Modeling and Solving Discrete Optimization Problems in Practice

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18.095 - Mathematics Lecture Series.

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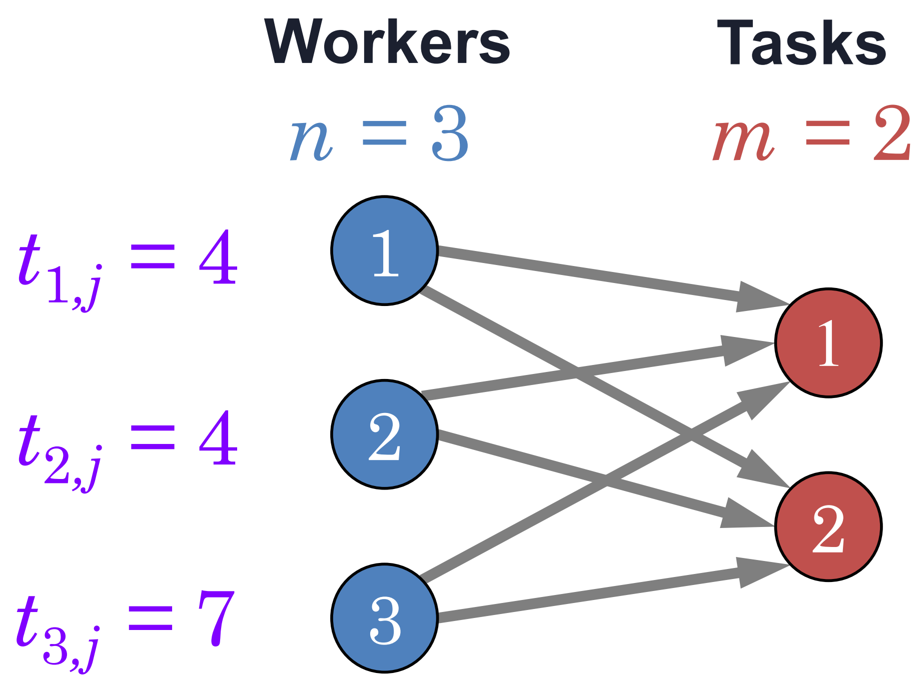
Combinatorial Example: Assignment Problem

- Assign n workers to m tasks to complete all tasks
- At most one task per worker
- Worker i takes $t_{i,j}$ hours to complete task j
- Minimize total time worked

- Graph:
  - Worker and task **nodes**
  - **Arcs** between worker and task **nodes**



Combinatorial Example: Assignment Problem

- Assign n workers to m tasks to complete all tasks
- At most one task per worker
- Worker i takes $t_{i,j}$ hours to complete task j
- Minimize total time worked
- Variables: $x_{i,j} = 1$ if worker i is assigned to task j and 0 o.w.

$$\min \quad \sum_{i=1}^{n} \sum_{j=1}^{m} t_{i,j} x_{i,j}$$

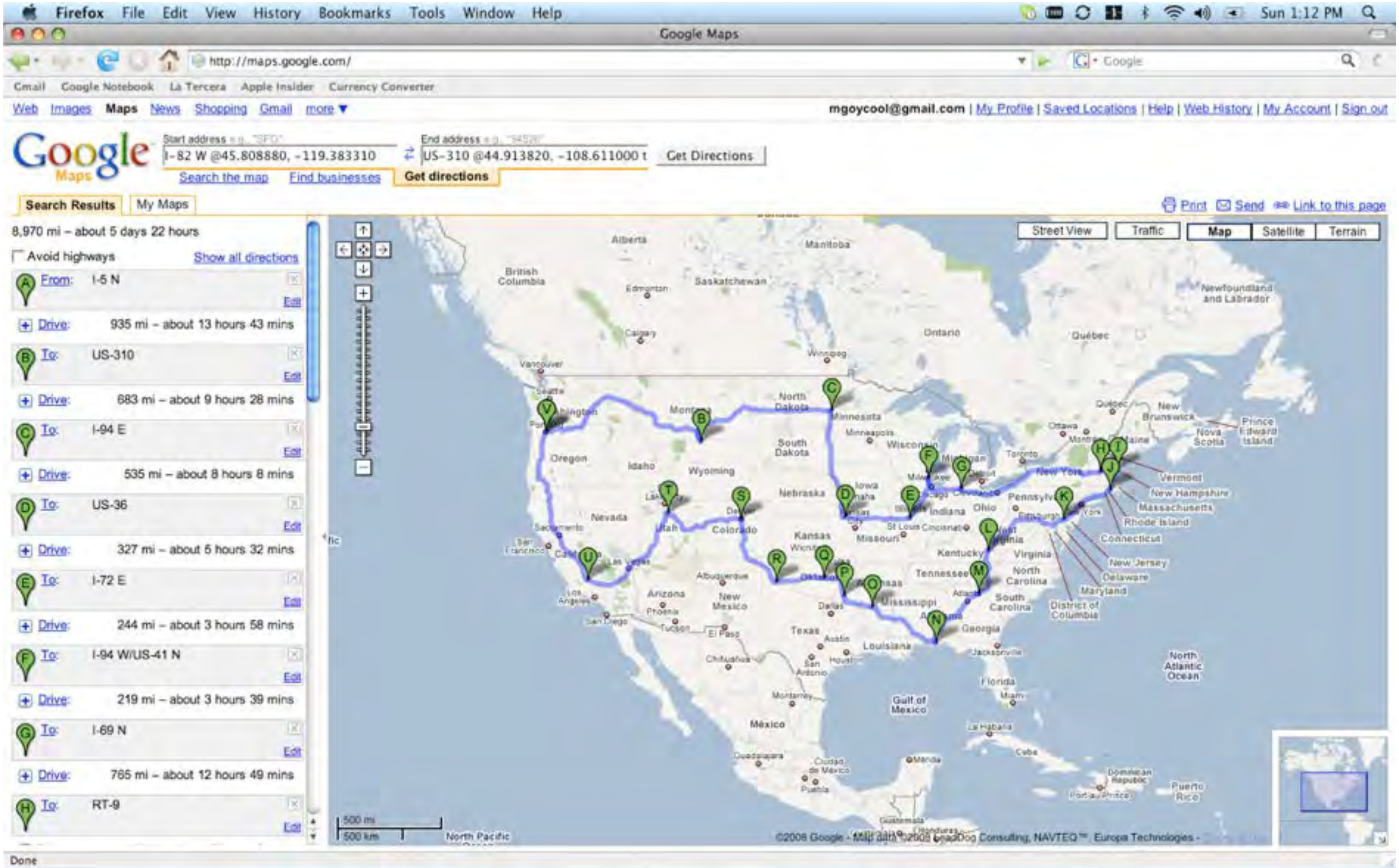
$$s.t.$$

$$\sum_{j=1}^{m} x_{i,j} \leq 1 \qquad \forall i \in \{1, \ldots, n\}$$ **Worker constraints**

$$\sum_{i=1}^{n} x_{i,j} \geq 1 \qquad \forall j \in \{1, \ldots, m\}$$ **Task constraints**

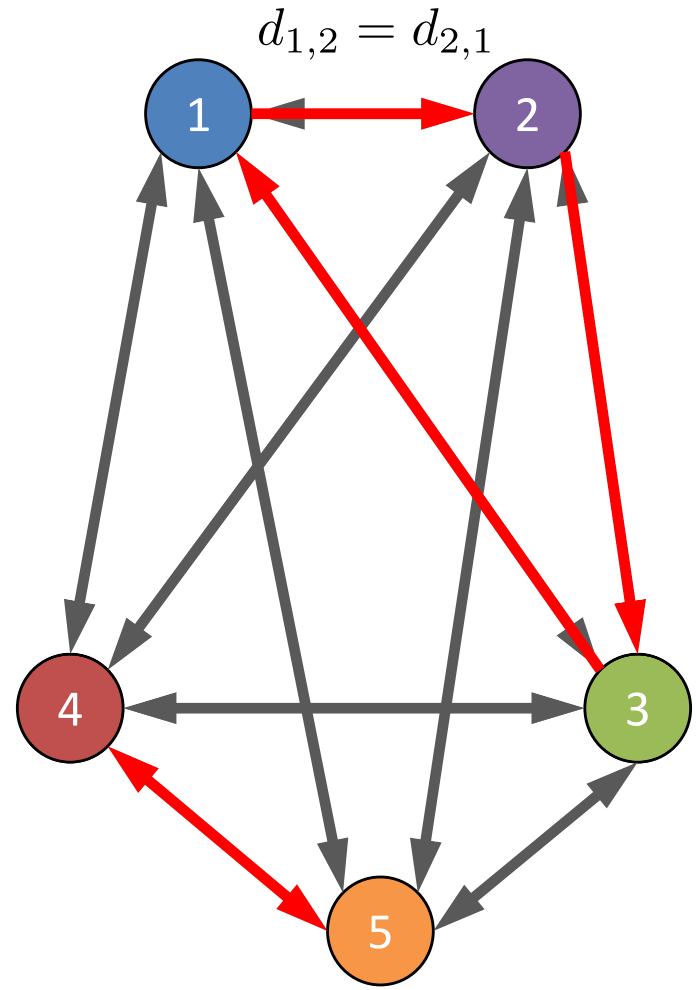
$$x_{i,j} \in \{0, 1\} \qquad \forall i \in \{1, \ldots, n\}, \; j \in \{1, \ldots, m\}$$

Traveling Salesman Problem : Visit all Cities Once



Formulation for Traveling Salesman Problem

$$[n] := \{1, \ldots, n\}$$



$$\min \sum_{i,j=1}^{n} d(i,j) x_{i,j}$$

$s.t.$

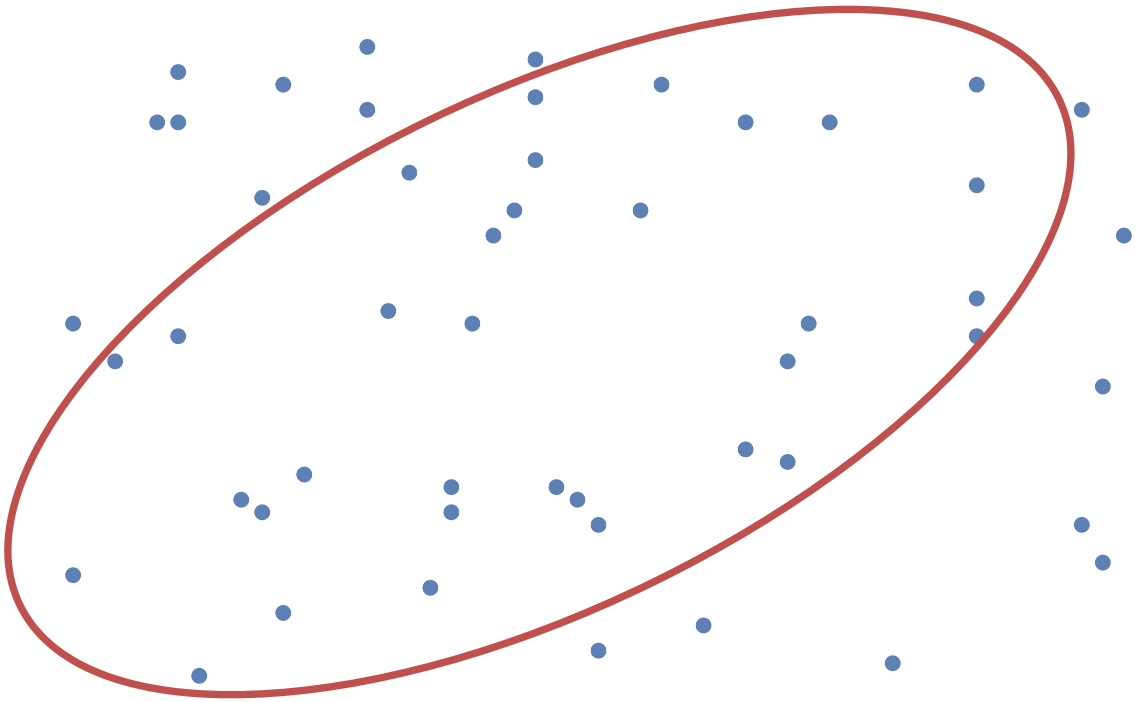
$$\sum_{j=1}^{n} x_{i,j} = 1 \qquad \forall i \in [n]$$
$$\sum_{i=1}^{n} x_{i,j} = 1 \qquad \forall j \in [n]$$
$$x_{i,i} = 0 \qquad \forall i \in [n]$$
$$x_{i,j} \in \{0,1\} \qquad \forall i,j \in [n]$$

Homework Question 1: Add missing constraints

Hint: You will need around 2^n inequalities

Mixed Integer Programming (MIP)

- Discrete and continuous variables or combinatorial constraints on continuous variables.
- Example: Find minimum volume ellipsoid that contains 90% of data points



MIP & Daily Fantasy Sports

The Greater Boston FOOD BANK

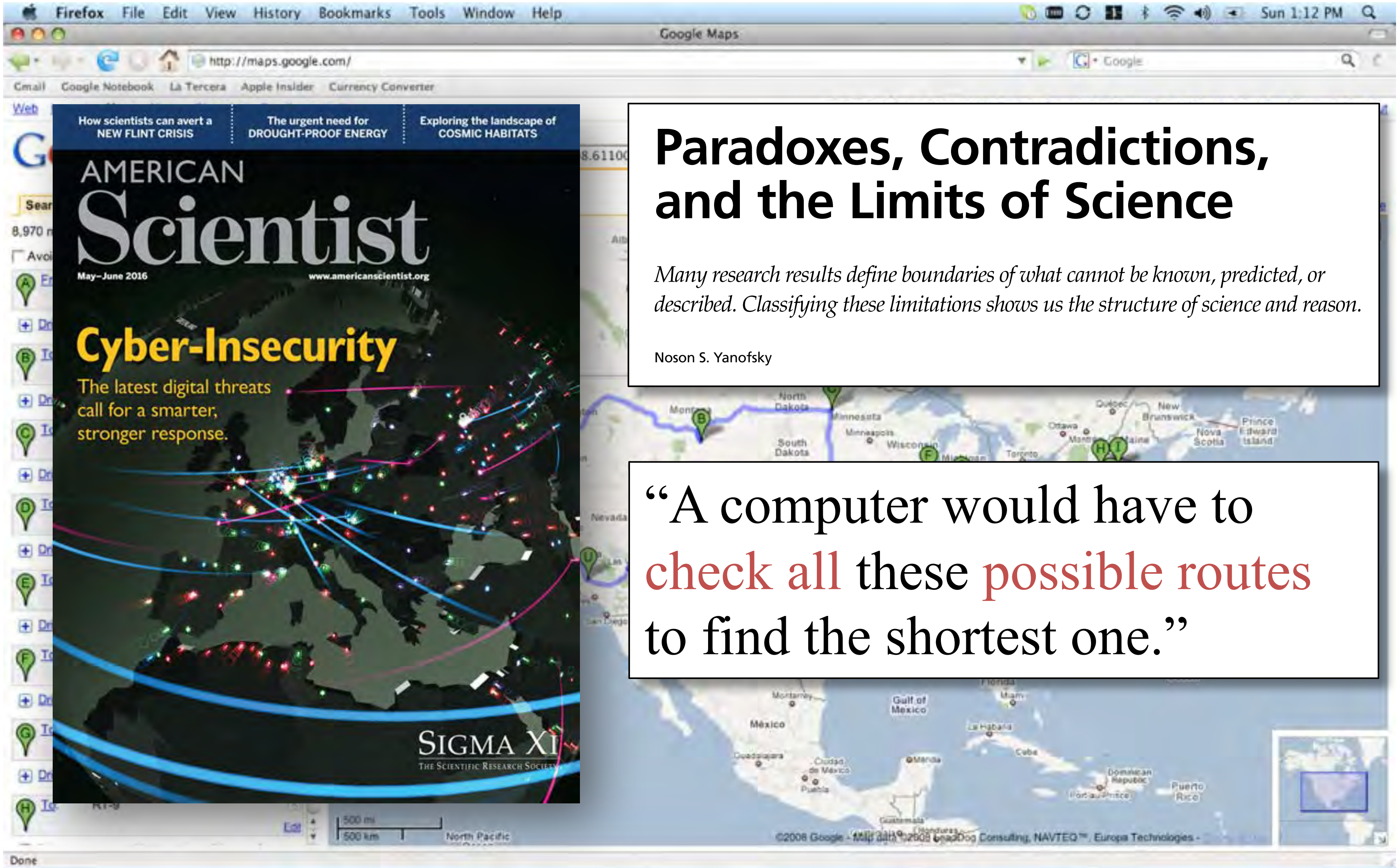
> $15K

Download Code from Github:

https://github.com/dscotthunter/Fantasy-Hockey-IP-Code

http://arxiv.org/pdf/1604.01455v1.pdf

How hard is MIP: Traveling Salesman Problem ?



"A computer would have to check all these possible routes to find the shortest one."

MIP = Avoid (Complete) Enumeration

- Number of tours for 49 cities $= 48!/2 \approx 10^{60}$
- Fastest supercomputer $\approx 10^{17}$flops
- Assuming one floating point operation per tour:
  $> 10^{35}$ years $\approx 10^{25}$ times the age of the universe!
- How long does it take on an iphone?
  - < 1 sec ! Dantzig, Fulkerson and Johnson ✍️ in 54'
  - Even theoretically hard MIPs "can" be solved:
    - Open-source solvers: GLPK, CBC, etc.
    - Commercial: Gurobi, CPLEX, etc.
    - Modeling Language: JuMP julia

Easy MIP through julia & JuMP

- julia : general purpose programming language
  - download https://julialang.org/downloads/ then click or run from command line
- JuMP : modeling language for optimization
- GNU GLPK : Open-source MIP solver
  - ```
    julia> Pkg.add("JuMP"); Pkg.add("GLPKMathProgInterface")
    ```
- Can also try JuliaBox on web
  - https://www.juliabox.com/
    ```

Easy MIP through julia & JuMP

- Assignment problem:

$$\min \quad \sum_{i=1}^{n} \sum_{j=1}^{m} t_{i,j} x_{i,j}$$

$$s.t.$$

$$\sum_{j=1}^{m} x_{i,j} \leq 1 \qquad \forall i \in \{1, \ldots, n\}$$

$$\sum_{i=1}^{n} x_{i,j} \geq 1 \qquad \forall j \in \{1, \ldots, m\}$$

$$x_{i,j} \in \{0,1\} \quad \forall i \in \{1, \ldots, n\},\ j \in \{1, \ldots, m\}$$

```
model = Model(solver=GLPKSolverMIP());
@variable(model, x[1:n,1:m], Bin);
@objective(model, Min, sum(t[i,j]*x[i,j] for i in 1:n, j in 1:m));
@constraint(model, [i=1:n], sum(x[i,j] for j in 1:m) <= 1);
@constraint(model, [j=1:m], sum(x[i,j] for i in 1:n) >= 1);
```

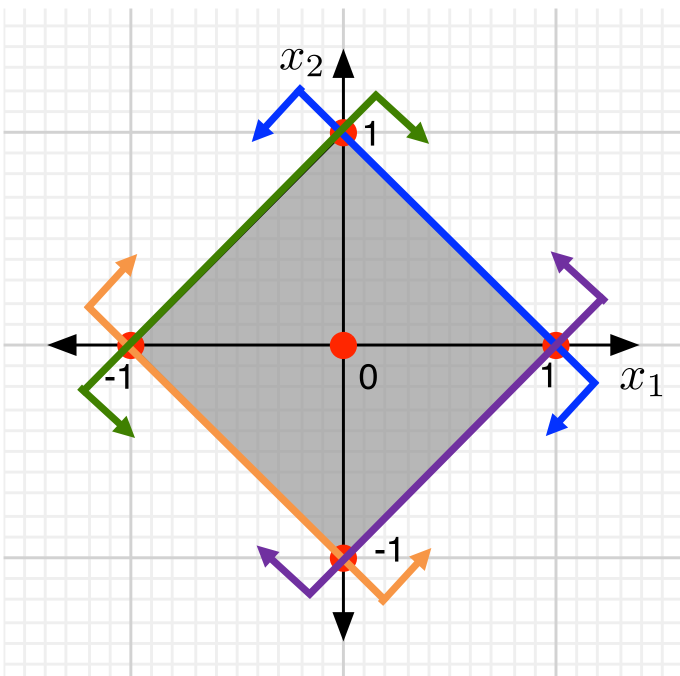
Homework Question 2: Solve problem with random cost

Complete file in website.

Solving MIPs: Step 1 = Linear Programming

$$\begin{aligned} \max \quad & x_2 \\ s.t. \quad & x_1 + x_2 \leq 1 \\ & -x_1 - x_2 \leq 1 \\ & +x_1 - x_2 \leq 1 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

← Linear Programming (LP) Relaxation



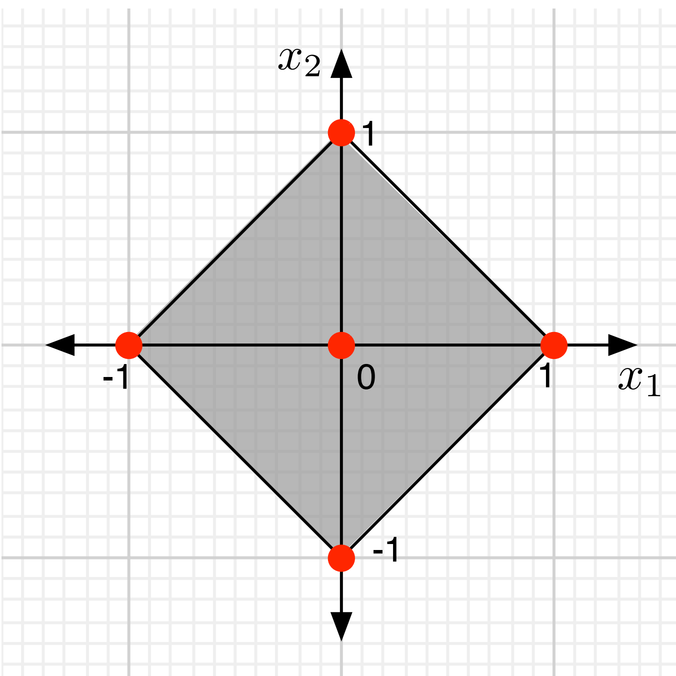
- Solving LPs is easy in theory and practice.
- One reason = LP duality
  - Suppose I guess optimum $x_1 = 0$ and $x_2 = 1$.
  - How do I prove that for all solutions of LP $x_2 \leq 1$?

$$\begin{array}{rl} & (1/2) \times (\ \ x_1 + x_2 \leq 1) \\ + & (1/2) \times (-x_1 + x_2 \leq 1) \\ \hline & x_2 \leq 1 \end{array}$$

Solving MIPs: Step 1 = Linear Programming

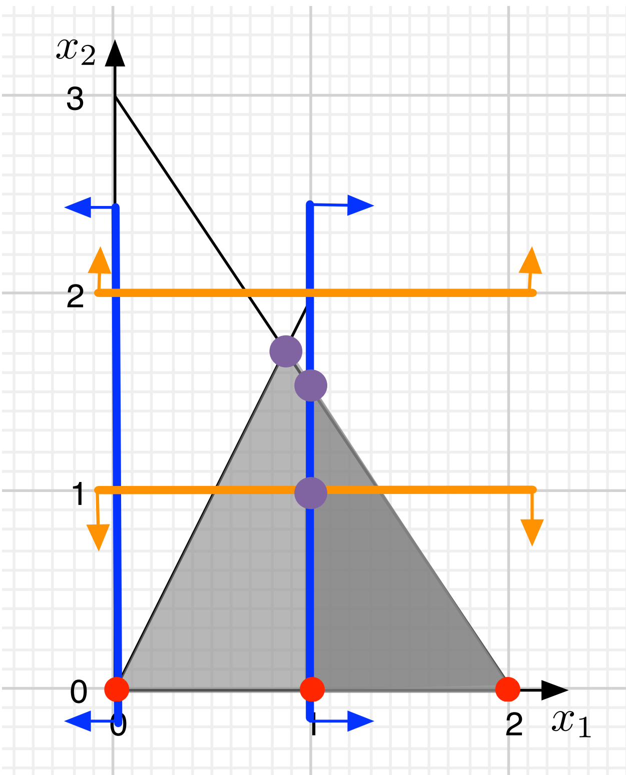
$$\begin{aligned} \max \quad & x_2 \\ s.t. \quad & x_1 + x_2 \leq 1 \\ & -x_1 - x_2 \leq 1 \\ & +x_1 - x_2 \leq 1 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

← Linear Programming (LP) Relaxation

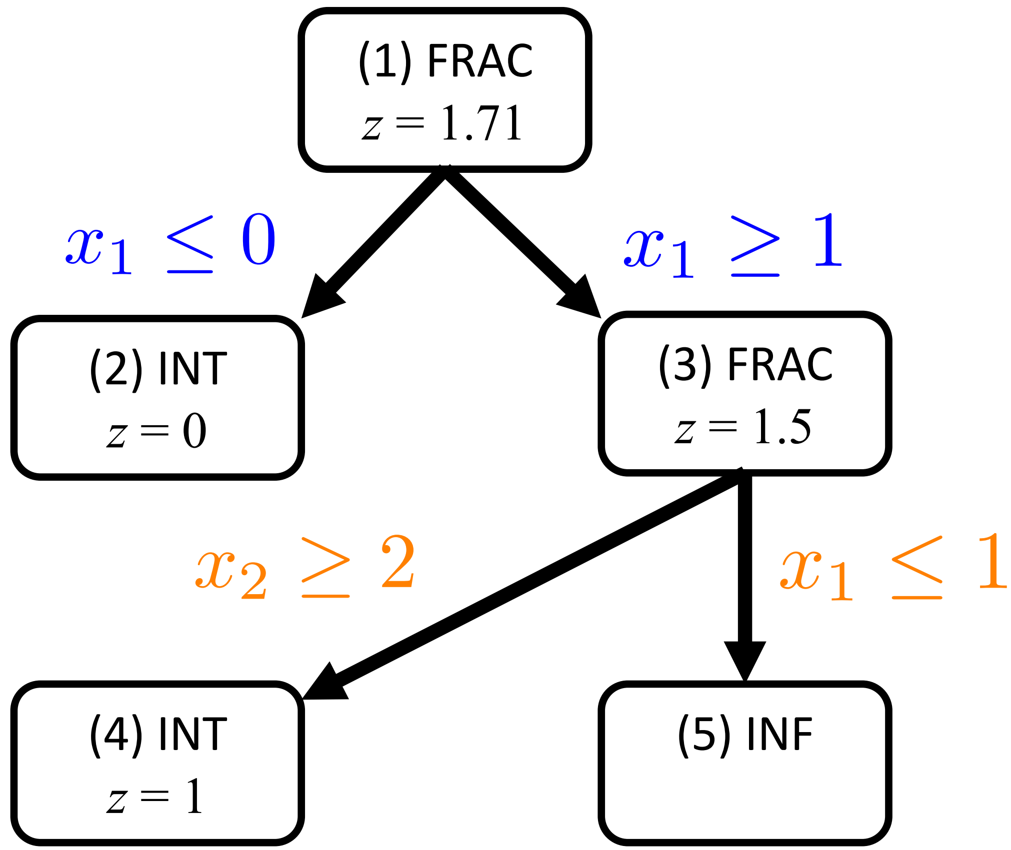


- LP relaxation always gives a (upper) bound on the MIP:
  - If solution of LP is "integer" then you solved the MIP
  - LP solvers return "corner" solution, which fixes "multiple optima" (e.g. $\max x_1 + x_2$)
  - Homework Question 3: Solve LP relaxation of assignment problem with JuMP. Is solution integer?

Solving MIPs: Step 2 = Branch-and-Bound



$$\max z := x_2$$
$$3x_1 + 2x_2 \le 6$$
$$-2x_1 + x_2 \le 0$$
$$x_1,\ x_2 \ge 0$$
$$x_1,\ x_2 \in \mathbb{Z}$$



← Linear Programming (LP) Relaxation

Homework Question 4:
Prove $x_2 \le 12 / 7$ for LP Relaxation.

Modern MIP Solvers = B&B++

- Really branch-and-cut:
  - Use cuts to improve LP relaxation.
- Elaborate heuristics: Rounding +++
- Preprocessing: fixing variables by logical implications.
- Advanced management of B&B tree.
- Extensive tuning of parameters and techniques.

Cutting Plane Example: Chátal-Gomory Cuts

$$P := \left\{ x \in \mathbb{R}^2 \;:\; \begin{array}{r} x_1 + x_2 \leq 3, \\ 5x_1 - 3x_2 \leq 3 \end{array} \right\}$$

$$\cap\!\!|$$

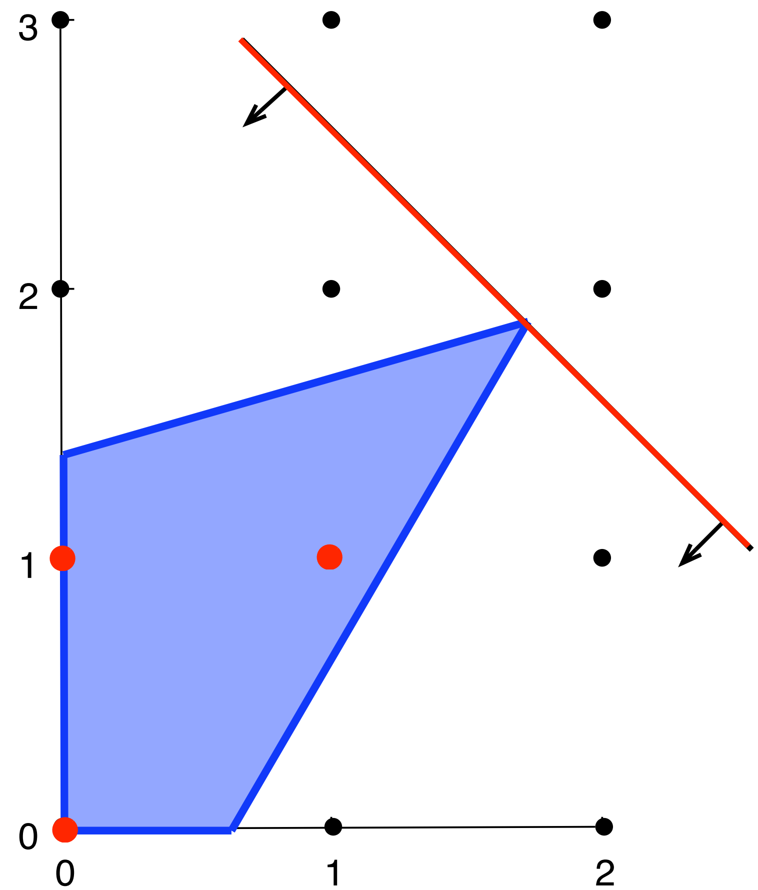
$$H := \{x \in \mathbb{R}^2 \;:\; \underbrace{4x_1 + 3x_2}_{\in \mathbb{Z}} \leq 10.5\}$$

if $x \in \mathbb{Z}^2$

$$4x_1 + 3x_2 \leq \lfloor 10.5 \rfloor$$

Valid for $H \cap \mathbb{Z}^2$

Valid for $P \cap \mathbb{Z}^2$

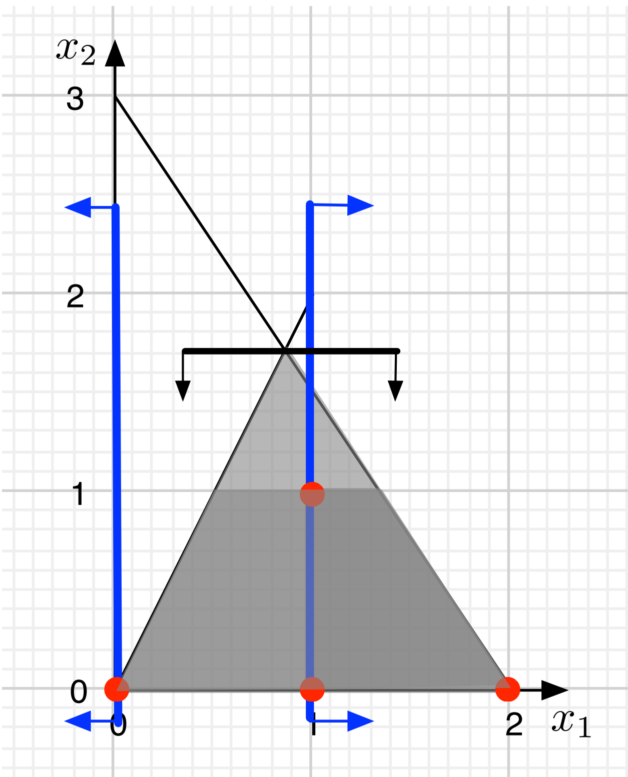


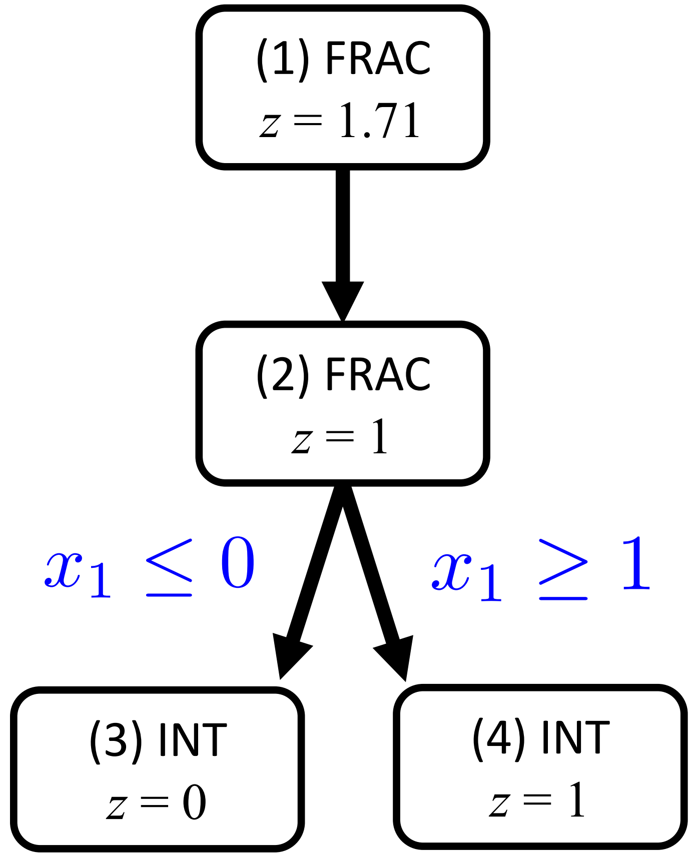
$$(27/8)(\;x_1 + \;x_2 \leq 3)$$

$$+\;(\;1/8)(5x_1 - 3x_2 \leq 3)$$

$$\Rightarrow 4x_1 + 3x_2 \leq 10.5$$

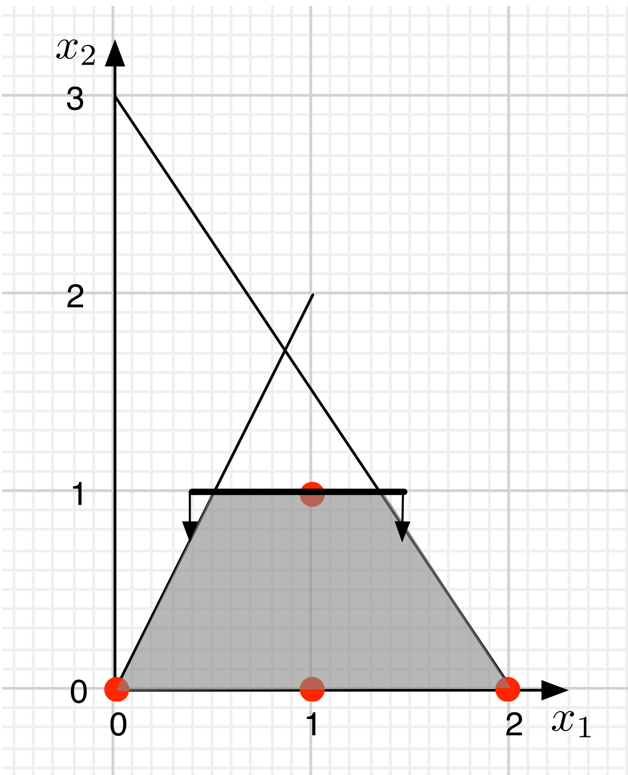
Branch-and-Bound and Cuts (Branch-and-Cut)





$$
\begin{aligned}
\max z &:= x_2 \\
3x_1 + 2x_2 &\leq 6 \quad x_2 \leq \lfloor 1.71 \rfloor = 1 \\
-2x_1 + x_2 &\leq 0 \\
x_1,\ x_2 &\geq 0 \\
x_1,\ x_2 &\in \mathbb{Z}
\end{aligned}
$$

Branch-and-Bound and Cuts (Branch-and-Cut)



$$\max z := x_2$$
$$3x_1 + 2x_2 \le 6 \quad x_2 \le \lfloor 1.71 \rfloor = 1$$
$$-2x_1 + x_2 \le 0$$
$$x_1,\ x_2 \ge 0$$
$$x_1,\ x_2 \in \mathbb{Z}$$

Homework Question 5:
Add two more Chátal-Gomory cuts so the LP relaxation with all cuts solves the MIP.

Hint:

$$\begin{array}{rl} & (1/3)(3x_1 + 2x_2 \le 6) \\ + & (1/3)(\qquad\quad x_2 \le 1) \\ \hline \end{array}$$

and

$$\begin{array}{rl} & (1/2)(-2x_1 + x_2 \le 0) \\ + & (1/2)(\qquad\quad x_2 \le 1) \\ \hline \end{array}$$

No Enumeration = Keep Adding Cuts

- Number of tours for 49 cities $= 48!/2 \approx 10^{60}$
- Fastest supercomputer $\approx 10^{17}$flops
- Assuming one floating point operation per tour:
  $> 10^{35}$ years $\approx 10^{25}$ times the age of the universe!
- How long does it take on an iphone?
  - < 1 sec ! Dantzig, Fulkerson and Johnson ✍ in 54'
  - This is how DFJ solved the problem by hand in 54'
  - In practice Branch-and-Cut is better.
  - More details in Concord TSP App
    - Cutting plane tutorial for TSP
    - http://www.math.uwaterloo.ca/tsp/iphone/

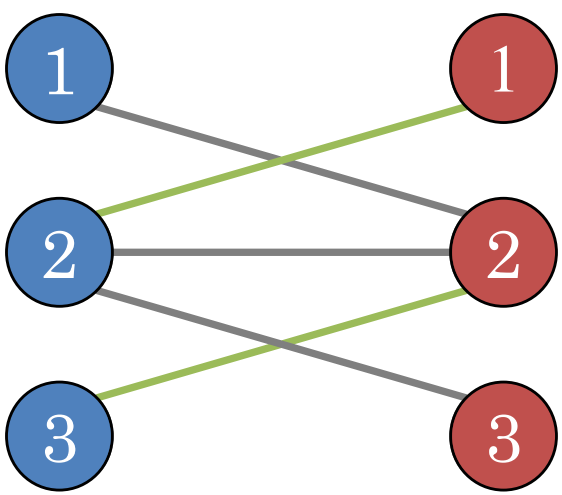
Easy Problems : LP Relaxation Always Integral

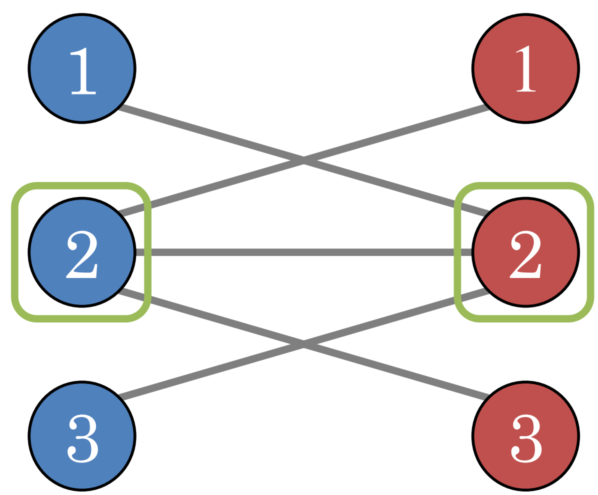
Consequence of LP duality: Kőnig's theorem

- Largest Matching
  - Pick edges, at most one edge per node

=

- Smallest Node Cover
  - Pick nodes that touch all edges





Classes and Links

- julia, JuMP and Optimization
  - https://github.com/JuliaOpt/JuMP.jl
  - http://www.juliaopt.org
- 15.053 Optimization Methods in Business Analytics
  - Modeling and computation
  - Instructor: James B. Orlin
  - Spring 2018: http://mit.edu/15.053/www/
- 18.453 Combinatorial Optimization
  - Theory and algorithms
  - Instructor: Michel Goemans
  - Spring **2017** : http://www-math.mit.edu/~goemans/18453S17/18453.html

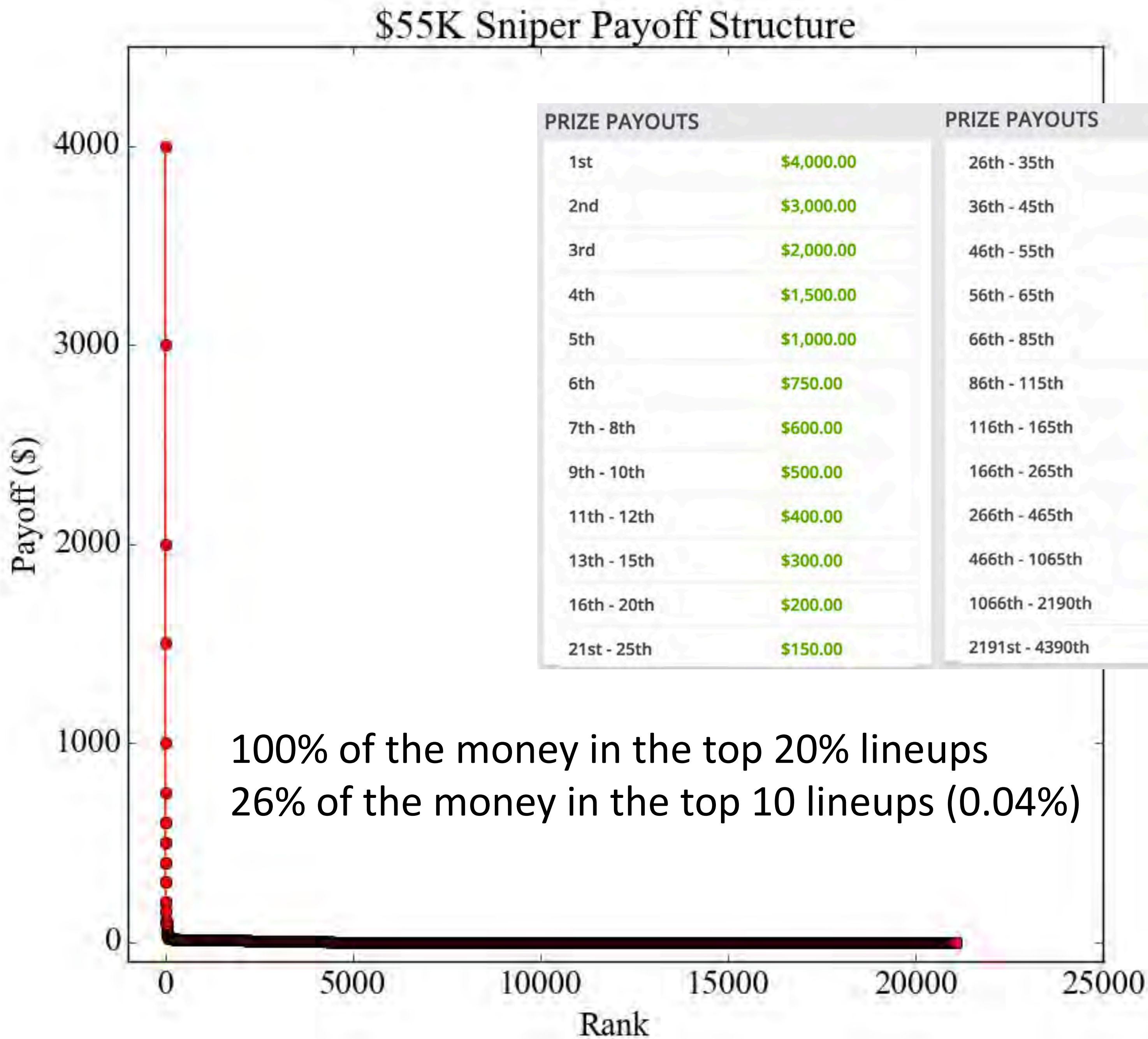
MIP & Daily Fantasy Sports

Example Entry

LINEUP

Avg. Rem. / Player: $0
Rem. Salary: $0

POS	PLAYER	OPP	FPPG	SALARY
C	Jussi Jokinen	Fla@Anh	3.1	$5,300
C	Brandon Sutter	Pit@Van	3.0	$4,400
W	Nikolaj Ehlers	Wpg@Tor	3.9	$4,800
W	Daniel Sedin	Pit@Van	3.8	$6,400
W	Radim Vrbata	Pit@Van	3.4	$5,800
D	Brian Campbell	Fla@Anh	2.6	$4,100
D	Morgan Rielly	Wpg@Tor	3.5	$4,200
G	Corey Crawford P	StL@Chi	6.3	$7,800
UTIL	Blake Wheeler	Wpg@Tor	4.8	$7,200



PRIZE PAYOUTS	
1st	$4,000.00
2nd	$3,000.00
3rd	$2,000.00
4th	$1,500.00
5th	$1,000.00
6th	$750.00
7th - 8th	$600.00
9th - 10th	$500.00
11th - 12th	$400.00
13th - 15th	$300.00
16th - 20th	$200.00
21st - 25th	$150.00

PRIZE PAYOUTS	
26th - 35th	$100.00
36th - 45th	$75.00
46th - 55th	$50.00
56th - 65th	$40.00
66th - 85th	$30.00
86th - 115th	$25.00
116th - 165th	$20.00
166th - 265th	$15.00
266th - 465th	$12.00
466th - 1065th	$10.00
1066th - 2190th	$8.00
2191st - 4390th	$6.00

100% of the money in the top 20% lineups
26% of the money in the top 10 lineups (0.04%)

Building a Lineup

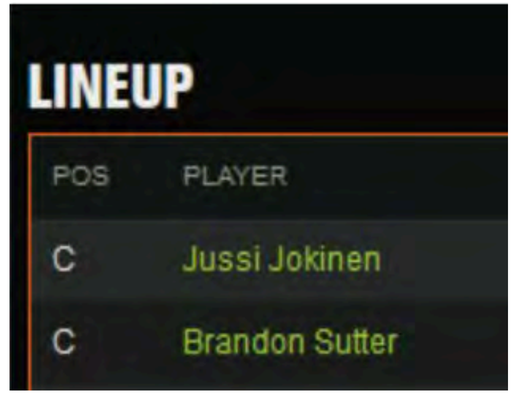
MIP Formulation

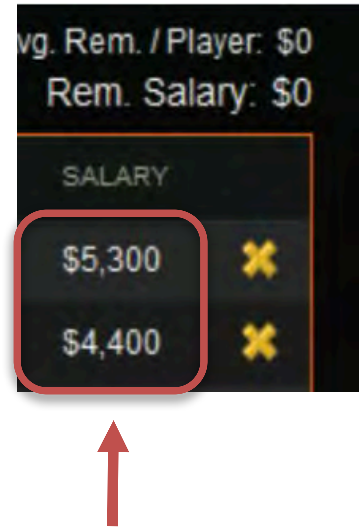
- L lineups : indexed by l
- 9 players per lineup: indexed by p
- Decision variables

$$x_{pl} = \begin{cases} 1, & \text{if player } p \text{ in lineup } l \\ 0, & \text{otherwise} \end{cases}$$

Basic Feasibility

- Basic constraints:
  - 9 different players
  - Salary less than $50,000





c_p

$$\sum_{p=1}^{N} c_p x_{pl} \leq \$50,000, \quad \text{(budget constraint)}$$

$$\sum_{p=1}^{N} x_{pl} = 9, \quad \text{(lineup size constraint)}$$

$$x_{pl} \in \{0, 1\}, \quad 1 \leq p \leq N.$$

Position Feasibility

- Between 2 and 3 centers
- Between 3 and 4 wingers
- Between 2 and 3 defensemen
- 1 goalie

Position constraints

$$2 \leq \sum_{p \in C} x_{pl} \leq 3, , \quad \text{(center constraint)}$$

$$3 \leq \sum_{u \in W} x_{pl} \leq 4, \quad \text{(winger constraint)}$$

$$2 \leq \sum_{u \in D} x_{pl} \leq 3, \quad \text{(defensemen constraint)}$$

$$\sum_{u \in G} x_{pl} = 1 \quad \text{(goalie constraint)}$$

Team Feasibility

- At least 3 different NHL teams

Team constraints

$$t_i \leq \sum_{p \in T_i} x_{pl}, \quad \forall\, i \in \{1,\ \ldots\ , N_T\}$$

$$\sum_{i=1}^{N_T} t_i \geq 3,$$

$$t_i \in \{0,\ 1\}\,, \quad \forall\, i \in \{1,\ \ldots\ , N_T\}\,.$$

Maximize Points

- Forecasted points for player p: f_p

RotoGrinders
The Daily Fantasy Games Authority

dfnerd

Score type	Points
Goal	3
Assist	2
Shot on Goal	0.5
Blocked Shot	0.5
Short Handed Point Bonus (Goal/Assist)	1
Shootout Goal	0.2
Hat Trick Bonus	1.5
Win (goalie only)	3
Save (goalie only)	0.2
Goal allowed (goalie only)	-1
Shutout Bonus (goalie only)	2

Table 1 **Points system for NHL contests in DraftKings.**

Points Objective Function

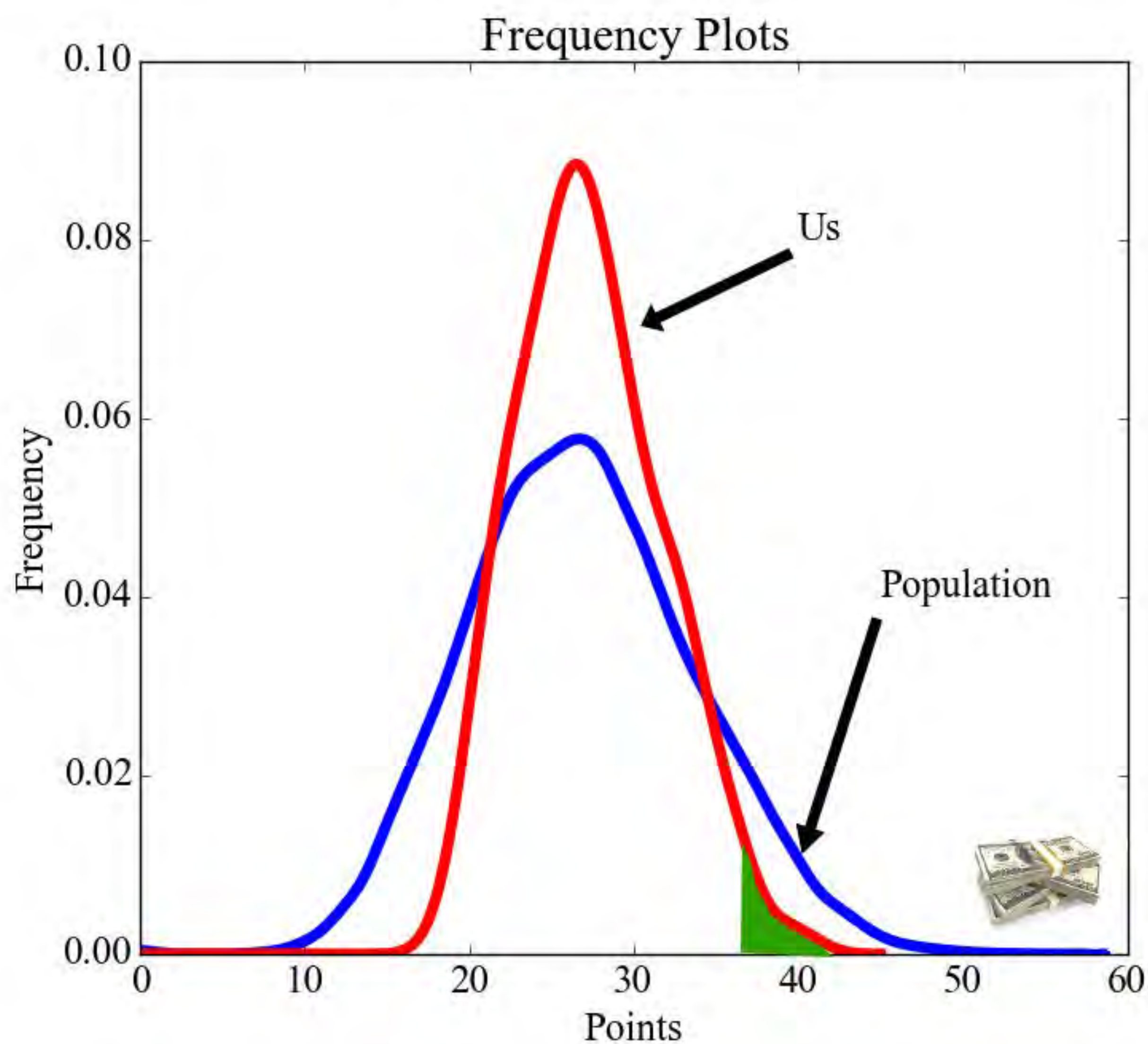
$$\sum_{p=1}^{N} f_p x_{pl}$$

Lineup

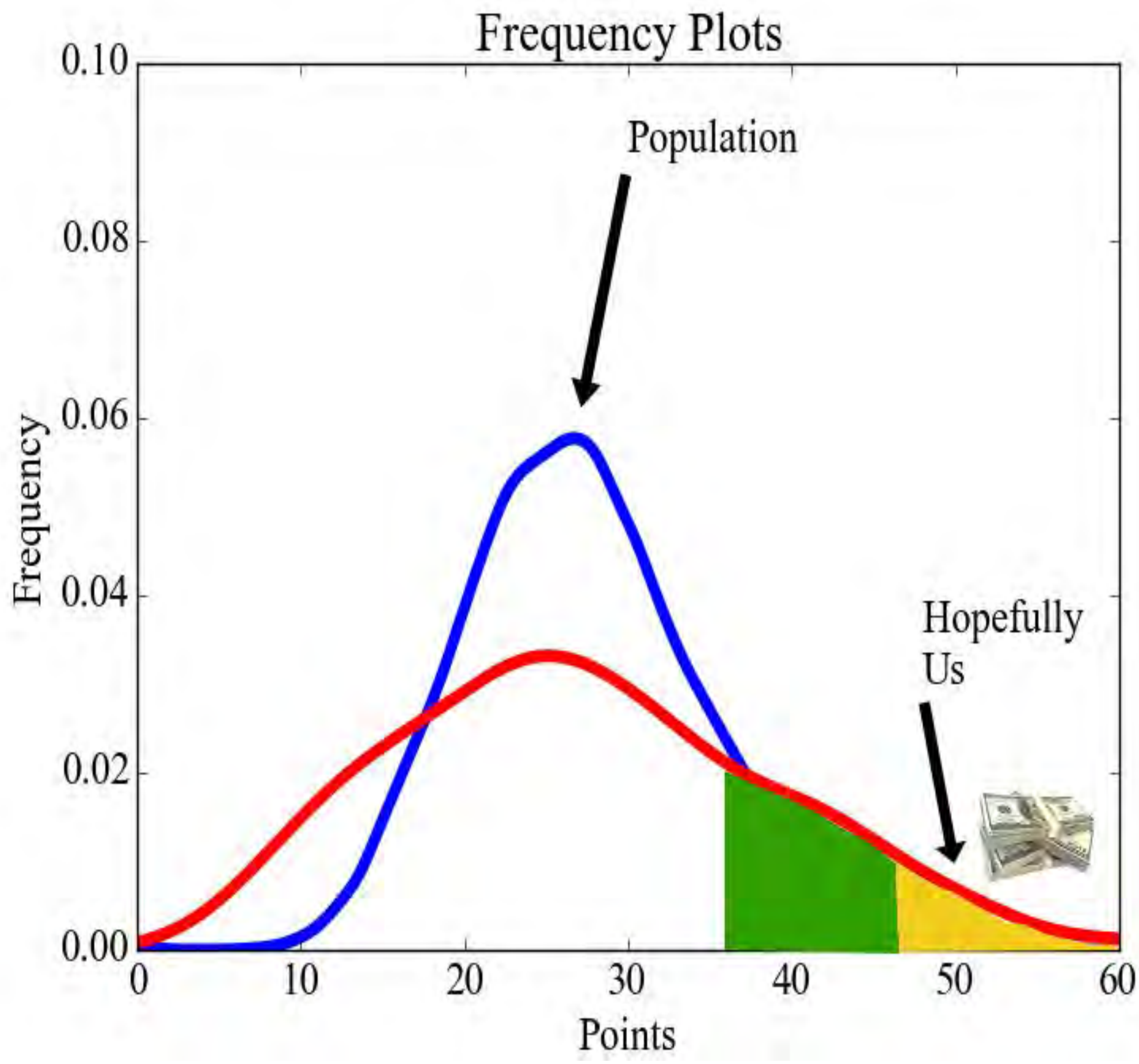
Projections:	5.4	2.5	3.4	3.0	3.2	4.2	3.5	3.4	5.7
	$9500	$2700	$4600	$3800	$4600	$6400	$5200	$5100	$8000
	W	UTIL	D	D	C	C	W	W	G

23 points on average

Need > 38 points for a chance to win



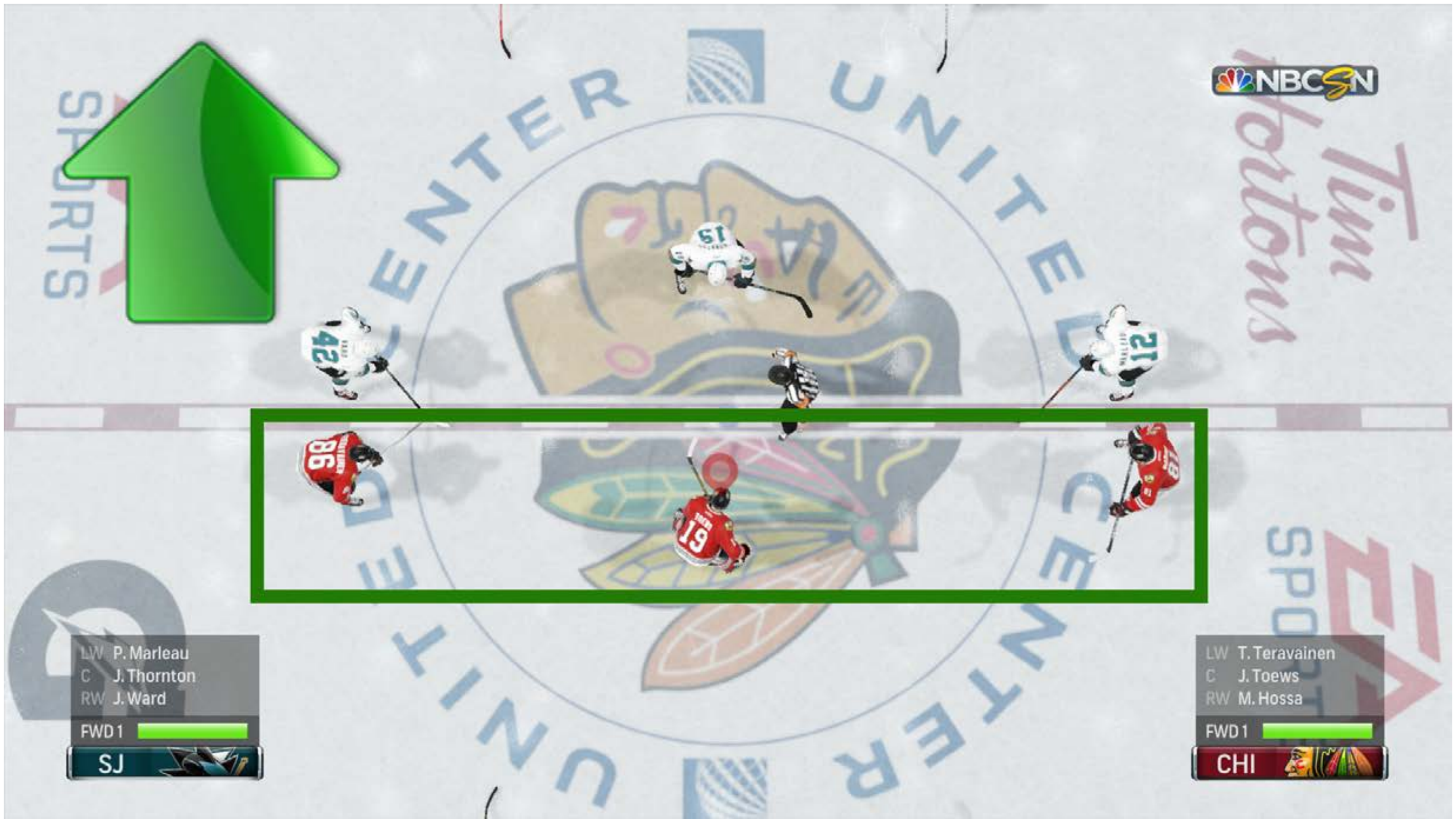
Increase variance to have a chance



Structural Correlations : Teams

Structural Correlations : Lines

- Goal = 3 pt, assist = 2 pt



Structural Correlations : Lines = Stacking

- At least 1 complete line (3 players per line)
- At least 2 partial lines (at least 2 players per line)

1 complete line constraint

$$3v_i \leq \sum_{p \in L_i} x_{pl}, \qquad \forall i \in \{1, \ldots, N_L\}$$

$$\sum_{i=1}^{N_L} v_i \geq 1$$

$$v_i \in \{0, 1\}, \qquad \forall i \in \{1, \ldots, N_L\}.$$

2 partial lines constraint

$$2w_i \leq \sum_{p \in L_i} x_{pl}, \qquad \forall i \in \{1, \ldots, N_L\}$$

$$\sum_{i=1}^{N_L} w_i \geq 2$$

$$w_i \in \{0, 1\}, \qquad \forall i \in \{1, \ldots, N_L\}.$$

Structural Correlations :
Goalie Against Opposing Players

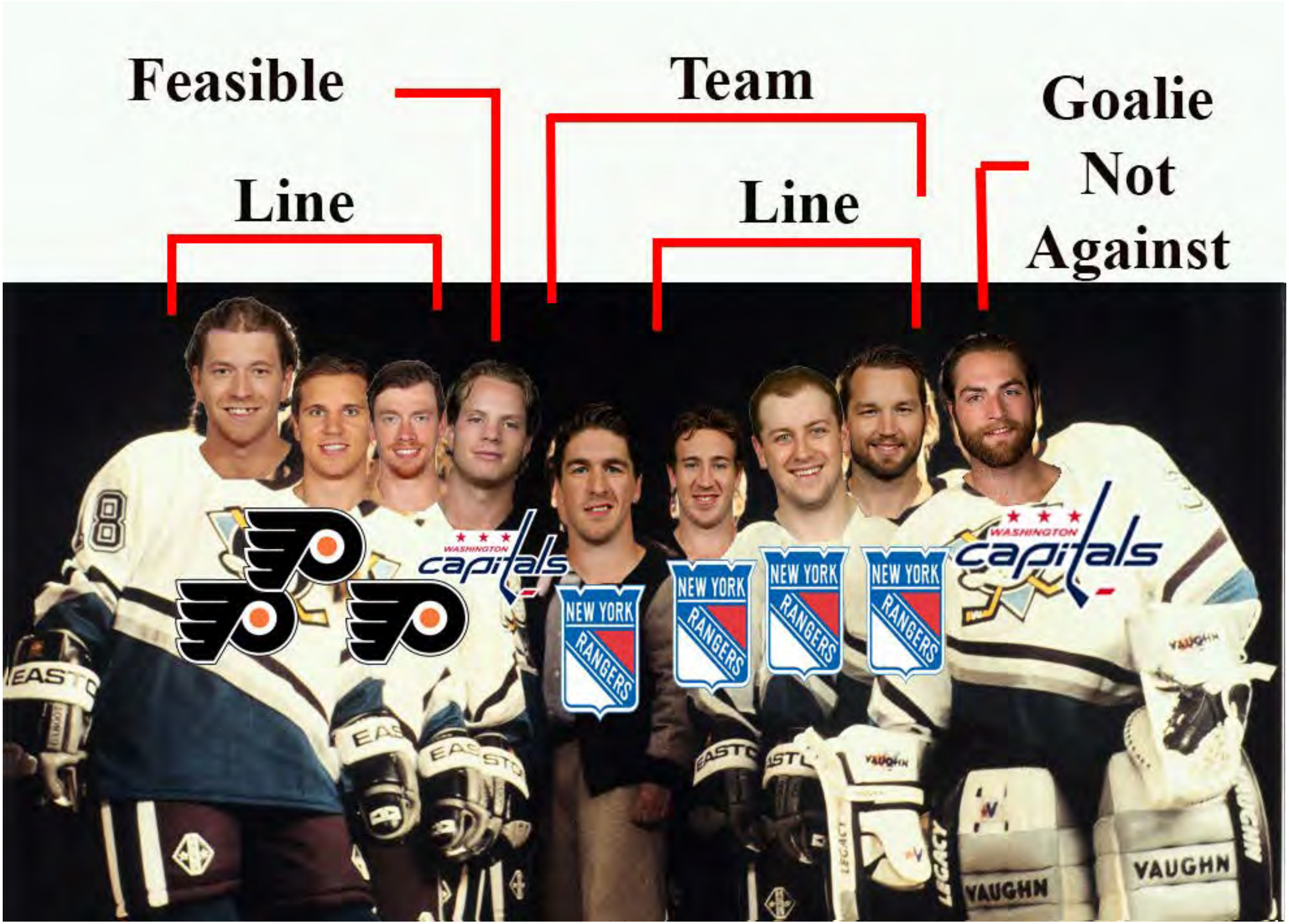
Structural Correlations : Goalie Against Opposing Players

- No skater against goalie

No skater against goalie constraint

$$6x_{pl} + \sum_{q \in Opponents_p} x_{ql} \leq 6, \quad \forall p \in G$$

Good, but not great chance



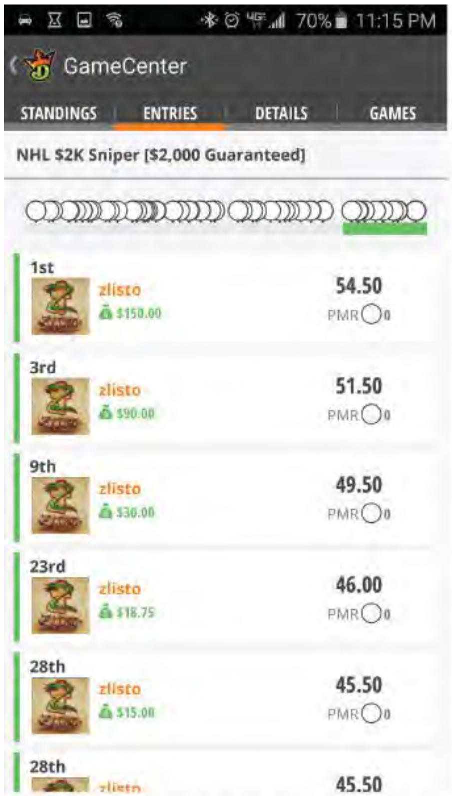
Play many diverse Lineups

- Make sure lineup l has no more than γ players in common with lineups 1 to l-1

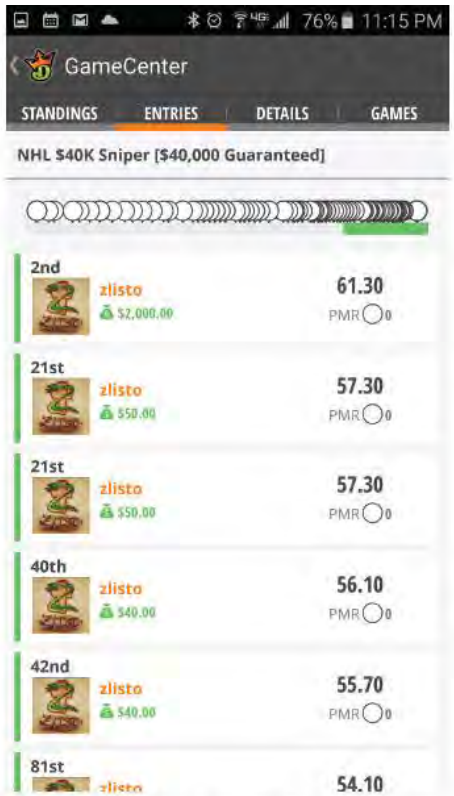
Diversity constraint

$$\sum_{p=1}^{N} x^*_{pk} x_{pl} \leq \gamma, k = 1, \ldots, l-1$$

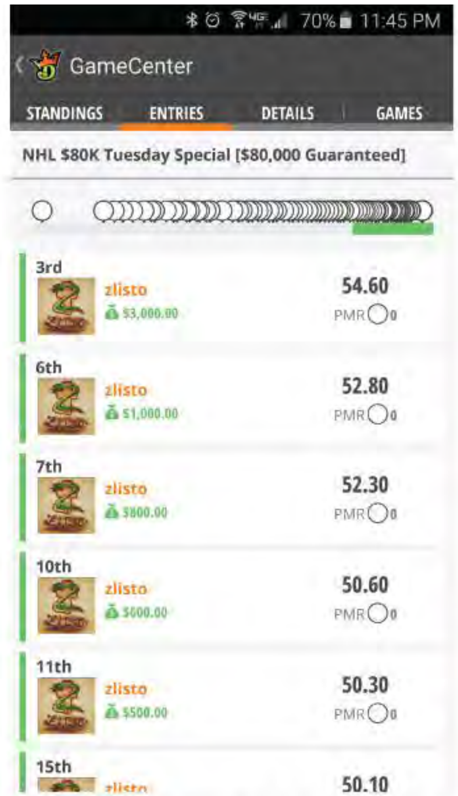
Were we able to do it?



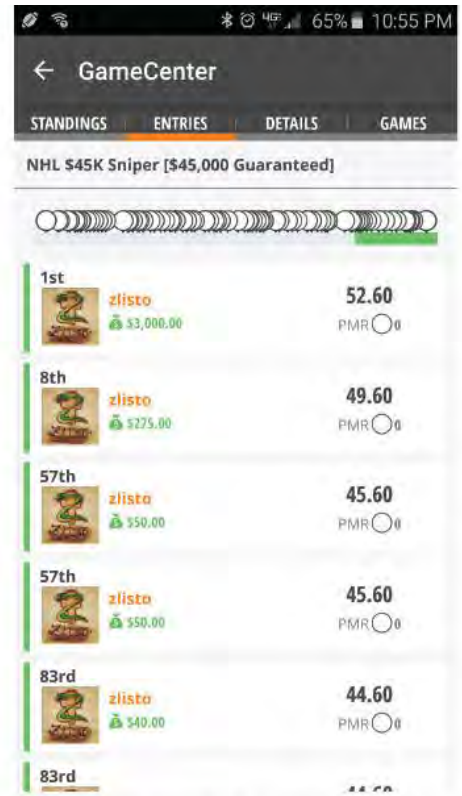
November 15, 2015



November 16, 2015



November 17, 2015



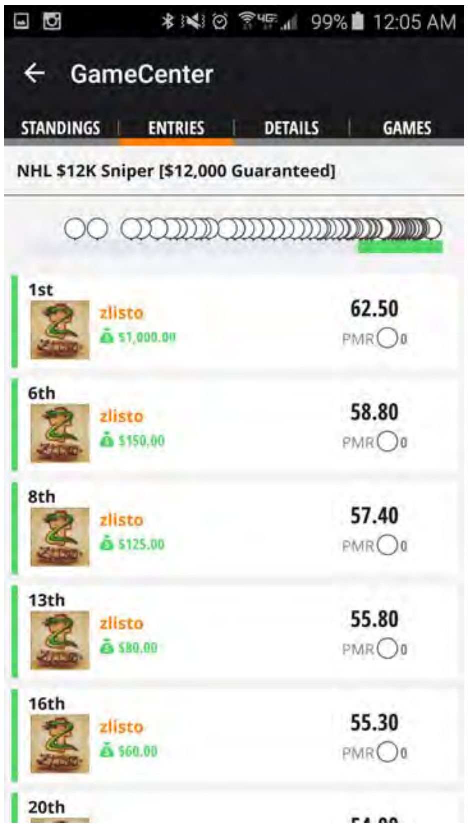
November 23, 2015

200 lineups

Policy Change

200 lineups -> 100 lineups

Were we able to continue it?



December 12, 2015

100 lineups

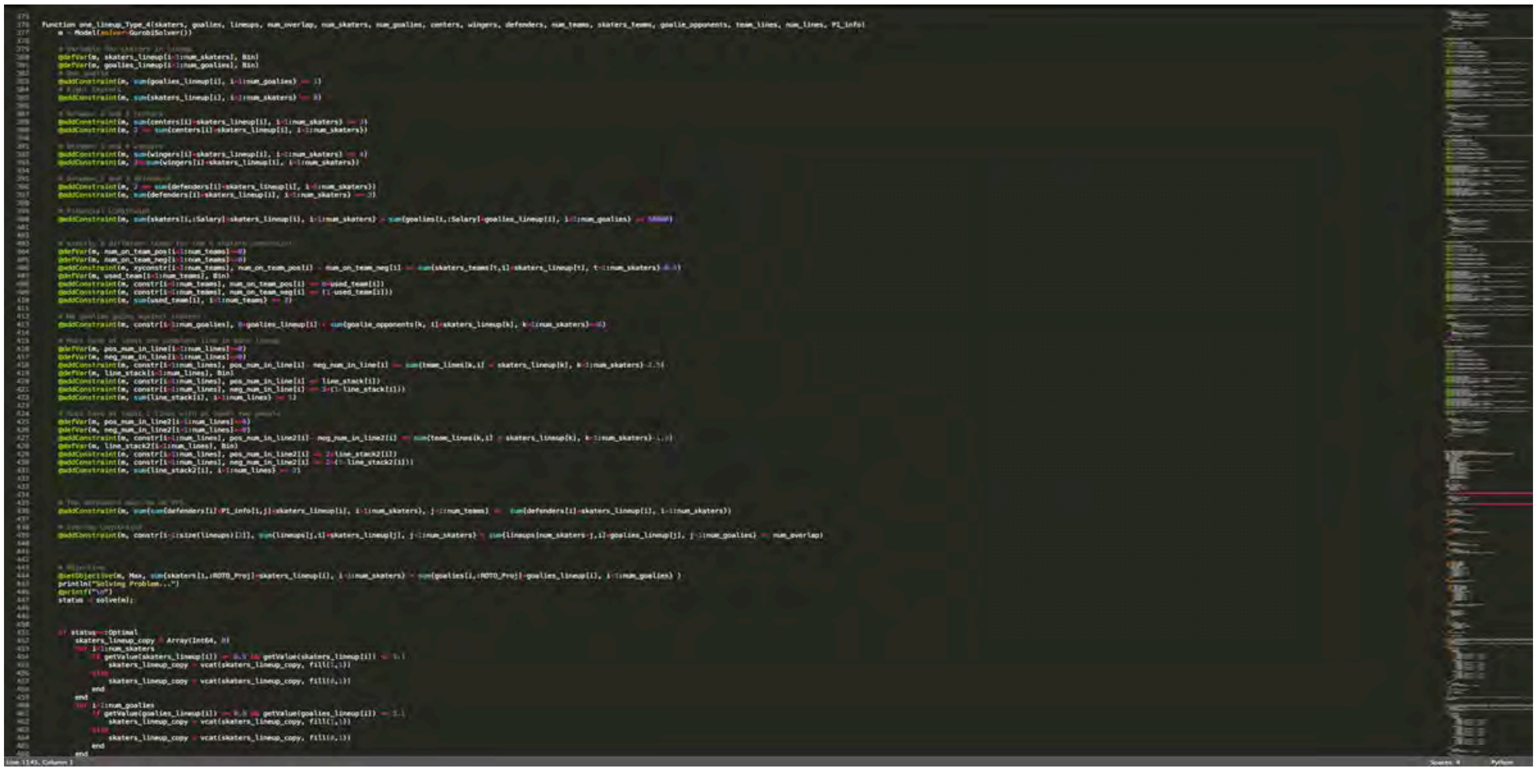
The Greater Boston FOOD BANK ®

> $15K

How can you do it?

Download Code from Github:

https://github.com/dscotthunter/Fantasy-Hockey-IP-Code



http://arxiv.org/pdf/1604.01455v1.pdf

Performance Time < 30 Minutes