The Chvátal-Gomory Closure of a Strictly Convex Body is a Polyhedron

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Joint work with

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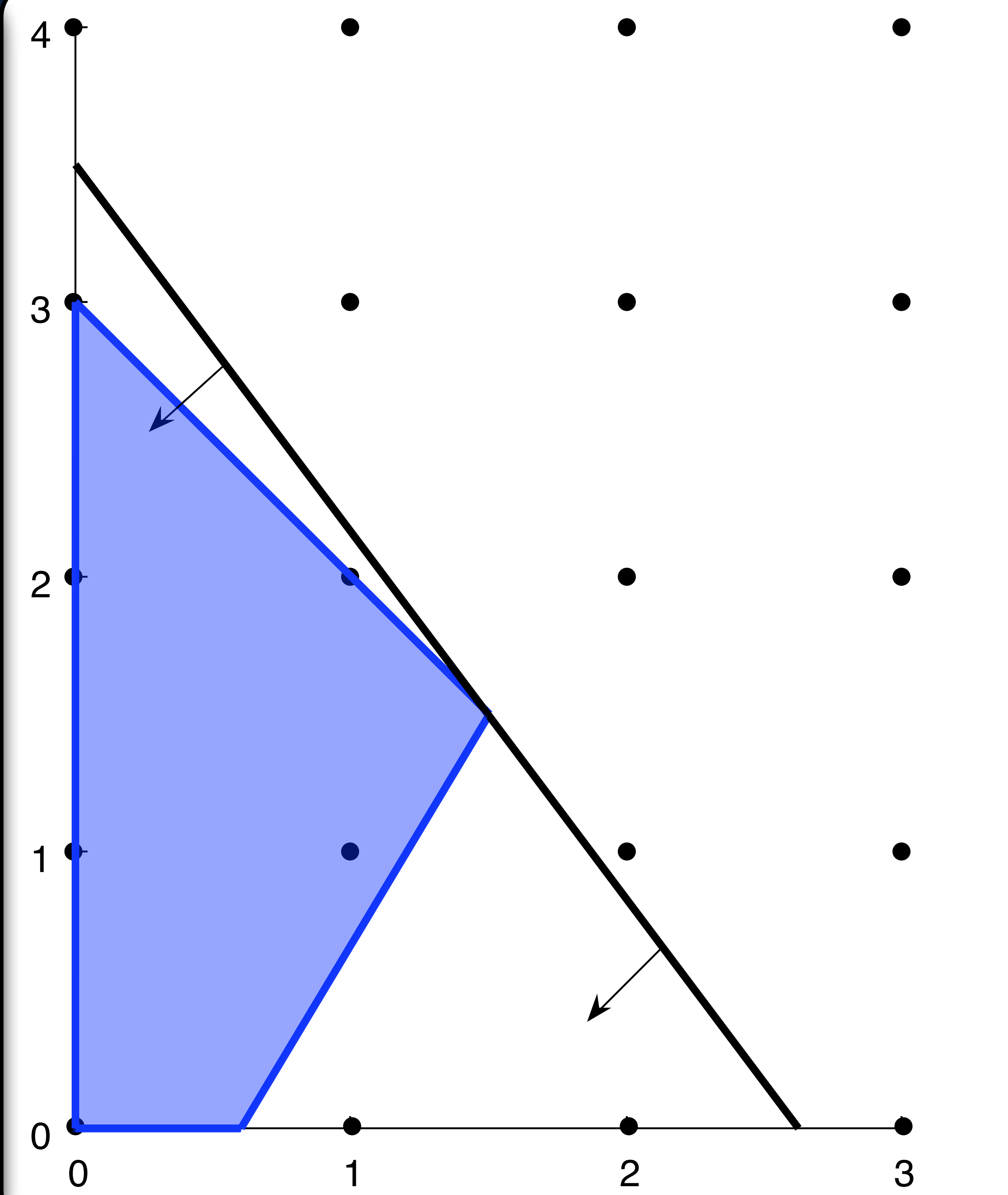
Outline

- Introduction
- Proof:
    - Step 1
    - Step 2
- Separation Lemma for Step 1
- Conclusions and Future Work

CG Cuts for Rational Polyhedra

$$P := \left\{ x \in \mathbb{R}^2 \ : \ \begin{array}{r} x_1 + x_2 \leq 3, \\ 5x_1 - 3x_2 \leq 3 \end{array} \right\}$$

$4x_1 + 3x_2 \leq 10.5$ Valid for P



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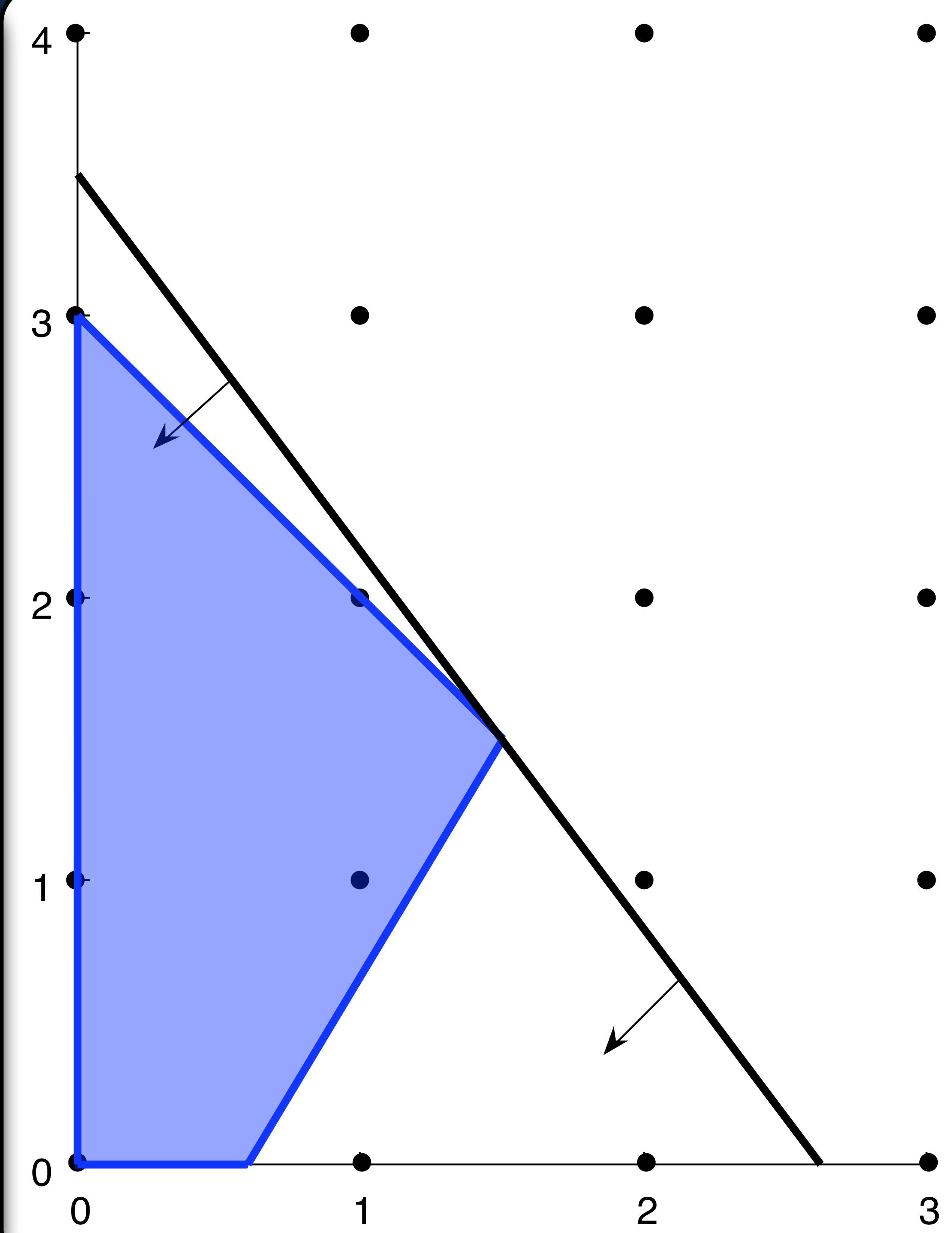

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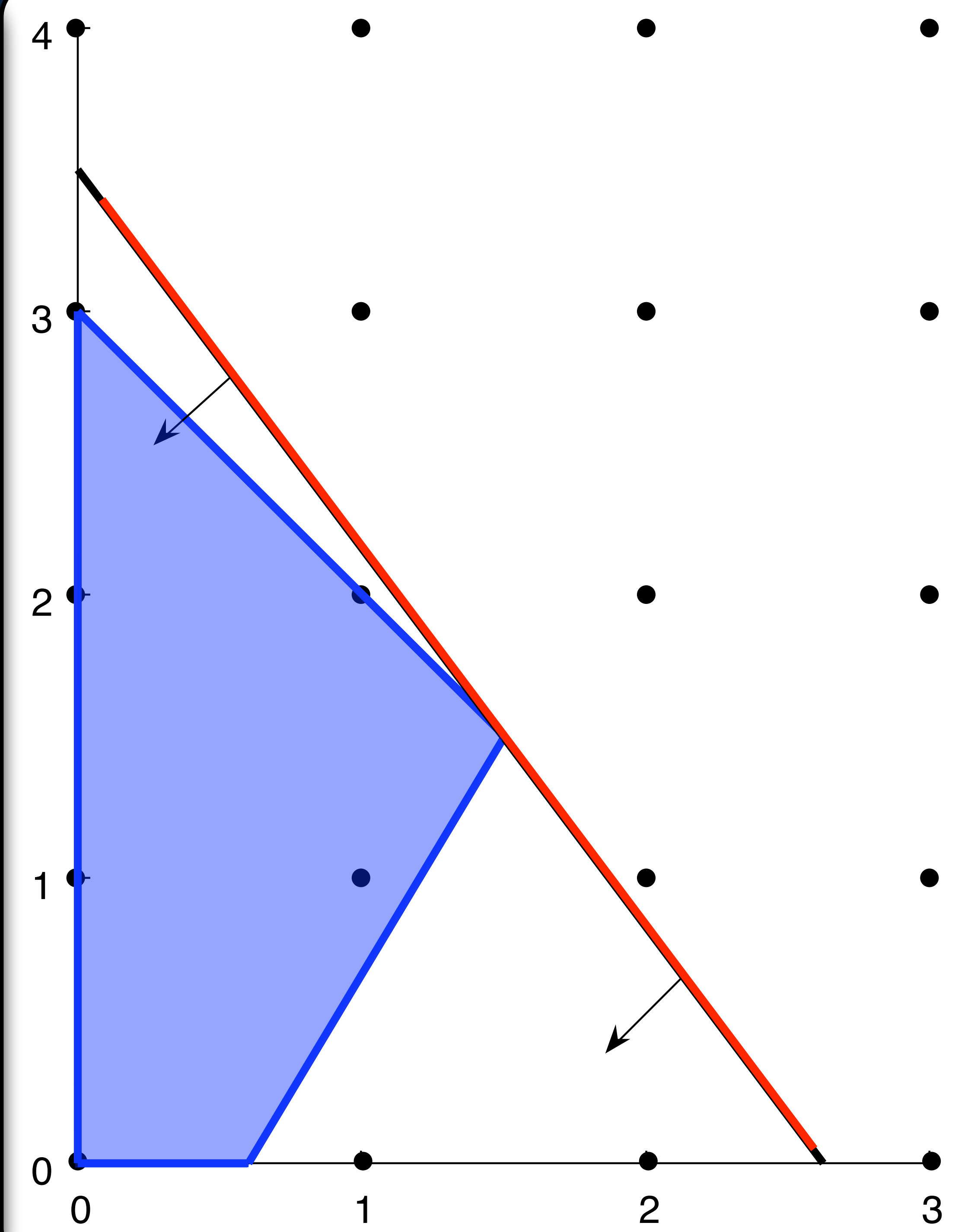

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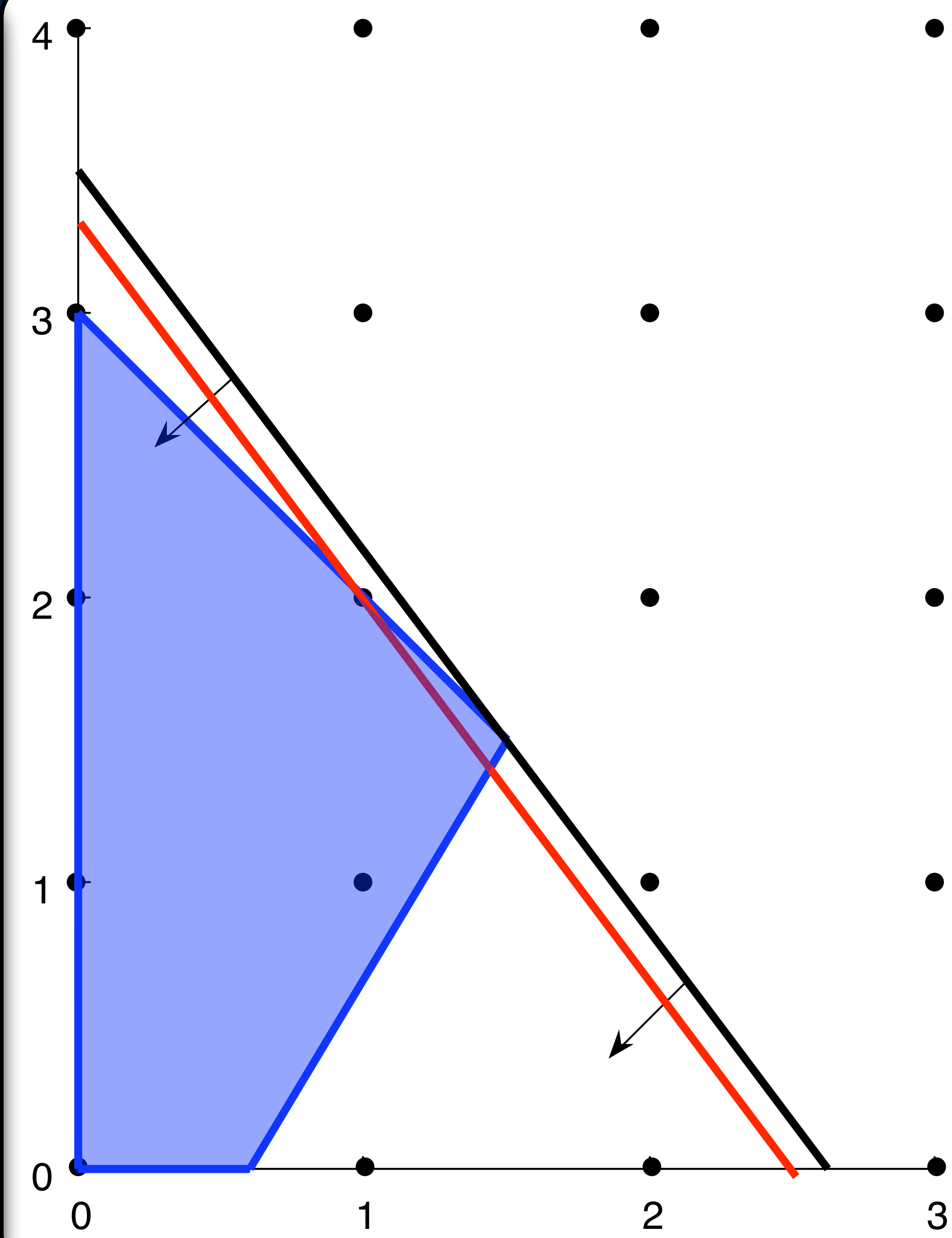
$\in \mathbb{Z}$ if $x \in \mathbb{Z}^n$

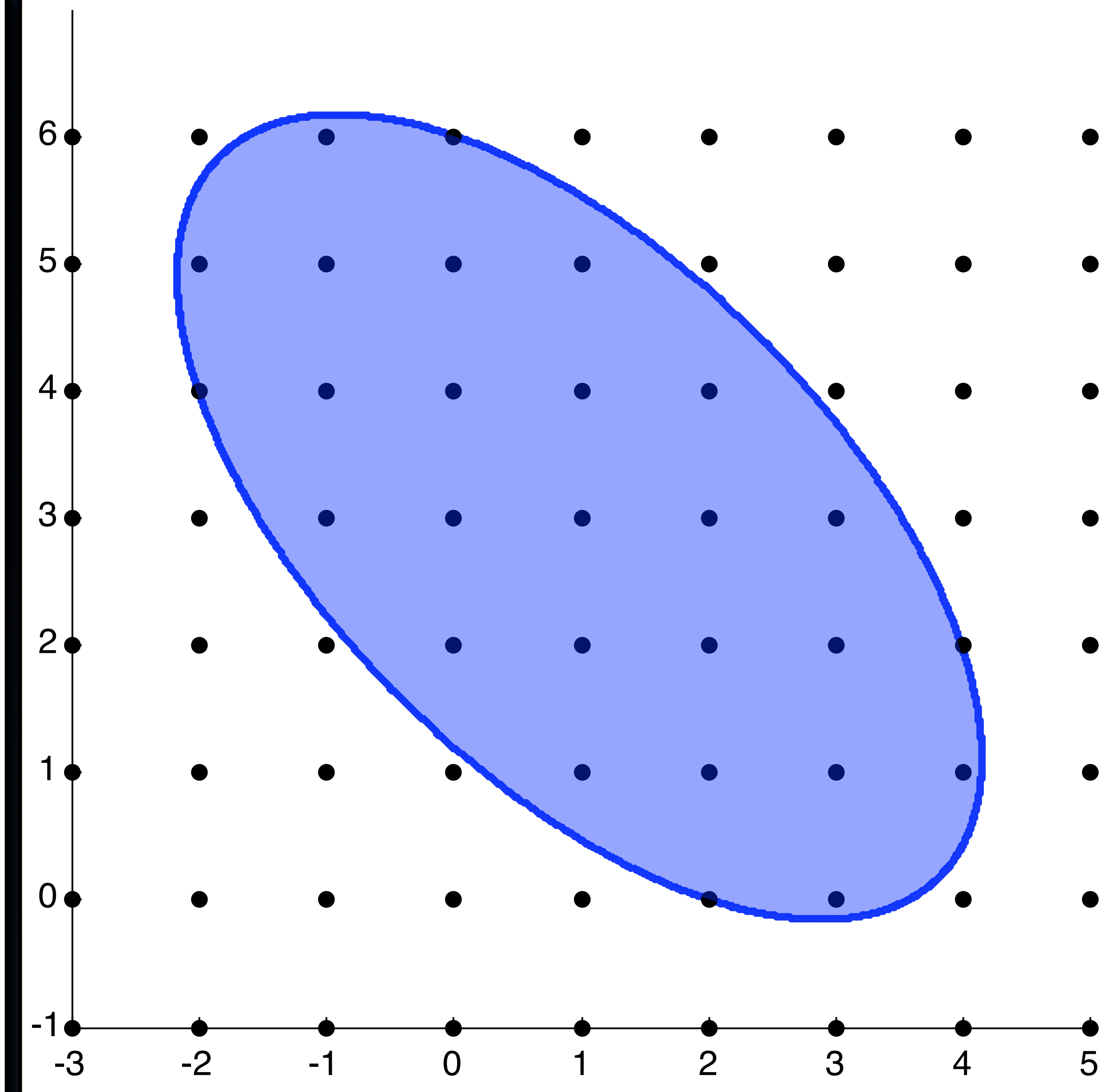
$$4x_1 + 3x_2 \leq \lfloor 10.5 \rfloor$$ Valid for $P \cap \mathbb{Z}^2$



CG Cuts for General Convex Sets

$$\sigma_C(a) := \sup\{\langle a, x\rangle \mid x \in C\}$$

$$\overbrace{\bigcap_{a\in\mathbb{Z}^n} \{x \in \mathbb{R}^n \ : \ \langle a, x\rangle \leq \sigma_C(a)\}}^{C}$$



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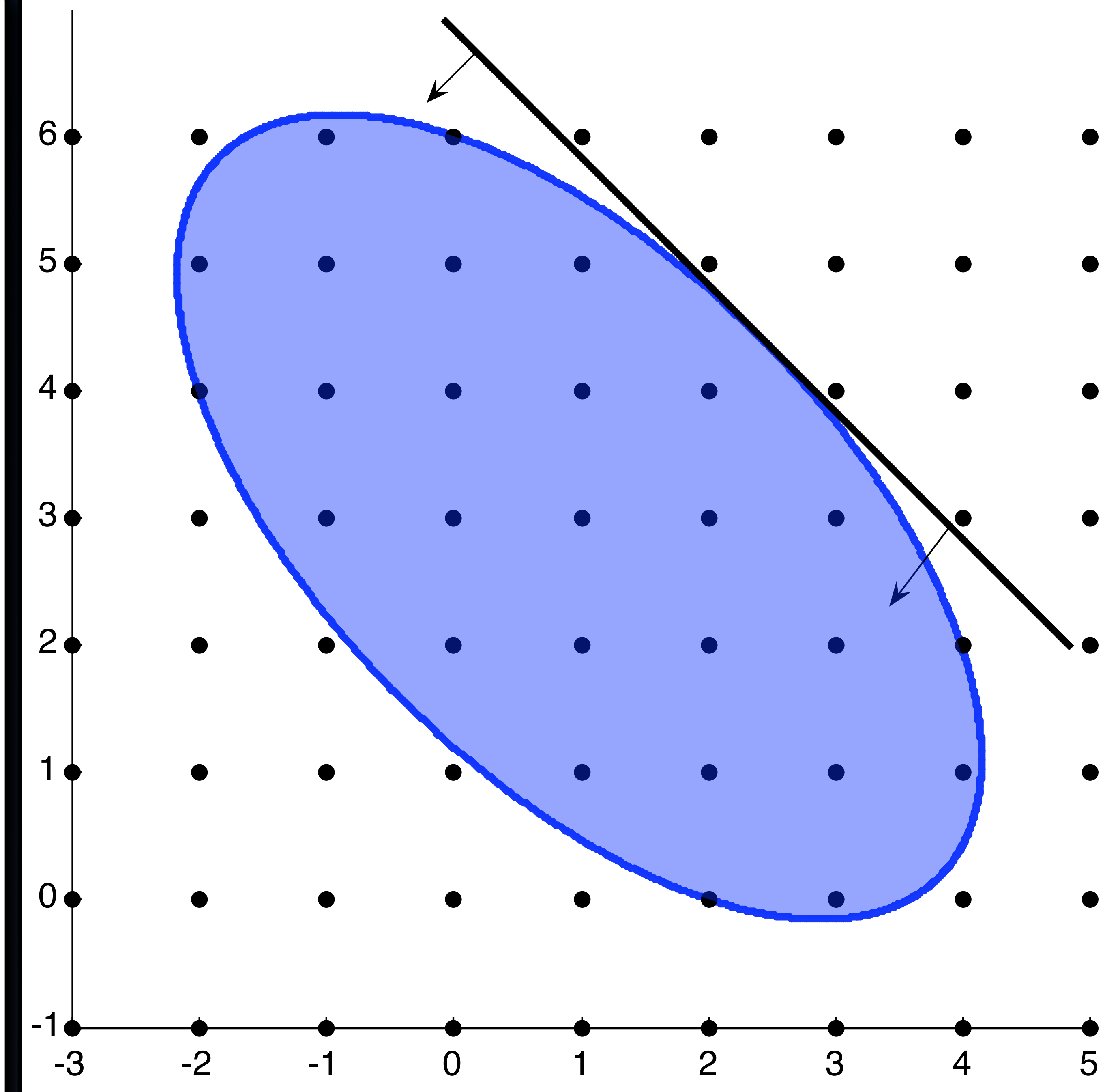

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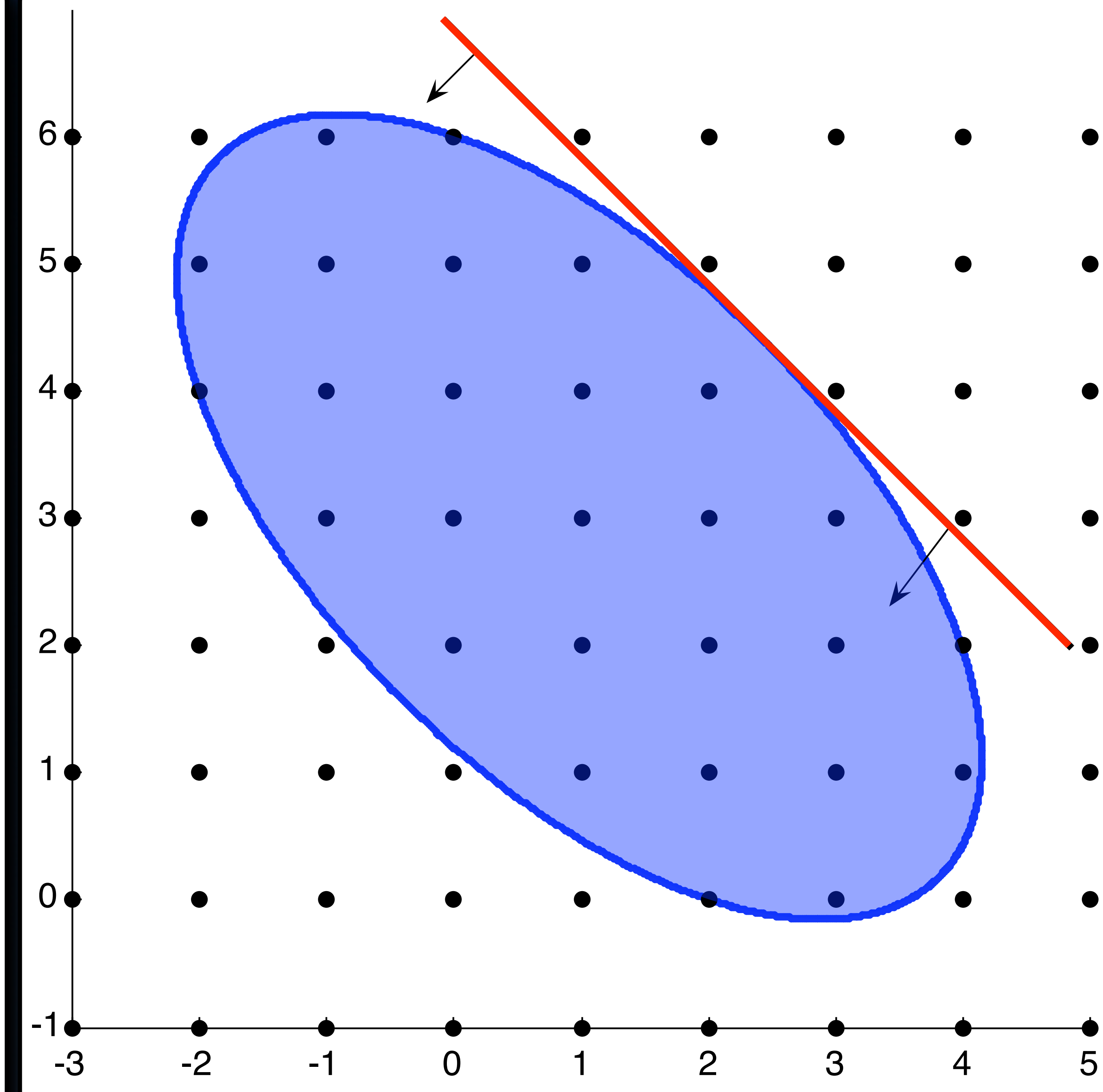

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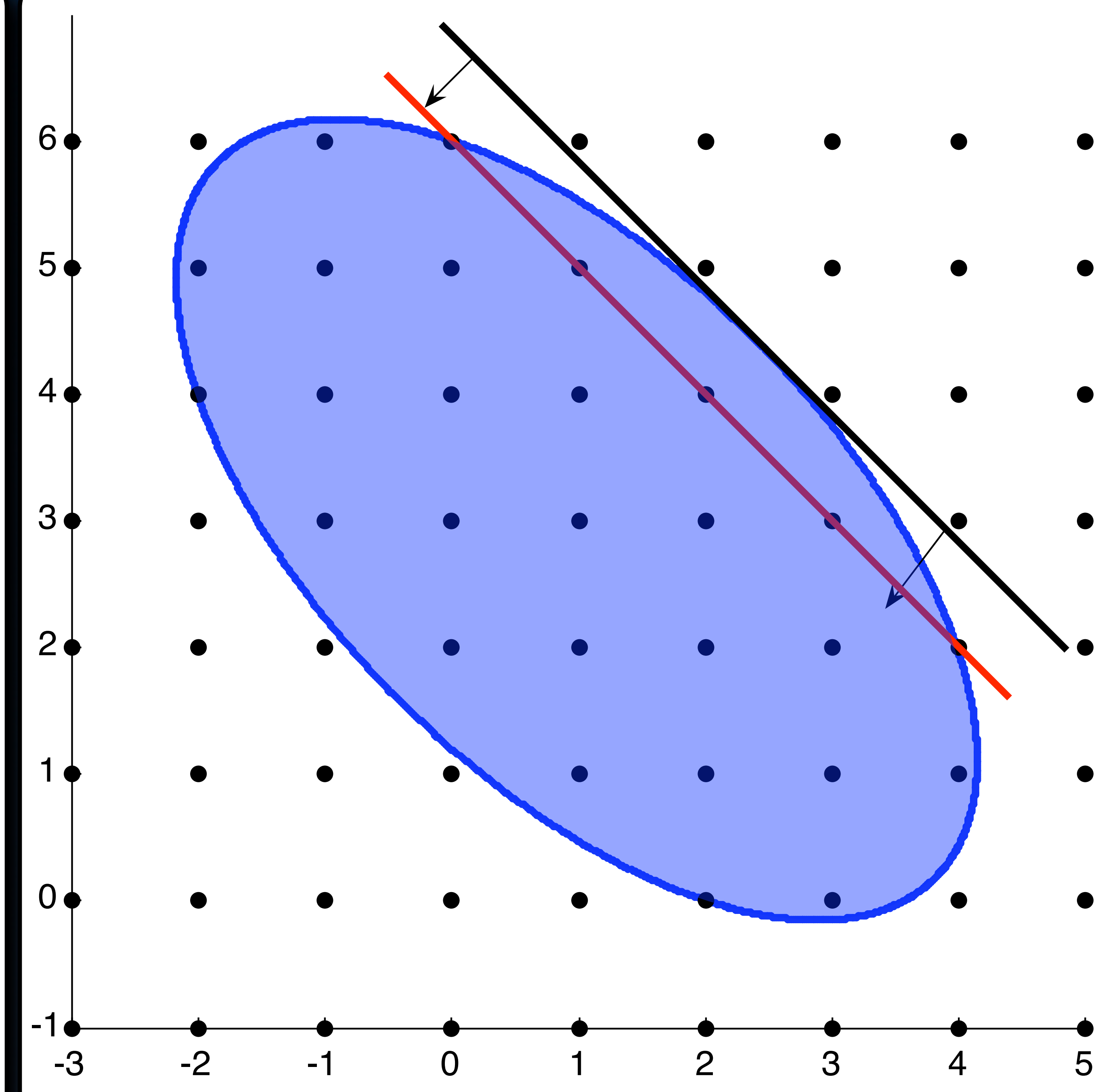
↓

$\langle a, x\rangle \leq \lfloor \sigma_C(a) \rfloor$ Valid for $C \cap \mathbb{Z}^n$



CG Closure of a Convex Set

$$\texttt{CGC}(D, C) := \bigcap_{a \in D} \{x \in \mathbb{R}^n \ : \ \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}$$

- CG Closure: $\texttt{CGC}(\mathbb{Z}^n, C)$
- Is CG closure a polyhedron?
  - Finite set $S \subset \mathbb{Z}^n$ s.t. $\texttt{CGC}(\mathbb{Z}^n, C) = \texttt{CGC}(S, C)$
  - Yes, for rational polyhedra (Schrijver, 1980)
  - What about other convex sets?

What we know for Convex Bodies

$$C^0 := C, \quad C^k := \texttt{CGC}(\mathbb{Z}^n, C^{k-1})$$

- There exists k s.t. $C^k = \mathrm{conv}(C \cap \mathbb{Z}^n)$ (Chvátal, 1973)
- Also for unbounded rational polyhedra (Schrijver, 1980).
- Result does not imply polyhedrality of C^1

Proof Outline: Generation Procedure

- Step 1: Construct a finite set $S^1 \subset \mathbb{Z}^n$ such that
  - $\texttt{CGC}(S^1, C) \subseteq C$
  - $\texttt{CGC}(S^1, C) \cap \mathrm{bd}(C) \subset \mathbb{Z}^n$

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- $\texttt{CGC}(\mathbb{Z}^n, C) = \texttt{CGC}(S^1, C) \cap \texttt{CGC}(S^2, C)$

Outline of Step 1

- Step 1: Construct a finite set $S^1 \subset \mathbb{Z}^n$ such that
  - $\texttt{CGC}(S^1, C) \subseteq C$ and $\texttt{CGC}(S^1, C) \cap \mathrm{bd}(C) \subset \mathbb{Z}^n$

(a) Separate non-integral points in $\mathrm{bd}(C)$.

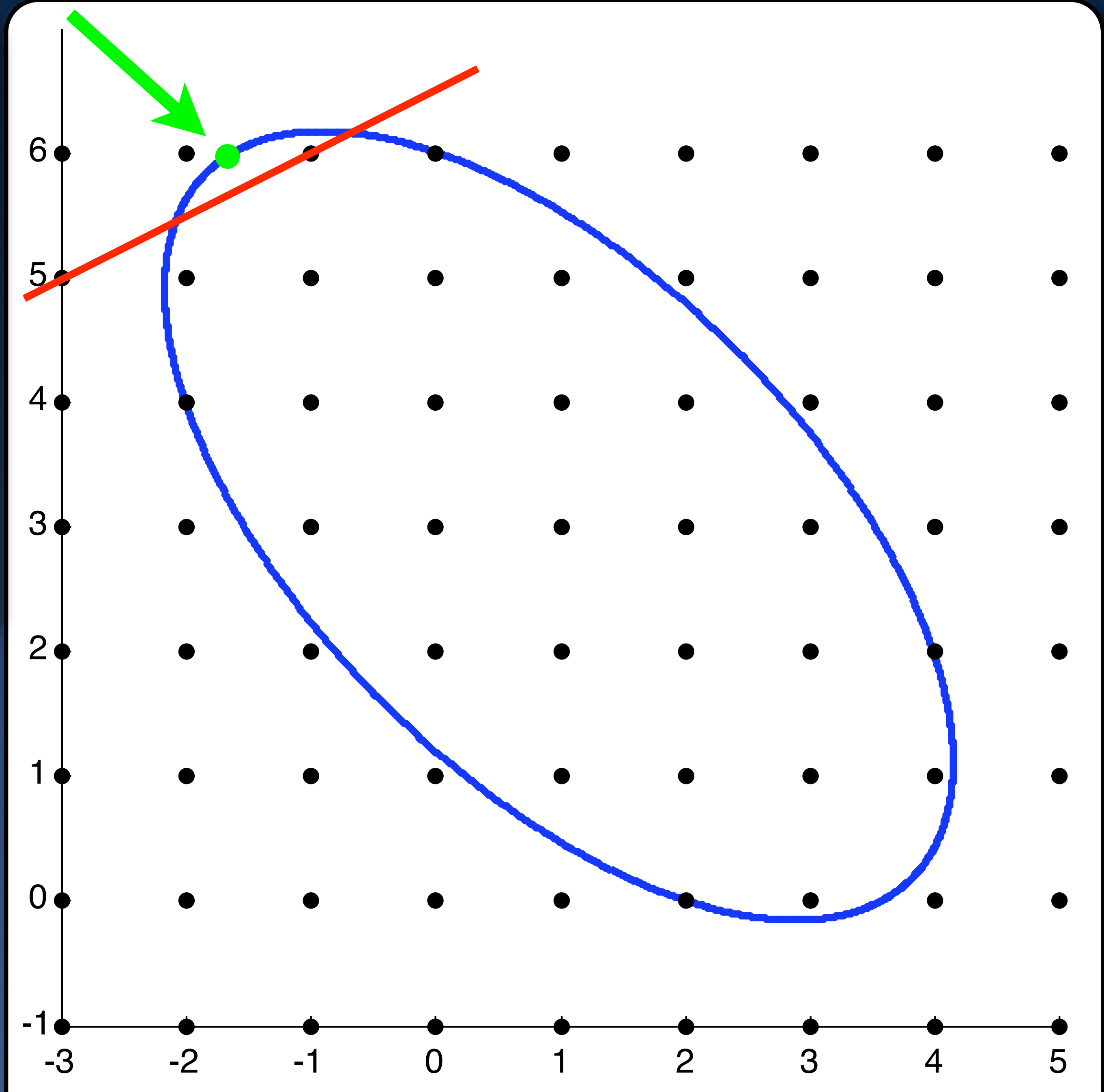
(b) Separate neighborhood of integral points in $\mathrm{bd}(C)$.

(c) Compactness argument to construct finite S^1.

Separate non-integral points in $\mathrm{bd}(C)$

$u \in \mathrm{bd}(C) \setminus \mathbb{Z}^n$ $\quad \exists\, a^u \in \mathbb{Z}^n$

$$\langle a^u, u \rangle > \lfloor \sigma_C(a^u) \rfloor$$



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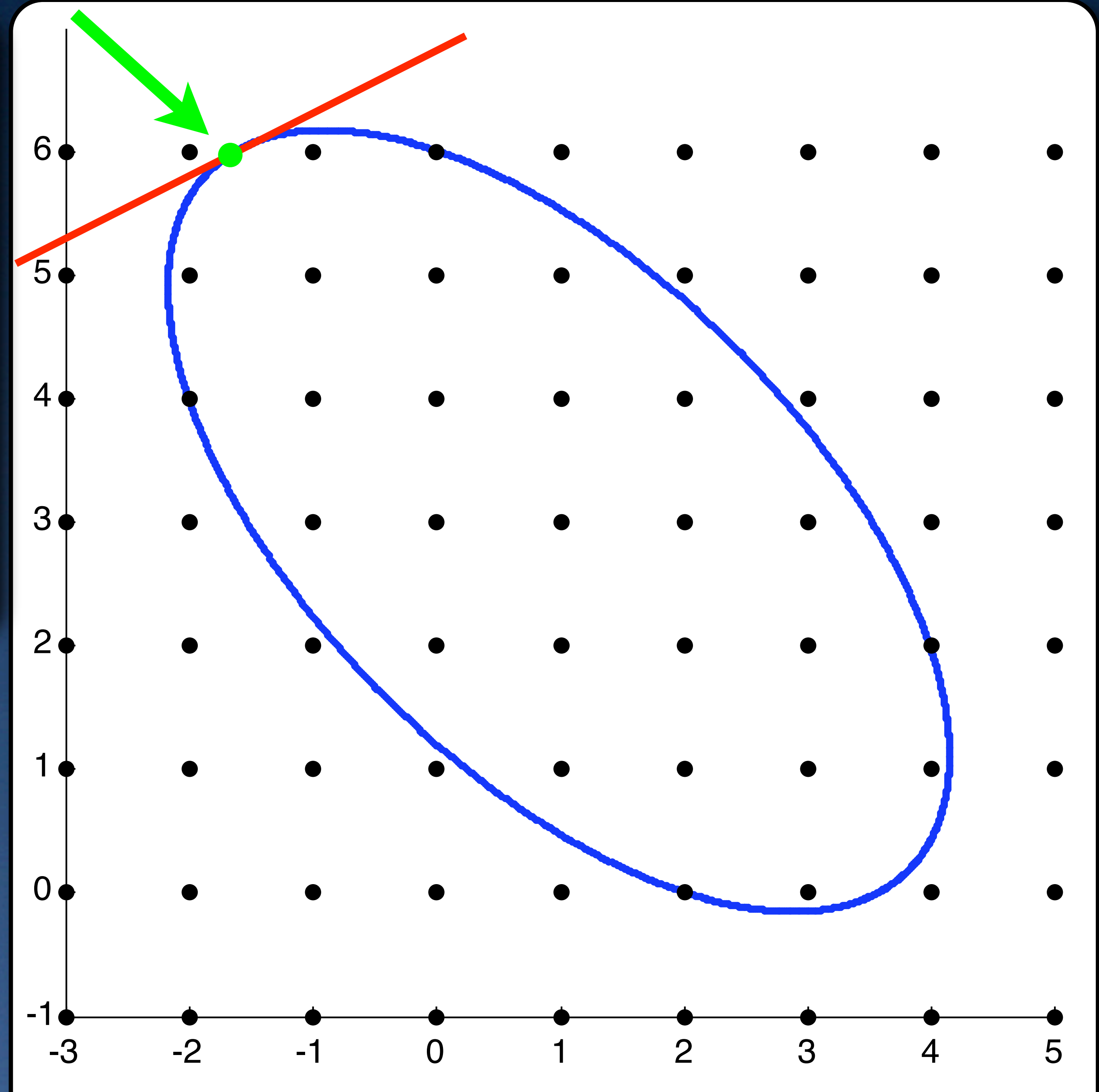
$$\langle s(u), u \rangle = \sigma_C(s(u))$$



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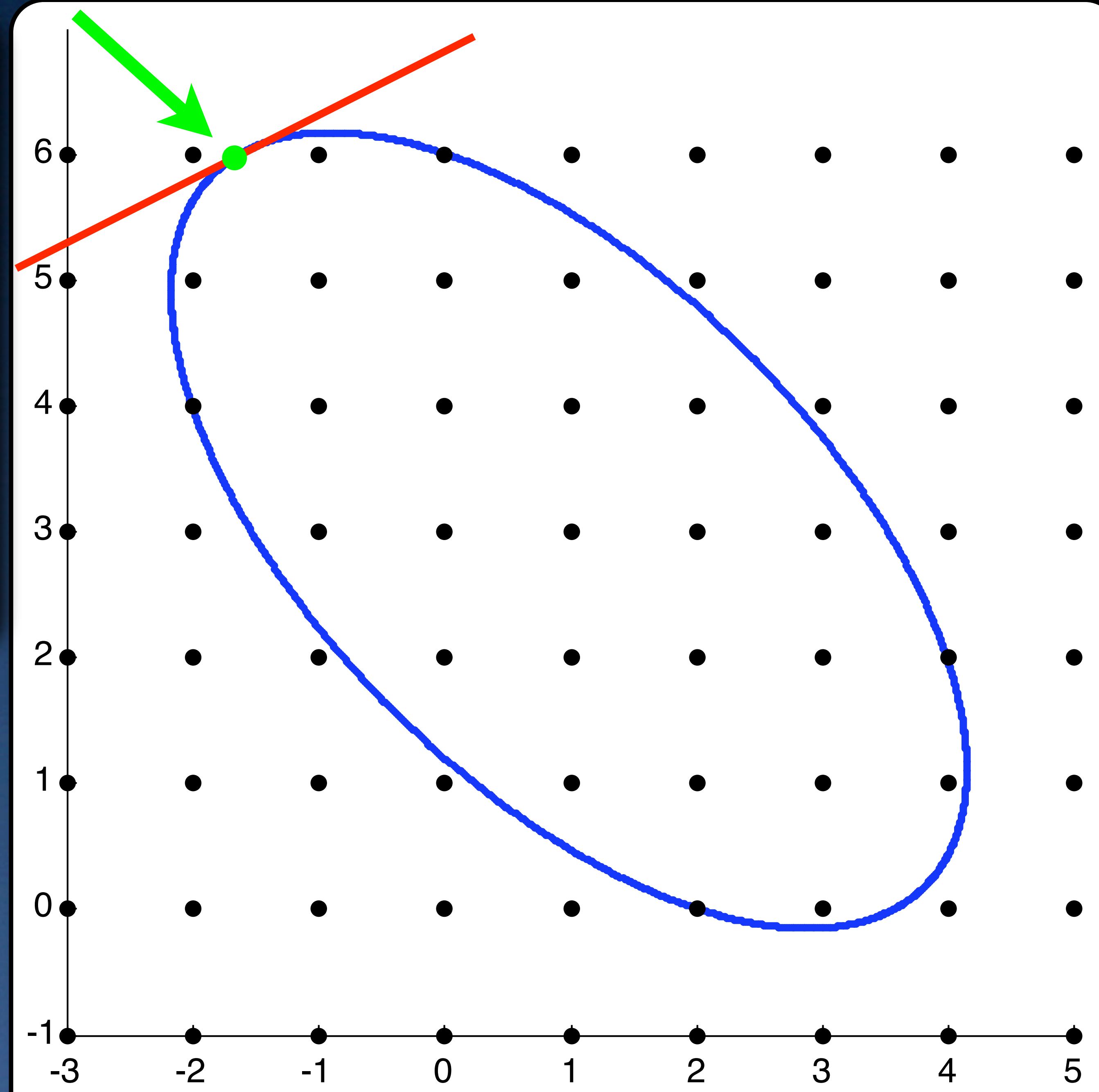

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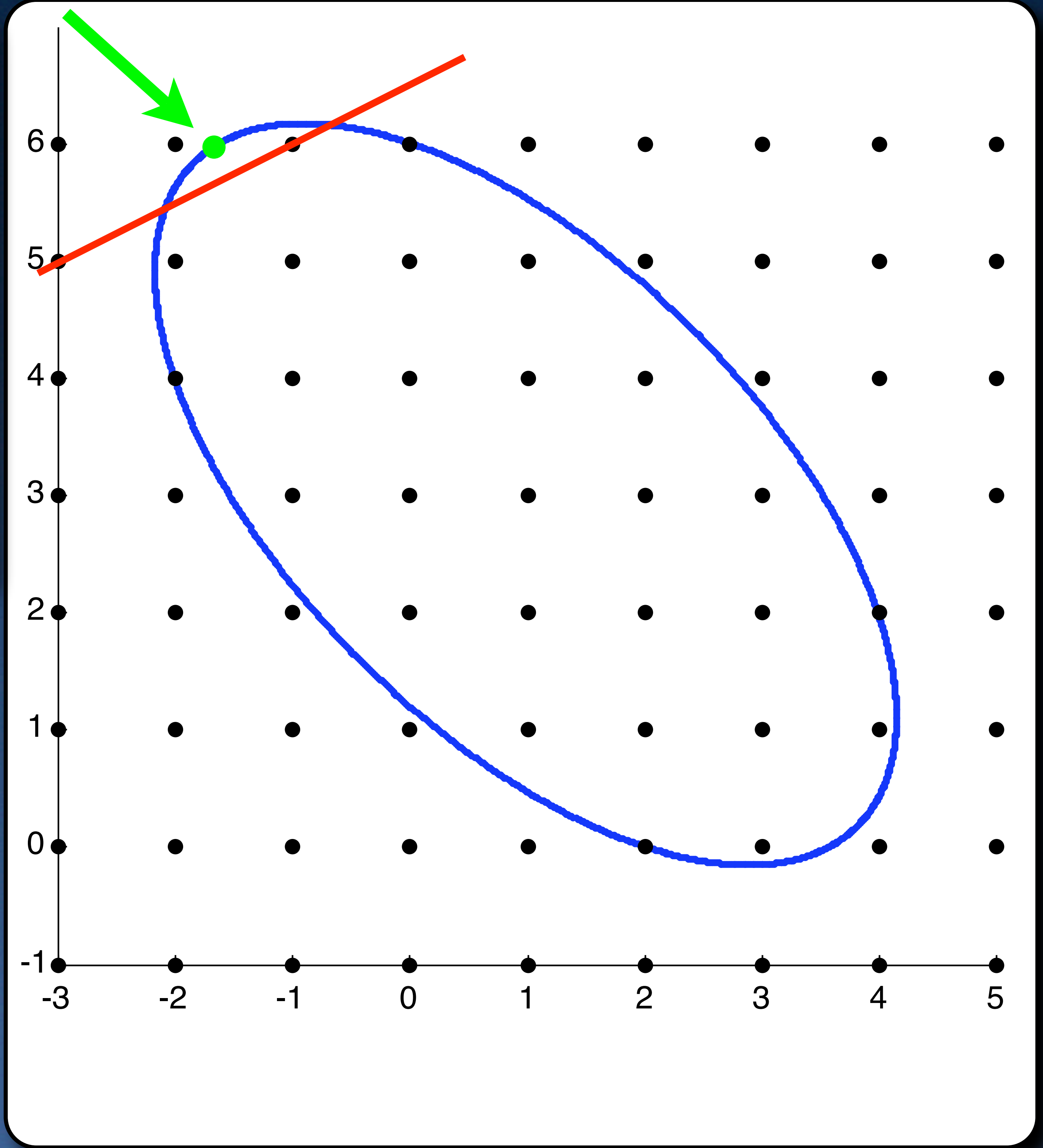

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$\lambda s(u) \notin \mathbb{Z}^n \quad \forall \lambda > 0 :$

$\lambda s(u) \in \mathbb{Z}^n \Rightarrow \sigma_C(\lambda s(u)) \in \mathbb{Z} :$

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$$C = \left\{ x \in \mathbb{R}^2 \;:\; \sqrt{x_1^2 + x_2^2} \leq 1 \right\}$$

$$u = (1/2, \sqrt{3}/2)^T \in \mathrm{bd}(C)$$

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$$C = \left\{ x \in \mathbb{R}^2 \ : \ \sqrt{x_1^2 + x_2^2} \leq 5 \right\}$$

$$u = (25/13, 60/13)^T \in \mathrm{bd}(C)$$

$$s(u) = (5, 12)^T, \ \sigma_C(s(u)) = 65$$

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$$\frac{s^i}{\|s^i\|} \xrightarrow{i \to \infty} \frac{s(u)}{\|s(u)\|}$$

$$\lim_{i \to \infty} \langle s^i, u \rangle - \lfloor \sigma_C(s^i) \rfloor > 0$$

Diophantine approx. of $s(u)$

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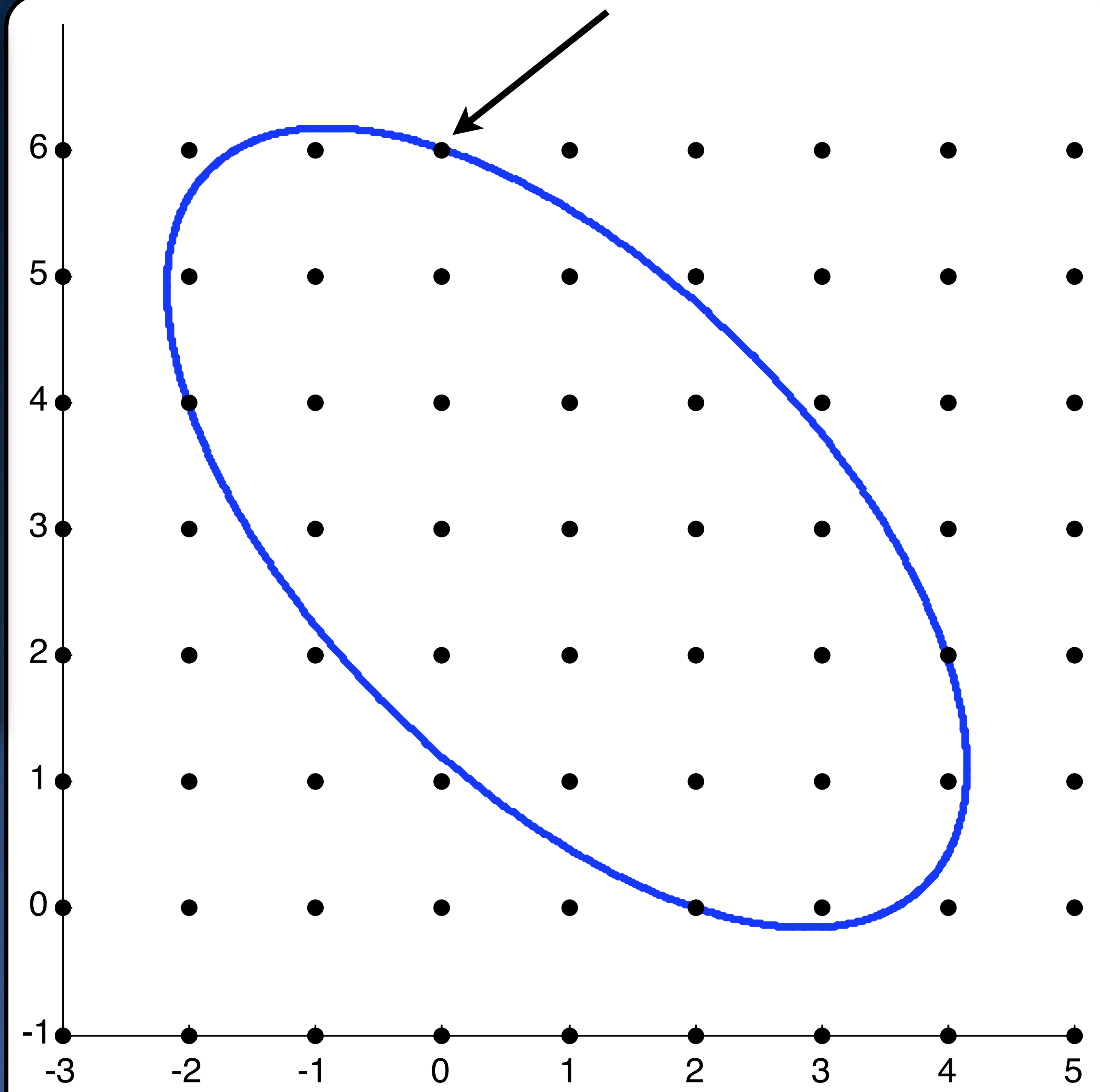
Separate neighborhood of integers

$u \in \mathrm{bd}(C) \cap \mathbb{Z}^n$



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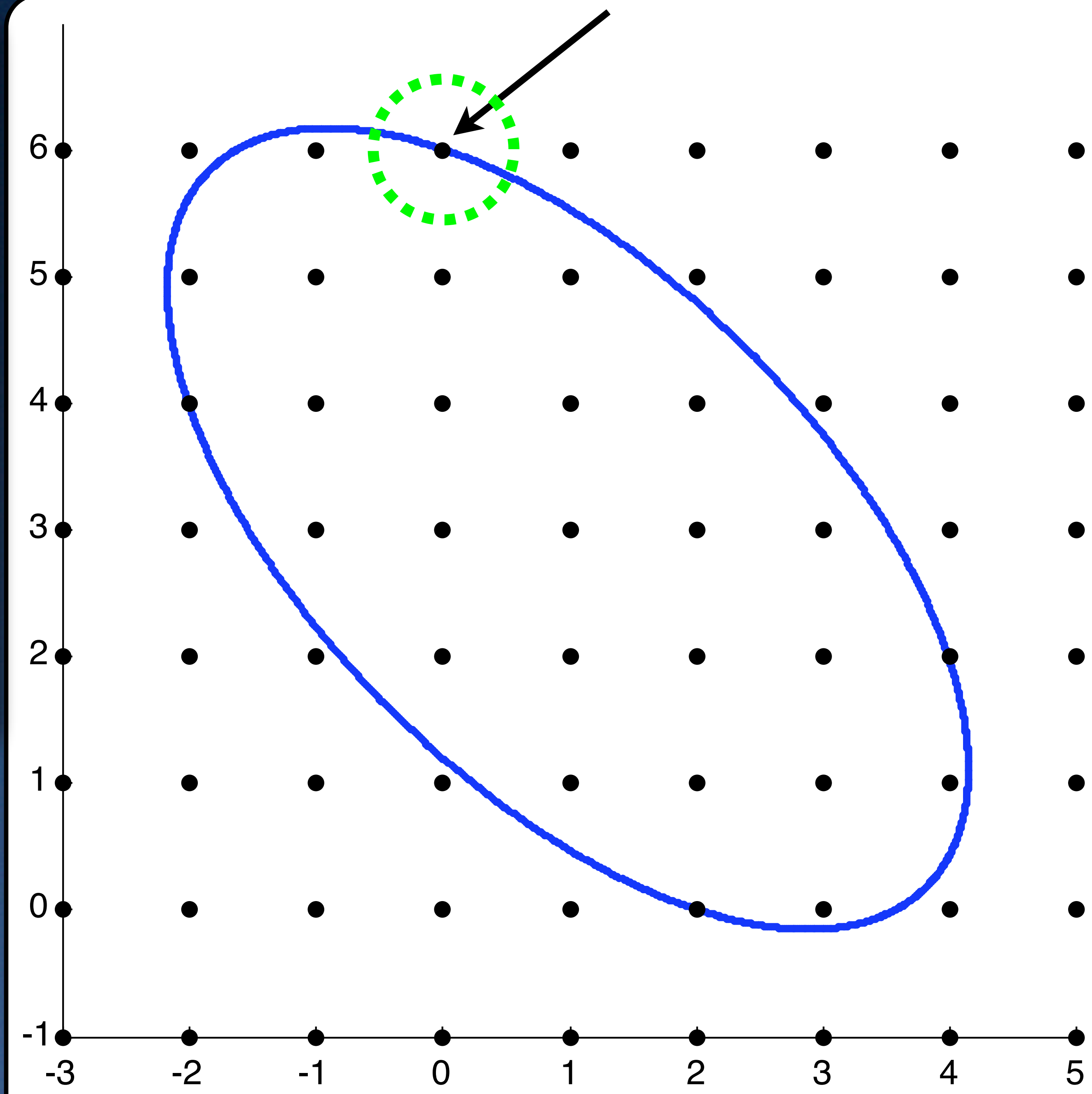
$\exists$ open neighborhood $\mathcal{N}$ of u and finite set $I \subset \mathbb{Z}^n$



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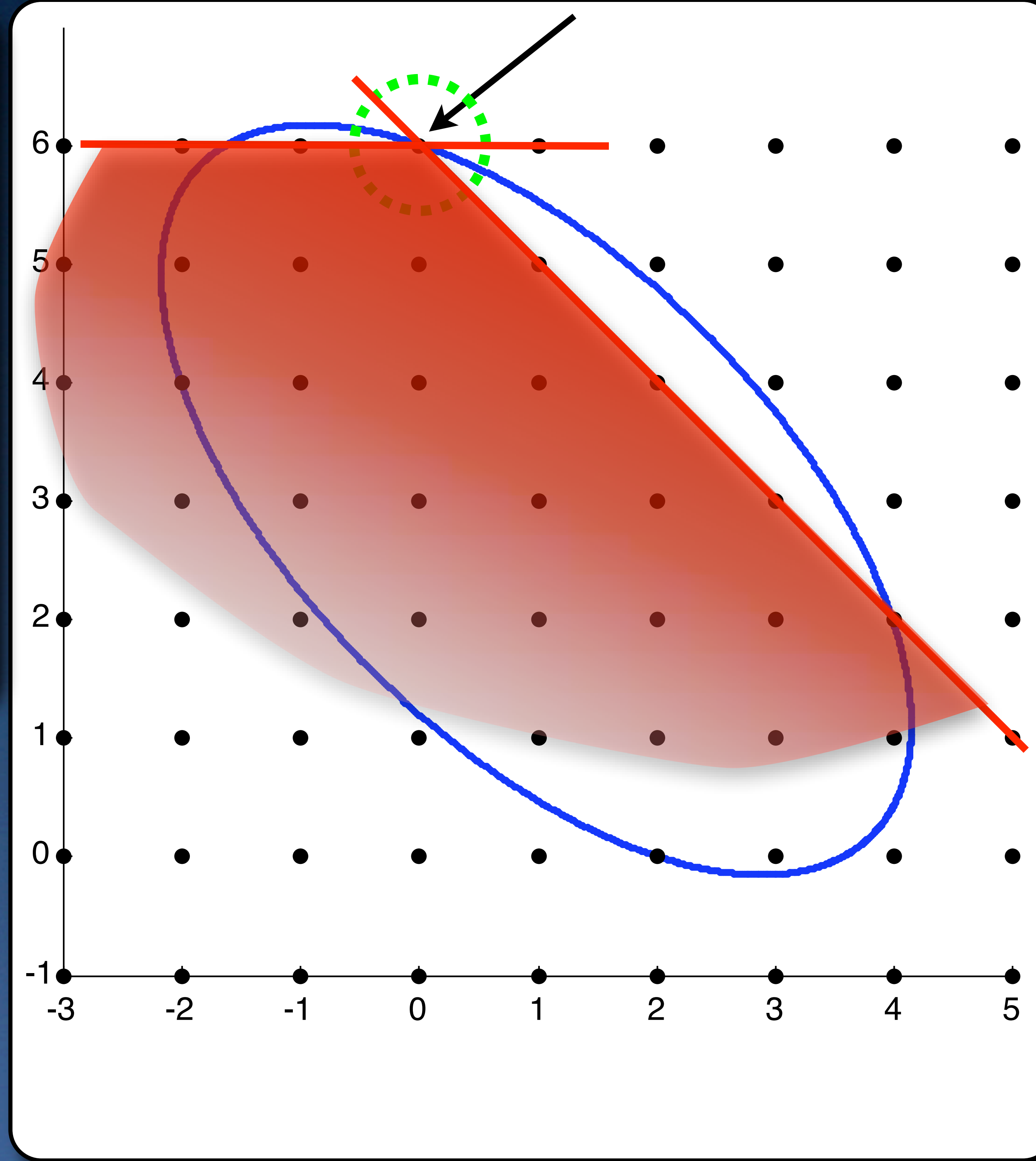

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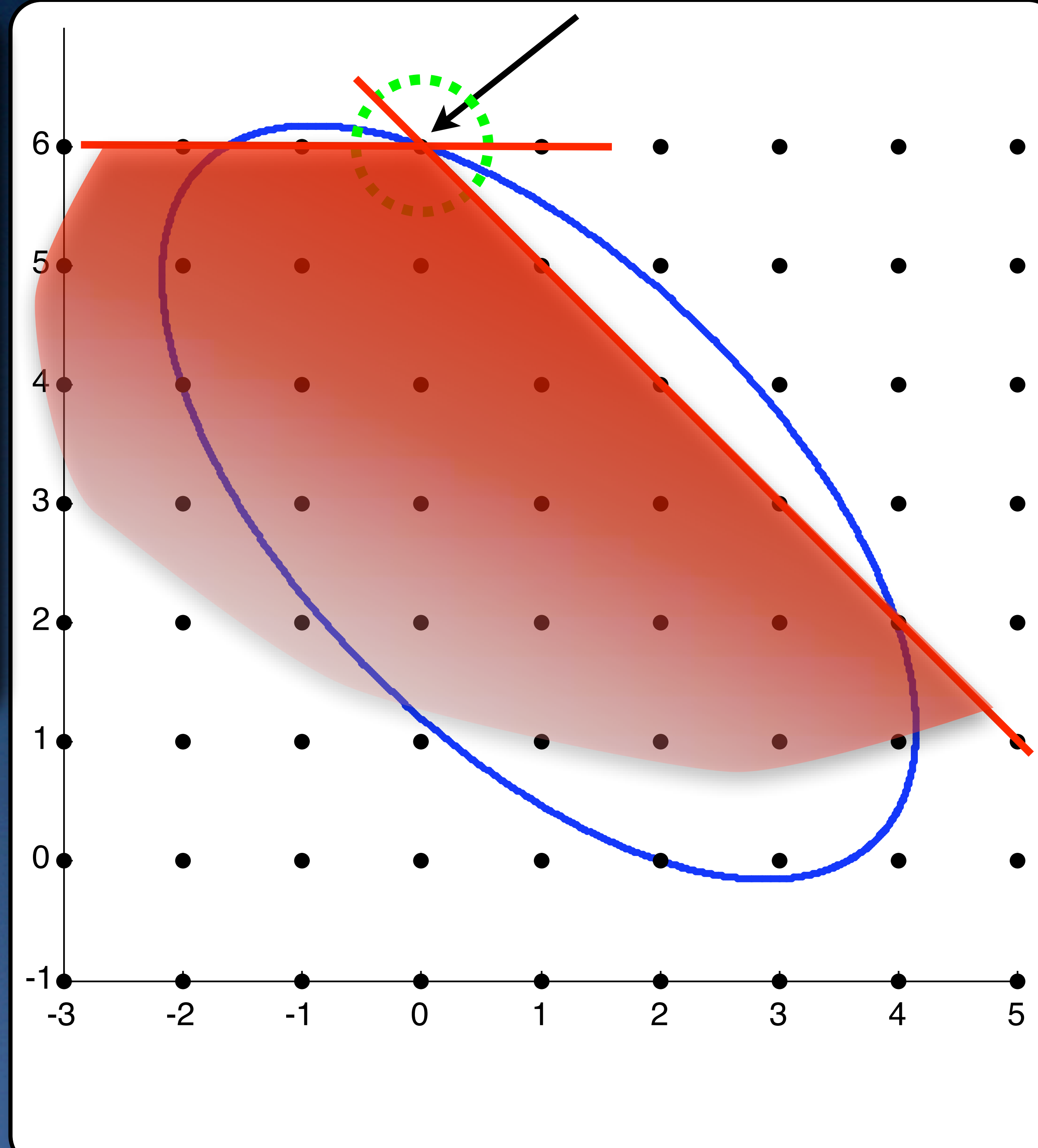
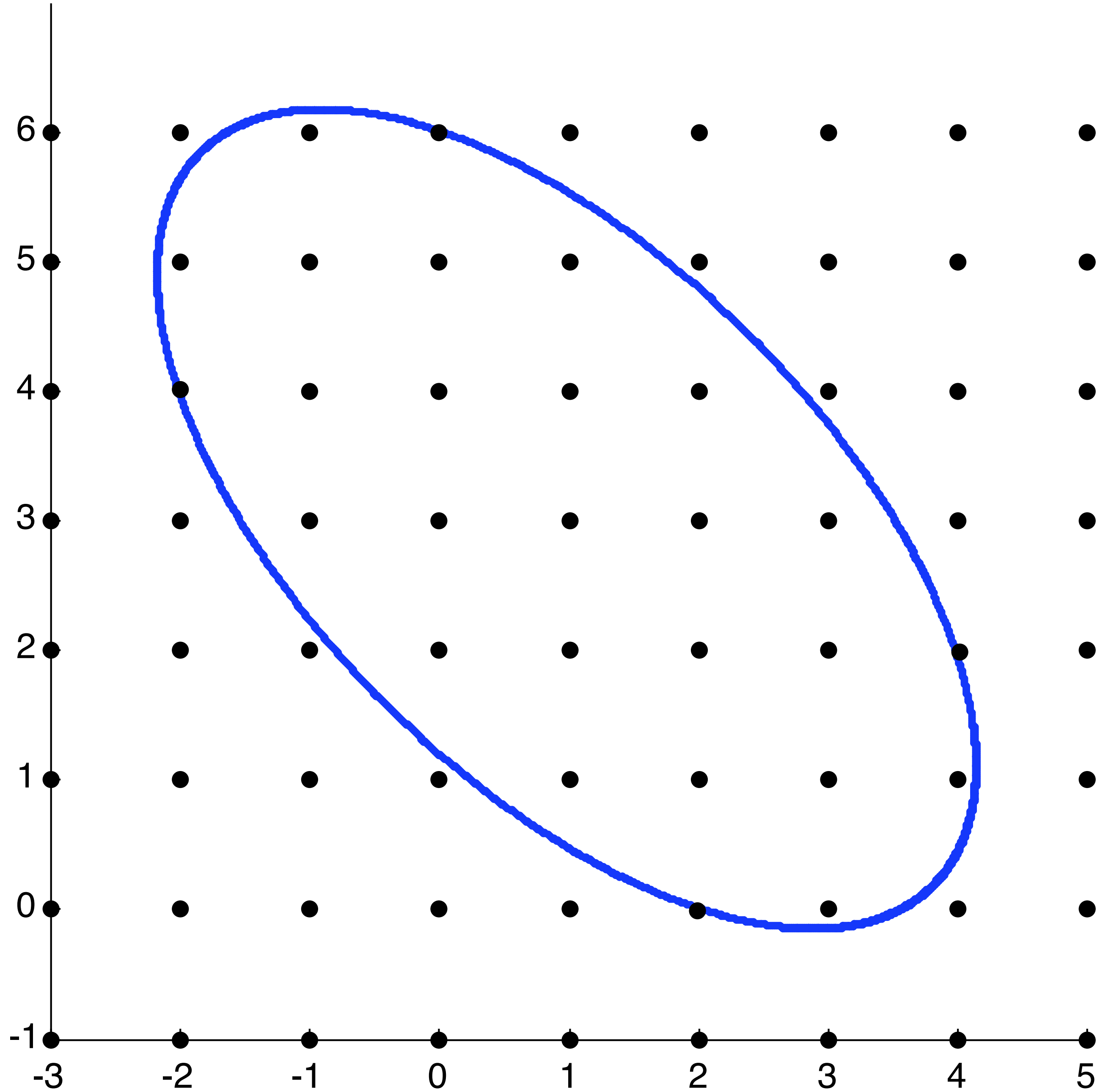
- Similar to non-integer separation + compactness argument



Compactness Argument

$$K := \mathrm{bd}(C) \setminus \bigcup_{v \in \mathrm{bd}(C) \cap \mathbb{Z}^n} \mathcal{N}_v$$



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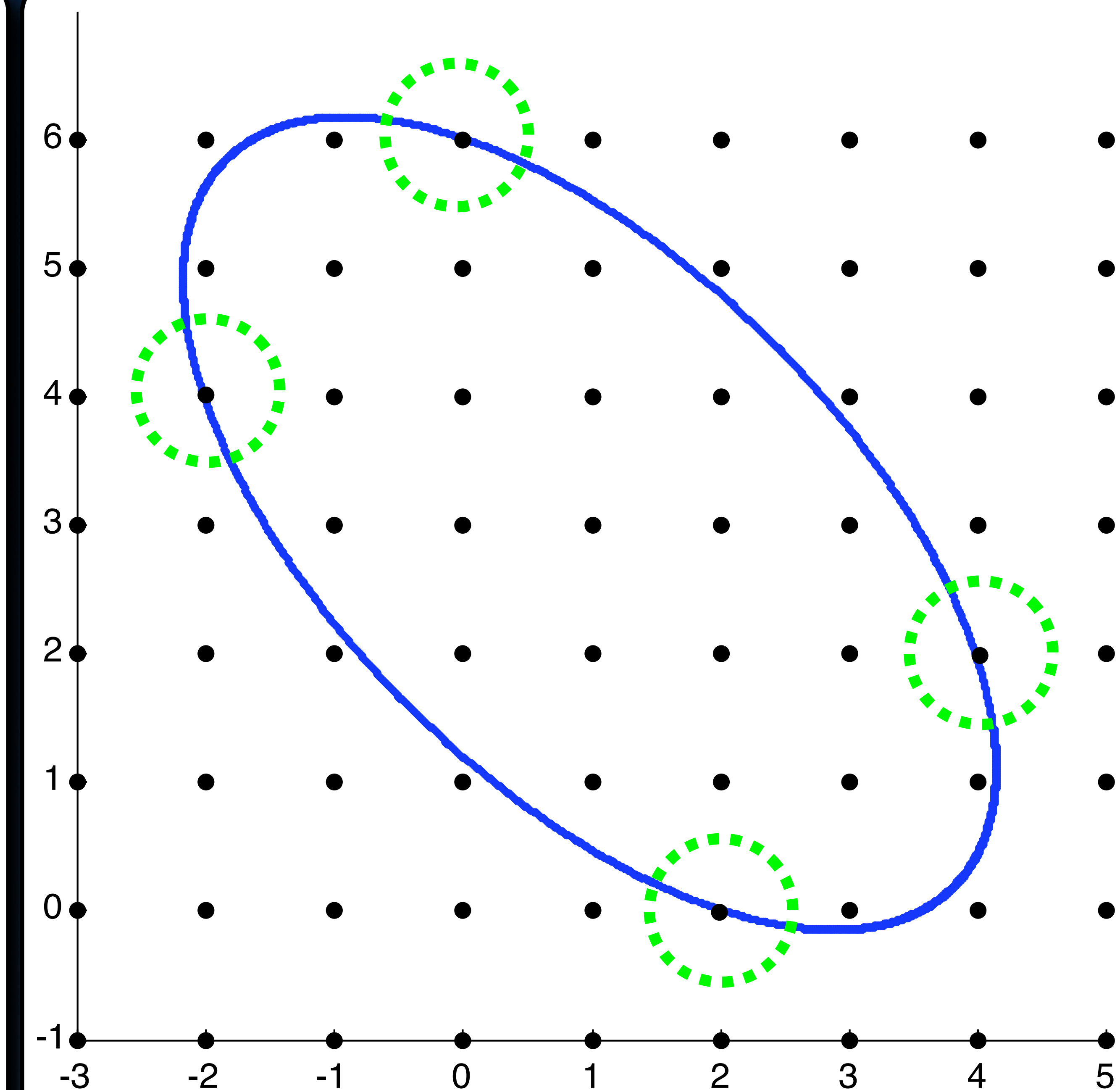

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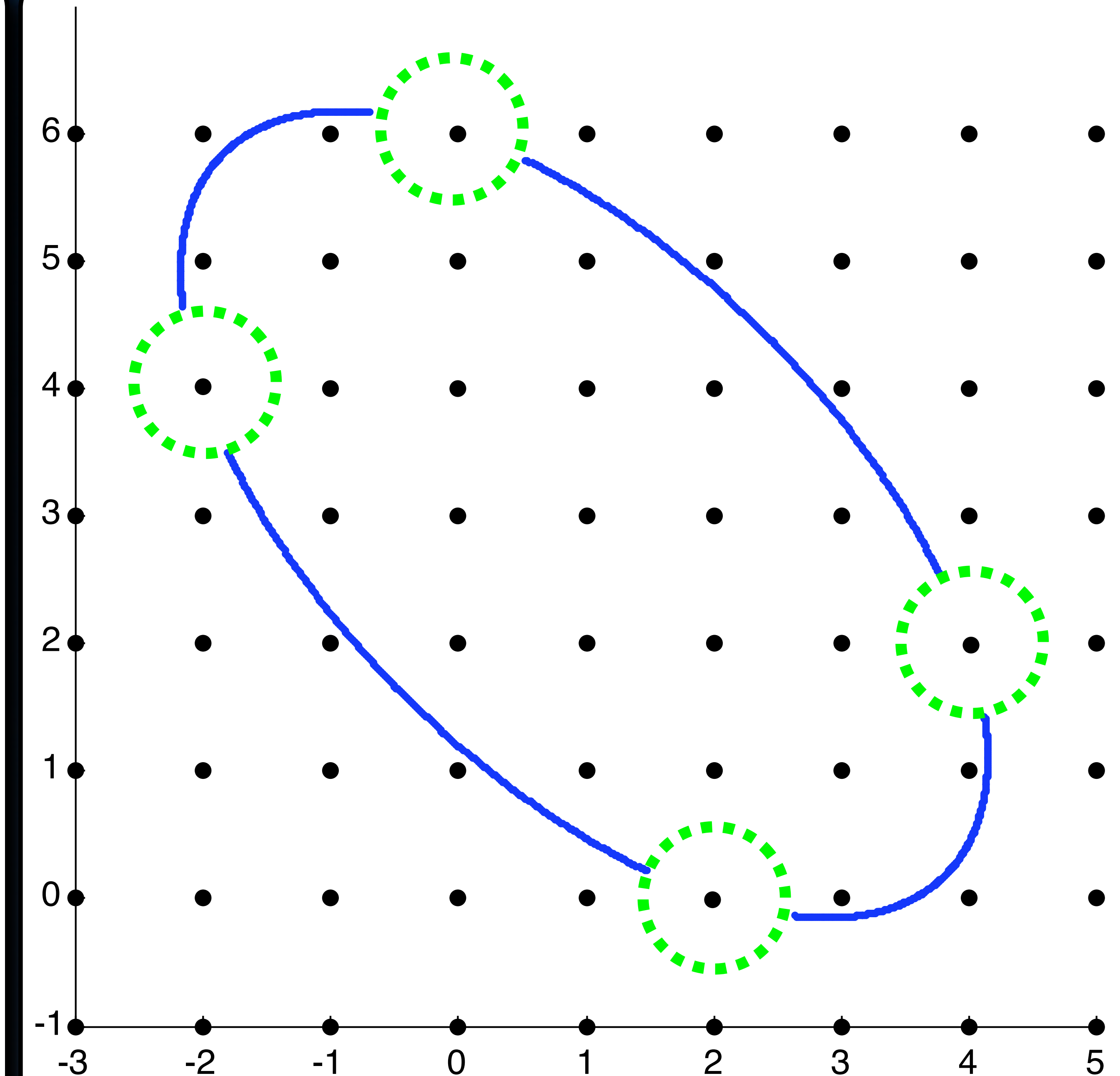

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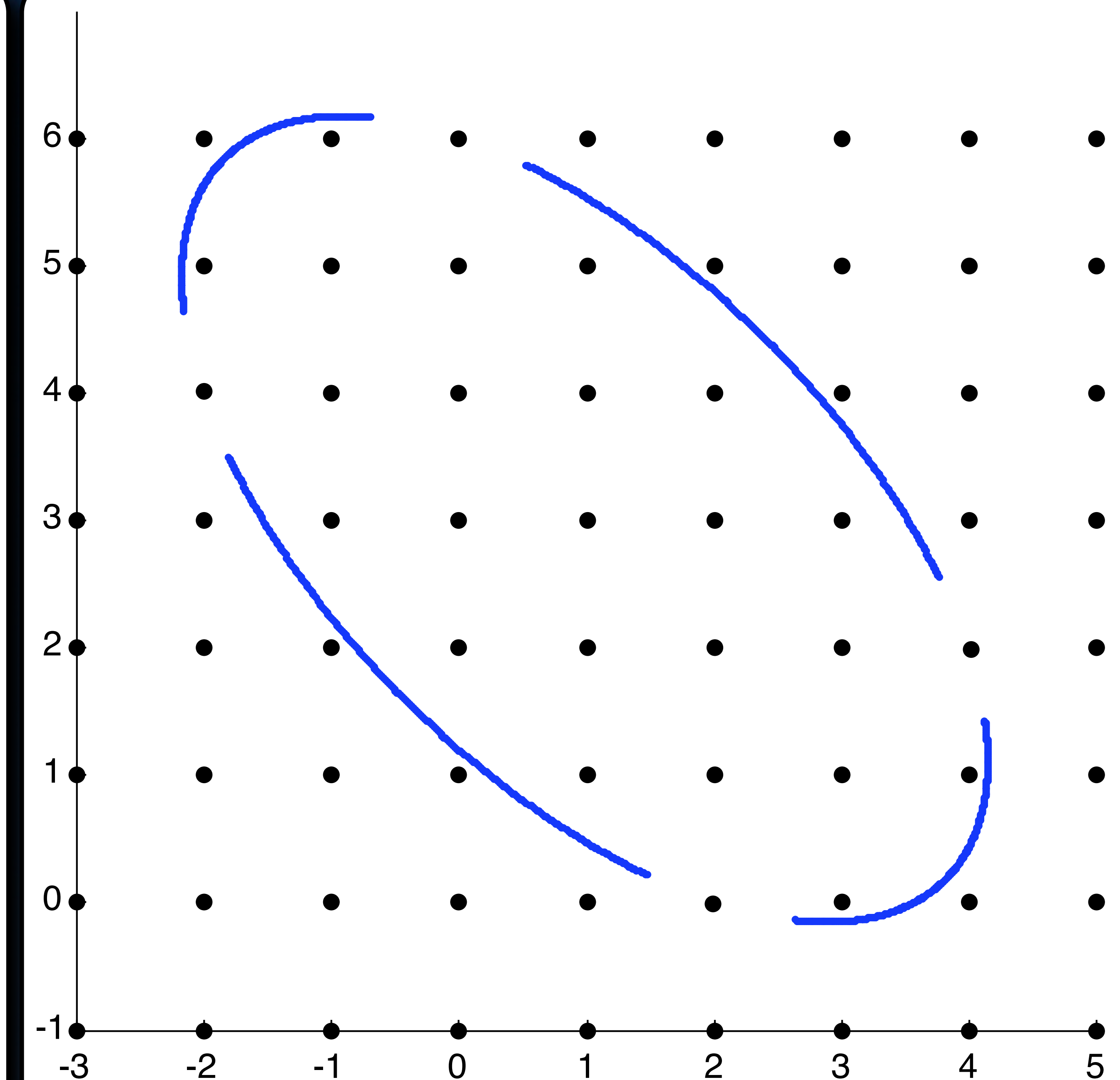

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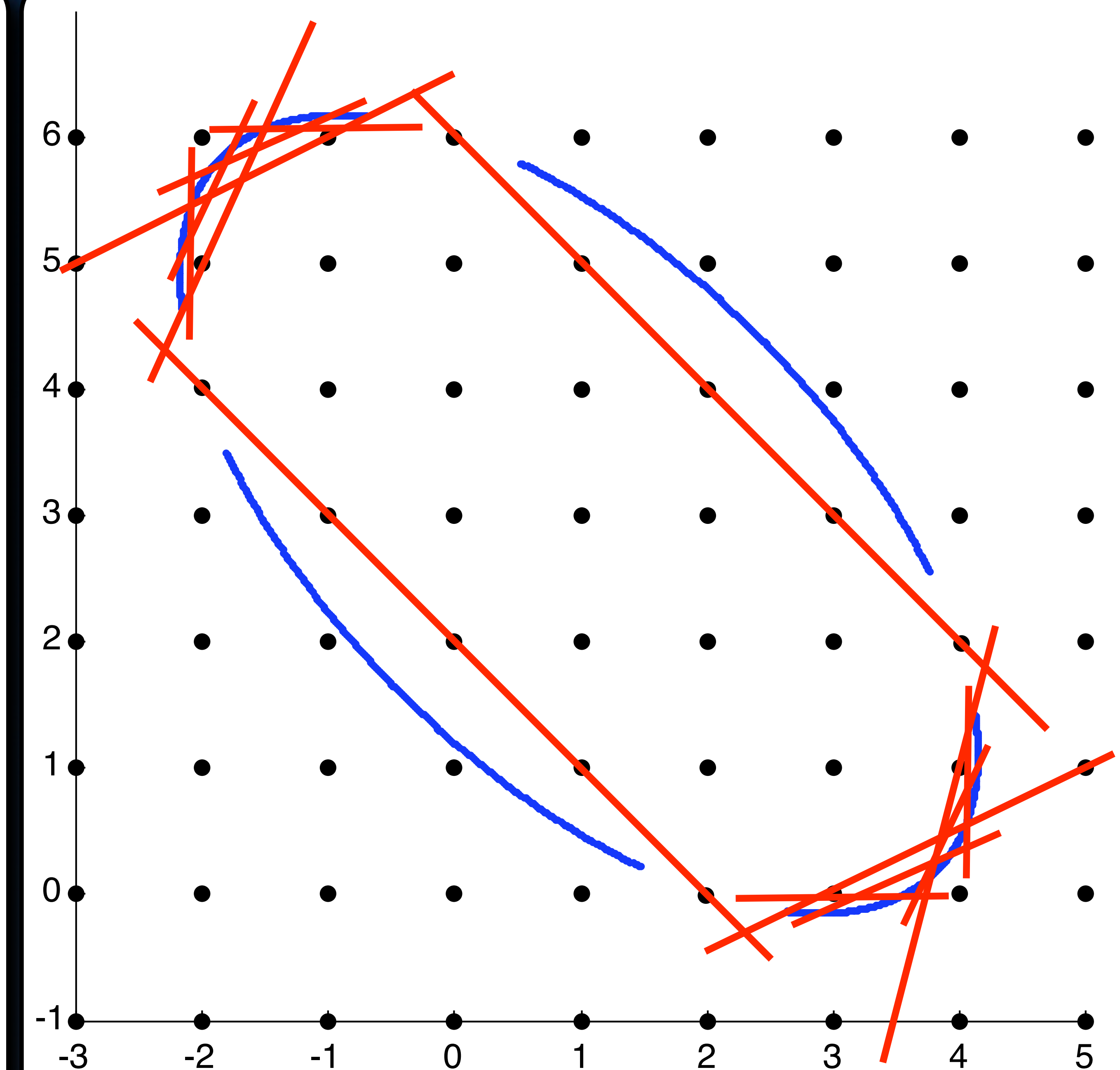
$$K \subset \bigcup_{u \in K} \mathcal{S}_u$$



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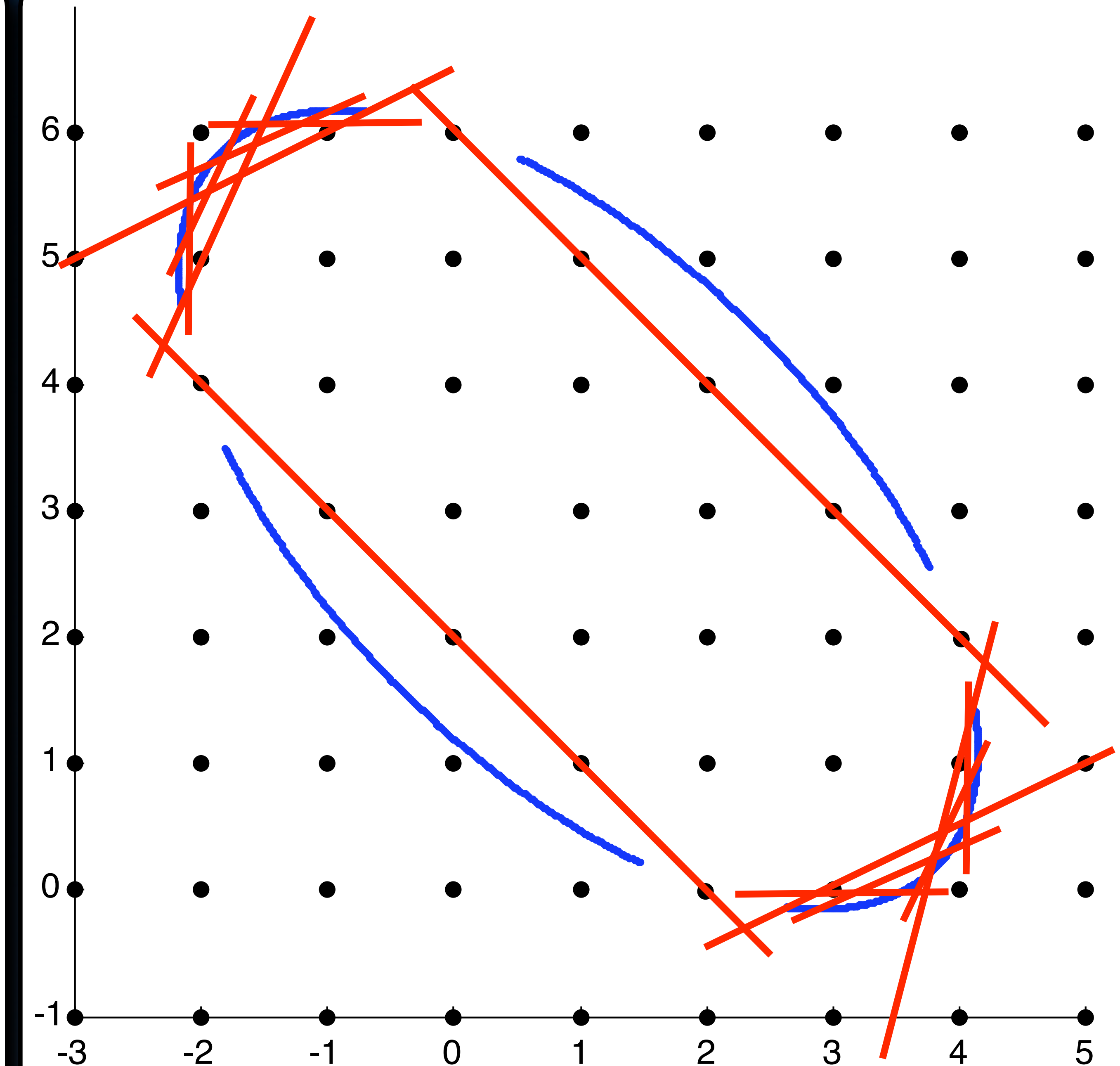
compact → $K \subset \bigcup_{u \in K} \mathcal{S}_u$

$$K \subset \bigcup_{i=1}^{m} \mathcal{S}_{u^i}$$



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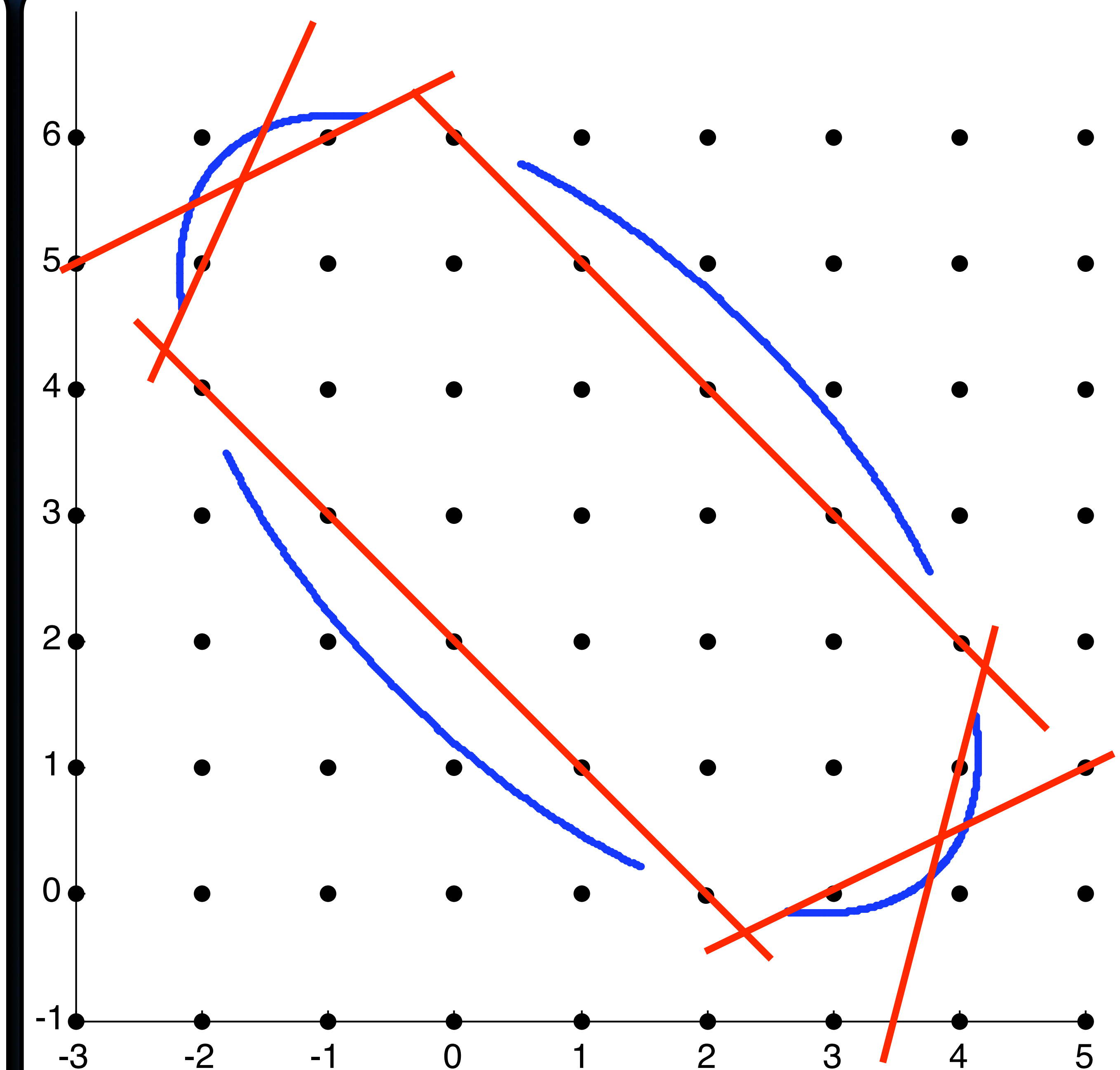

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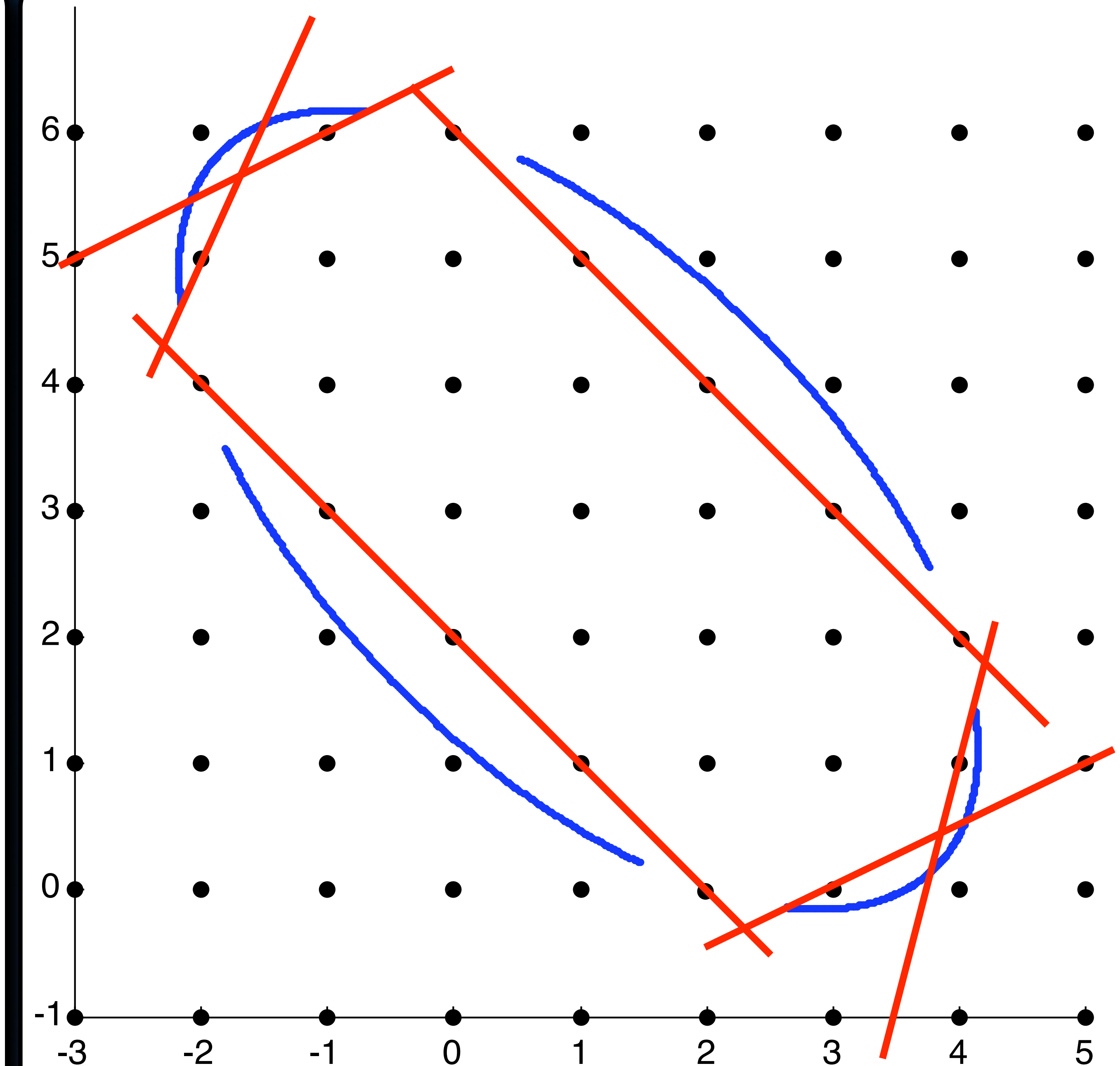

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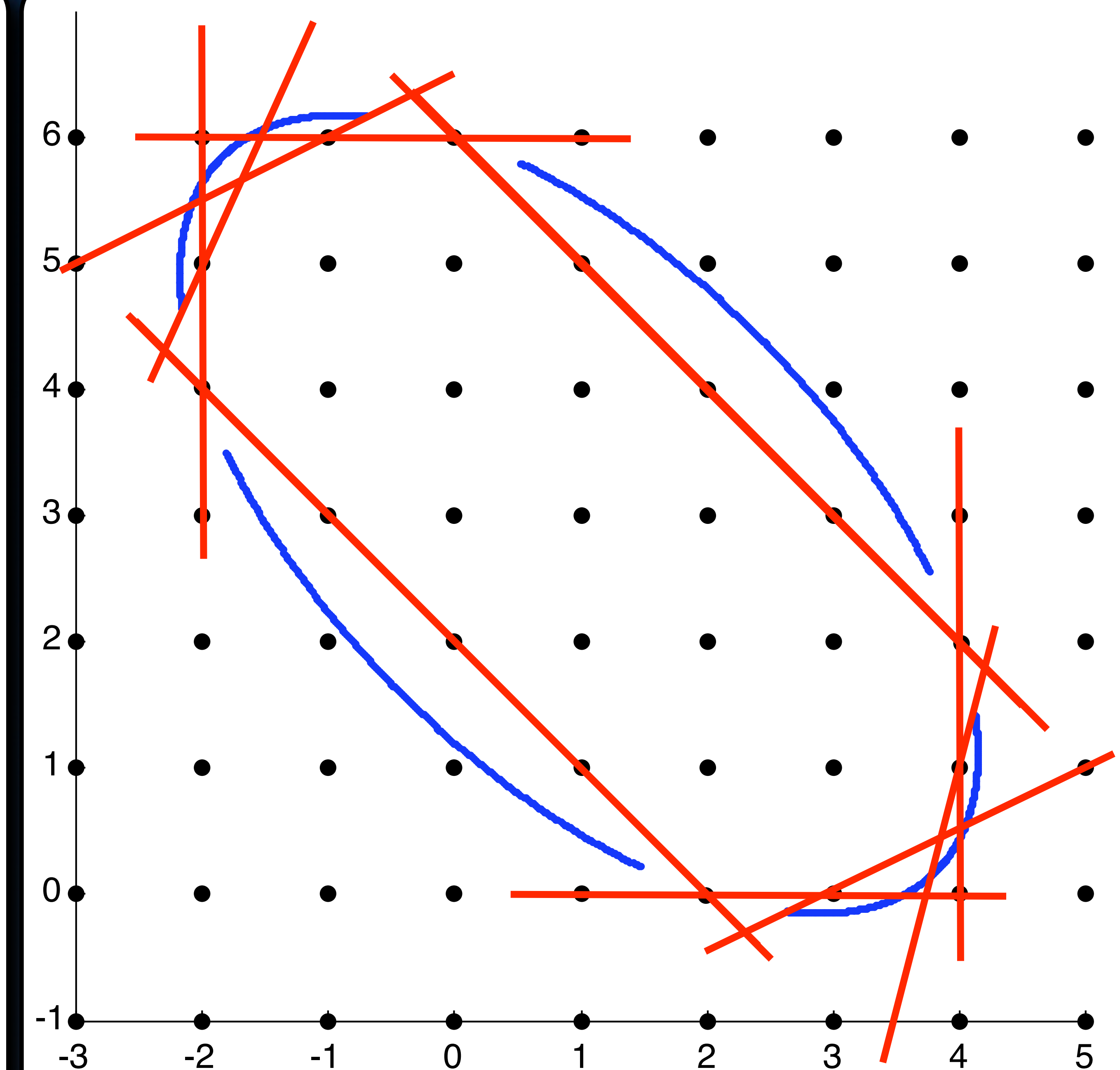

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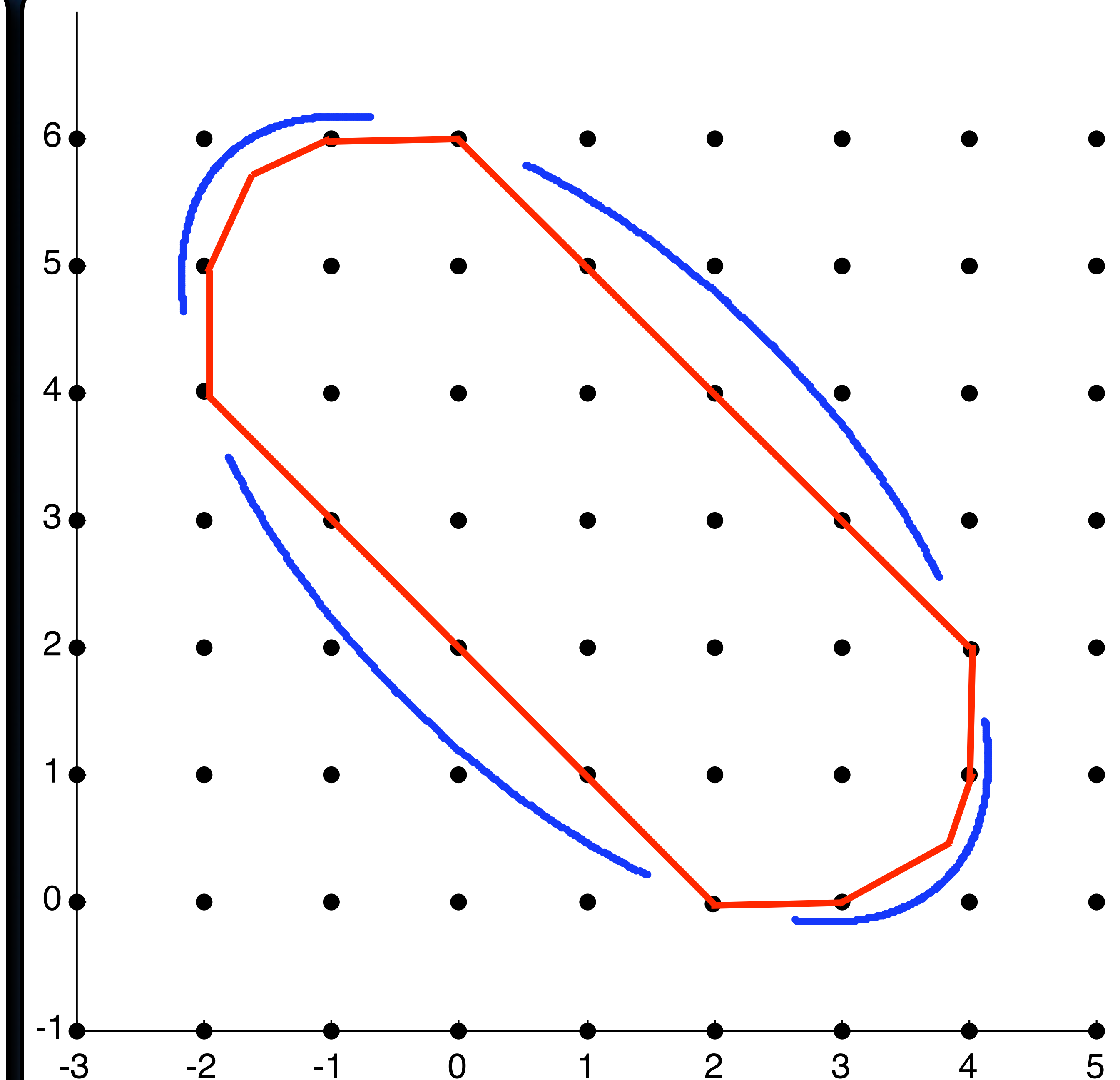

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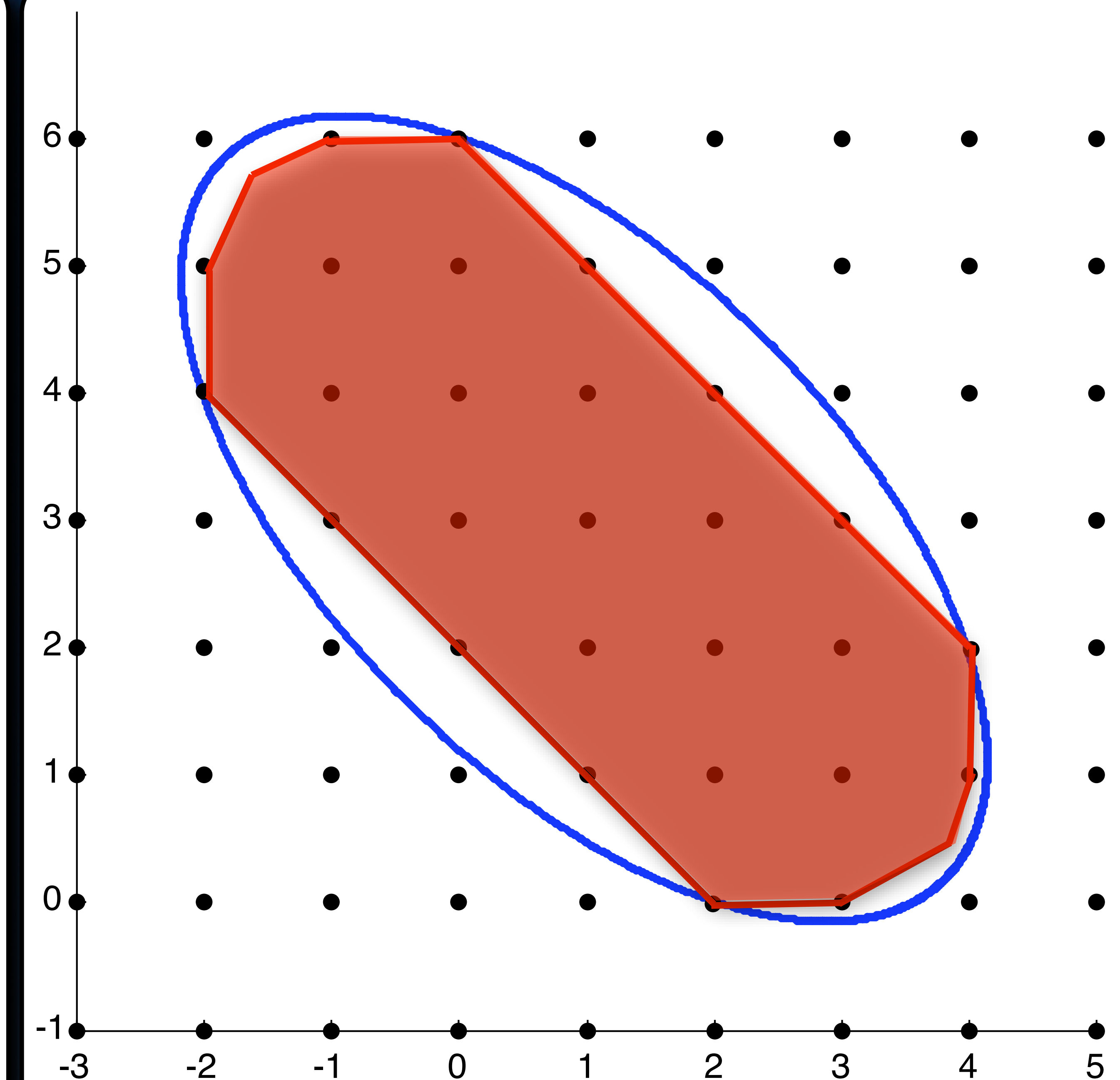

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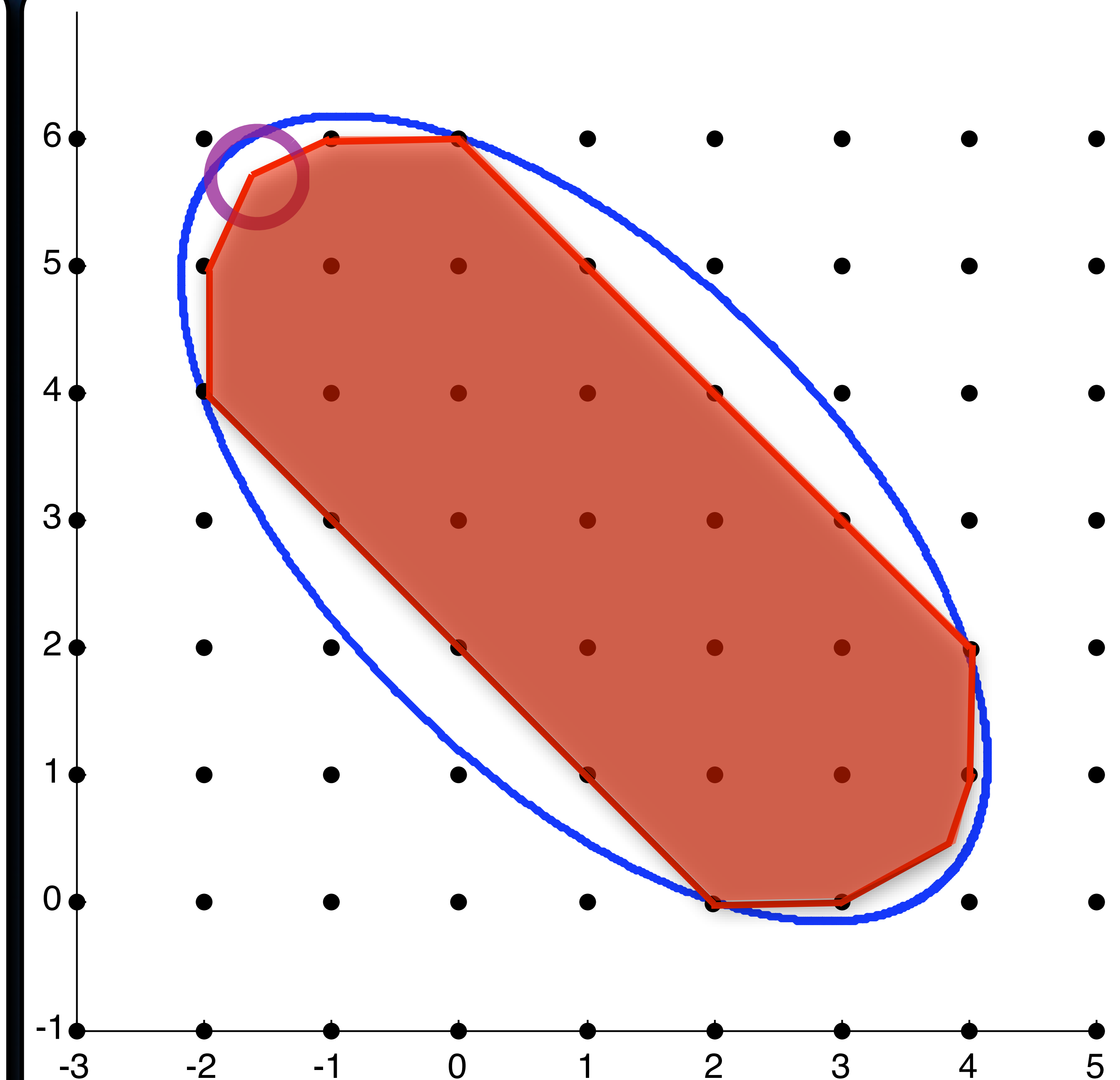

Compactness Argument

$$K := \mathrm{bd}(C) \setminus \bigcup_{v \in \mathrm{bd}(C) \cap \mathbb{Z}^n} \mathcal{N}_v$$

$$K \subset \bigcup_{i=1}^{m} \mathcal{S}_{u^i}$$

$$v \in \mathrm{bd}(C) \cap \mathbb{Z}^n$$

$$\mathtt{CGC}(I_v, C) \cap \mathcal{N}_v \cap \mathrm{bd}(C) = \{v\}$$

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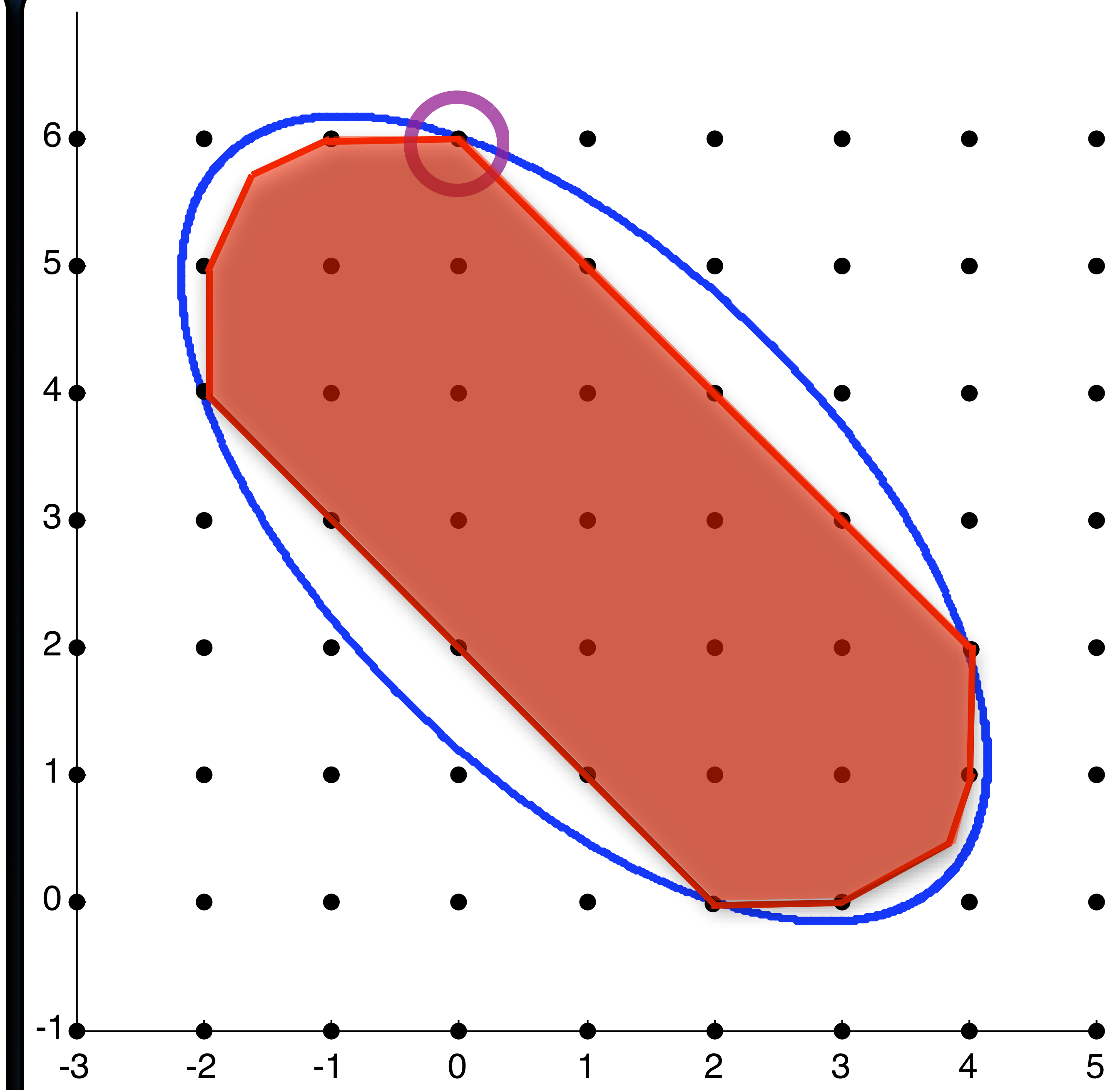
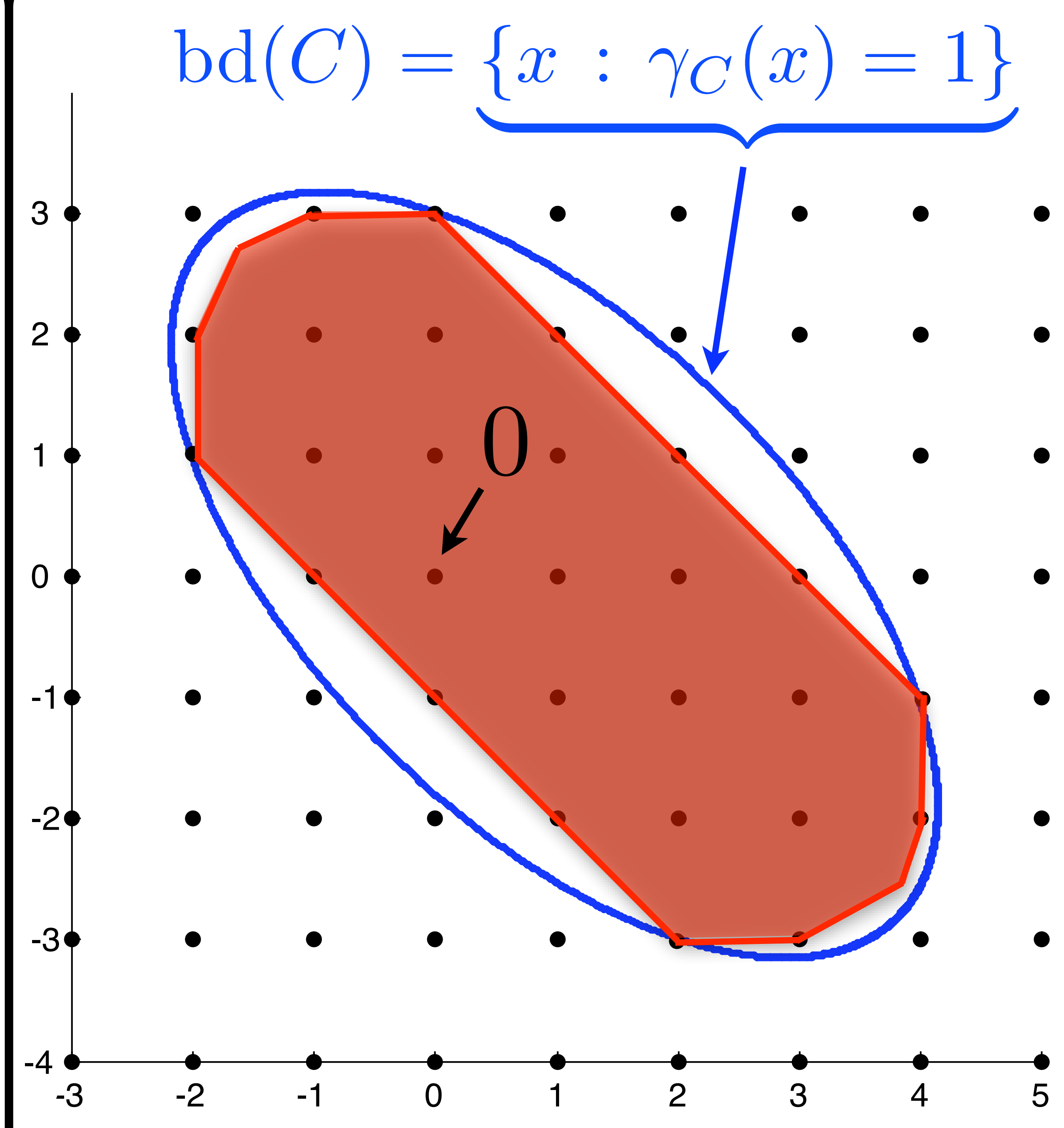


Step 2 : Separate $\mathtt{CGC}(S^1, C) \setminus \mathtt{CGC}(\mathbb{Z}^n, C)$



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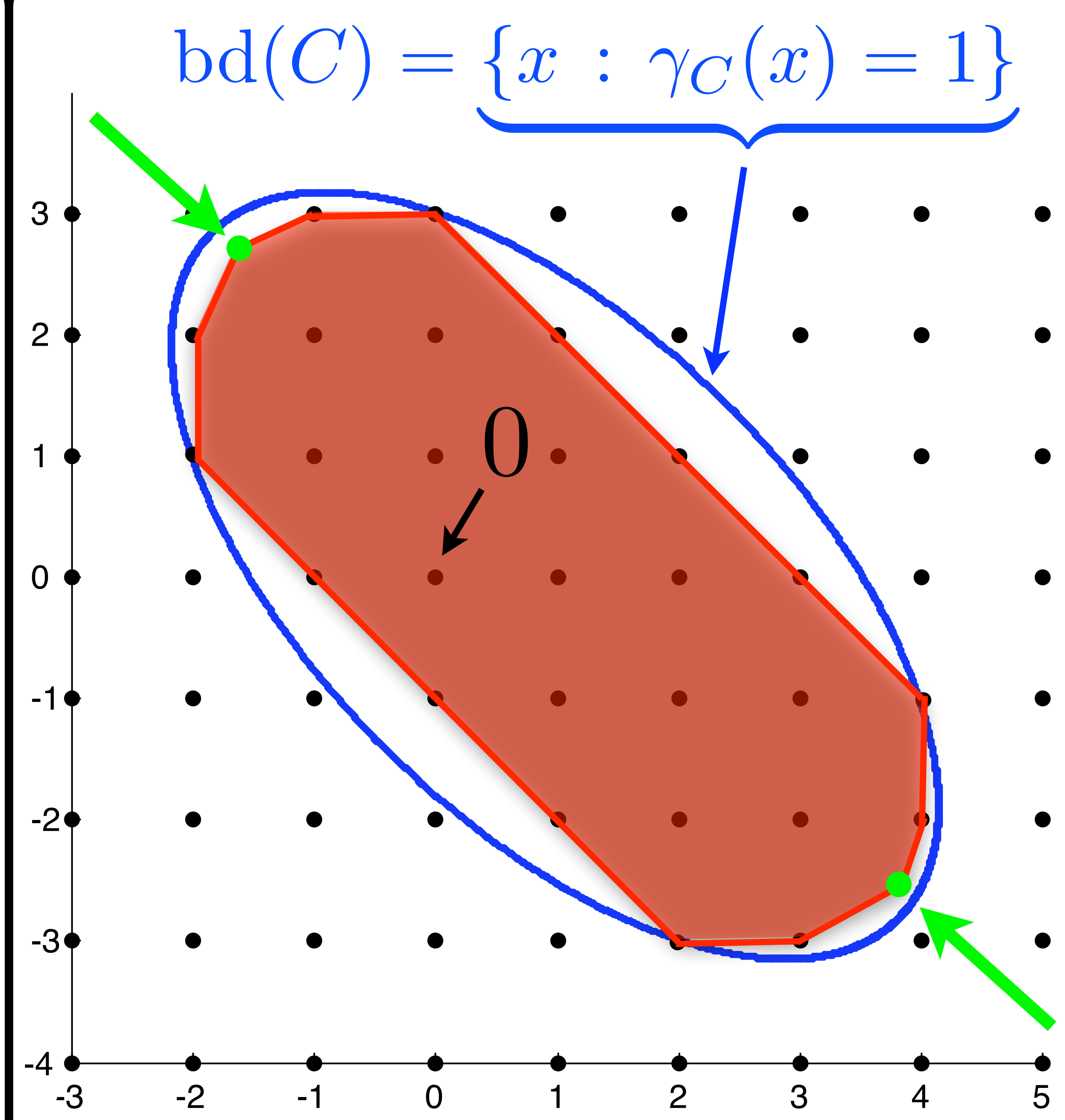
$$V := \mathrm{Ext}\left(\texttt{CGC}(S^1, C)\right) \setminus \mathbb{Z}^n$$



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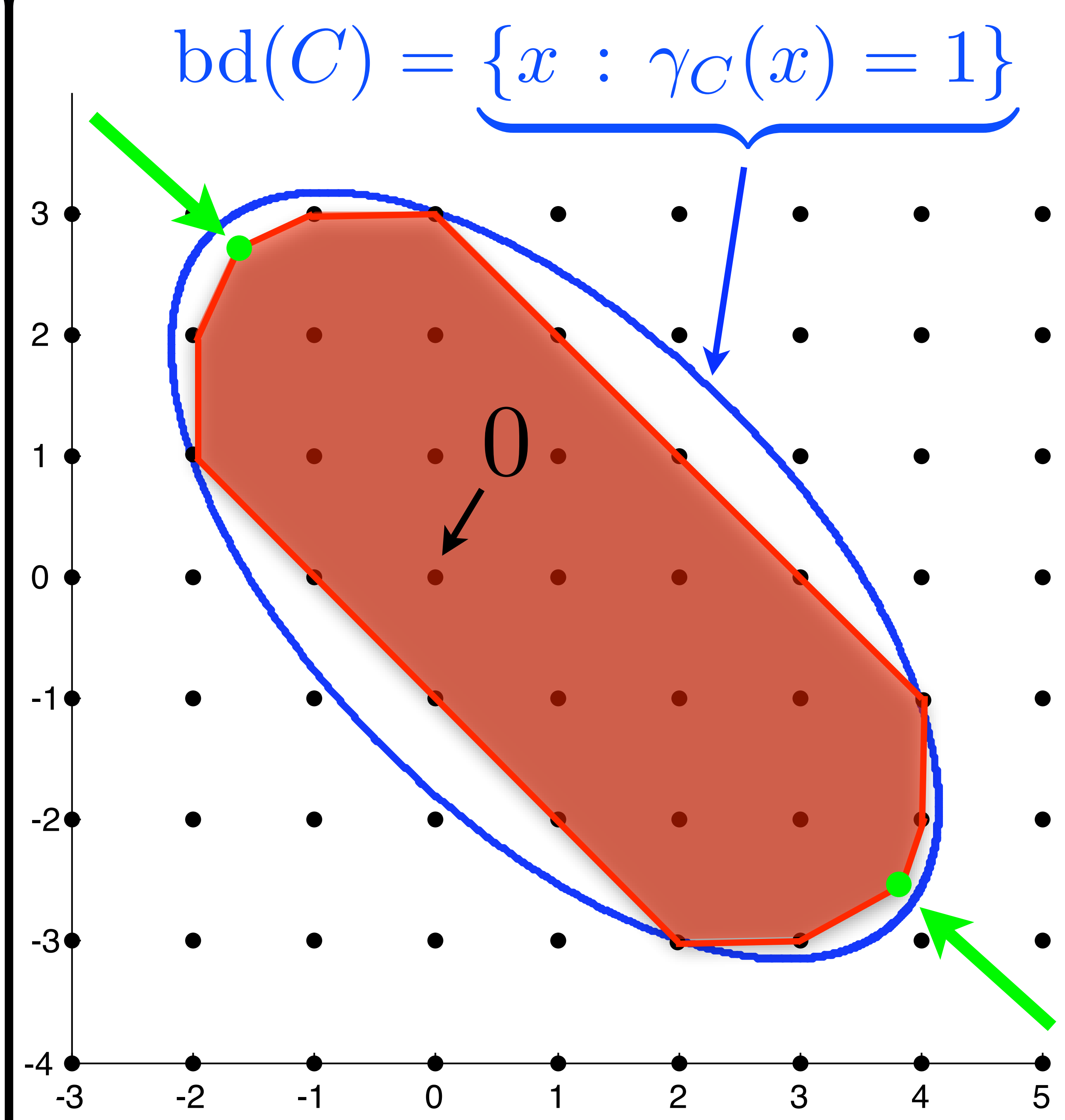

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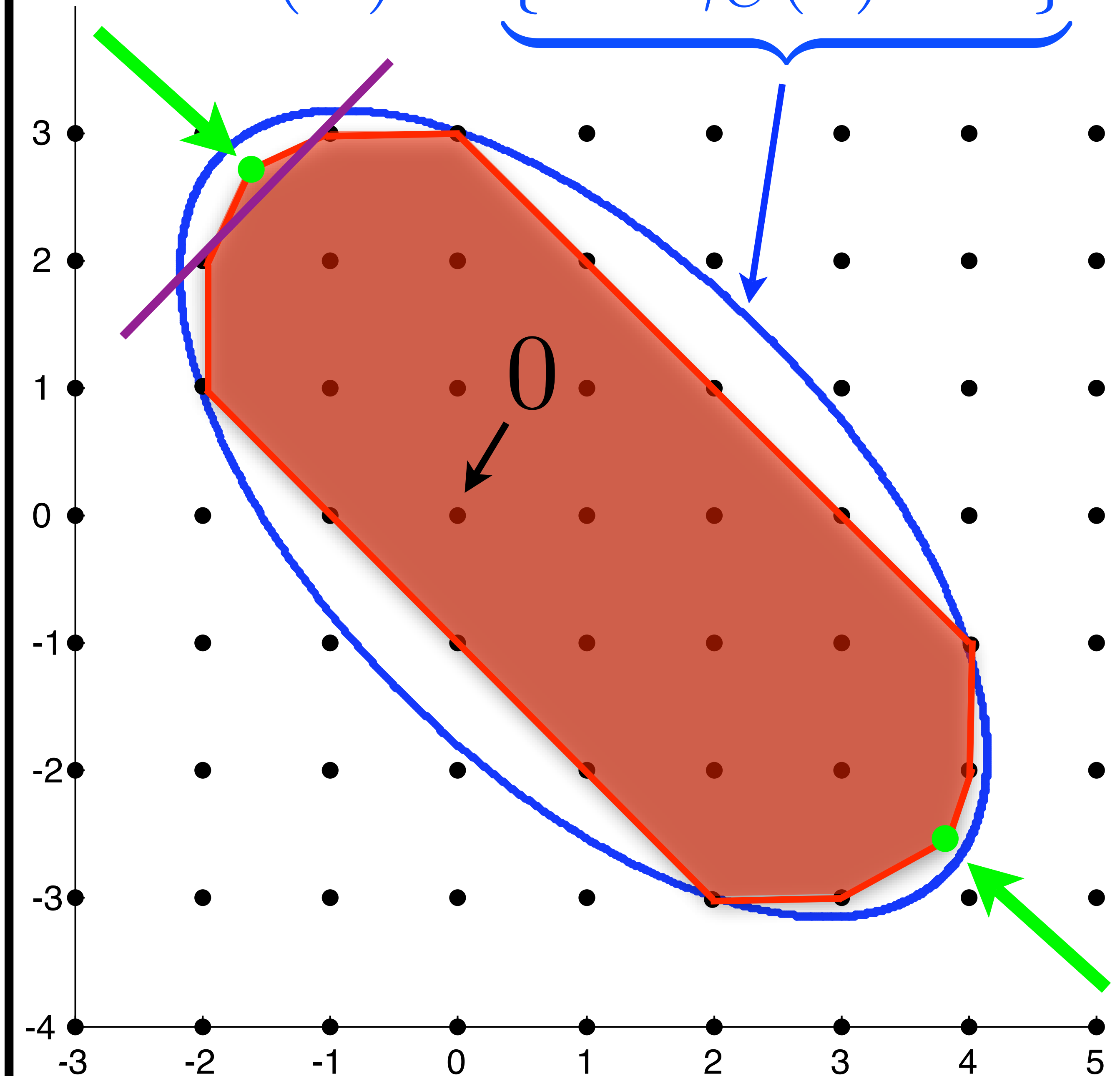

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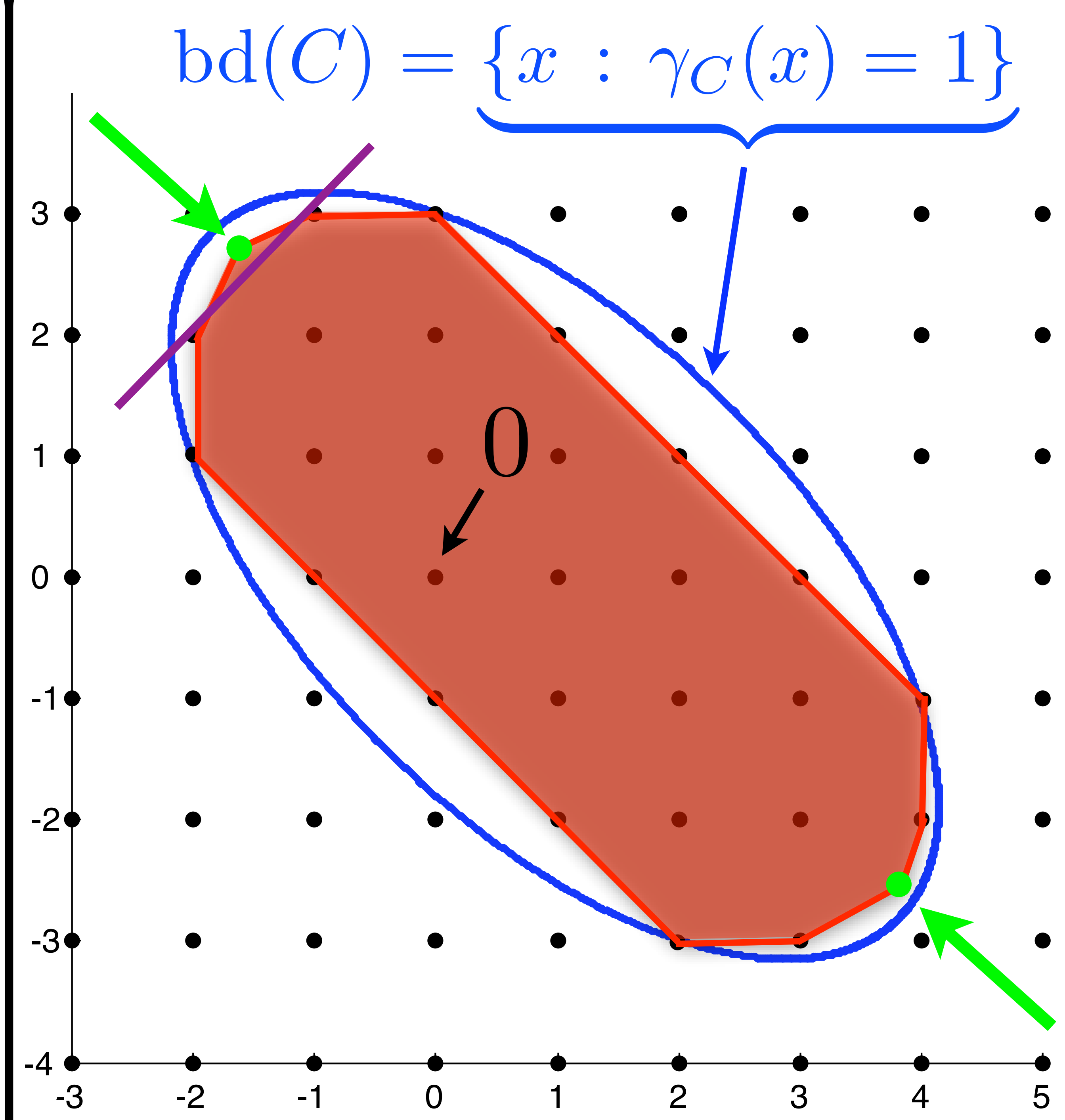
$$\Rightarrow \sigma_C(a) < \frac{1}{1 - \gamma_C(v)} \leq 1/\varepsilon$$

$$\Rightarrow a \in (1/\varepsilon)B^\circ$$



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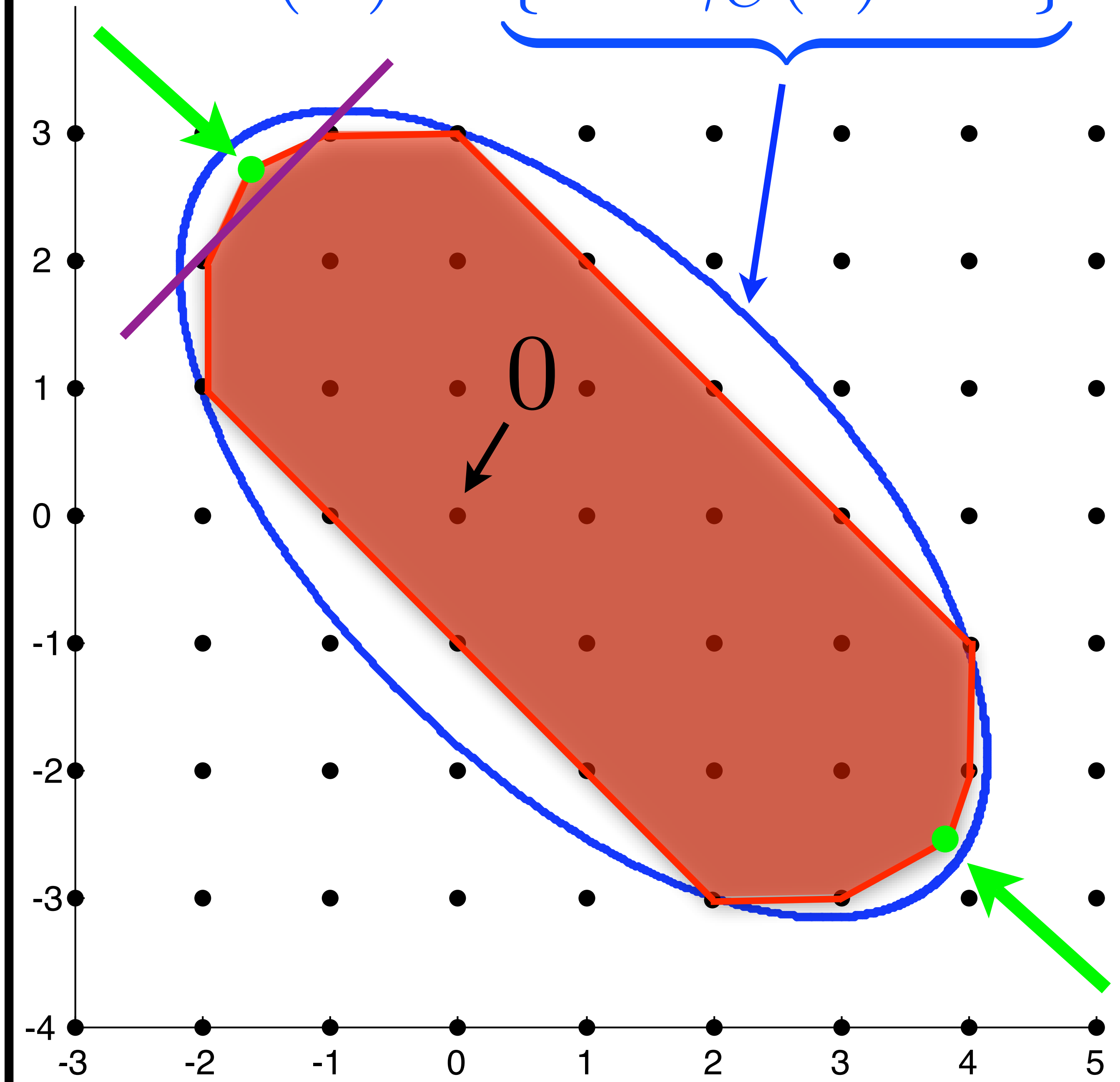

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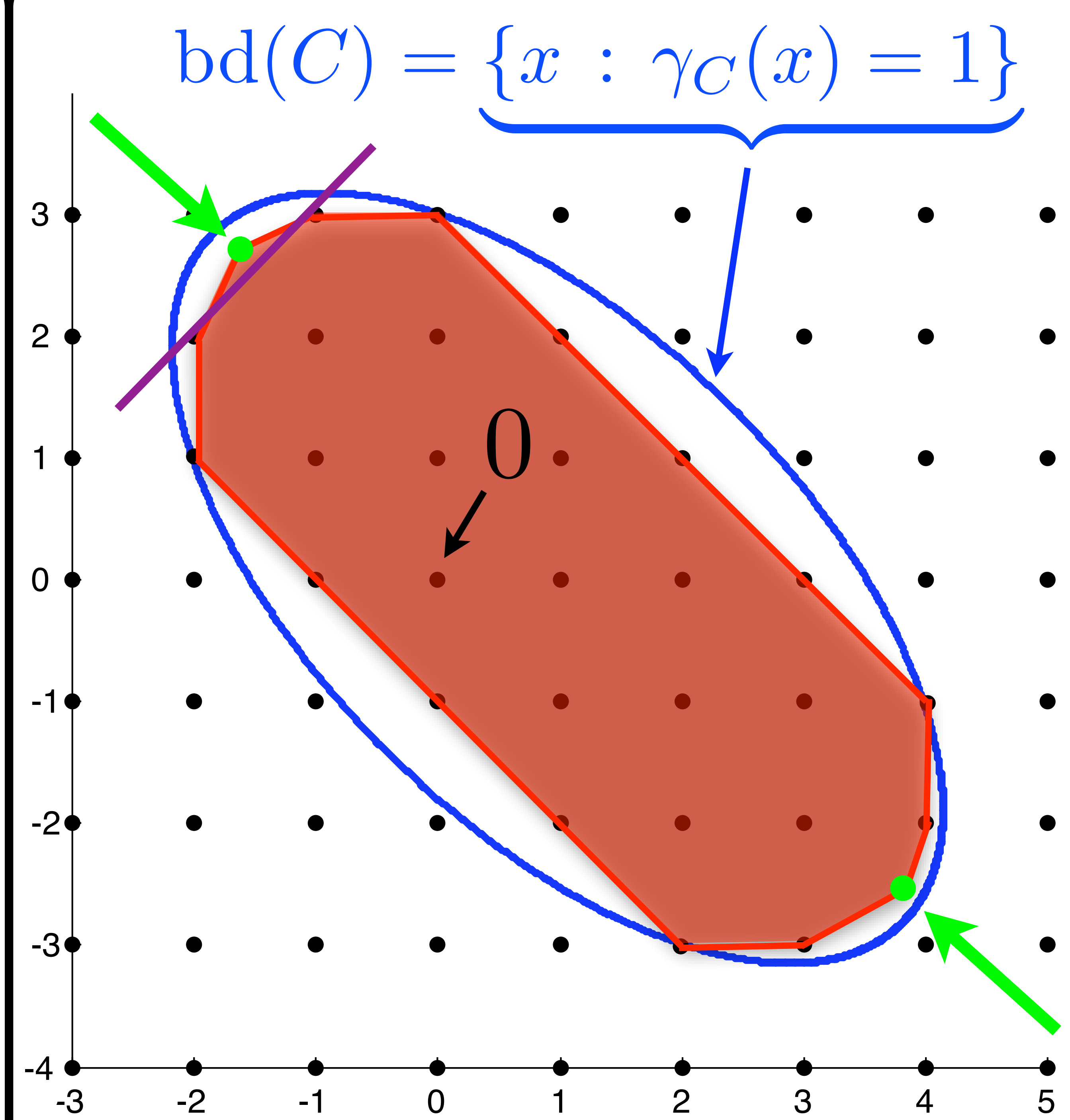
$$S^2 = (1/\varepsilon)B^\circ \cap \mathbb{Z}^n$$



Non-Integral Separation Outline

$$\frac{s^i}{\|s^i\|} \xrightarrow{i \to \infty} \frac{s(u)}{\|s(u)\|}, \qquad \lim_{i \to \infty} \langle s^i, u \rangle - \lfloor \sigma_C(s^i) \rfloor > 0$$

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- Let $s^i := p^i + e^l$ for $u_l \notin \mathbb{Z}^n$, then:
    - $0 \geq \langle s^i, u \rangle - \sigma_C(s^i) \xrightarrow{i \to \infty} 0$
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$$0 \geq \langle s^i, u \rangle - \sigma_C(s^i) \xrightarrow{i \to \infty} 0$$

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$$\begin{aligned}
\sigma_C\left(s^i\right) - \left\langle s^i, u\right\rangle &= \left\langle s^i, m_C\left(\bar{s}^i\right)\right\rangle - \left\langle s^i, m_C(\bar{s})\right\rangle \\
&= \left\|s^i\right\| \left(\left\langle \bar{s}^i, m_C\left(\bar{s}^i\right)\right\rangle - \left\langle \bar{s}^i, m_C(\bar{s})\right\rangle\right) \\
&\leq \left\|s^i\right\| \left(\left\langle \bar{s}^i, m_C\left(\bar{s}^i\right)\right\rangle - \left\langle \bar{s}^i, m_C(\bar{s})\right\rangle\right. \\
&\qquad \left. + \langle \bar{s}, m_C(\bar{s})\rangle - \left\langle \bar{s}, m_C\left(\bar{s}^i\right)\right\rangle\right) \\
&= \left\|s^i\right\| \left\langle \bar{s}^i - \bar{s}, m_C\left(\bar{s}^i\right) - m_C(\bar{s})\right\rangle \\
&\leq \left\|s^i\right\| \left\|\bar{s}^i - \bar{s}\right\| \left\|m_C(\bar{s}^i) - m_C(\bar{s})\right\| \xrightarrow{i\to\infty} 0
\end{aligned}$$

Separation Lemma

$$F\left(\sigma_C\left(s^i\right)\right) \xrightarrow{i\to\infty} F\left(u_l\right) > 0$$

$$\left\|p^i - q_i s(u)\right\| \xrightarrow{i\to\infty} 0,\ 0 \geq \langle s^i, u\rangle - \sigma_C(s^i) \xrightarrow{i\to\infty} 0,\ \langle s, u\rangle \in \mathbb{Z}^n$$

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$$\begin{aligned}
\sigma_C\left(s^i\right) &\geq \left\langle s^i, u\right\rangle = \left\langle p^i, u\right\rangle + \left\langle e^l, u\right\rangle \\
&= \left\langle p^i - q_i s, u\right\rangle + \langle q_i s, u\rangle + \left\langle e^l, u\right\rangle \\
&\geq -\left\|p^i - q_i s\right\| \|u\| + \langle q_i s, u\rangle + \left\langle e^l, u\right\rangle \\
&\geq \left\langle q_i s + e^l, u\right\rangle - \varepsilon = \langle q_i s, u\rangle + u_l - \varepsilon
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$$\sigma_C\left(s^i\right) \leq \langle q_i s, u\rangle + u_l + \varepsilon$$

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$$\sigma_C\left(s^i\right) \xrightarrow{i\to\infty} \langle q_i s, u\rangle + u_l$$

Conclusions and Future Work

- Only dependence on strict convexity:
  - Separation Lemma.
- Non-Constructive because of compactness argument in step 1.
- Current/Future work:
  - Intersection of strictly convex and rational polytope.
  - Conic Representable Sets.