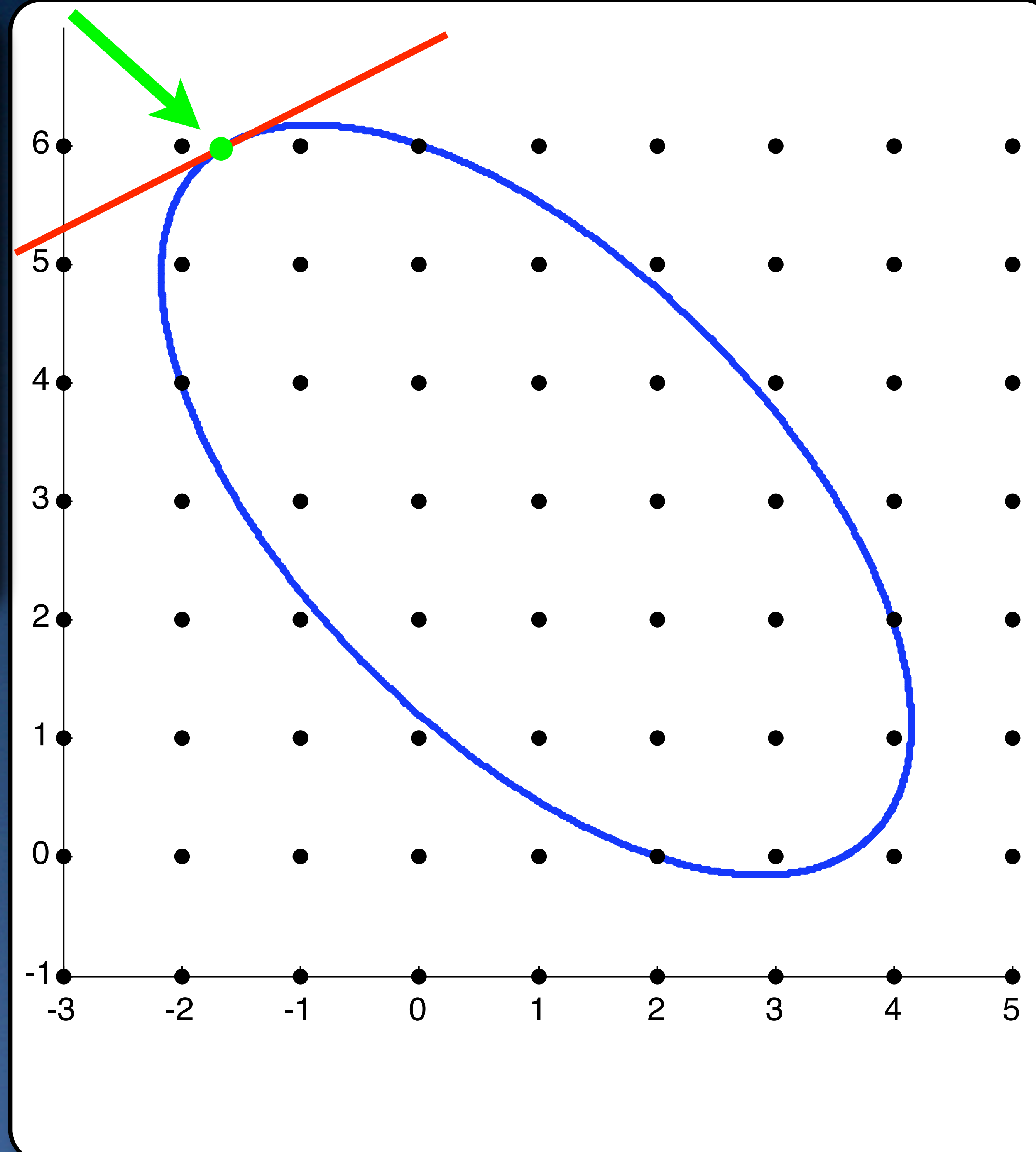

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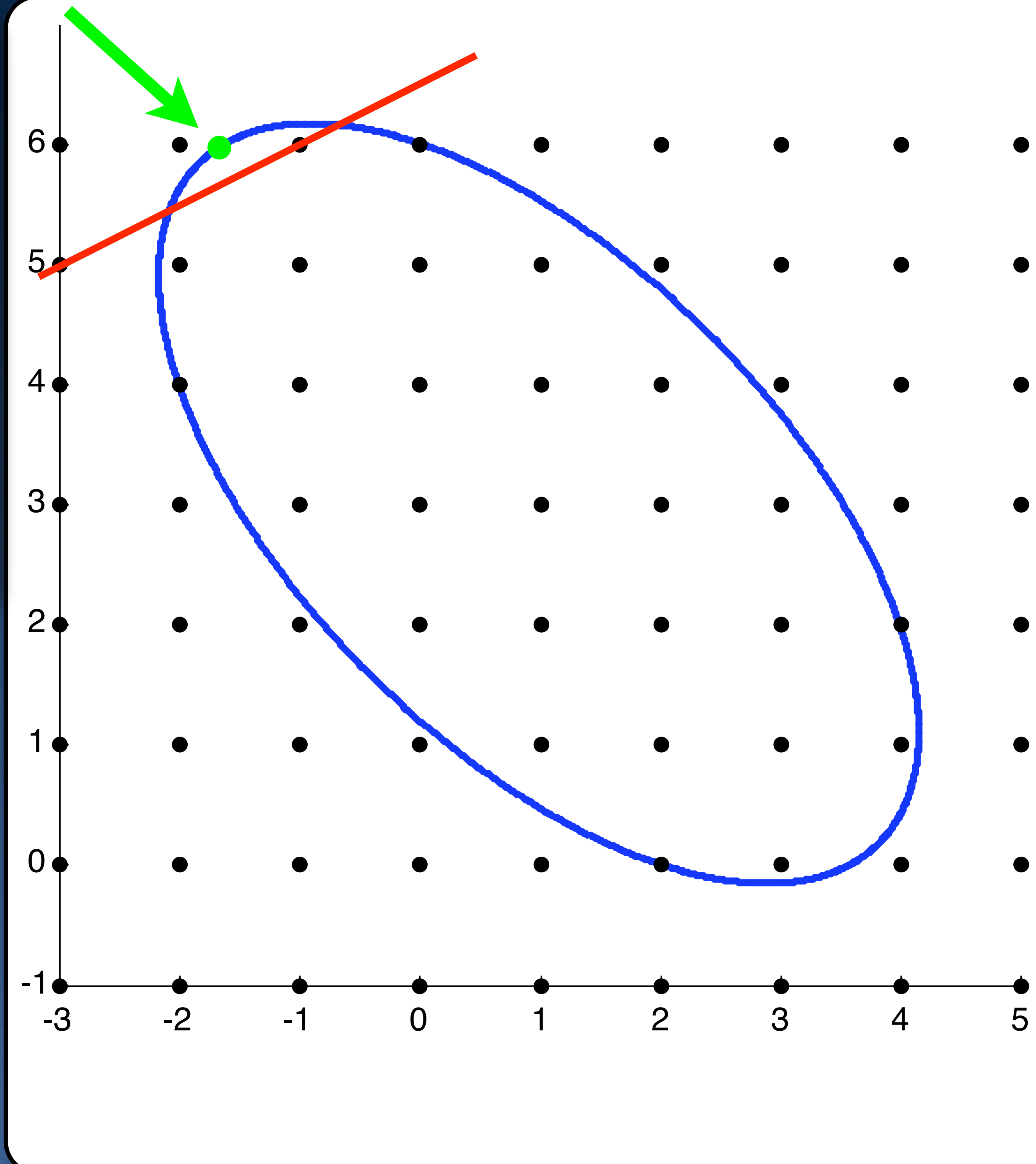

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$\lambda s(u) \notin \mathbb{Z}^n \quad \forall \lambda > 0 :$

$\lambda s(u) \in \mathbb{Z}^n \Rightarrow \sigma_C(\lambda s(u)) \in \mathbb{Z} :$

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$$C = \left\{ x \in \mathbb{R}^2 \;:\; \sqrt{x_1^2 + x_2^2} \leq 1 \right\}$$

$$u = (1/2, \sqrt{3}/2)^T \in \mathrm{bd}(C)$$

$$s(u) = (1/2, \sqrt{3}/2)^T$$

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$$C = \left\{ x \in \mathbb{R}^2 \;:\; \sqrt{x_1^2 + x_2^2} \leq 5 \right\}$$

$$u = (25/13, 60/13)^T \in \mathrm{bd}(C)$$

$$s(u) = (5, 12)^T,\; \sigma_C(s(u)) = 65$$

Separate points in $\mathrm{bd}(C)$

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$$\underbrace{\langle s(u), u \rangle}_{\in \mathbb{Z}^n} = \underbrace{\sigma_C\left(s(u)\right)}_{\notin \mathbb{Z}}$$

$$\frac{s^i}{\|s^i\|} \xrightarrow{i \to \infty} \frac{s(u)}{\|s(u)\|}$$

$$\lim_{i \to \infty} \langle s^i, u \rangle - \lfloor \sigma_C(s^i) \rfloor > 0$$

Diophantine approx. of $s(u)$

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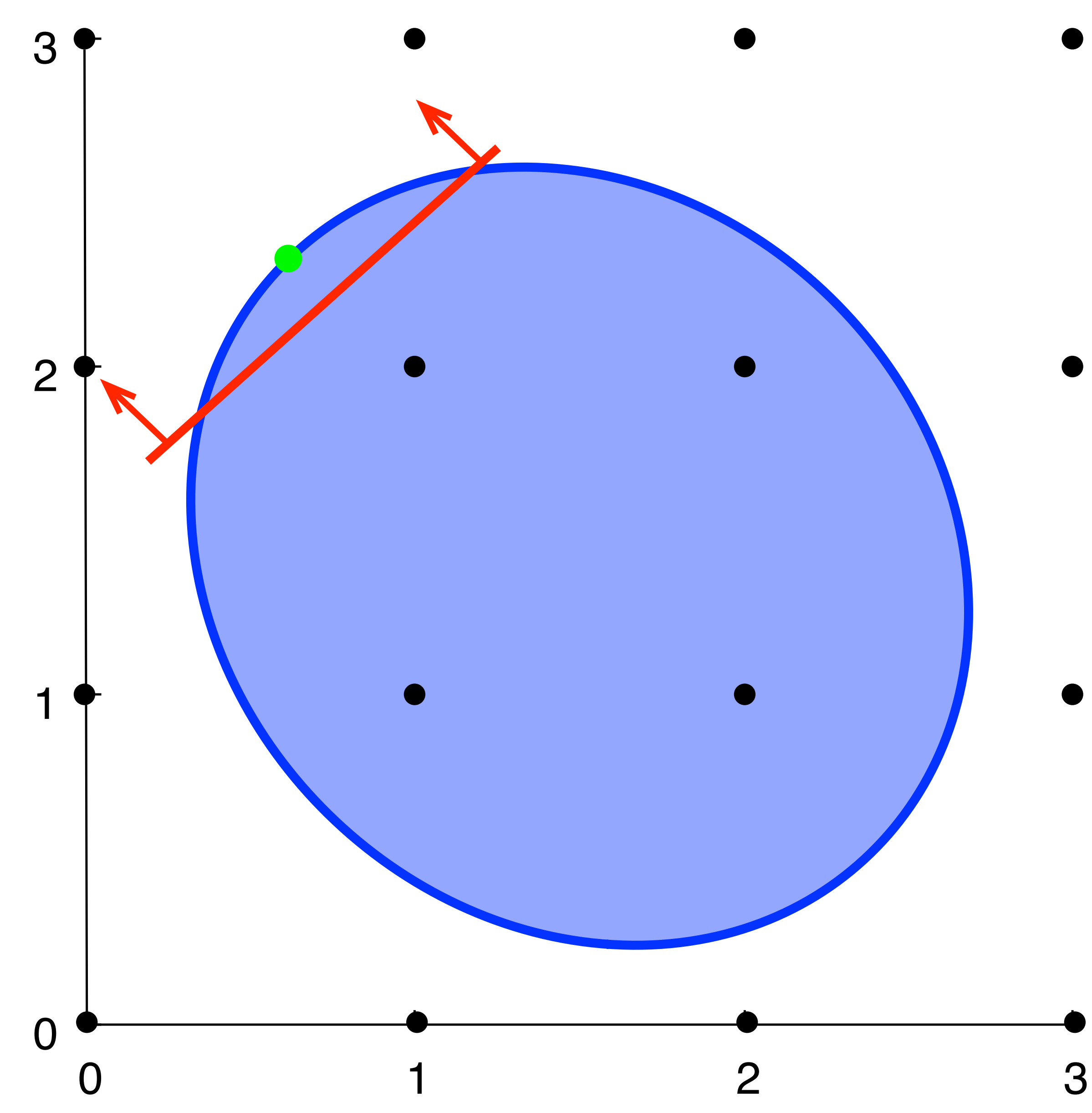
Compactness Argument

$$K := \mathrm{bd}(C)$$

$$S_u := \{x \,:\, \langle a^u, x\rangle > \lfloor \sigma_C(a^u) \rfloor\}$$



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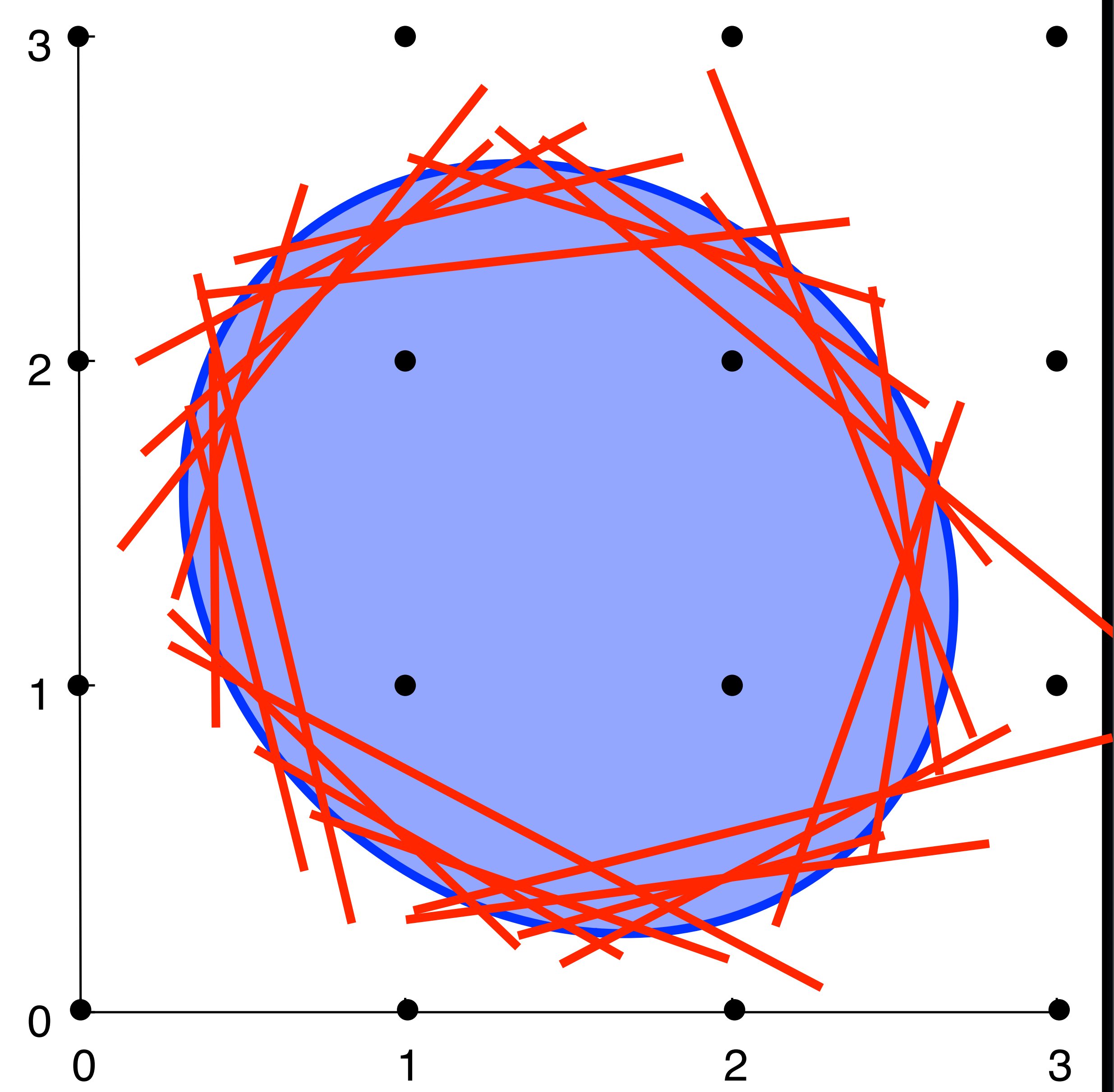
$$K \subset \bigcup_{u \in K} S_u$$



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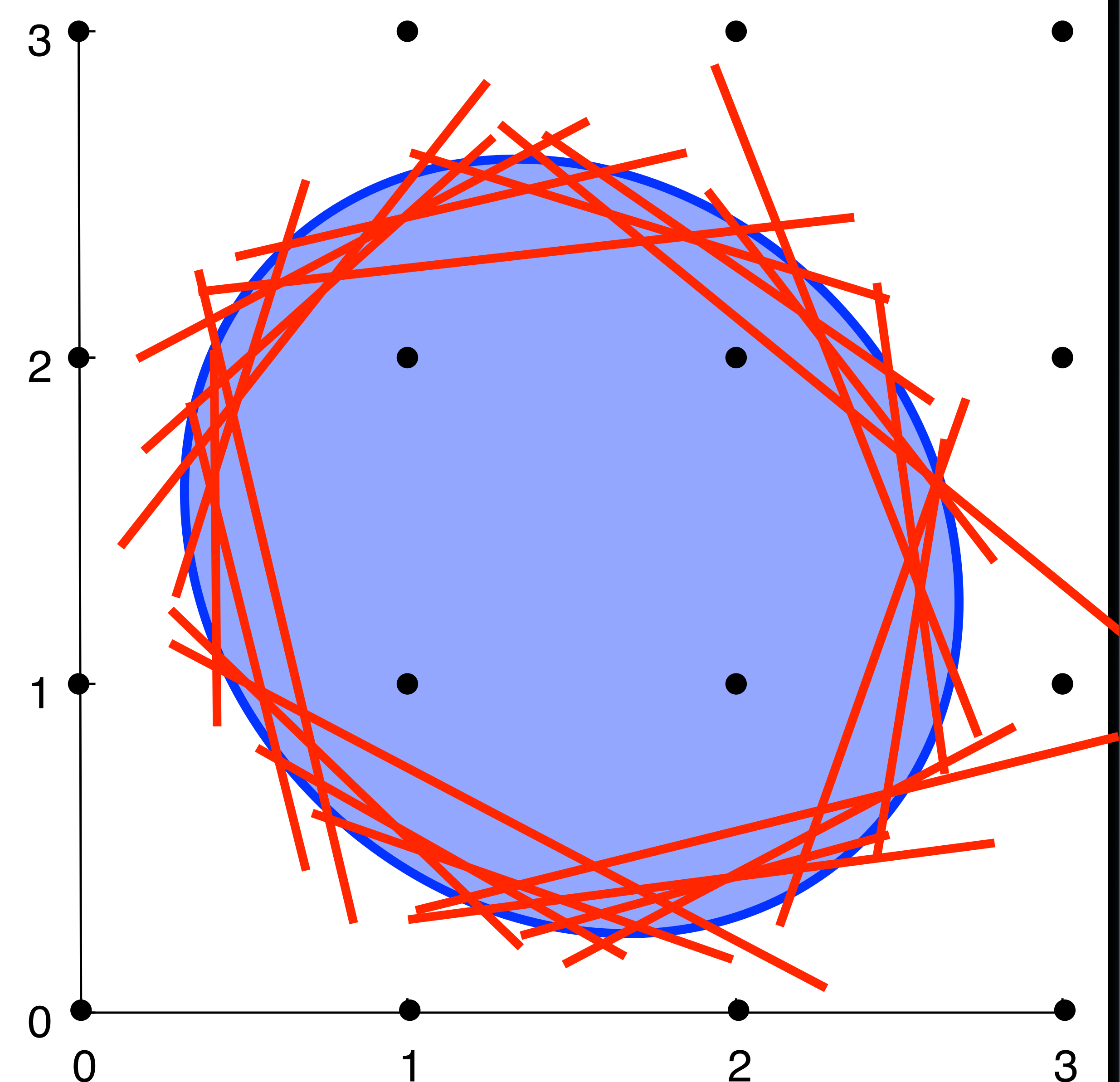
compact $\longrightarrow$ $K \subset \bigcup_{u \in K} S_u$



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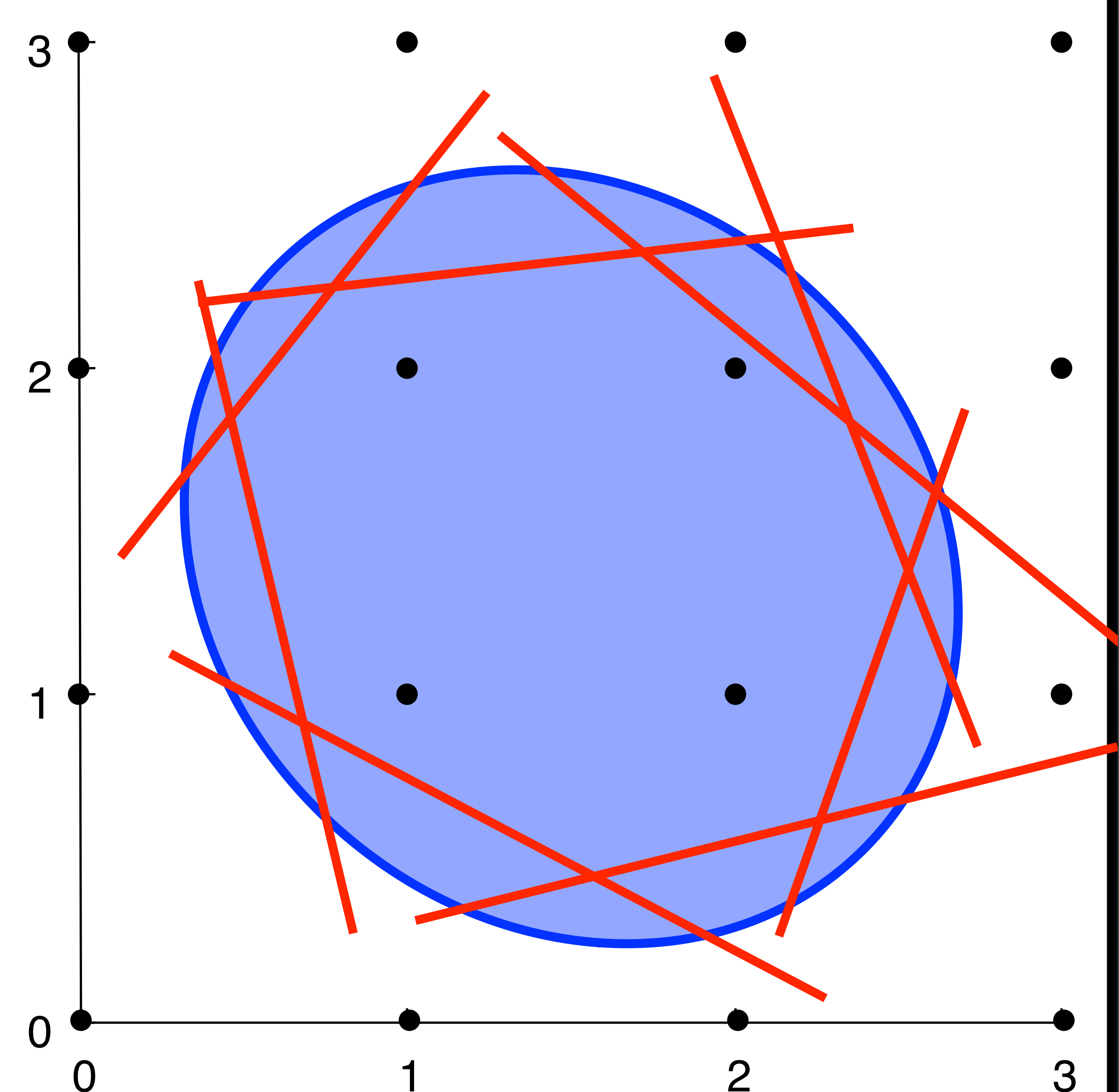
$\downarrow$

$K \subset \bigcup_{i=1}^{m} S_{u^i}$



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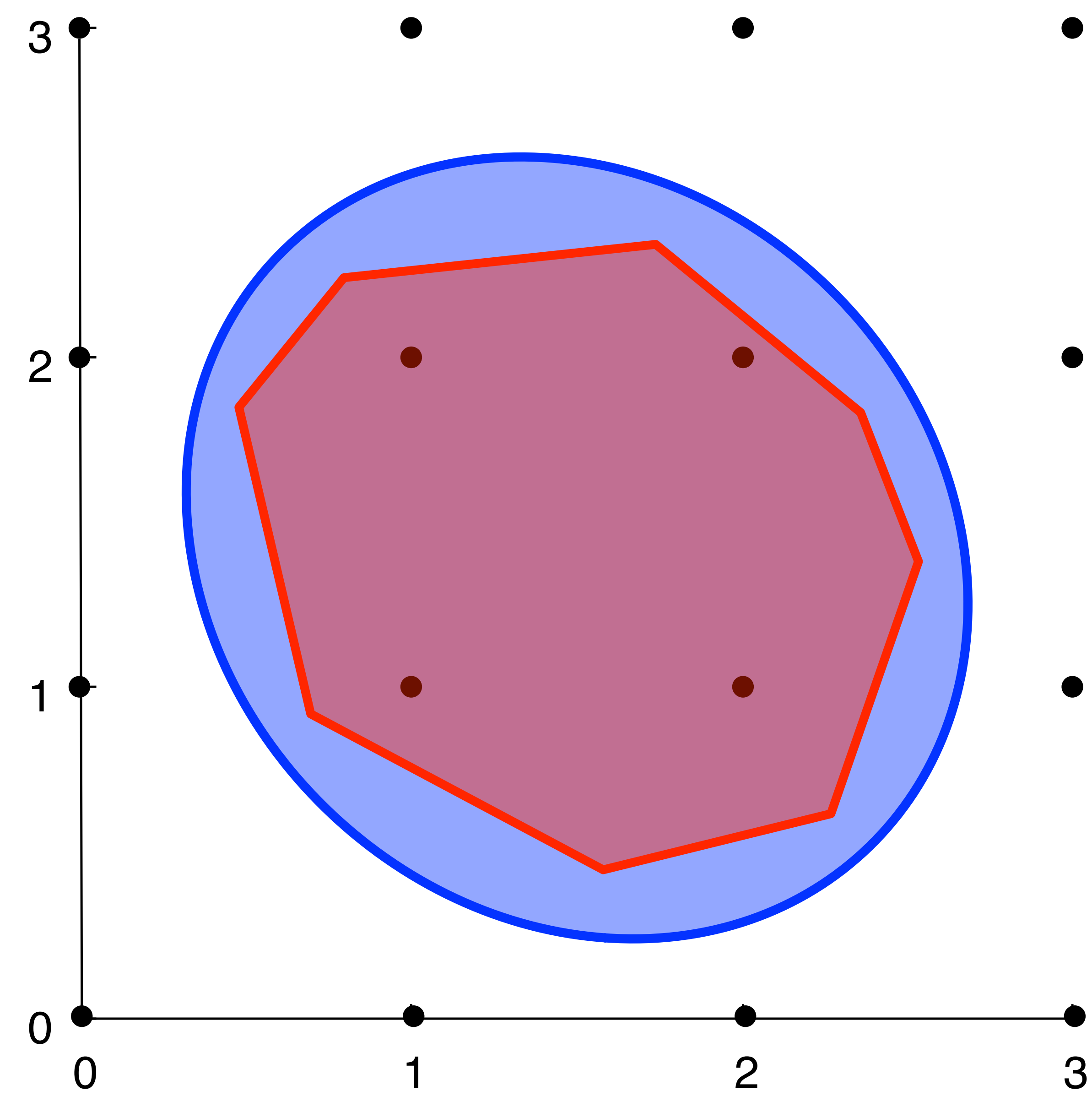
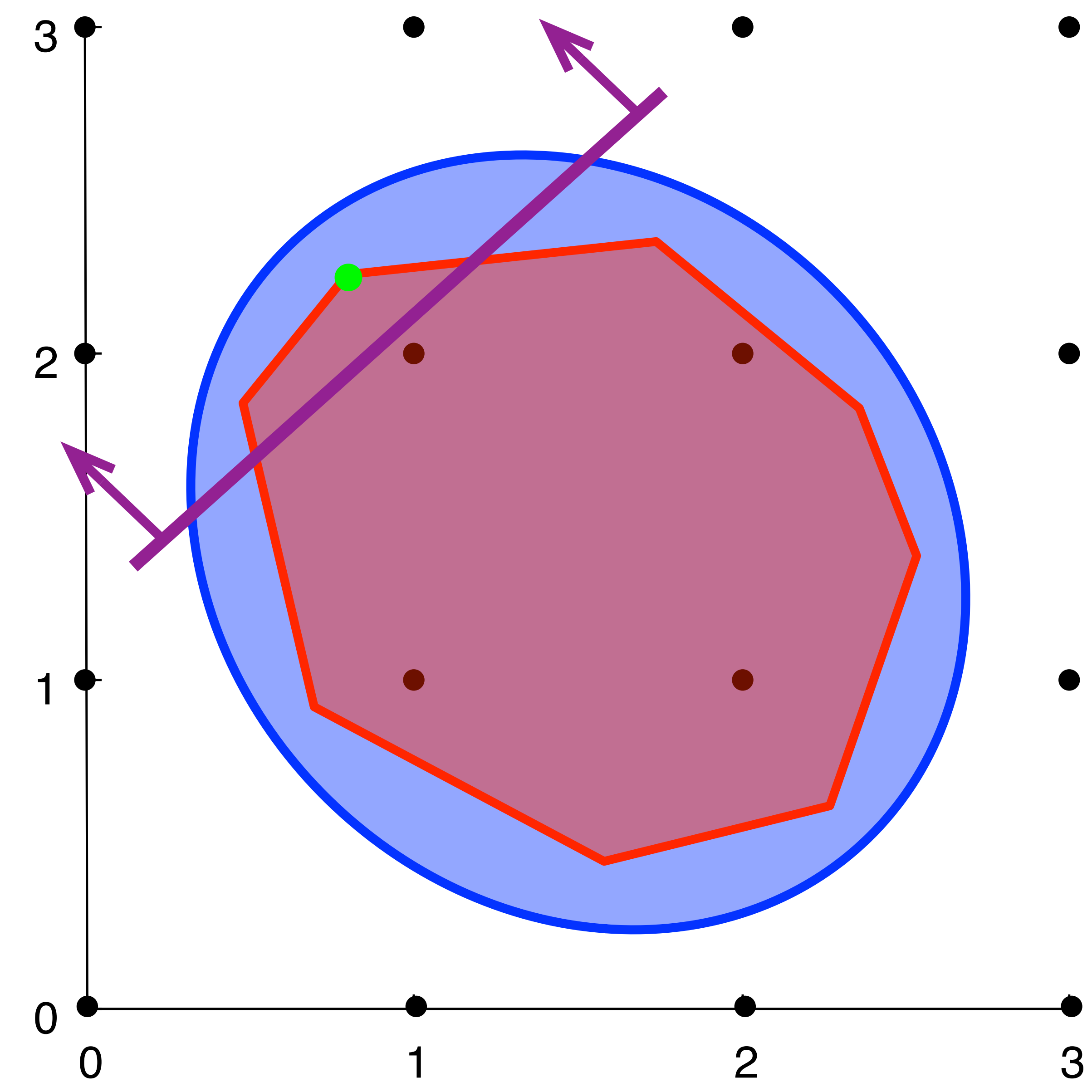
$$S^1 = \bigcup_{i=1}^{m} \left\{ a^{u^i} \right\}$$



Step 2 : Separate $\mathtt{CGC}(S^1, C) \setminus \mathtt{CGC}(C)$

$$V := \operatorname{Ext}\left(\mathtt{CGC}(S^1, C)\right) \setminus \mathbb{Z}^n$$

$$\langle a, v\rangle > \lfloor \sigma_C(a) \rfloor$$



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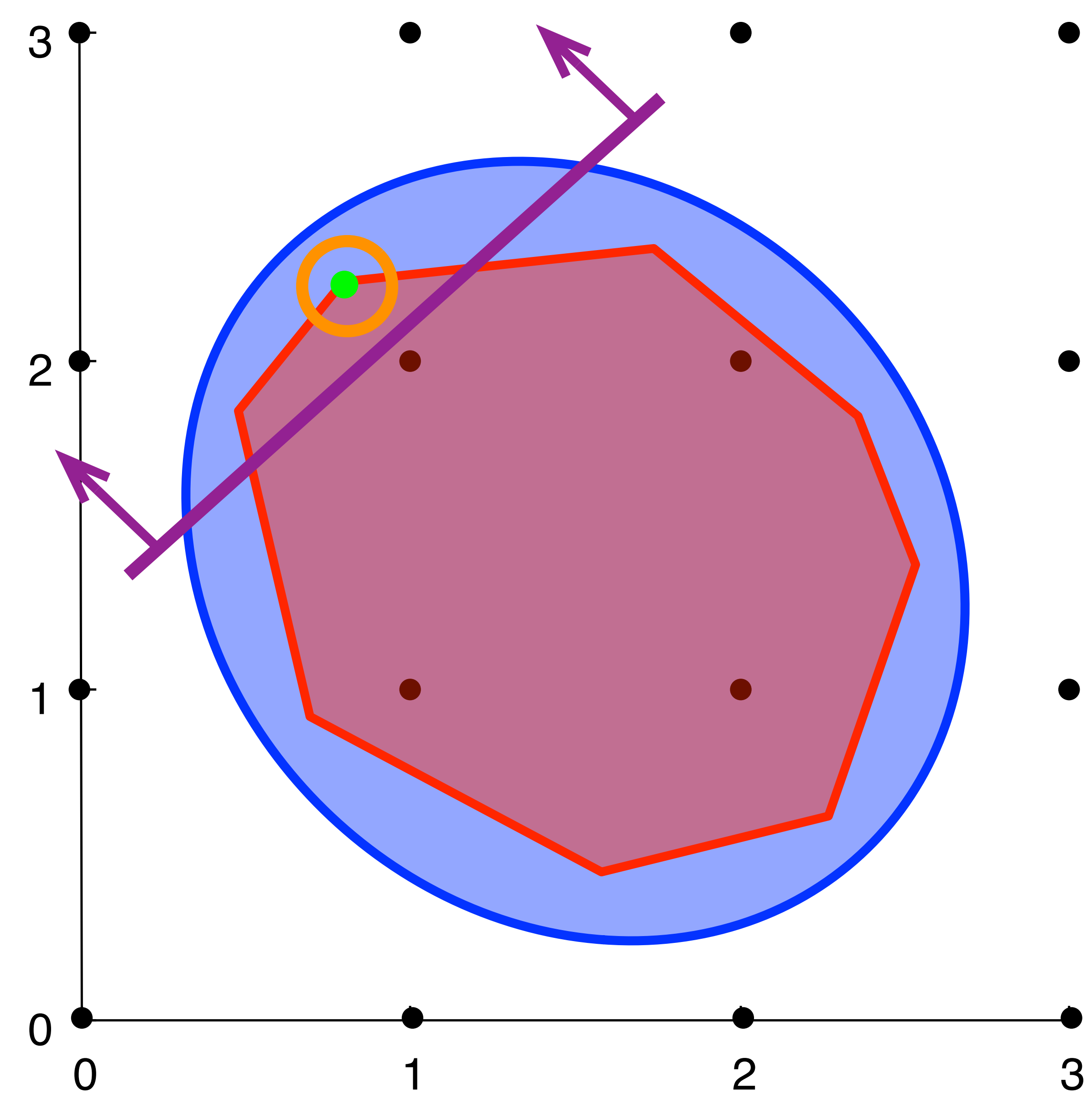

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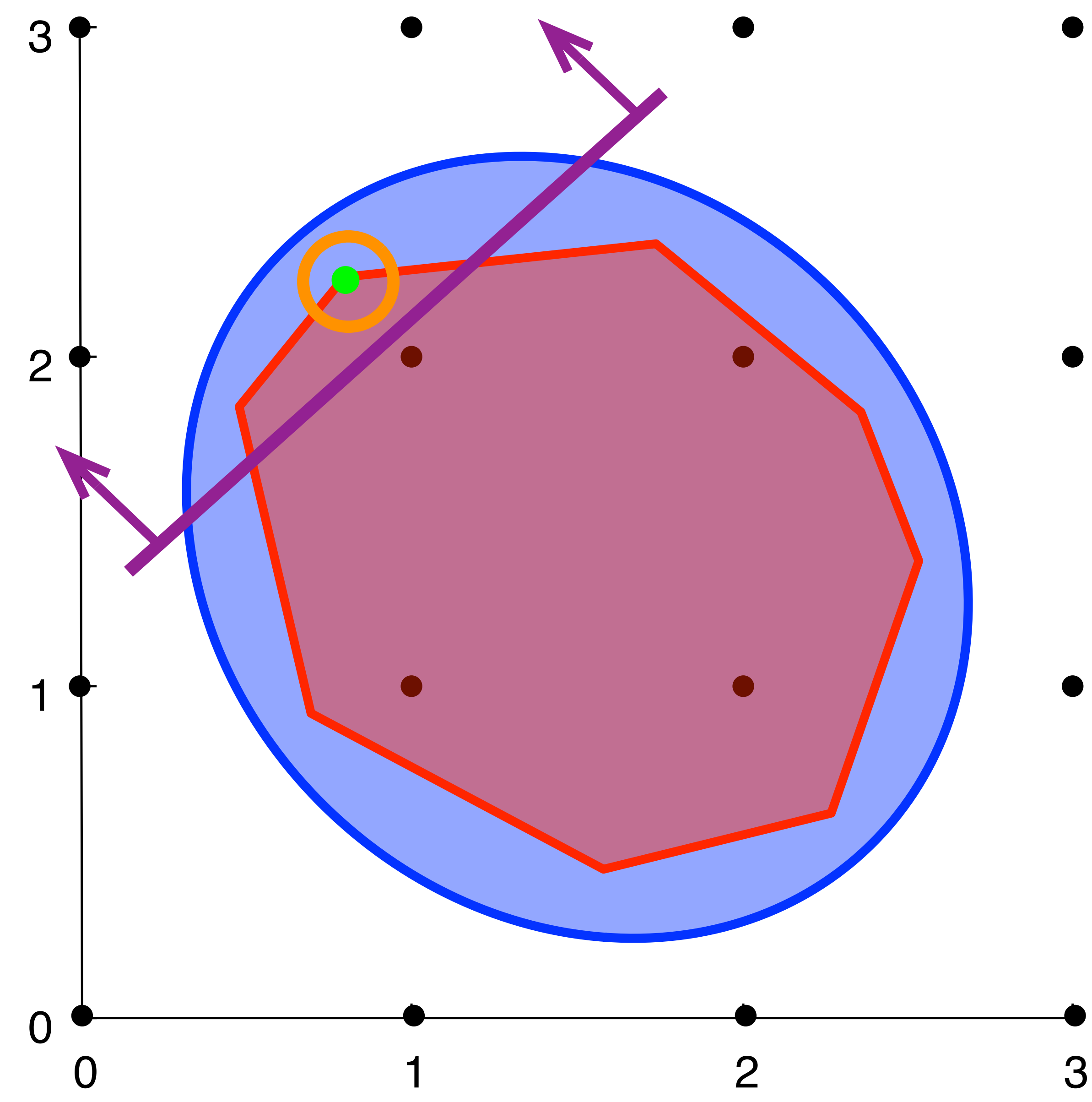
$$\|a\| \geq \frac{1}{\varepsilon} \Rightarrow$$

$$\begin{aligned}\lfloor \sigma_C(a) \rfloor &\geq \sigma_C(a) - 1\\ &\geq \sigma_{v+\varepsilon B^n}(a) - 1\\ &= \langle v, a\rangle + \varepsilon\|a\| - 1\\ &\geq \langle v, a\rangle\end{aligned}$$



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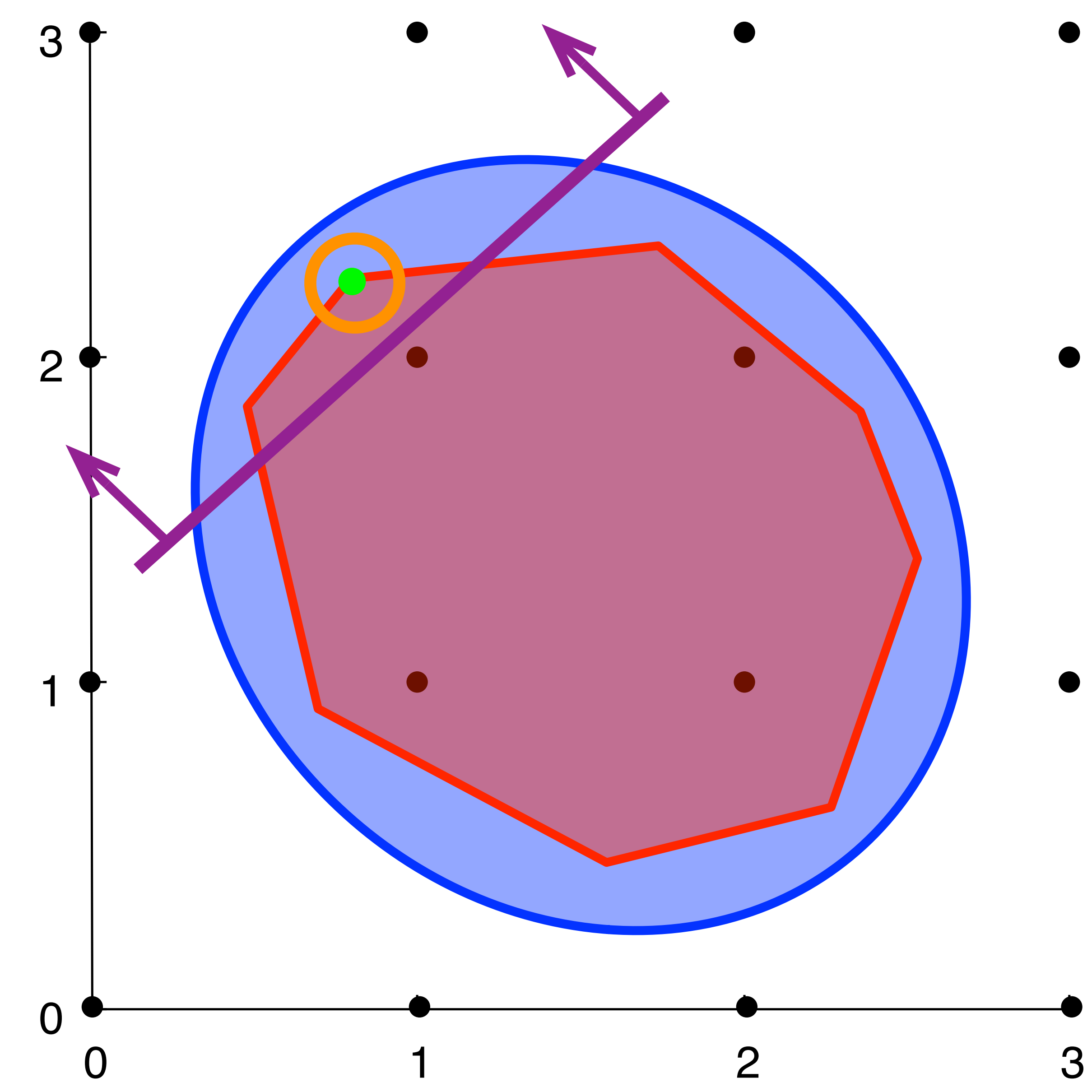
$$S^2 = (1/\varepsilon)B \cap \mathbb{Z}^n$$



Conclusions and Future Work

- Non-Constructive because of compactness argument in step 1.
- General compact convex sets including non-rational polytopes done:
  - Dadush, Dey and V. MB39: Monday 11:00, C - Room 9B, Level 3
- Open Problems:
  - Simpler Proof (Circle in $\mathbb{R}^2$?).
  - Constructive/Algorithmic proof.