Mixed Integer Programming (MIP) for Causal Inference and Beyond

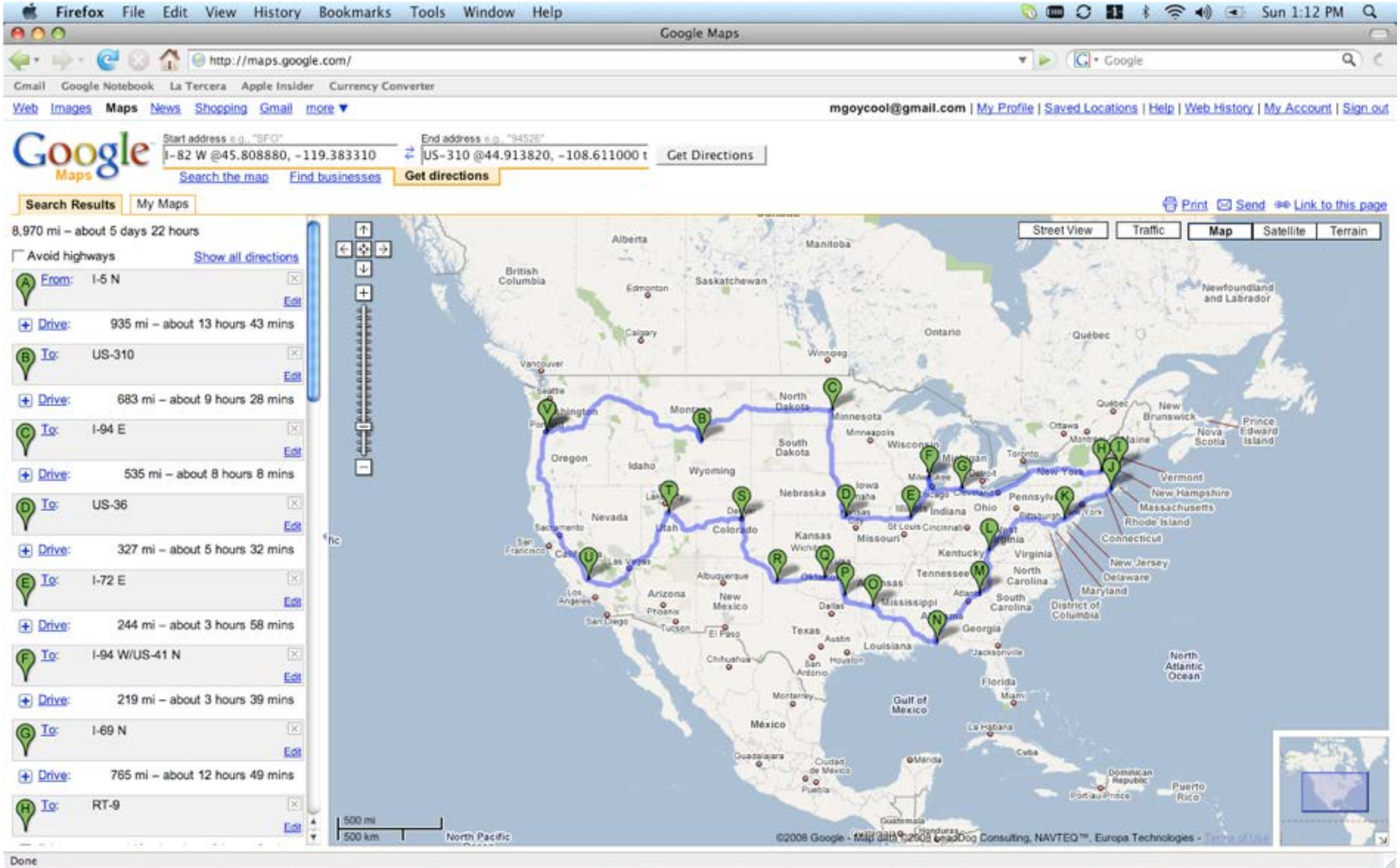
Juan Pablo Vielma

Massachusetts Institute of Technology

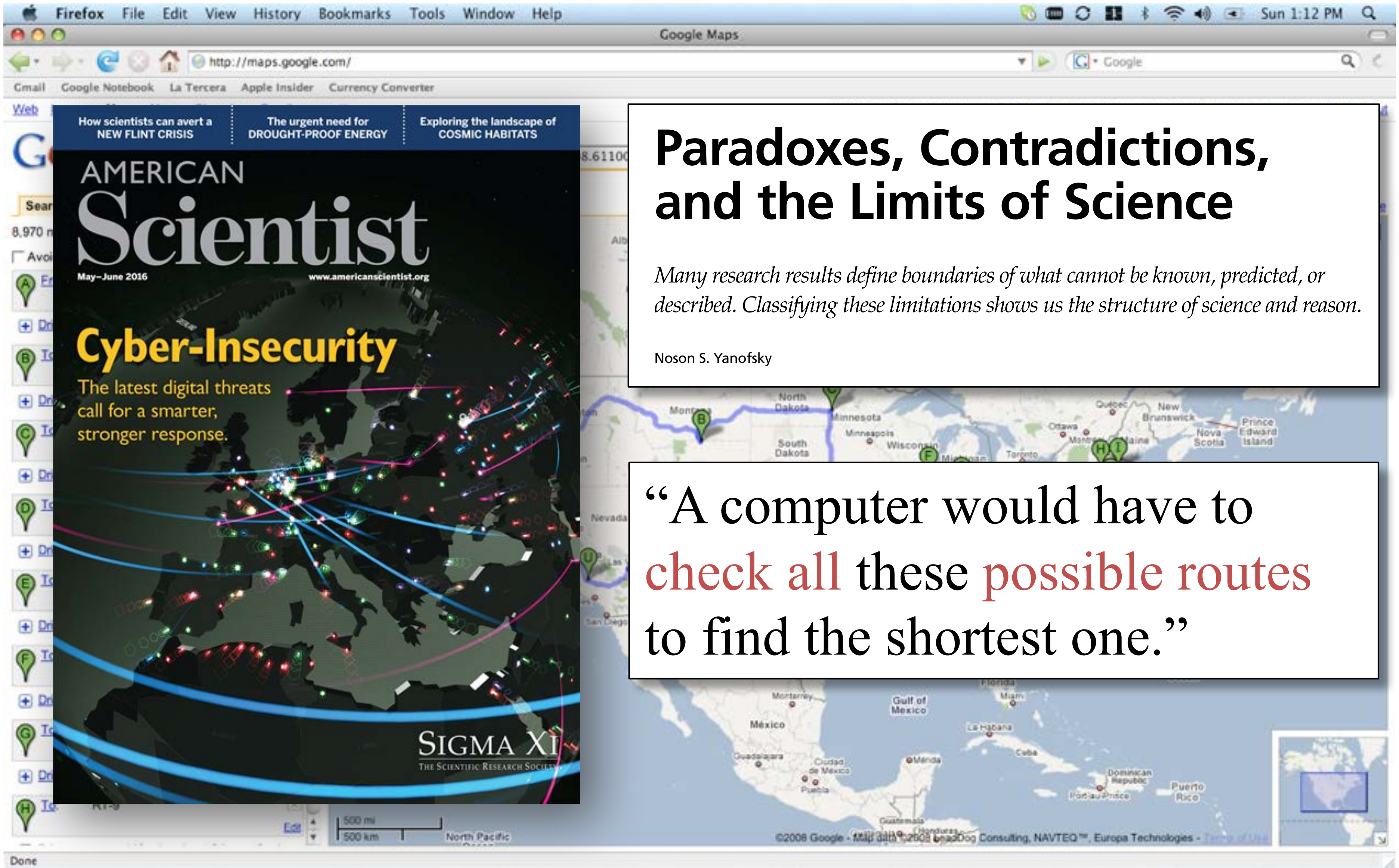
Columbia Business School

New York, NY, October, 2016.

Traveling Salesman Problem (TSP): Visit Cities Fast



Traveling Salesman Problem (TSP): Visit Cities Fast



Paradoxes, Contradictions, and the Limits of Science

Many research results define boundaries of what cannot be known, predicted, or described. Classifying these limitations shows us the structure of science and reason.

Noson S. Yanofsky

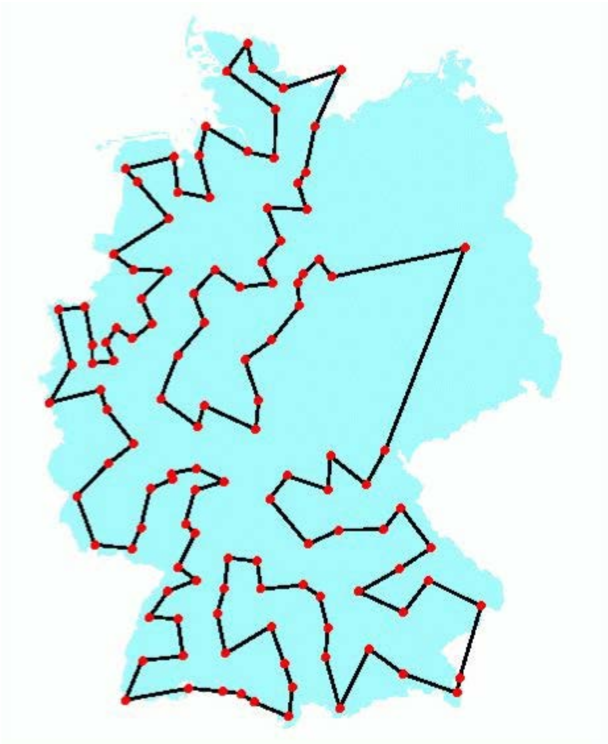
"A computer would have to check all these possible routes to find the shortest one."

MIP = Avoid Enumeration

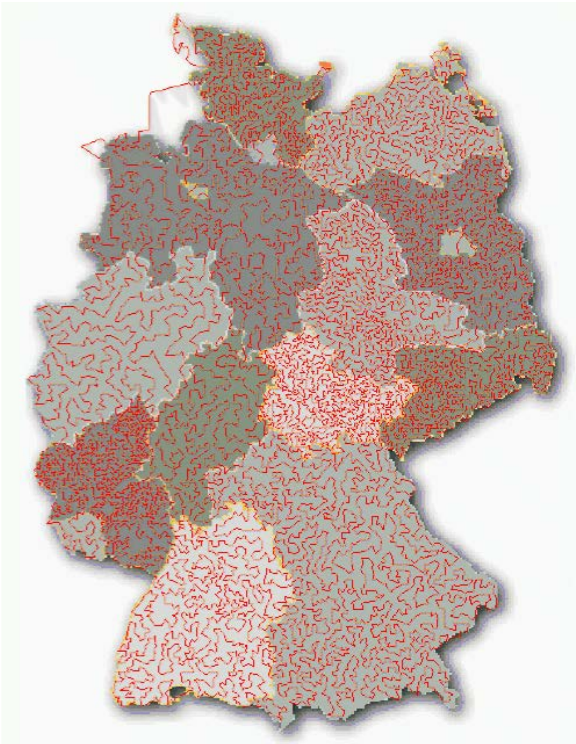
- Number of tours for 49 cities $= 48!/2 \approx 10^{60}$
- Fastest supercomputer $\approx 10^{17}$flops
- Assuming one floating point operation per tour:
  $> 10^{35}$ years $\approx 10^{25}$ times the age of the universe!
- How long does it take on an iphone?
  - Less than a second!
  - 4 iterations of **cutting plane** method!
  - Dantzig, Fulkerson and Johnson 1954 did it by hand!
  - For more info see tutorial in ConcordeTSP app
  - **Cutting planes** are the key for effectively solving (even NP-hard) MIP problems in practice.

Using IP to visit Germany

45 cities(1832)



120 cities(1977)



15,112 cities(2004)

http://www.math.uwaterloo.ca/tsp/d15sol/dhistory.html

50+ Years of MIP = Significant Solver Speedups

- Algorithmic Improvements (Machine Independent):
  - CPLEX v1.2 (1991) – v11 (2007): 29,000x speedup
  - Gurobi v1 (2009) – v6.5 (2015): 48.7x speedup
  - Commercial, but free for academic use
- (Reasonably) effective free / open source solvers:
  - GLPK, CBC and SCIP (free only for non-commercial)
- Easy to use, fast and versatile modeling languages
  - Julia based JuMP modelling language
- Linear MIP solvers very mature and effective:
  - Convex nonlinear MIP getting there (quadratic nearly there)

Matching

Treated Units: $\mathcal{T} = \{t_1, \ldots, t_T\}$

Control Units: $\mathcal{C} = \{c_1, \ldots, c_C\}$

Observed Covariates: $\mathcal{P} = \{p_1, \ldots, p_P\}$

Covariate Values: $\mathbf{x}^t = \left(x_p^t\right)_{p \in \mathcal{P}}, \quad t \in \mathcal{T}$

$$\mathbf{x}^c = \left(x_p^c\right)_{p \in \mathcal{P}}, \quad c \in \mathcal{C}$$

Maximum Cardinality Exact Matching

$$\max \quad \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} m_{t,c}$$

$$s.t.$$

$$\sum_{t \in \mathcal{T}} m_{t,c} \leq 1, \qquad \forall c \in \mathcal{C}$$

$$\sum_{c \in \mathcal{C}} m_{t,c} \leq 1, \qquad \forall t \in \mathcal{T}$$

$$m_{t,c} = 0 \qquad \forall t, c \quad \mathbf{x}^t \neq \mathbf{x}^c$$

$$0 \leq m_{t,c} \leq 1 \quad \sout{m_{t,c} \in \{0,1\}} \quad \forall t \in \mathcal{T}, \quad c \in \mathcal{C}.$$

- Solve time for truncated Lalonde CPS 429 = 0.001 s
- Why? Can solve relaxation.

Maximum Cardinality Marginal Means

$$\max \quad \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} m_{t,c}$$

$$s.t.$$

$$
\begin{aligned}
\sum_{t \in \mathcal{T}} m_{t,c} \leq 1, & \qquad \forall c \in \mathcal{C} \\
\sum_{c \in \mathcal{C}} m_{t,c} \leq 1, & \qquad \forall t \in \mathcal{T} \\
\sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} \mathbf{x}_p^t m_{t,c} = \sum_{c \in \mathcal{C}} \sum_{t \in \mathcal{T}} \mathbf{x}_p^c m_{t,c} & \qquad \forall p \in \mathcal{P} \\
m_{t,c} \in \{0, 1\} & \qquad \forall t \in \mathcal{T}, \quad c \in \mathcal{C}.
\end{aligned}
$$

- Solve time for truncated Lalonde CPS 429 = 444 s
- Why? One reason = relaxation has fractions.

Maximum Cardinality Fine Balance, Take 1

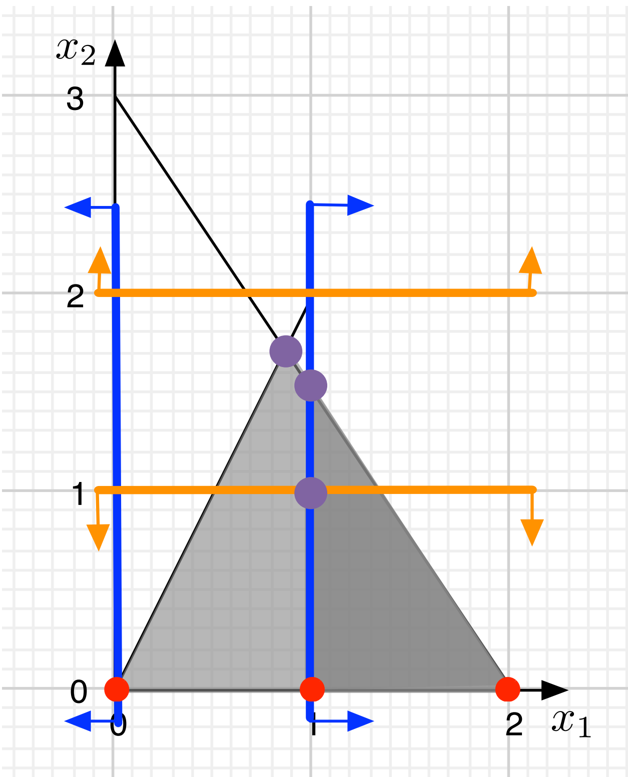
$$\max \quad \sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} m_{t,c}$$

$s.t.$

$$\begin{aligned}
\sum_{t\in\mathcal{T}} m^p_{t,c} &\le 1, && \forall c\in\mathcal{C},\ p\in\mathcal{P}\\
\sum_{c\in\mathcal{C}} m^p_{t,c} &\le 1, && \forall t\in\mathcal{T},\ p\in\mathcal{P}\\
{m^p}_{t,c} &= 0 && \forall t,c \quad \mathbf{x}^t_p \ne \mathbf{x}^c_p,\ p\in\mathcal{P}\\
\sum_{t\in\mathcal{T}} m^p_{t,c} &= \sum_{t\in\mathcal{T}} m^q_{t,c} && \forall c\in\mathcal{C},\ p,q\in\mathcal{P}\\
\sum_{c\in\mathcal{C}} m^p_{t,c} &= \sum_{c\in\mathcal{C}} m^q_{t,c} && \forall t\in\mathcal{T},\ p,q\in\mathcal{P}\\
m^p_{t,c} &\in \{0,1\} && \forall t\in\mathcal{T},\ c\in\mathcal{C},\ p\in\mathcal{P}.
\end{aligned}$$

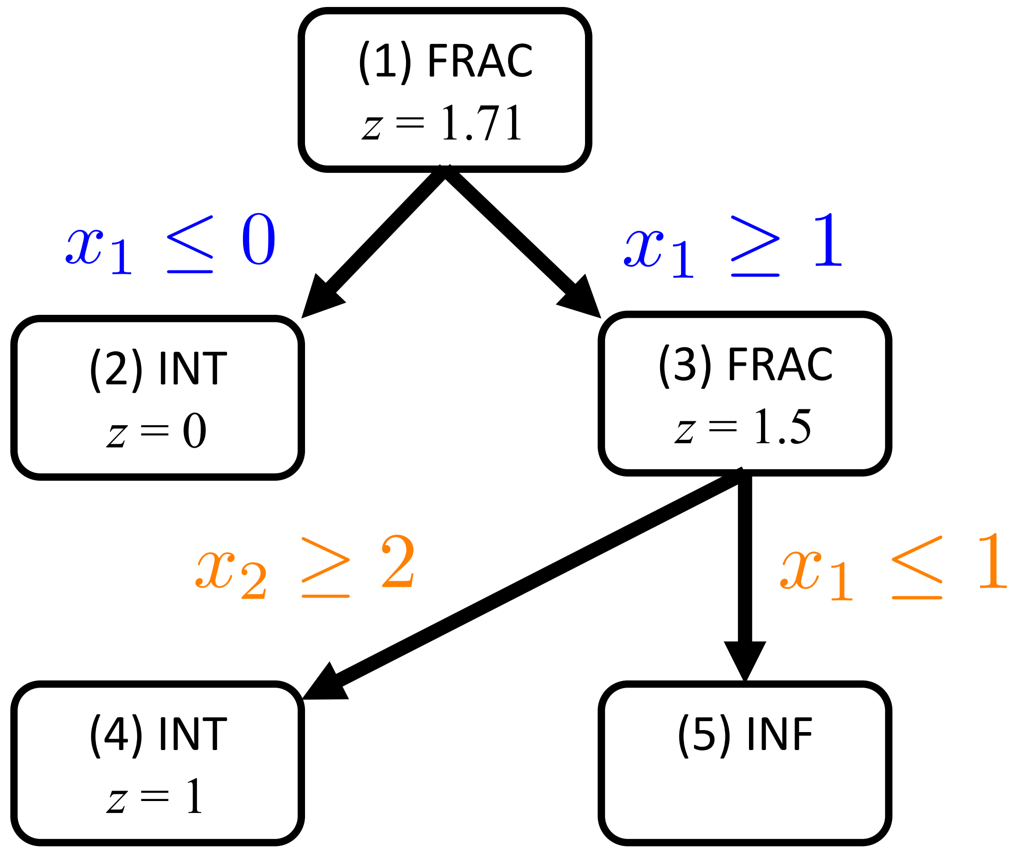
- Solve time for truncated Lalonde CPS 429 = 0.81 s
- Why? One reason = relaxation has “fewer” fractions.

Basic Branch-and-Bound

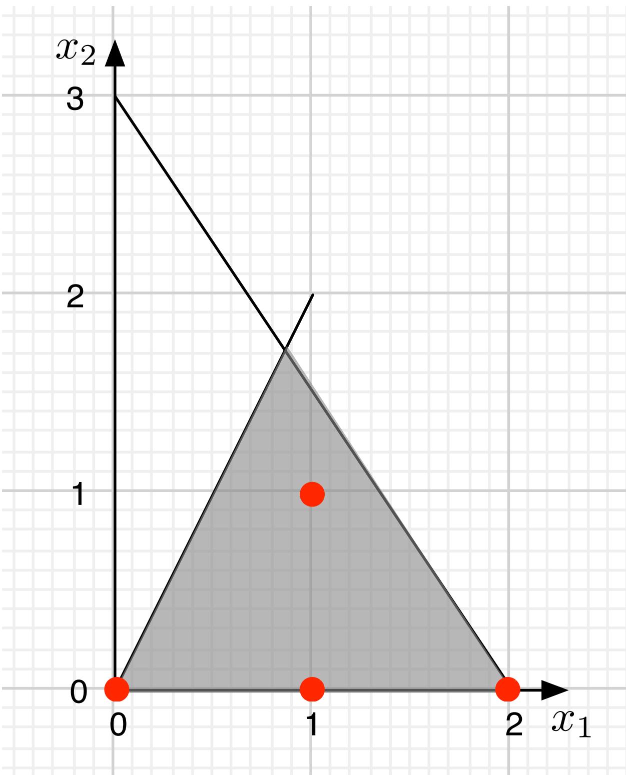


$$\max z := x_2$$
$$3x_1 + 2x_2 \leq 6$$
$$-2x_1 + x_2 \leq 0$$
$$x_1,\ x_2 \geq 0$$
$$x_1,\ x_2 \in \mathbb{Z}$$

← Linear Programming (LP) Relaxation

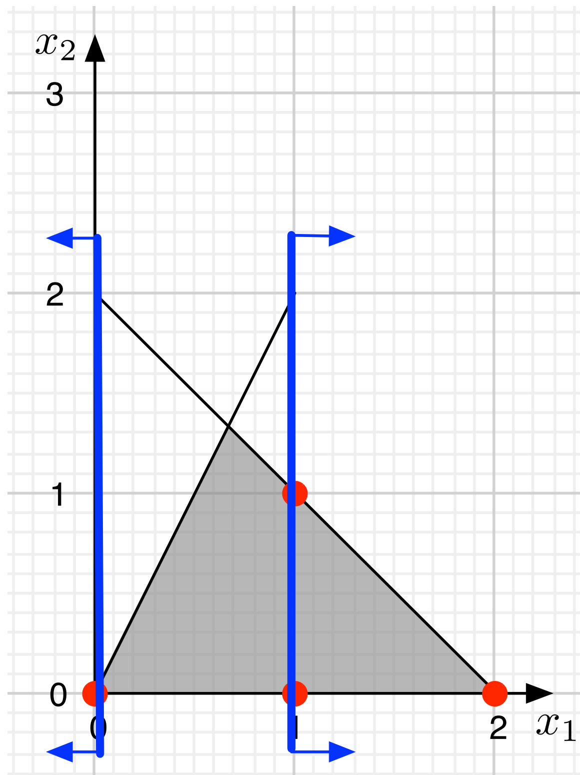


Stronger Formulations = Faster Solves ?



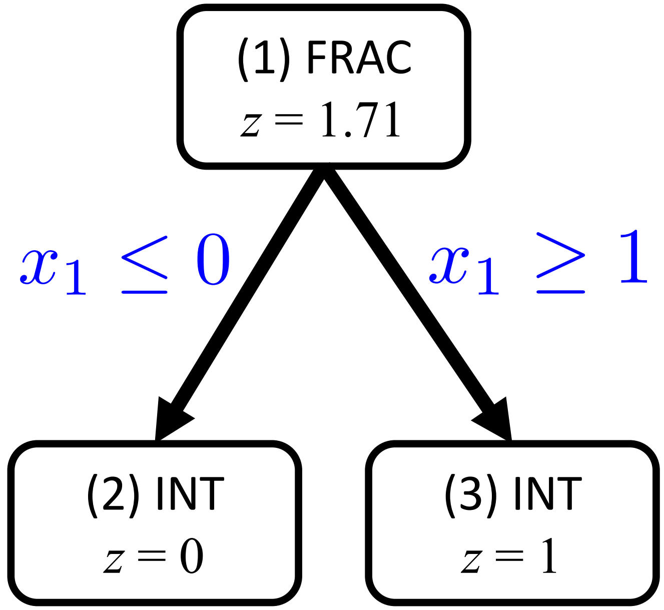
$$\begin{aligned} \max z := x_2 & \\ 3x_1 + 2x_2 &\leq 6 \\ -2x_1 + x_2 &\leq 0 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

$$x_1,\ x_2 \in \mathbb{Z}$$



$$\begin{aligned} \max z := x_2 & \\ x_1 + x_2 &\leq 2 \\ -2x_1 + x_2 &\leq 0 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

$$x_1,\ x_2 \in \mathbb{Z}$$



Cutting Plane Example: Chátal-Gomory Cuts

$$P := \left\{ x \in \mathbb{R}^2 \ : \ \begin{array}{r} x_1 + x_2 \leq 3, \\ 5x_1 - 3x_2 \leq 3 \end{array} \right\}$$

$$\subseteq$$

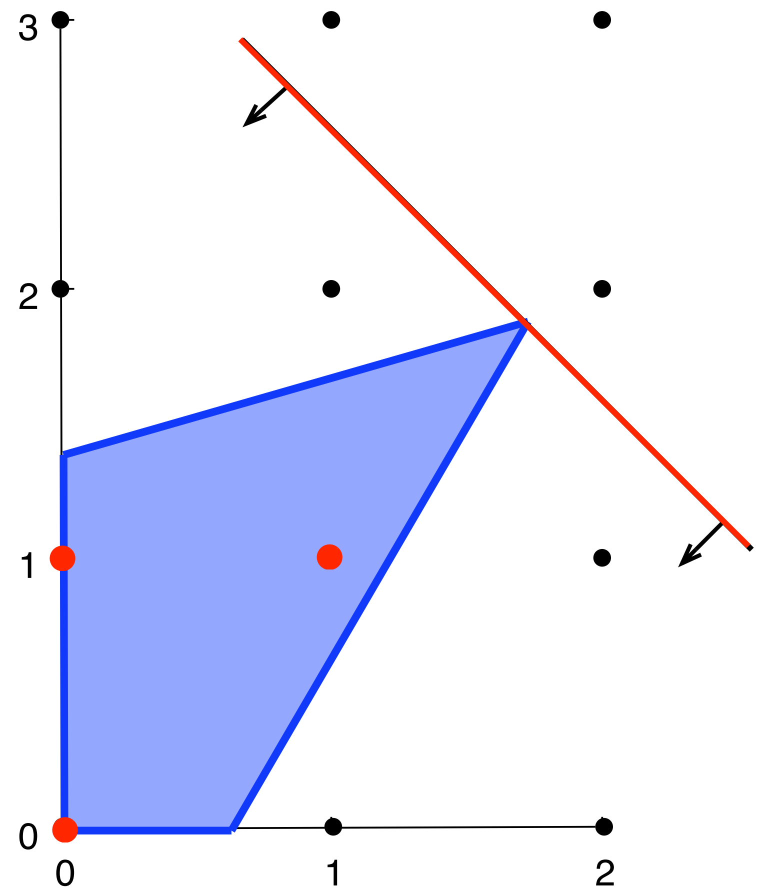
$$H := \{ x \in \mathbb{R}^2 \ : \ \underbrace{4x_1 + 3x_2}_{\in \mathbb{Z}} \leq 10.5 \}$$

if $x \in \mathbb{Z}^2$

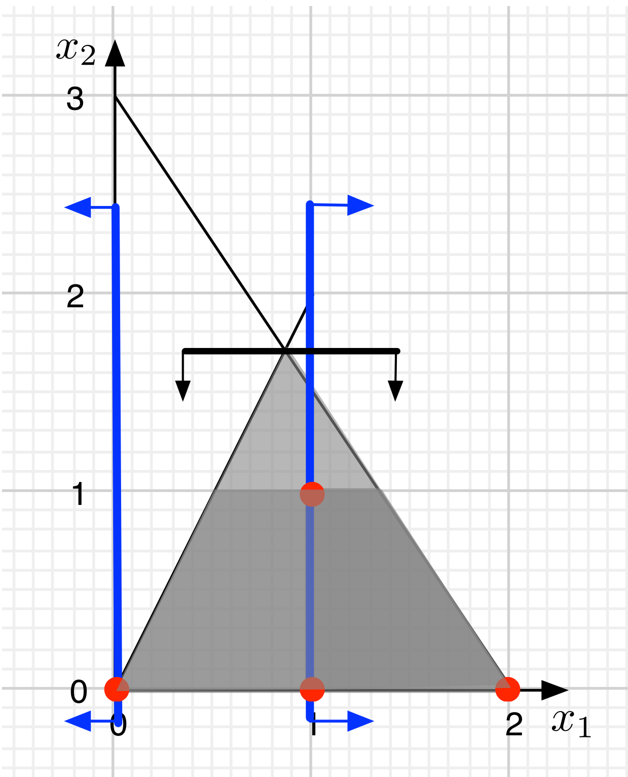
$$4x_1 + 3x_2 \leq \lfloor 10.5 \rfloor$$

Valid for $H \cap \mathbb{Z}^2$

Valid for $P \cap \mathbb{Z}^2$



Branch-and-Bound and Cuts (Branch-and-Cut)



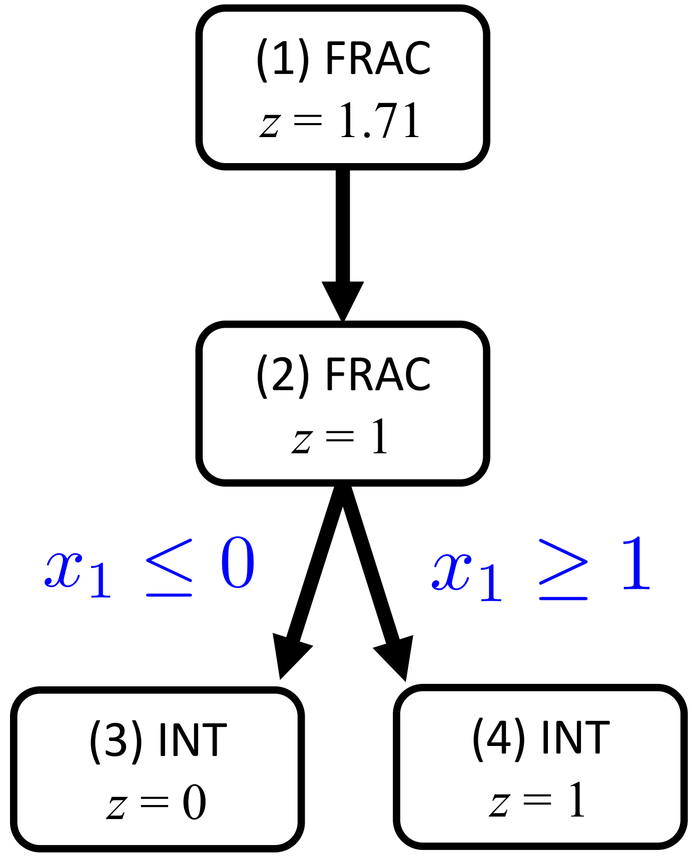
$$\max z := x_2$$

$$3x_1 + 2x_2 \leq 6 \quad x_2 \leq \lfloor 1.71 \rfloor = 1$$

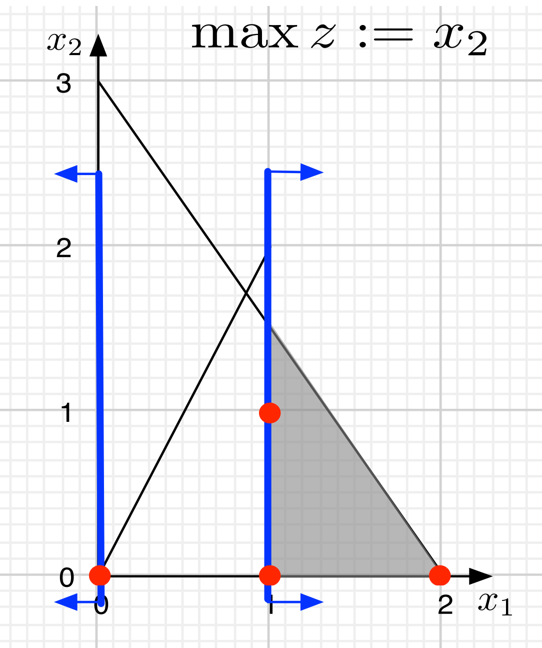
$$-2x_1 + x_2 \leq 0$$

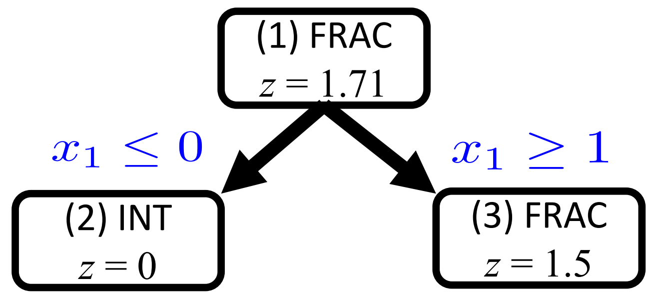
$$x_1,\, x_2 \geq 0$$

$$x_1,\, x_2 \in \mathbb{Z}$$



Partial Solves and GAP





$$GAP = 100 \times \frac{|\text{bestnode} - \text{bestinteger}|}{10^{-10} + |\text{bestinteger}|}$$

$$BB\,GAP = 100 \times \frac{|1.5 - 0|}{10^{-10} + |0|} = 1.5 \times 10^{12}$$

$$BB^{+}\,GAP = 100 \times \frac{|1.5 - 1|}{10^{-10} + |1|} = 50\%$$

```
                      Nodes                                              Cuts/
        Node  Left     Objective  IInf  Best Integer    Best Bound    ItCnt     Gap

            0     0     1415.0000    16                 1415.0000      143
            0     0     1446.4286    45                ZeroHalf: 33    317
*     0+    0                            3130.0000     1446.4286      317   53.79%
      0     2     1446.4286    34        3130.0000     1446.4286      317   53.79%
      Elapsed real time =   0.18 sec. (tree size =  0.01 MB, solutions = 1)
*    28    28      integral     0        3060.0000     1449.7619     1168   52.62%
*    34    32      integral     0        3045.0000     1449.7619     1320   52.39%
```

Maximum Cardinality Marginal Means

$$\max \quad \sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} m_{t,c}$$

$$s.t.$$

$$\sum_{t\in\mathcal{T}} m_{t,c} \leq 1, \qquad \forall c \in \mathcal{C}$$

$$\sum_{c\in\mathcal{C}} m_{t,c} \leq 1, \qquad \forall t \in \mathcal{T}$$

$$\sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} \mathbf{x}_p^t m_{t,c} = \sum_{c\in\mathcal{C}}\sum_{t\in\mathcal{T}} \mathbf{x}_p^c m_{t,c} \qquad \forall p \in \mathcal{P}$$

$$m_{t,c} \in \{0,1\} \qquad \forall t \in \mathcal{T}, \quad c \in \mathcal{C}.$$

- Solve time for truncated Lalonde CPS 429 = 444 s
- Optimal = 17, LP Relaxation = 19.35

Maximum Cardinality Marginal Means

"Black Magic"

```
Root relaxation: objective 1.935024e+01, 1947 iterations, 0.16 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0   19.35024    0   10   -0.00000   19.35024     -      -    0s
     0     0   19.30273    0   20   -0.00000   19.30273     -      -    1s
    .
    .
    .
     0     2   19.29748    0   25   -0.00000   19.29748     -      -    4s
     4     5   19.20830    3   23   -0.00000   19.29273     -   1407    5s
    .
    .
    .
  1140   763   18.56894   18   23   -0.00000   19.16557     -    340   51s
H 1149   726                         4.0000000   19.16557   379%   344   51s
    .
    .
    .
*23876  7827              71        17.0000000   18.00366  5.90%   150  443s

Cutting planes:  Gomory: 1  Flow cover: 1

Explored 25301 nodes (3601039 simplex iterations) in 443.89 seconds

Optimal solution found (tolerance 1.00e-04)
Best objective 1.700000000000e+01, best bound 1.700000000000e+01, gap 0.0%
```

Maximum Cardinality Fine Balance, Take 1

$$\max \quad \sum_{t\in\mathcal{T}} \sum_{c\in\mathcal{C}} m_{t,c}$$

$s.t.$

$$\begin{aligned}
\sum_{t\in\mathcal{T}} m^p_{t,c} &\leq 1, && \forall c \in \mathcal{C},\ p \in \mathcal{P} \\
\sum_{c\in\mathcal{C}} m^p_{t,c} &\leq 1, && \forall t \in \mathcal{T},\ p \in \mathcal{P} \\
{m^p}_{t,c} &= 0 && \forall t, c \quad \mathbf{x}^t_p \neq \mathbf{x}^c_p,\ p \in \mathcal{P} \\
\sum_{t\in\mathcal{T}} m^p_{t,c} &= \sum_{t\in\mathcal{T}} m^q_{t,c} && \forall c \in \mathcal{C},\ p, q \in \mathcal{P} \\
\sum_{c\in\mathcal{C}} m^p_{t,c} &= \sum_{c\in\mathcal{C}} m^q_{t,c} && \forall t \in \mathcal{T},\ p, q \in \mathcal{P} \\
m^p_{t,c} &\in \{0, 1\} && \forall t \in \mathcal{T},\ c \in \mathcal{C},\ p \in \mathcal{P}.
\end{aligned}$$

- Solve time for truncated Lalonde CPS 429 = 0.81 s
- Optimal = 10, LP Relaxation = 11

Maximum Cardinality Fine Balance, Take 1

```
Root relaxation: objective 1.100000e+01, 1490 iterations, 0.03 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0   11.00000    0  158   -0.00000   11.00000      -      -    0s
     0     0   11.00000    0  164   -0.00000   11.00000      -      -    0s
     0     0   11.00000    0  164   -0.00000   11.00000      -      -    0s
H    0     0                      10.0000000   11.00000  10.0%     -    0s
    .
    .
    .
     0     2   11.00000    0  111   10.00000   11.00000  10.0%     -    0s

Cutting planes:  Zero half: 2

Explored 8 nodes (9933 simplex iterations) in 0.78 seconds
Optimal solution found (tolerance 1.00e-04)
Best objective 1.000000000000e+01, best bound 1.000000000000e+01, gap 0.0%
```

Maximum Cardinality Fine Balance, Take 1

$$\max \quad \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} m_{t,c}$$

$s.t.$

$$\begin{aligned}
\sum_{t \in \mathcal{T}} m^p_{t,c} &\le 1, && \forall c \in \mathcal{C},\ p \in \mathcal{P} \\
\sum_{c \in \mathcal{C}} m^p_{t,c} &\le 1, && \forall t \in \mathcal{T},\ p \in \mathcal{P} \\
m^p{}_{t,c} &= 0 && \forall t, c \quad \mathbf{x}^t_p \ne \mathbf{x}^c_p,\ p \in \mathcal{P} \\
\sum_{t \in \mathcal{T}} m^p_{t,c} &= \sum_{t \in \mathcal{T}} m^q_{t,c} && \forall c \in \mathcal{C},\ p, q \in \mathcal{P} \\
\sum_{c \in \mathcal{C}} m^p_{t,c} &= \sum_{c \in \mathcal{C}} m^q_{t,c} && \forall t \in \mathcal{T},\ p, q \in \mathcal{P} \\
m^p_{t,c} &\in \{0, 1\} && \forall t \in \mathcal{T},\ c \in \mathcal{C},\ p \in \mathcal{P}.
\end{aligned}$$

- Solve time for truncated Lalonde CPS 429 = 0.81 s
- Scalability? Size = $|\mathcal{T}| \times |\mathcal{C}| \times |\mathcal{P}|$

Maximum Cardinality Fine Balance, Take 2

$$\mathcal{K}(p) = \{\mathbf{x}_p^c\}_{c\in\mathcal{P}} \cup \{\mathbf{x}_p^t\}_{t\in\mathcal{T}}$$

$$\mathcal{C}_{p,k} = \{c \in \mathcal{C} \,:\, \mathbf{x}_p^c = k\}$$

$$\mathcal{T}_{p,k} = \{t \in \mathcal{T} \,:\, \mathbf{x}_p^t = k\}$$

$$\begin{aligned}
\max \quad & \sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} m_{t,c} \\
s.t. \quad & \\
& \sum_{t\in\mathcal{T}} m_{t,c} \leq 1, && \forall c \in \mathcal{C} \\
& \sum_{c\in\mathcal{C}} m_{t,c} \leq 1, && \forall t \in \mathcal{T} \\
& \sum_{t\in\mathcal{T}_{p,k}}\sum_{c\notin\mathcal{C}_{p,k}} m_{t,c} = \sum_{t\notin\mathcal{T}_{p,k}}\sum_{c\in\mathcal{C}_{p,k}} m_{t,c} && \forall p \in \mathcal{P}, k \in \mathcal{K}(p) \\
& m_{t,c} \in \{0,1\} && \forall t \in \mathcal{T}, \quad c \in \mathcal{C}.
\end{aligned}$$

- Solve time for truncated Lalonde CPS 429 = 0.61 s
- Scalability? Size = $|\mathcal{T}| \times |\mathcal{C}| + |\mathcal{P}|$

Take 1 Revisited

$$\max \quad \sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} m_{t,c}$$

$$s.t.$$

$$\sum_{t\in\mathcal{T}} m^p_{t,c} \le 1, \qquad \forall c\in\mathcal{C},\ p\in\mathcal{P}$$

$$\sum_{c\in\mathcal{C}} m^p_{t,c} \le 1, \qquad \forall t\in\mathcal{T},\ p\in\mathcal{P}$$

$$m^p{}_{t,c} = 0 \qquad \forall t,c \quad \mathbf{x}^t_p \ne \mathbf{x}^c_p,\ p\in\mathcal{P}$$

$$\sum_{t\in\mathcal{T}} m^p_{t,c} = \sum_{t\in\mathcal{T}} m^q_{t,c} \qquad \forall c\in\mathcal{C},\ p,q\in\mathcal{P}$$

$$\sum_{c\in\mathcal{C}} m^p_{t,c} = \sum_{c\in\mathcal{C}} m^q_{t,c} \qquad \forall t\in\mathcal{T},\ p,q\in\mathcal{P}$$

$$m^p_{t,c} \in \{0,1\} \qquad \forall t\in\mathcal{T},\ c\in\mathcal{C},\ p\in\mathcal{P}.$$

- Solve time for truncated Lalonde CPS 429 = 0.81 s
- Scalability? Size = $|\mathcal{T}| \times |\mathcal{C}| \times |\mathcal{P}|$

Take 1 Revisited

$$\max \quad \sum_{t \in \mathcal{T}} x_t$$

$$s.t.$$

$$\sum_{t \in \mathcal{T}_{p,k}} m^p_{t,c} = y_c, \qquad \forall p \in \mathcal{P}, \quad k \in \mathcal{K}(p), \quad c \in \mathcal{C}_{p,k}$$

$$\sum_{c \in \mathcal{C}_{p,k}} m^p_{t,c} = x_t, \qquad \forall \mathcal{S} \in \mathcal{F}, \quad k \in \mathcal{K}(p), \quad t \in \mathcal{T}_{p,k}$$

$$m^p_{t,c} \in \{0,1\} \quad \forall p \in \mathcal{P}, \quad k \in \mathcal{K}(p), \quad t \in \mathcal{T}_{p,k}, \quad c \in \mathcal{C}_{p,k}$$

$$x_t \in \{0,1\} \quad \forall t \in \mathcal{T}$$

$$y_c \in \{0,1\} \quad \forall c \in \mathcal{C}.$$

Maximum Cardinality Fine Balance, Take 3

$$\max \sum_{t \in \mathcal{T}} x_t$$

$$\mathcal{K}(p) = \{\mathbf{x}_p^c\}_{c \in \mathcal{P}} \cup \{\mathbf{x}_p^t\}_{t \in \mathcal{T}}$$

$$\mathcal{C}_{p,k} = \{c \in \mathcal{C} \,:\, \mathbf{x}_p^c = k\}$$

$$\mathcal{T}_{p,k} = \{t \in \mathcal{T} \,:\, \mathbf{x}_p^t = k\}$$

$s.t.$

$$\sum_{t \in \mathcal{T}} x_t = \sum_{c \in \mathcal{C}} y_c,$$

$$\sum_{t \in \mathcal{T}_{p,k}} x_t = \sum_{c \in \mathcal{C}_{p,k}} y_c, \quad \forall p \in \mathcal{P}, \quad k \in \mathcal{K}(p)$$

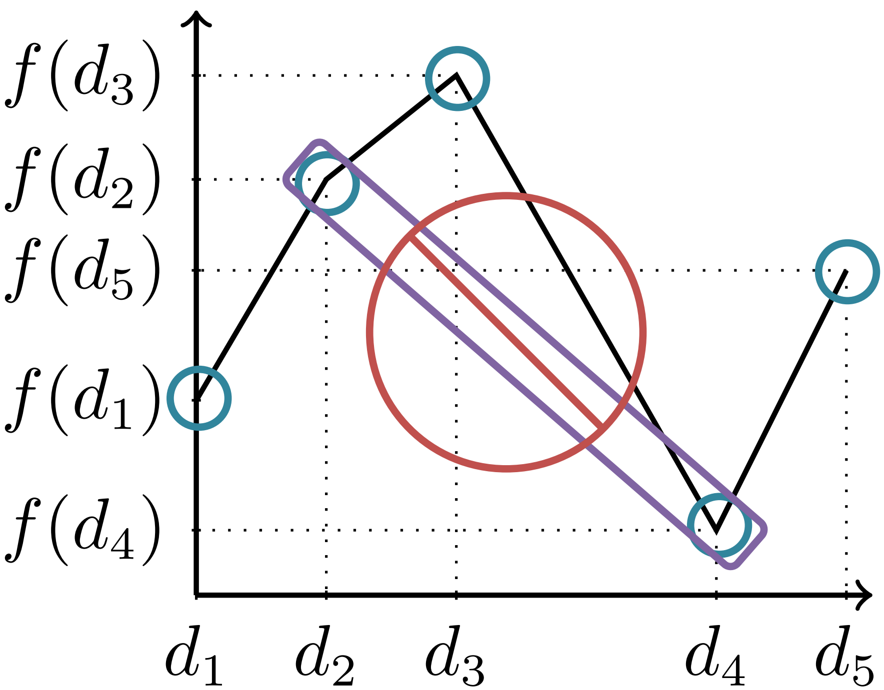
$$x_t \in \{0,1\} \qquad \forall t \in \mathcal{T}$$

$$y_c \in \{0,1\} \qquad \forall c \in \mathcal{C}.$$

- Solve time for truncated Lalonde CPS 429 = 0.006 s
- Scalability? Size = $|\mathcal{P}| \times (|\mathcal{T}| + |\mathcal{C}|)$

Simple Formulation for Univariate Functions

$$z = f(x)$$



$$\text{Size} = O(\#\text{ of segments})$$

Non-Ideal: Fractional Extreme Points

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^{5} \lambda_j, \quad \lambda_j \geq 0$$

$$y \in \{0,1\}^4, \quad \sum_{i=1}^{4} y_i = 1$$

$$0 \leq \lambda_1 \leq y_1$$

$$0 \leq \lambda_2 \leq y_1 + y_2$$

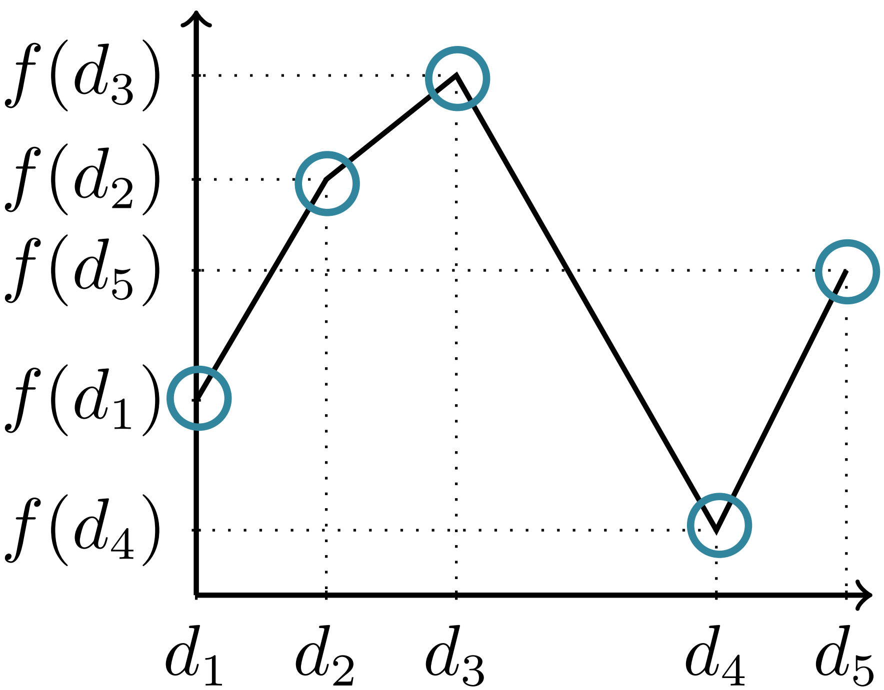
$$0 \leq \lambda_3 \leq y_2 + y_3$$

$$0 \leq \lambda_4 \leq y_3 + y_4$$

$$0 \leq \lambda_5 \leq y_4$$

Advanced Formulation for Univariate Functions

$$z = f(x)$$



$$\text{Size} = O(\log_2 \# \text{ of segments})$$

Ideal: Integral Extreme Points

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^{5} \lambda_j, \quad \lambda_j \geq 0$$

$$y \in \{0,1\}^2$$

$$0 \leq \lambda_1 + \lambda_5 \leq 1 - y_1$$

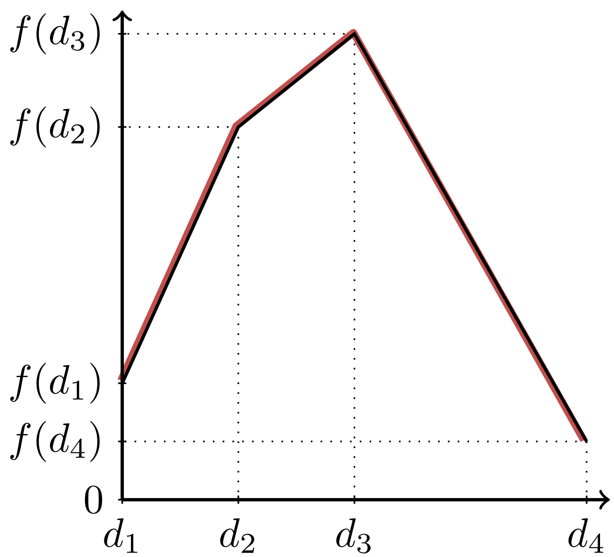
$$0 \leq \lambda_3 \leq y_1$$

$$0 \leq \lambda_4 + \lambda_5 \leq 1 - y_2$$

$$0 \leq \lambda_1 + \lambda_2 \leq y_2$$

Extended Formulation for PWL Functions

$$S = \mathrm{gr}(f) = \bigcup_{i=1}^{k} \left\{ (x,z) \in \mathbb{R}^2 \ : \ \begin{array}{r} d_i \le x \le d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$



MC Formulation:

$$d_i y_i \le x^i \le d_{i+1} y_i \quad \forall i \in [k]$$

$$m_i x^i + c_i y_i = z^i \quad \forall i \in [k]$$

$$\sum_{i=1}^{k} x^i = x$$

$$\sum_{i=1}^{k} z^i = z$$

$$\sum_{i=1}^{k} y_i = 1$$

$$y \in \{0,1\}^k$$

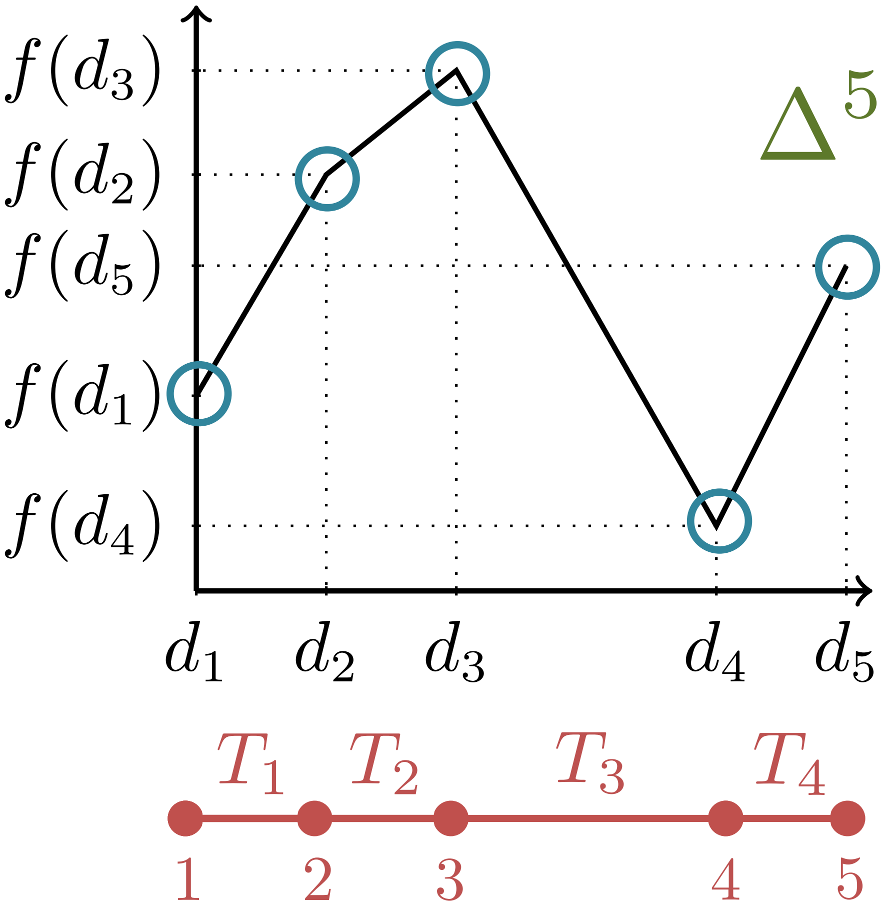
Abstracting Univariate Functions

$$P_i := \{\lambda \in \Delta^5 : \lambda_j = 0 \quad \forall j \notin T_i\}$$

$$z = f(x)$$

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$T_i := \{i, i+1\} \quad i \in \{1, \ldots, 4\}$$

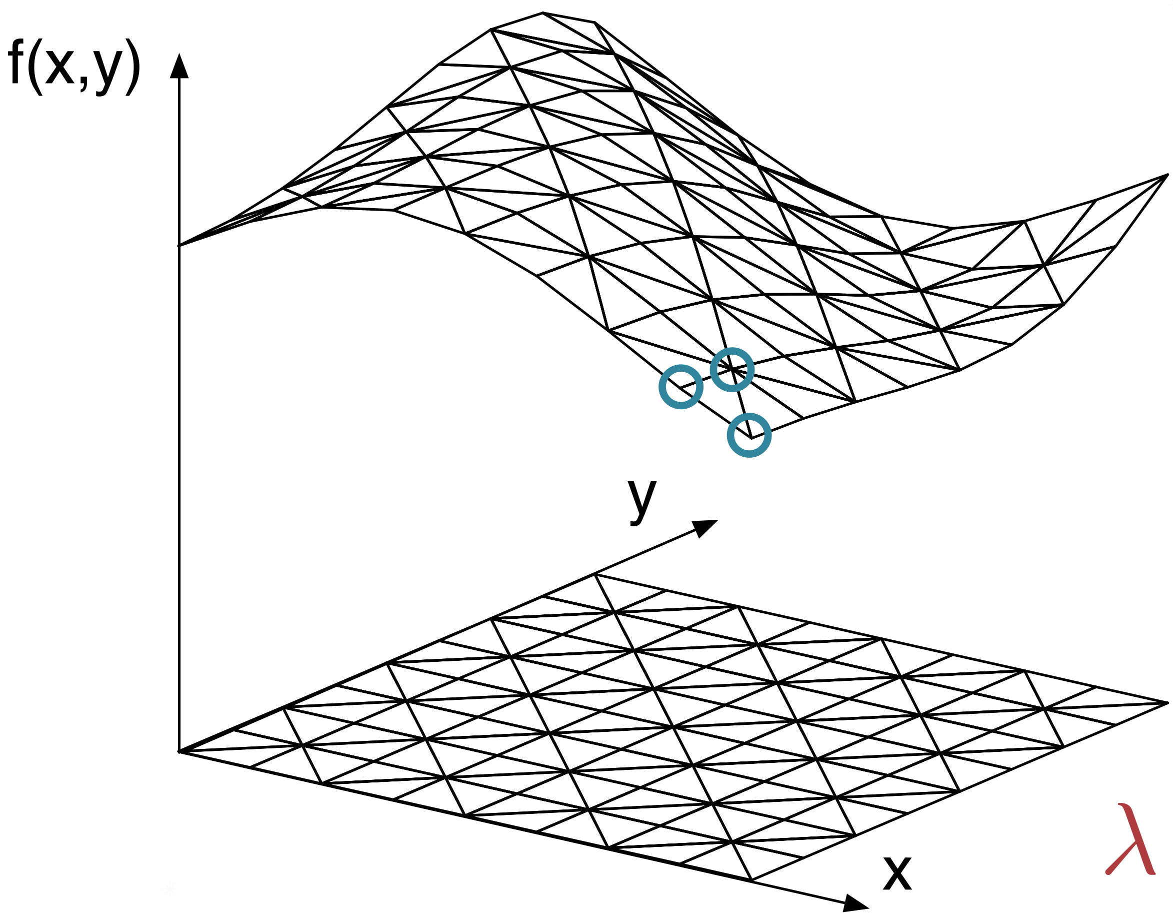


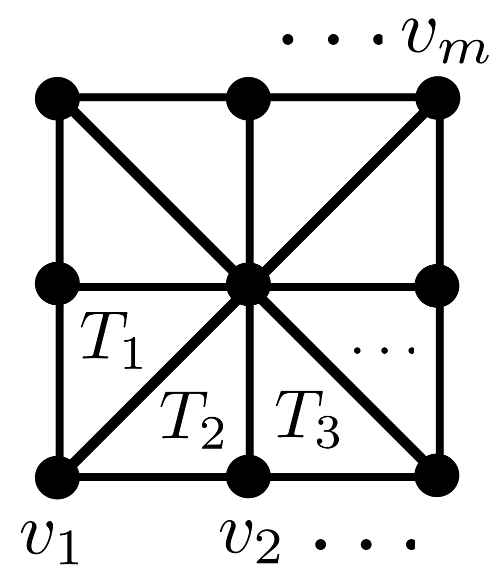
$$\Delta^5 = \left\{ 1 = \sum_{j=1}^{5} \lambda_j, \quad \lambda_j \geq 0 \right\}$$

$$\lambda \in \bigcup_{i=1}^{4} P_i \subseteq \Delta^5$$

Abstraction Works for Multivariate Functions

$$P_i := \{\lambda \in \Delta^m \ : \ \lambda_j = 0 \quad \forall v_j \notin T_i\}$$

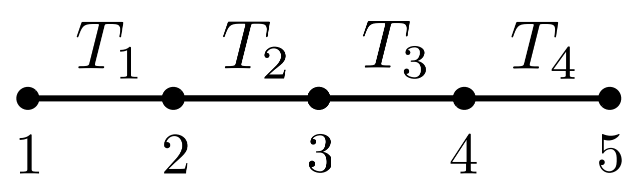


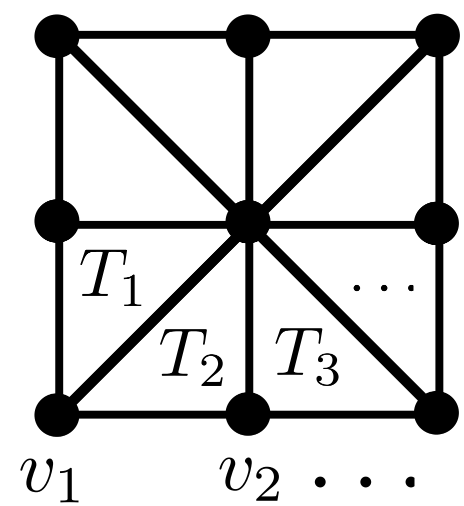


$$\lambda \in \bigcup_{i=1}^{n} P_i \subseteq \Delta^m$$

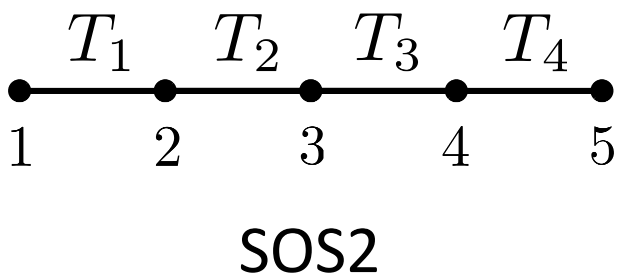
Complete Abstraction

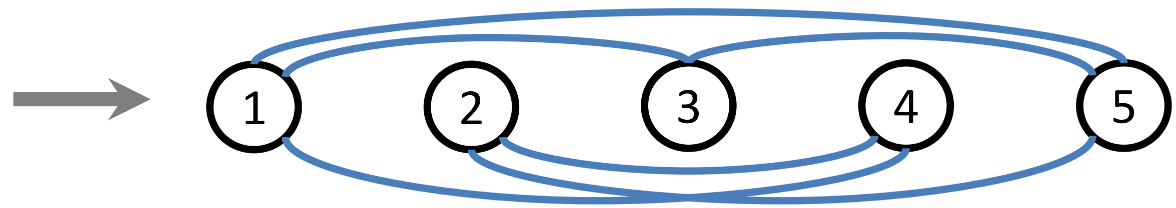
- $\Delta^V := \left\{\lambda \in \mathbb{R}_+^V : \sum_{v \in V} \lambda_v = 1\right\},$
- $P_i = \left\{\lambda \in \Delta^V : \lambda_v = 0 \quad \forall v \notin T_i\right\}$
- $\lambda \in \bigcup_{i=1}^{n} P_i$
- $T_i =$ cliques of a graph



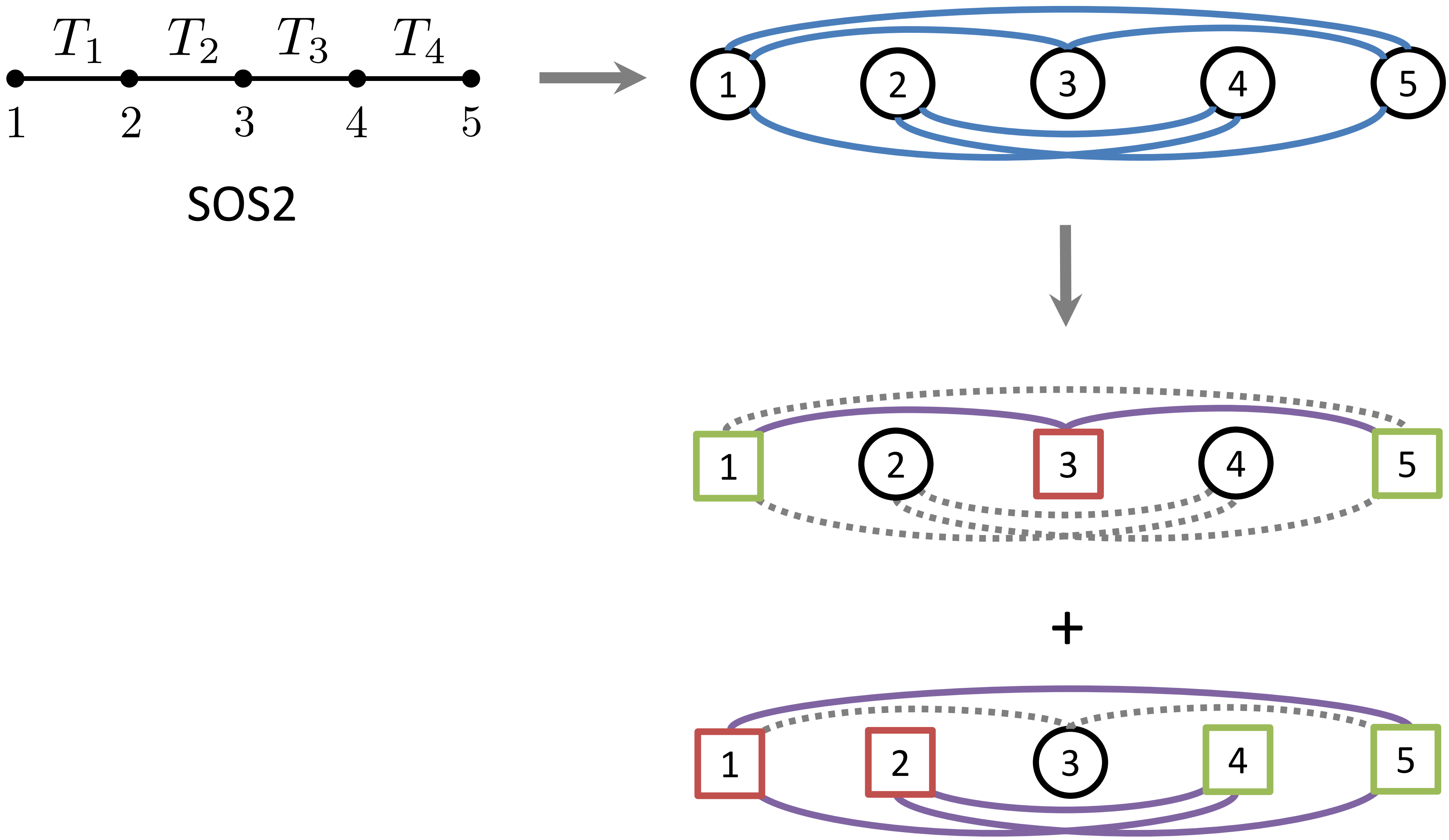


From Cliques to (Complement) Conflict Graph

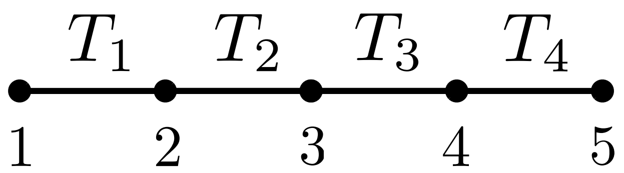




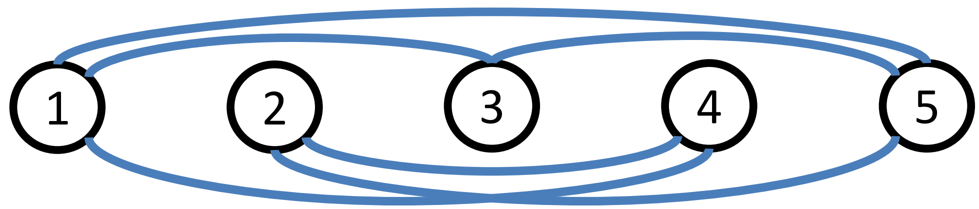
From Conflict Graph to Bi-clique Cover

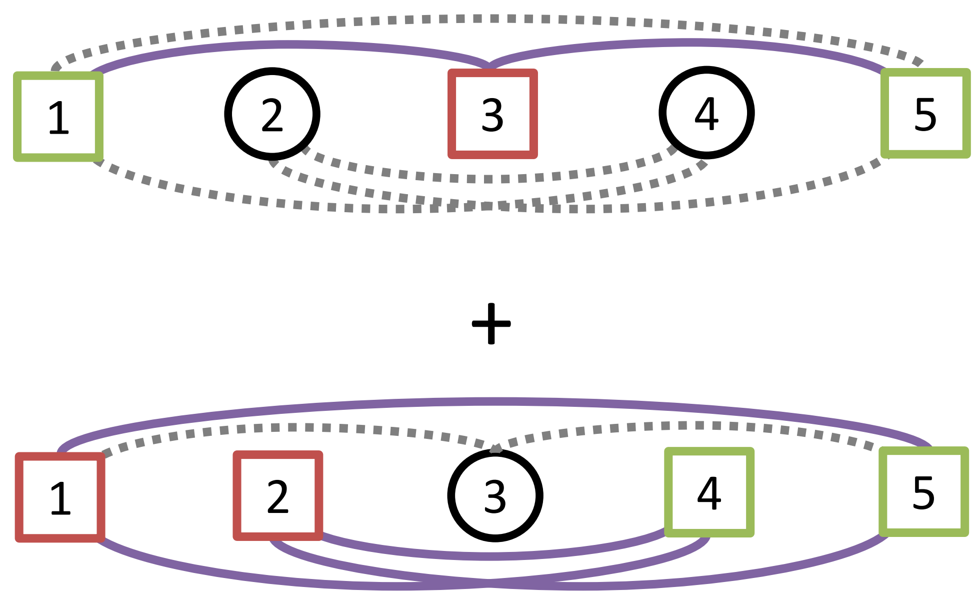


From Bi-clique Cover to Formulation



SOS2





$$
\begin{aligned}
0 &\le \lambda_1 + \lambda_5 \le 1 - y_1 \\
0 &\le \lambda_3 \quad\quad\;\; \le y_1 \\
0 &\le \lambda_4 + \lambda_5 \le 1 - y_2 \\
0 &\le \lambda_1 + \lambda_2 \le y_2
\end{aligned}
$$

Ideal Formulation from Bi-clique Cover

- Conflict Graph $G = (V, E)$

$$E = \{(u, v) : u, v \in V,\ u \neq v, \quad \nexists i \text{ s.t. } u, v \in T_i\}$$

- Bi-clique cover $\{(A^j, B^j)\}_{j=1}^{t}, \quad A^j, B^j \subseteq V$

$$\forall \{u, v\} \in E \quad \exists j \text{ s.t. } u \in A^j \ \wedge v \in B^j$$

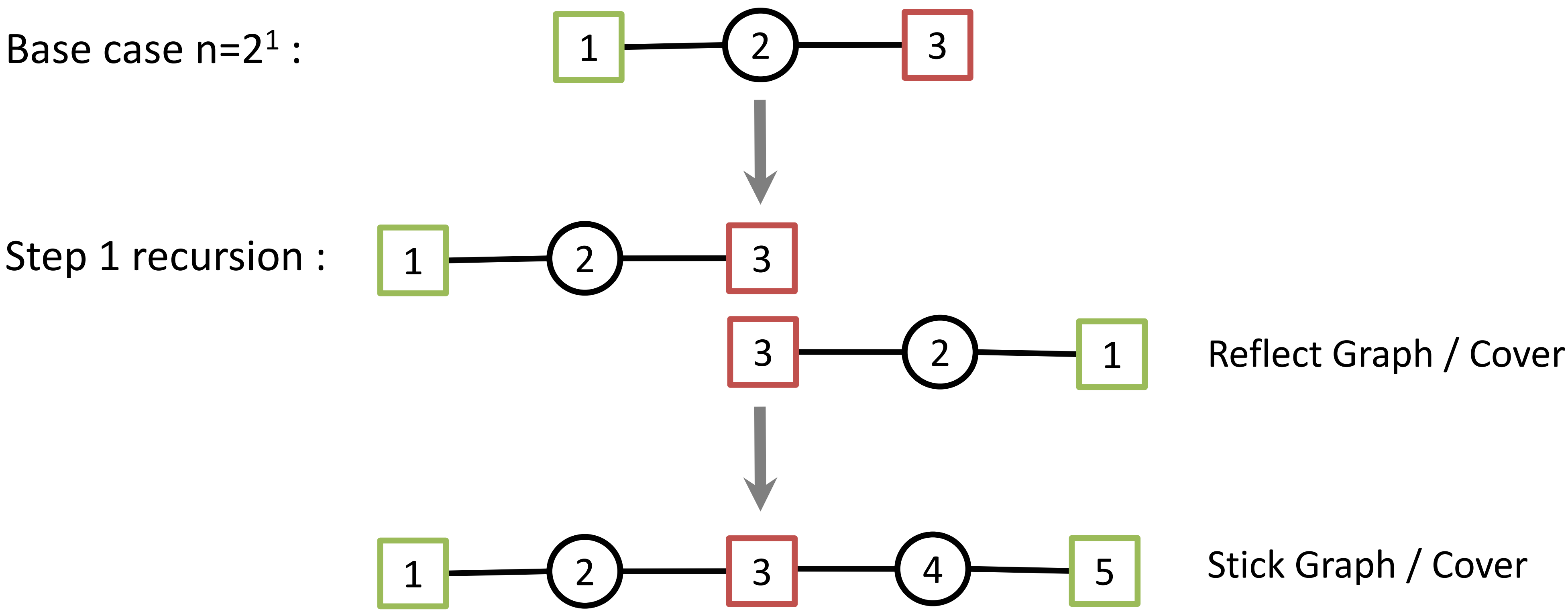
- Formulation

$$\sum_{v \in A^j} \lambda_v \leq 1 - y_j \quad \forall j \in [t]$$

$$\sum_{v \in B^j} \lambda_v \leq y_j \quad \forall j \in [t]$$

$$y \in \{0, 1\}^t$$

Recursive Construction of Cover for SOS2, Step 1

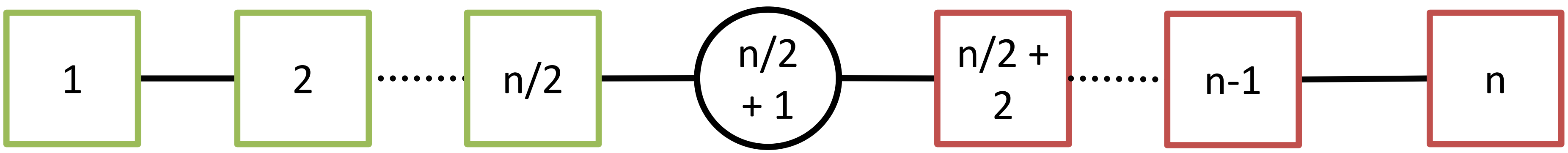


Repeat for all bi-cliques from 2^{k-1} to cover all edges within first and last half of conflict graph

Recursive Construction of Cover for SOS2, Step 2

Only edges missing are those between
first and last half of conflict graph

Step 2 : Add one more bi-clique

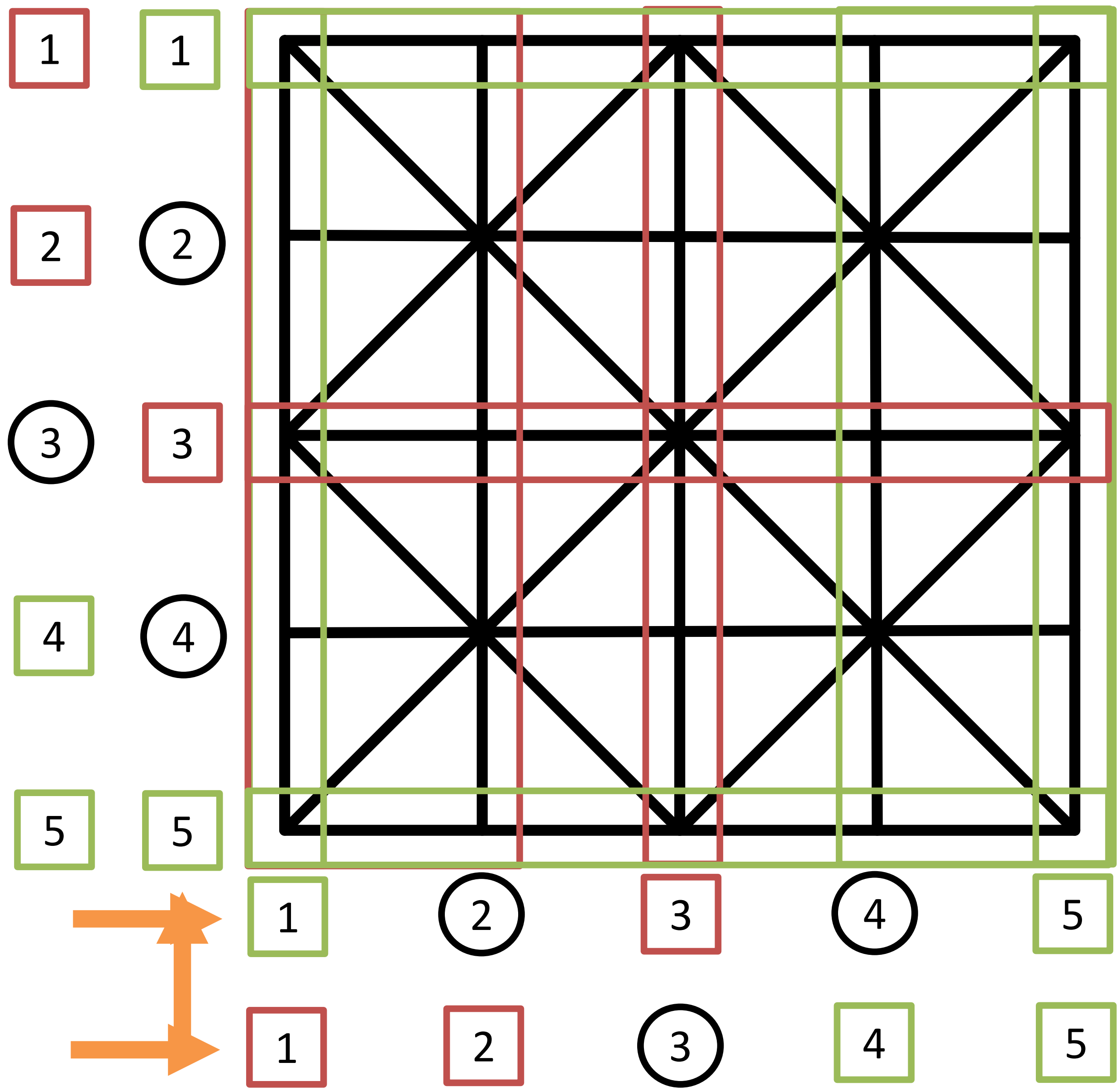


$$\text{Cover has } \log_2 n \text{ bi-cliques.}$$

For non-power of two just delete extra nodes.

Grid Triangulations: Step 1 = SOS2 for Inter-Box

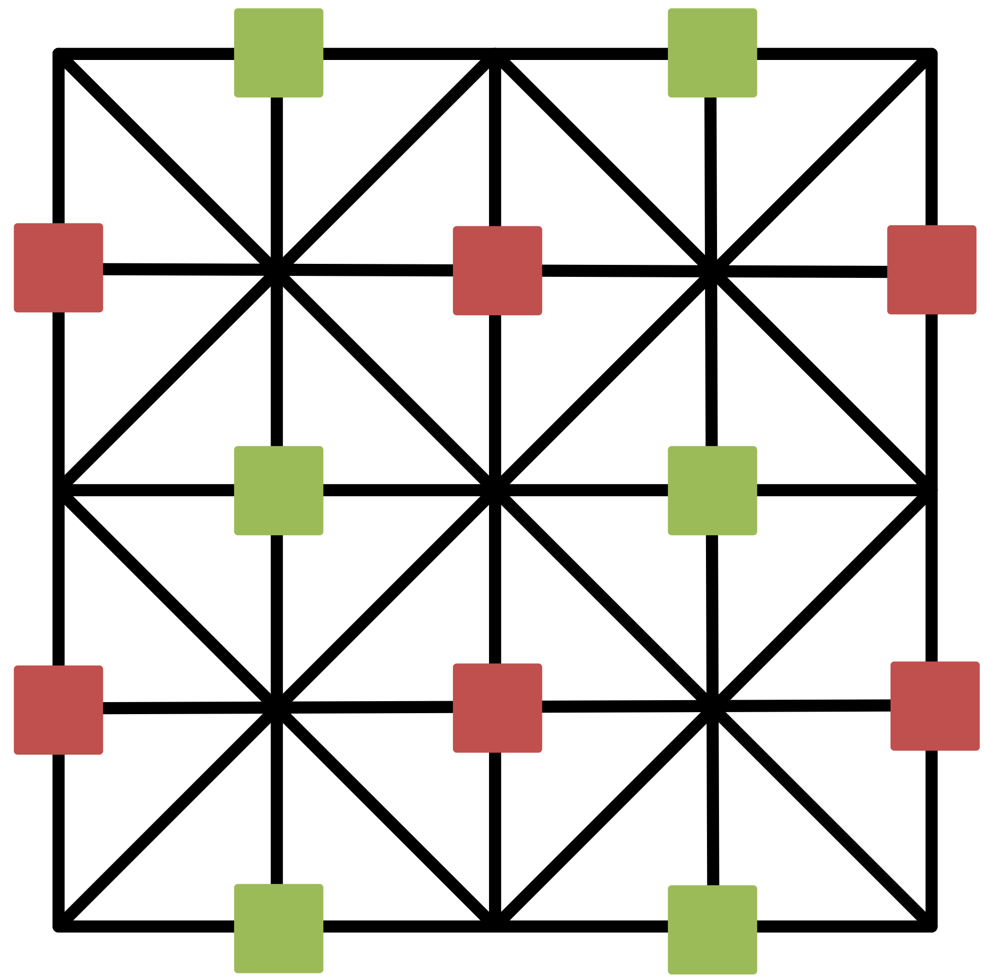
Covers all arcs between boxes



Grid Triangulations: Step 2 = Ad-hoc Intra-Box

Covers all arcs within boxes

Sometimes 1 additional cover



Grid Triangulations: Step 2 = Ad-hoc Intra-Box

Sometimes **2** additional covers

Sometimes more, but always less than **9**

Simple rules to get (near) optimal in Fall '16