

Mixed Integer Programming (MIP) for Causal Inference and Beyond

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New York, NY, October, 2016.

Traveling Salesman Problem (TSP): Visit Cities Fast

Firefox File Edit View History Bookmarks Tools Window Help
Google Maps
http://maps.google.com/

Start address e.g. "BFO" | End address e.g. "H452H"
I-82 W @45.808880, -119.383310 | US-310 @44.913820, -108.611000 | Get Directions

Search Results | My Maps

8,970 mi – about 5 days 22 hours

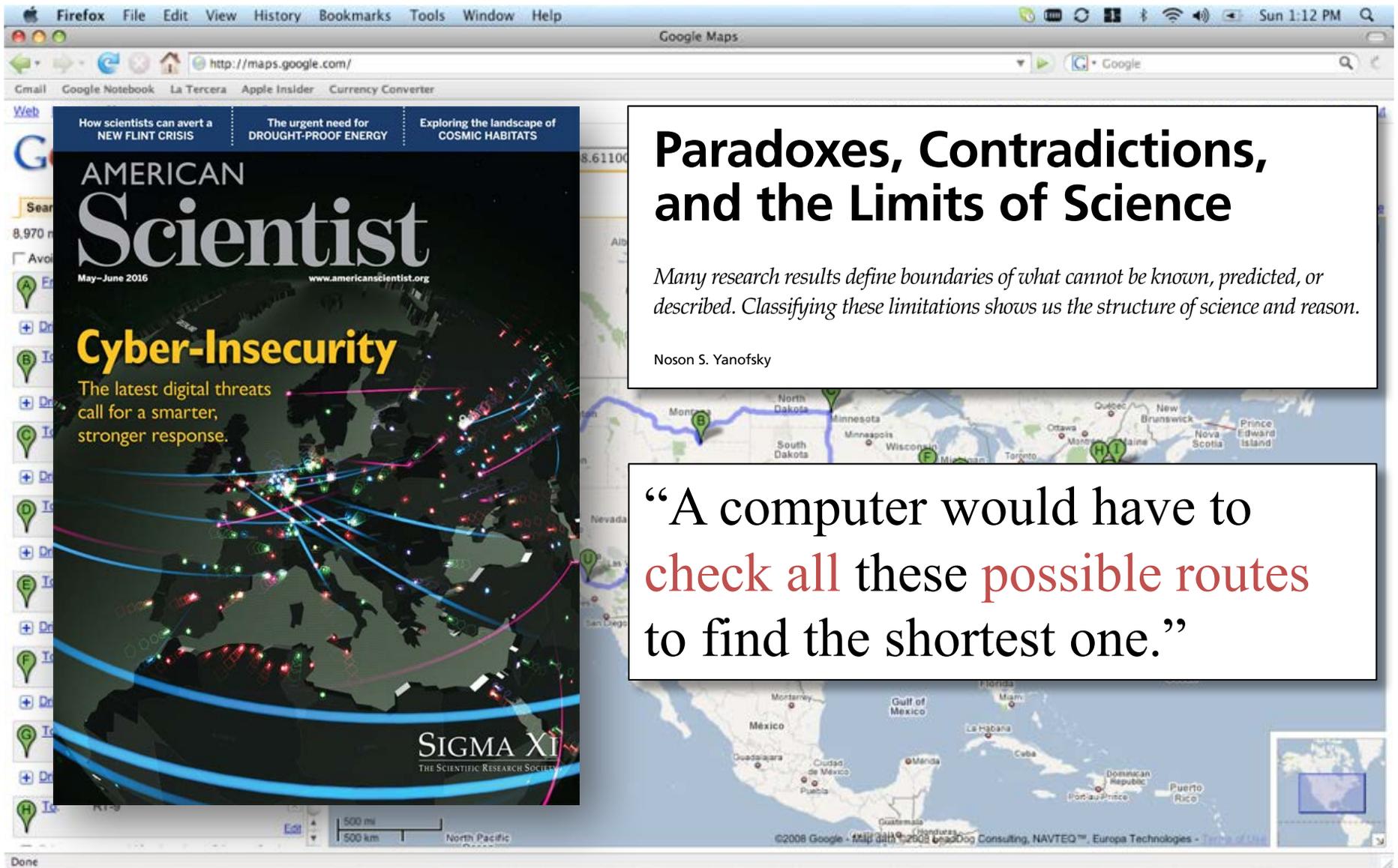
Avoid highways [Show all directions](#)

- A From:** I-5 N [Edit](#)
- + Drive:** 935 mi – about 13 hours 43 mins
- B To:** US-310 [Edit](#)
- + Drive:** 683 mi – about 9 hours 28 mins
- C To:** I-94 E [Edit](#)
- + Drive:** 535 mi – about 8 hours 8 mins
- D To:** US-36 [Edit](#)
- + Drive:** 327 mi – about 5 hours 32 mins
- E To:** I-72 E [Edit](#)
- + Drive:** 244 mi – about 3 hours 58 mins
- F To:** I-94 W/US-41 N [Edit](#)
- + Drive:** 219 mi – about 3 hours 39 mins
- G To:** I-69 N [Edit](#)
- + Drive:** 765 mi – about 12 hours 49 mins
- H To:** RT-9 [Edit](#)

Map controls: Street View, Traffic, Map, Satellite, Terrain

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Traveling Salesman Problem (TSP): Visit Cities Fast



The image is a screenshot of a Firefox browser window. The address bar shows 'http://maps.google.com/'. The browser's menu bar includes 'File', 'Edit', 'View', 'History', 'Bookmarks', 'Tools', 'Window', and 'Help'. The page content is split into two main sections. On the left, there is a preview of the cover of 'AMERICAN Scientist' magazine, dated May-June 2016. The cover features a dark background with a map of North America overlaid with a complex network of colorful lines (red, blue, green, yellow) representing connections or routes. The main headline on the cover is 'Cyber-Insecurity' in large yellow letters, with a sub-headline 'The latest digital threats call for a smarter, stronger response.' Below this, it says 'SIGMA XI THE SCIENTIFIC RESEARCH SOCIETY'. At the top of the magazine cover, there are three smaller headlines: 'How scientists can avert a NEW FLINT CRISIS', 'The urgent need for DROUGHT-PROOF ENERGY', and 'Exploring the landscape of COSMIC HABITATS'. On the right side of the browser window, there is a white text box with a black border. The text inside reads: 'Paradoxes, Contradictions, and the Limits of Science' in large black font. Below this, in a smaller italicized font, it says: 'Many research results define boundaries of what cannot be known, predicted, or described. Classifying these limitations shows us the structure of science and reason.' At the bottom of this box, it says 'Noson S. Yanofsky'. Below the text box is a map of North America, showing the United States and parts of Canada and Mexico. The map has several colored markers (green, red, blue) and lines connecting them, illustrating a Traveling Salesman Problem route. The map includes labels for various states and provinces, such as North Dakota, Minnesota, Wisconsin, Illinois, Indiana, Michigan, Ohio, Pennsylvania, New York, and others. At the bottom of the browser window, there is a scale bar showing 500 miles and 500 kilometers, and a copyright notice: '©2008 Google - Map data ©2008 GeoLog Consulting, NAVTEQ™, Europa Technologies - Terms of Use'.

AMERICAN
Scientist
May-June 2016
www.americanscientist.org

Cyber-Insecurity
The latest digital threats call for a smarter, stronger response.

SIGMA XI
THE SCIENTIFIC RESEARCH SOCIETY

Paradoxes, Contradictions, and the Limits of Science

Many research results define boundaries of what cannot be known, predicted, or described. Classifying these limitations shows us the structure of science and reason.

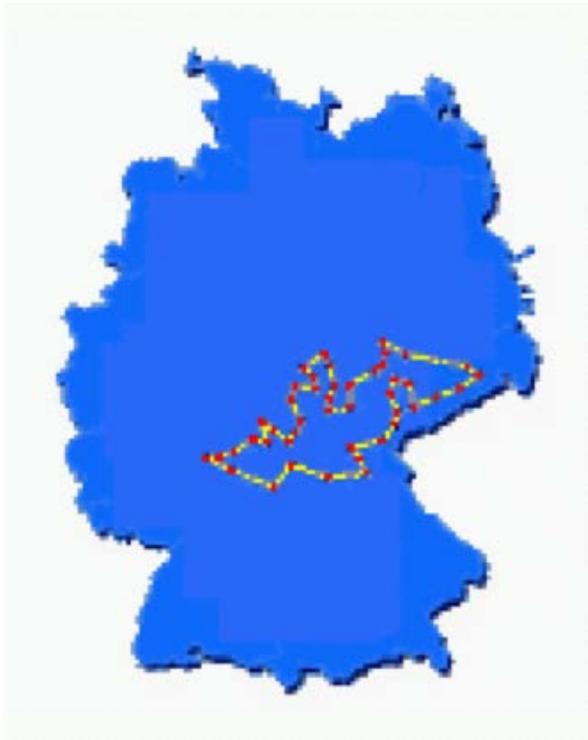
Noson S. Yanofsky

“A computer would have to check all these possible routes to find the shortest one.”

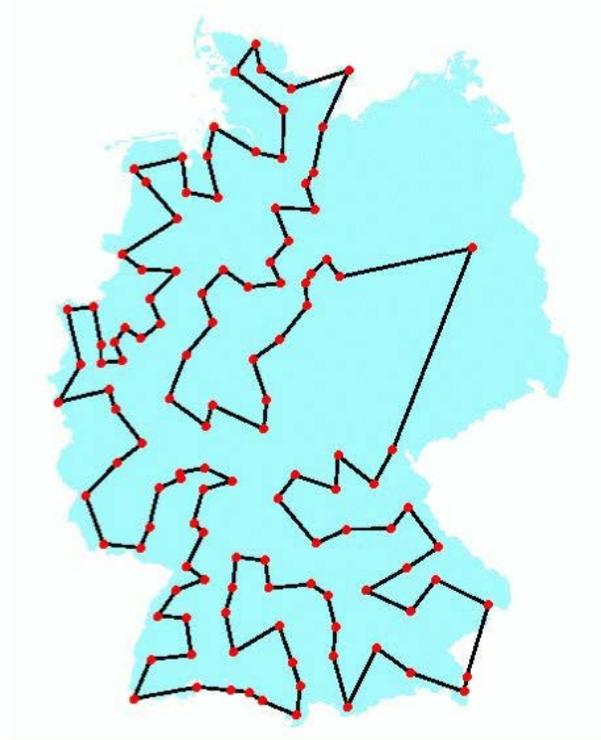
MIP = Avoid Enumeration

- Number of tours for 49 cities = $48!/2 \approx 10^{60}$
- Fastest supercomputer $\approx 10^{17}$ flops
- Assuming one floating point operation per tour:
> 10^{35} years $\approx 10^{25}$ times the age of the universe!
- How long does it take on an iphone?
 - Less than a second!
 - 4 iterations of **cutting plane** method!
 - Dantzig, Fulkerson and Johnson 1954 did it by hand!
 - For more info see tutorial in ConcordeTSP app
 - **Cutting planes** are the key for effectively solving (even NP-hard) MIP problems in practice.

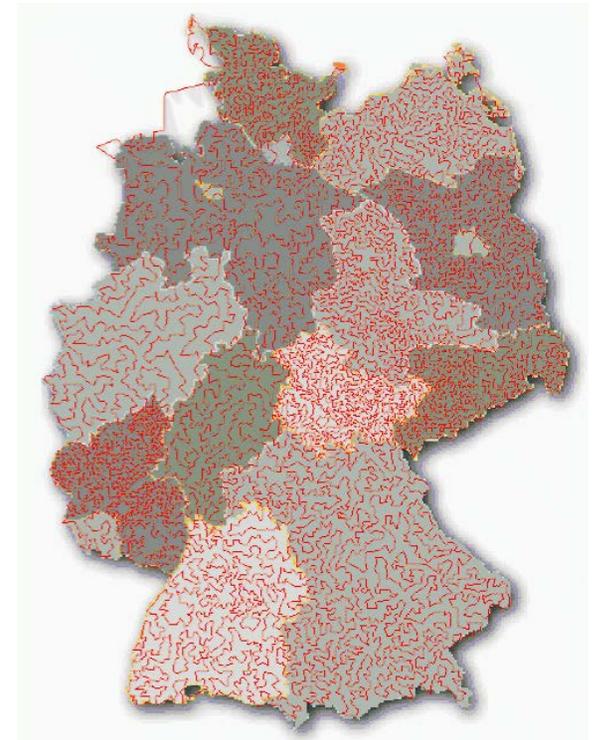
Using IP to visit Germany



45 cities(1832)



120 cities(1977)



15,112 cities(2004)

<http://www.math.uwaterloo.ca/tsp/d15sol/dhistory.html>

50+ Years of MIP = Significant Solver Speedups

- Algorithmic Improvements (Machine Independent):
 - CPLEX v1.2 (1991) – v11 (2007): 29,000x speedup
 - Gurobi v1 (2009) – v6.5 (2015): 48.7x speedup
 - Commercial, but free for academic use
- (Reasonably) effective free / open source solvers:
 - GLPK, CBC and SCIP (free only for non-commercial)
- Easy to use, fast and versatile modeling languages
 - Julia based JuMP modelling language
- Linear MIP solvers very mature and effective:
 - Convex nonlinear MIP getting there (quadratic nearly there)

Matching

Treated Units: $\mathcal{T} = \{t_1, \dots, t_T\}$

Control Units: $\mathcal{C} = \{c_1, \dots, c_C\}$

Observed Covariates: $\mathcal{P} = \{p_1, \dots, p_P\}$

Covariate Values: $\mathbf{x}^t = (x_p^t)_{p \in \mathcal{P}}, \quad t \in \mathcal{T}$

$\mathbf{x}^c = (x_p^c)_{p \in \mathcal{P}}, \quad c \in \mathcal{C}$

Maximum Cardinality Exact Matching

$$\max \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} m_{t,c}$$

s.t.

$$\sum_{t \in \mathcal{T}} m_{t,c} \leq 1, \quad \forall c \in \mathcal{C}$$

$$\sum_{c \in \mathcal{C}} m_{t,c} \leq 1, \quad \forall t \in \mathcal{T}$$

$$m_{t,c} = 0 \quad \forall t, c \quad \mathbf{x}^t \neq \mathbf{x}^c$$

$$0 \leq m_{t,c} \leq 1 \quad \cancel{m_{t,c} \in \{0, 1\}} \quad \forall t \in \mathcal{T}, \quad c \in \mathcal{C}.$$

- Solve time for truncated Lalonde CPS 429 = 0.001 s
- Why? Can solve relaxation.

Maximum Cardinality Marginal Means

$$\max \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} m_{t,c}$$

s.t.

$$\sum_{t \in \mathcal{T}} m_{t,c} \leq 1, \quad \forall c \in \mathcal{C}$$

$$\sum_{c \in \mathcal{C}} m_{t,c} \leq 1, \quad \forall t \in \mathcal{T}$$

$$\sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} \mathbf{x}_p^t m_{t,c} = \sum_{c \in \mathcal{C}} \sum_{t \in \mathcal{T}} \mathbf{x}_p^c m_{t,c} \quad \forall p \in \mathcal{P}$$

$$m_{t,c} \in \{0, 1\} \quad \forall t \in \mathcal{T}, \quad c \in \mathcal{C}.$$

- Solve time for truncated Lalonde CPS 429 = 444 s
- Why? One reason = relaxation has fractions.

Maximum Cardinality Fine Balance, Take 1

$$\max \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} m_{t,c}$$

s.t.

$$\sum_{t \in \mathcal{T}} m_{t,c}^p \leq 1,$$

$$\forall c \in \mathcal{C}, p \in \mathcal{P}$$

$$\sum_{c \in \mathcal{C}} m_{t,c}^p \leq 1,$$

$$\forall t \in \mathcal{T}, p \in \mathcal{P}$$

$$m_{t,c}^p = 0$$

$$\forall t, c \quad \mathbf{x}_p^t \neq \mathbf{x}_p^c, p \in \mathcal{P}$$

$$\sum_{t \in \mathcal{T}} m_{t,c}^p = \sum_{t \in \mathcal{T}} m_{t,c}^q$$

$$\forall c \in \mathcal{C}, p, q \in \mathcal{P}$$

$$\sum_{c \in \mathcal{C}} m_{t,c}^p = \sum_{c \in \mathcal{C}} m_{t,c}^q$$

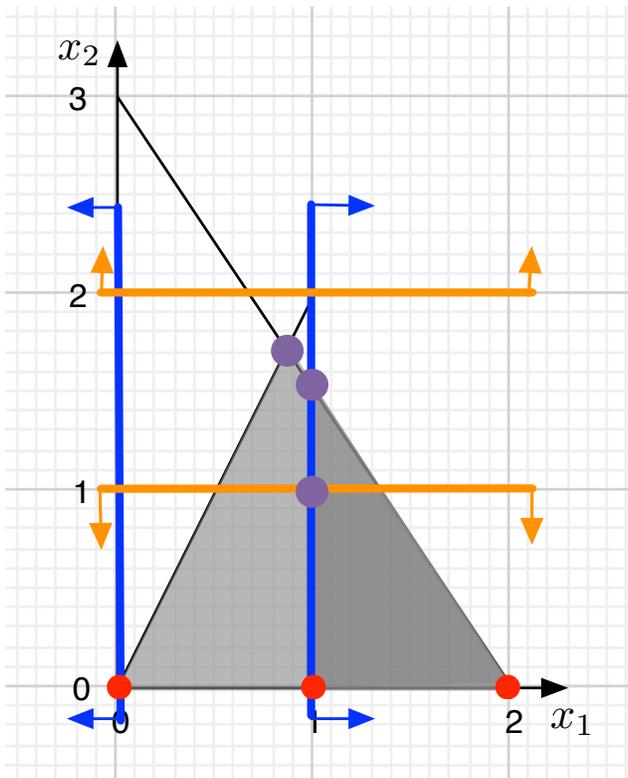
$$\forall t \in \mathcal{T}, p, q \in \mathcal{P}$$

$$m_{t,c}^p \in \{0, 1\}$$

$$\forall t \in \mathcal{T}, c \in \mathcal{C}, p \in \mathcal{P}.$$

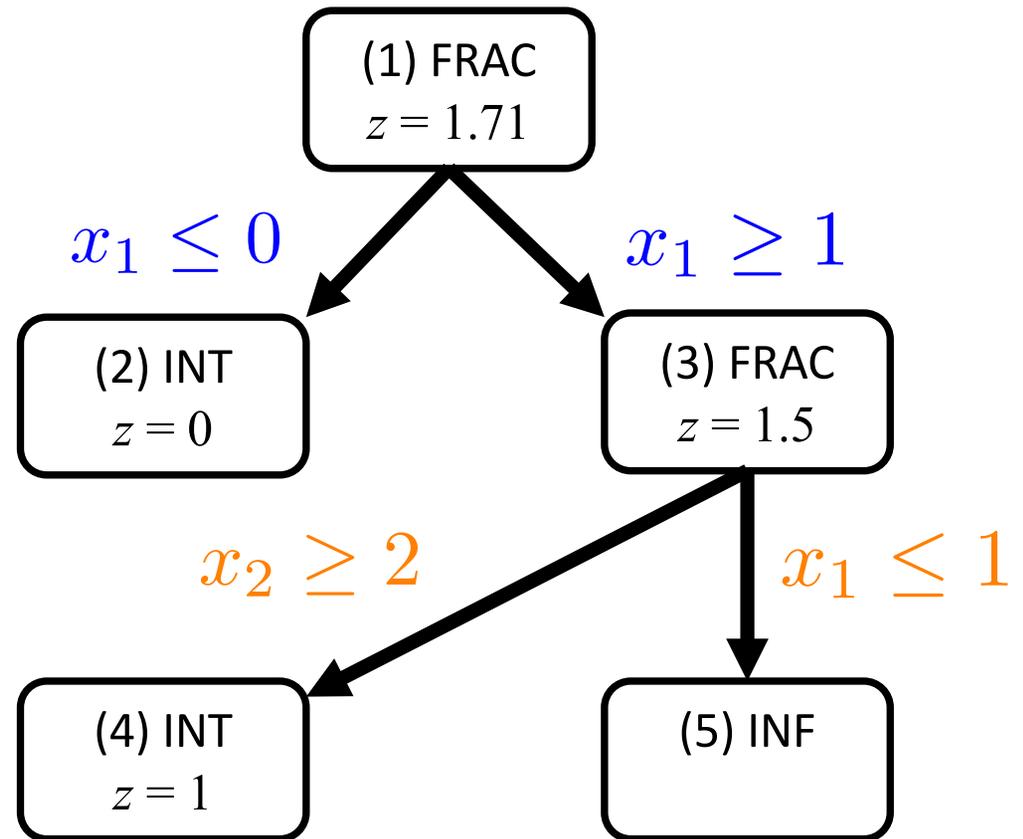
- Solve time for truncated Lalonde CPS 429 = 0.81 s
- Why? One reason = relaxation has “fewer” fractions.

Basic Branch-and-Bound

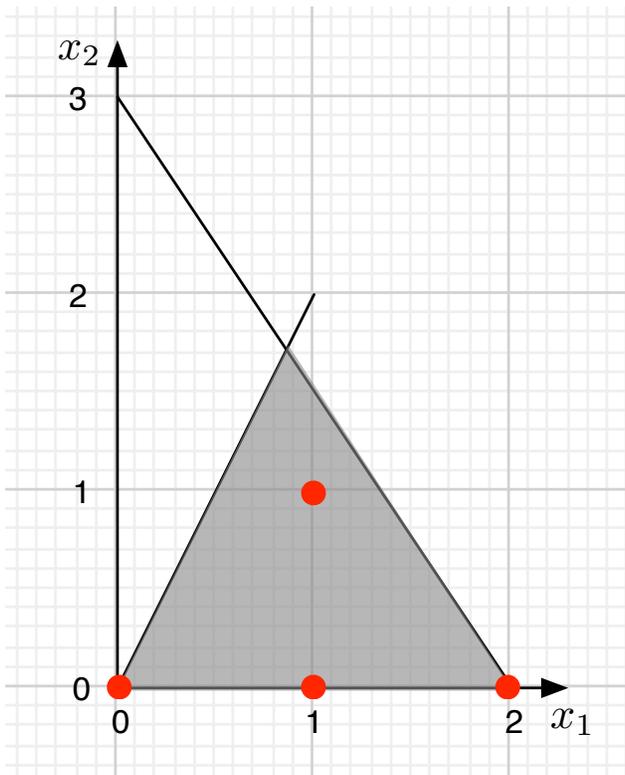


$$\begin{aligned} \max z &:= x_2 \\ 3x_1 + 2x_2 &\leq 6 \\ -2x_1 + x_2 &\leq 0 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\in \mathbb{Z} \end{aligned}$$

← Linear Programming (LP) Relaxation

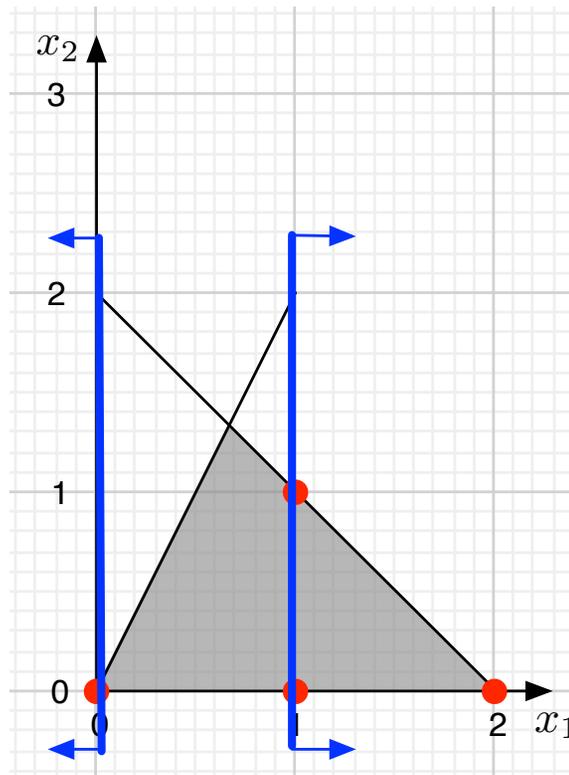


Stronger Formulations = Faster Solves ?



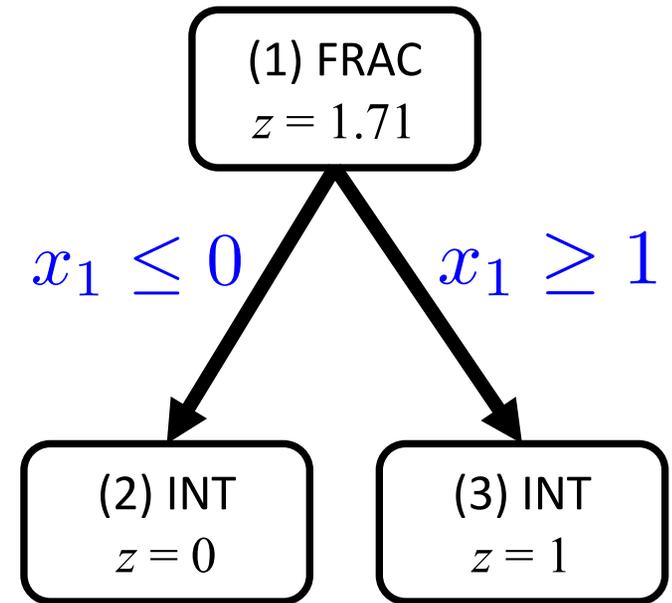
$$\begin{aligned} \max z &:= x_2 \\ 3x_1 + 2x_2 &\leq 6 \\ -2x_1 + x_2 &\leq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$x_1, x_2 \in \mathbb{Z}$$



$$\begin{aligned} \max z &:= x_2 \\ x_1 + x_2 &\leq 2 \\ -2x_1 + x_2 &\leq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$x_1, x_2 \in \mathbb{Z}$$



Cutting Plane Example: Chátal-Gomory Cuts

$$P := \left\{ x \in \mathbb{R}^2 : \begin{array}{l} x_1 + x_2 \leq 3, \\ 5x_1 - 3x_2 \leq 3 \end{array} \right\}$$

\cap

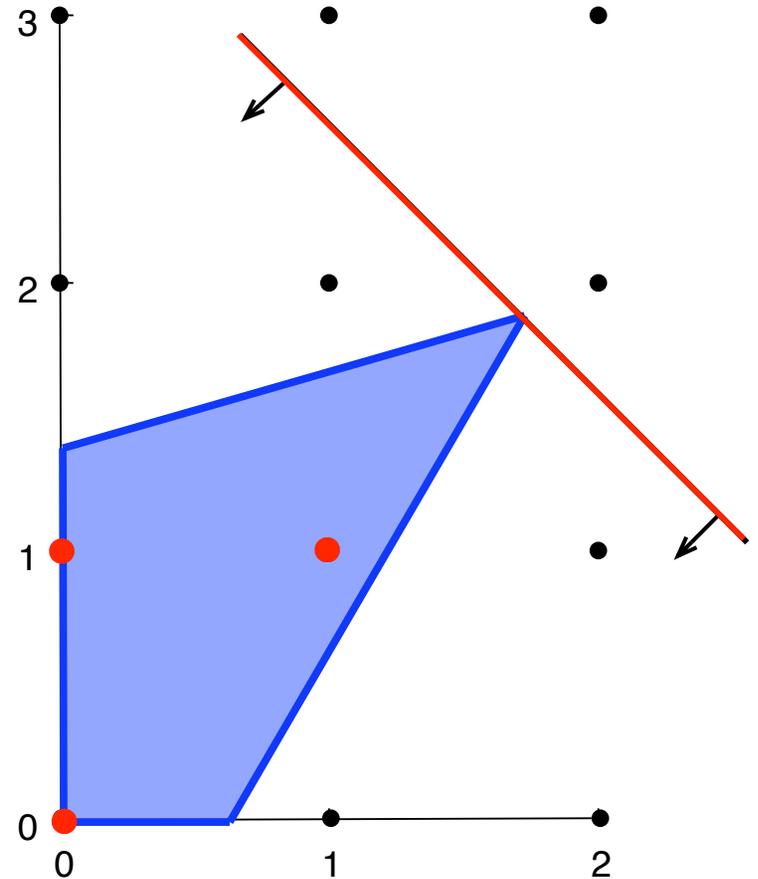
$$H := \{ x \in \mathbb{R}^2 : \underbrace{4x_1 + 3x_2}_{\in \mathbb{Z}} \leq 10.5 \}$$

if $x \in \mathbb{Z}^2$

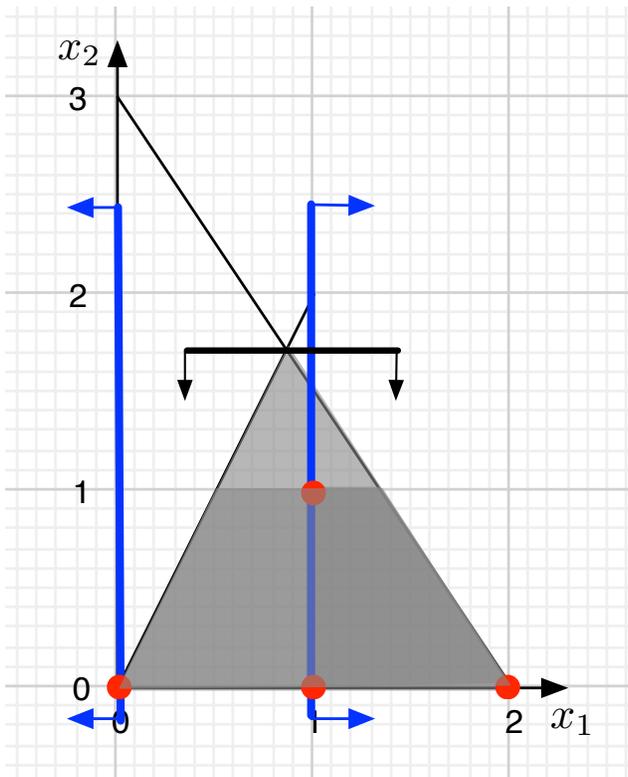
$$4x_1 + 3x_2 \leq \lfloor 10.5 \rfloor$$

Valid for $H \cap \mathbb{Z}^2$

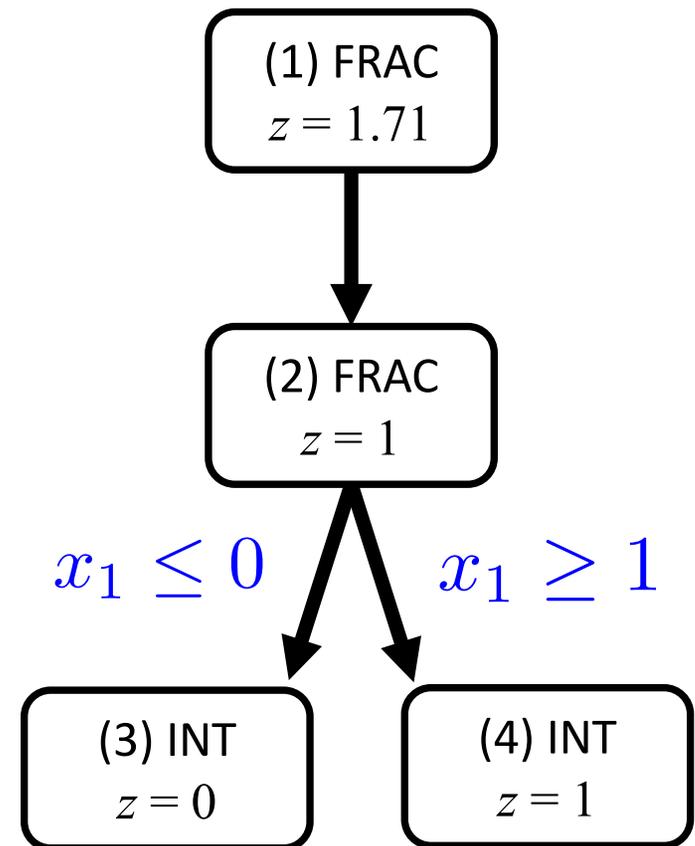
Valid for $P \cap \mathbb{Z}^2$



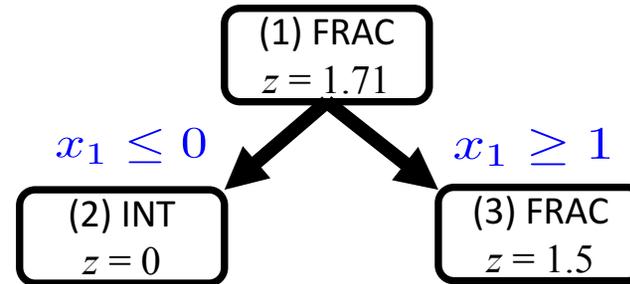
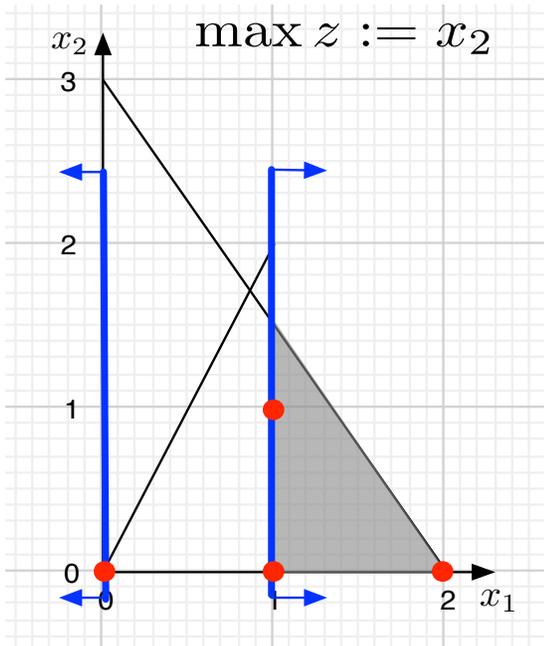
Branch-and-Bound and Cuts (Branch-and-Cut)



$$\begin{aligned} \max z &:= x_2 \\ 3x_1 + 2x_2 &\leq 6 & x_2 &\leq \lfloor 1.71 \rfloor = 1 \\ -2x_1 + x_2 &\leq 0 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\in \mathbb{Z} \end{aligned}$$



Partial Solves and GAP



$$GAP = 100 \times \frac{|\text{bestnode} - \text{bestinteger}|}{10^{-10} + |\text{bestinteger}|}$$

$$BBGAP = 100 \times \frac{|1.5 - 0|}{10^{-10} + |0|} = 1.5 \times 10^{12}$$

$$BB^+ GAP = 100 \times \frac{|1.5 - 1|}{10^{-10} + |1|} = 50\%$$

	Node	Left	Objective	Nodes	IInf	Best Integer	Best Bound	Cuts/ ItCnt	Gap
		0	0	1415.0000	16		1415.0000	143	
		0	0	1446.4286	45		ZeroHalf: 33	317	
*	0+	0				3130.0000	1446.4286	317	53.79%
	0	2	1446.4286		34	3130.0000	1446.4286	317	53.79%
Elapsed real time = 0.18 sec. (tree size = 0.01 MB, solutions = 1)									
*	28	28	integral		0	3060.0000	1449.7619	1168	52.62%
*	34	32	integral		0	3045.0000	1449.7619	1320	52.39%

Maximum Cardinality Marginal Means

$$\max \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} m_{t,c}$$

s.t.

$$\sum_{t \in \mathcal{T}} m_{t,c} \leq 1, \quad \forall c \in \mathcal{C}$$

$$\sum_{c \in \mathcal{C}} m_{t,c} \leq 1, \quad \forall t \in \mathcal{T}$$

$$\sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} \mathbf{x}_p^t m_{t,c} = \sum_{c \in \mathcal{C}} \sum_{t \in \mathcal{T}} \mathbf{x}_p^c m_{t,c} \quad \forall p \in \mathcal{P}$$

$$m_{t,c} \in \{0, 1\} \quad \forall t \in \mathcal{T}, \quad c \in \mathcal{C}.$$

- Solve time for truncated Lalonde CPS 429 = 444 s
- Optimal = 17, LP Relaxation = 19.35

Maximum Cardinality Marginal Means

Root relaxation: objective 1.935024e+01, 1947 iterations, 0.16 seconds

Nodes		Current Node				Objective Bounds			Work	
Expl	Unexpl	Obj	Depth	IntInf	Incumbent	BestBd	Gap	It/Node	Time	
0	0	19.35024	0	10	-0.00000	19.35024	-	-	0s	
0	0	19.30273	0	20	-0.00000	19.30273	-	-	1s	
.	
0	2	19.29748	0	25	-0.00000	19.29748	-	-	4s	
4	5	19.20830	3	23	-0.00000	19.29273	-	1407	5s	
.	
1140	763	18.56894	18	23	-0.00000	19.16557	-	340	51s	
H 1149	726				4.0000000	19.16557	379%	344	51s	
.	
*23876	7827			71	17.0000000	18.00366	5.90%	150	443s	

“Black Magic”



Cutting planes: Gomory: 1 Flow cover: 1

Explored 25301 nodes (3601039 simplex iterations) in 443.89 seconds

Optimal solution found (tolerance 1.00e-04)

Best objective 1.700000000000e+01, best bound 1.700000000000e+01, gap 0.0%



Better



Feasible Solution

Optimal
Solution

Bound

Maximum Cardinality Fine Balance, Take 1

$$\max \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} m_{t,c}$$

s.t.

$$\sum_{t \in \mathcal{T}} m_{t,c}^p \leq 1,$$

$$\forall c \in \mathcal{C}, p \in \mathcal{P}$$

$$\sum_{c \in \mathcal{C}} m_{t,c}^p \leq 1,$$

$$\forall t \in \mathcal{T}, p \in \mathcal{P}$$

$$m_{t,c}^p = 0$$

$$\forall t, c \quad \mathbf{x}_p^t \neq \mathbf{x}_p^c, p \in \mathcal{P}$$

$$\sum_{t \in \mathcal{T}} m_{t,c}^p = \sum_{t \in \mathcal{T}} m_{t,c}^q$$

$$\forall c \in \mathcal{C}, p, q \in \mathcal{P}$$

$$\sum_{c \in \mathcal{C}} m_{t,c}^p = \sum_{c \in \mathcal{C}} m_{t,c}^q$$

$$\forall t \in \mathcal{T}, p, q \in \mathcal{P}$$

$$m_{t,c}^p \in \{0, 1\}$$

$$\forall t \in \mathcal{T}, c \in \mathcal{C}, p \in \mathcal{P}.$$

- Solve time for truncated Lalonde CPS 429 = 0.81 s
- Optimal = 10, LP Relaxation = 11

Maximum Cardinality Fine Balance, Take 1

Root relaxation: objective 1.100000e+01, 1490 iterations, 0.03 seconds

Nodes		Current Node			Objective Bounds			Work	
Expl	Unexpl	Obj	Depth	IntInf	Incumbent	BestBd	Gap	It/Node	Time
0	0	11.00000	0	158	-0.00000	11.00000	-	-	0s
0	0	11.00000	0	164	-0.00000	11.00000	-	-	0s
0	0	11.00000	0	164	-0.00000	11.00000	-	-	0s
H	0				10.0000000	11.00000	10.0%	-	0s
.									
.									
.									
0	2	11.00000	0	111	10.00000	11.00000	10.0%	-	0s

Cutting planes: Zero half: 2

Explored 8 nodes (9933 simplex iterations) in 0.78 seconds

Optimal solution found (tolerance 1.00e-04)

Best objective 1.00000000000000e+01, best bound 1.00000000000000e+01, gap 0.0%



Maximum Cardinality Fine Balance, Take 1

$$\max \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} m_{t,c}$$

s.t.

$$\sum_{t \in \mathcal{T}} m_{t,c}^p \leq 1,$$

$$\forall c \in \mathcal{C}, p \in \mathcal{P}$$

$$\sum_{c \in \mathcal{C}} m_{t,c}^p \leq 1,$$

$$\forall t \in \mathcal{T}, p \in \mathcal{P}$$

$$m_{t,c}^p = 0$$

$$\forall t, c \quad \mathbf{x}_p^t \neq \mathbf{x}_p^c, p \in \mathcal{P}$$

$$\sum_{t \in \mathcal{T}} m_{t,c}^p = \sum_{t \in \mathcal{T}} m_{t,c}^q$$

$$\forall c \in \mathcal{C}, p, q \in \mathcal{P}$$

$$\sum_{c \in \mathcal{C}} m_{t,c}^p = \sum_{c \in \mathcal{C}} m_{t,c}^q$$

$$\forall t \in \mathcal{T}, p, q \in \mathcal{P}$$

$$m_{t,c}^p \in \{0, 1\}$$

$$\forall t \in \mathcal{T}, c \in \mathcal{C}, p \in \mathcal{P}.$$

- Solve time for truncated Lalonde CPS 429 = 0.81 s
- Scalability? Size = $|\mathcal{T}| \times |\mathcal{C}| \times |\mathcal{P}|$

Maximum Cardinality Fine Balance, Take 2

$$\max \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} m_{t,c}$$

s.t.

$$\sum_{t \in \mathcal{T}} m_{t,c} \leq 1,$$

$$\sum_{c \in \mathcal{C}} m_{t,c} \leq 1,$$

$$\sum_{t \in \mathcal{T}_{p,k}} \sum_{c \notin \mathcal{C}_{p,k}} m_{t,c} = \sum_{t \notin \mathcal{T}_{p,k}} \sum_{c \in \mathcal{C}_{p,k}} m_{t,c} \quad \forall p \in \mathcal{P}, k \in \mathcal{K}(p)$$

$$m_{t,c} \in \{0, 1\} \quad \forall t \in \mathcal{T}, c \in \mathcal{C}.$$

$$\mathcal{K}(p) = \{\mathbf{x}_p^c\}_{c \in \mathcal{P}} \cup \{\mathbf{x}_p^t\}_{t \in \mathcal{T}}$$

$$\mathcal{C}_{p,k} = \{c \in \mathcal{C} : \mathbf{x}_p^c = k\}$$

$$\mathcal{T}_{p,k} = \{t \in \mathcal{T} : \mathbf{x}_p^t = k\}$$

$$\forall c \in \mathcal{C}$$

$$\forall t \in \mathcal{T}$$

$$\forall p \in \mathcal{P}, k \in \mathcal{K}(p)$$

$$\forall t \in \mathcal{T}, c \in \mathcal{C}.$$

- Solve time for truncated Lalonde CPS 429 = 0.61 s
- Scalability? Size = $|\mathcal{T}| \times |\mathcal{C}| + |\mathcal{P}|$

Take 1 Revisited

$$\max \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} m_{t,c}$$

s.t.

$$\begin{aligned} \sum_{t \in \mathcal{T}} m_{t,c}^p &\leq 1, & \forall c \in \mathcal{C}, p \in \mathcal{P} \\ \sum_{c \in \mathcal{C}} m_{t,c}^p &\leq 1, & \forall t \in \mathcal{T}, p \in \mathcal{P} \\ m_{t,c}^p &= 0 & \forall t, c \quad \mathbf{x}_p^t \neq \mathbf{x}_p^c, p \in \mathcal{P} \\ \sum_{t \in \mathcal{T}} m_{t,c}^p &= \sum_{t \in \mathcal{T}} m_{t,c}^q & \forall c \in \mathcal{C}, p, q \in \mathcal{P} \\ \sum_{c \in \mathcal{C}} m_{t,c}^p &= \sum_{c \in \mathcal{C}} m_{t,c}^q & \forall t \in \mathcal{T}, p, q \in \mathcal{P} \\ m_{t,c}^p &\in \{0, 1\} & \forall t \in \mathcal{T}, c \in \mathcal{C}, p \in \mathcal{P}. \end{aligned}$$

- Solve time for truncated Lalonde CPS 429 = 0.81 s
- Scalability? Size = $|\mathcal{T}| \times |\mathcal{C}| \times |\mathcal{P}|$

Take 1 Revisited

$$\max \sum_{t \in \mathcal{T}} x_t$$

s.t.

$$\sum_{t \in \mathcal{T}_{p,k}} m_{t,c}^p = y_c, \quad \forall p \in \mathcal{P}, \quad k \in \mathcal{K}(p), \quad c \in \mathcal{C}_{p,k}$$

$$\sum_{c \in \mathcal{C}_{p,k}} m_{t,c}^p = x_t, \quad \forall \mathcal{S} \in \mathcal{F}, \quad k \in \mathcal{K}(p), \quad t \in \mathcal{T}_{p,k}$$

$$m_{t,c}^p \in \{0, 1\} \quad \forall p \in \mathcal{P}, \quad k \in \mathcal{K}(p), \quad t \in \mathcal{T}_{p,k}, \quad c \in \mathcal{C}_{p,k}$$

$$x_t \in \{0, 1\} \quad \forall t \in \mathcal{T}$$

$$y_c \in \{0, 1\} \quad \forall c \in \mathcal{C}.$$

Maximum Cardinality Fine Balance, Take 3

$$\max \sum_{t \in \mathcal{T}} x_t$$

s.t.

$$\sum_{t \in \mathcal{T}} x_t = \sum_{c \in \mathcal{C}} y_c,$$

$$\sum_{t \in \mathcal{T}_{p,k}} x_t = \sum_{c \in \mathcal{C}_{p,k}} y_c, \quad \forall p \in \mathcal{P}, \quad k \in \mathcal{K}(p)$$

$$x_t \in \{0, 1\} \quad \forall t \in \mathcal{T}$$

$$y_c \in \{0, 1\} \quad \forall c \in \mathcal{C}.$$

$$\mathcal{K}(p) = \{\mathbf{x}_p^c\}_{c \in \mathcal{C}} \cup \{\mathbf{x}_p^t\}_{t \in \mathcal{T}}$$

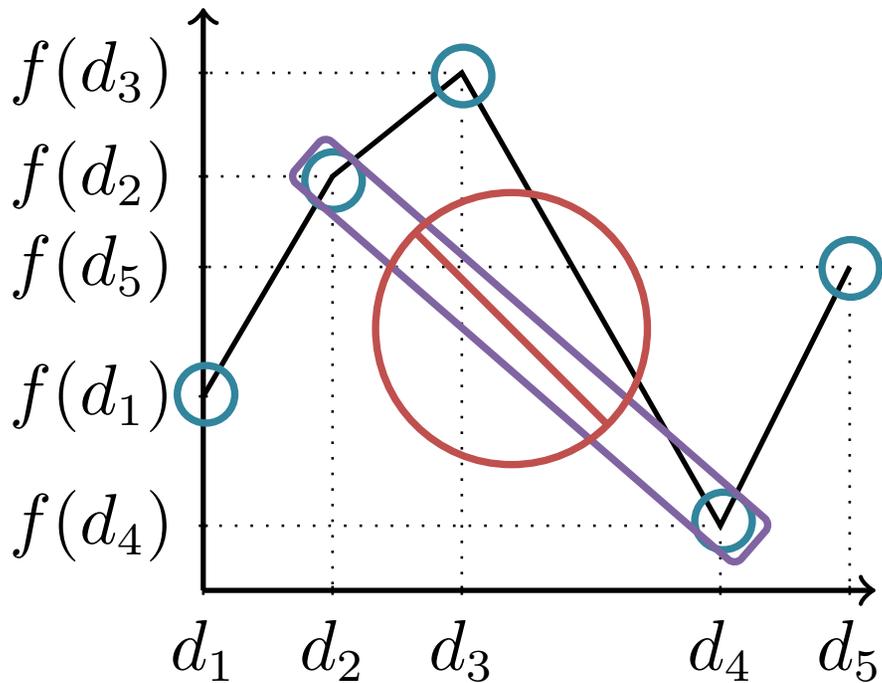
$$\mathcal{C}_{p,k} = \{c \in \mathcal{C} : \mathbf{x}_p^c = k\}$$

$$\mathcal{T}_{p,k} = \{t \in \mathcal{T} : \mathbf{x}_p^t = k\}$$

- Solve time for truncated Lalonde CPS 429 = 0.006 s
- Scalability? Size = $|\mathcal{P}| \times (|\mathcal{T}| + |\mathcal{C}|)$

Simple Formulation for Univariate Functions

$$z = f(x)$$



Size = $O(\# \text{ of segments})$

Non-Ideal: Fractional Extreme Points

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^5 \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^5 \lambda_j, \quad \lambda_j \geq 0$$

$$y \in \{0, 1\}^4, \quad \sum_{i=1}^4 y_i = 1$$

$$0 \leq \lambda_1 \leq y_1$$

$$0 \leq \lambda_2 \leq y_1 + y_2$$

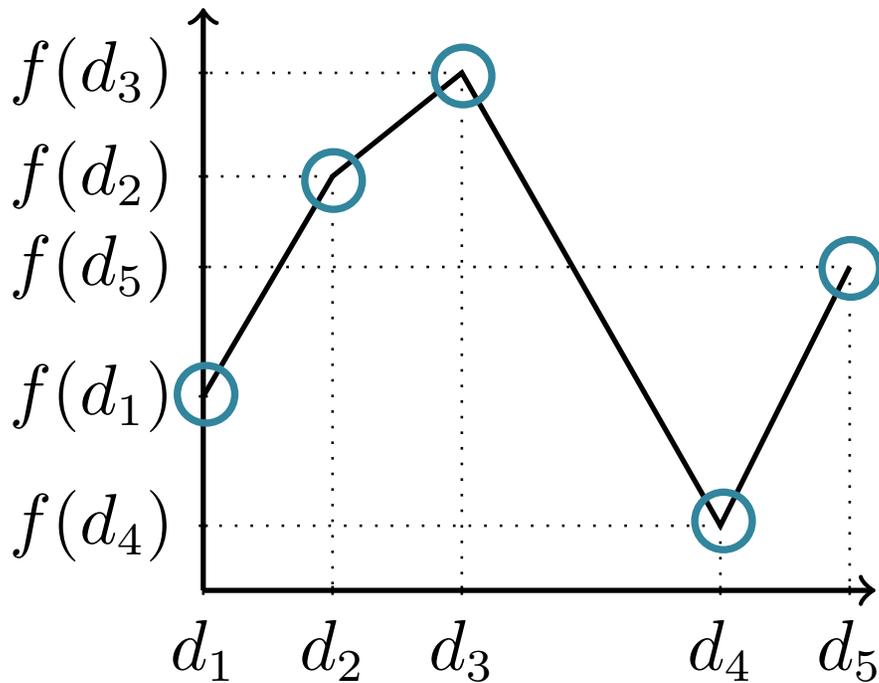
$$0 \leq \lambda_3 \leq y_2 + y_3$$

$$0 \leq \lambda_4 \leq y_3 + y_4$$

$$0 \leq \lambda_5 \leq y_4$$

Advanced Formulation for Univariate Functions

$$z = f(x)$$



Size = $O(\log_2 \# \text{ of segments})$

Ideal: Integral Extreme Points

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^5 \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^5 \lambda_j, \quad \lambda_j \geq 0$$

$$y \in \{0, 1\}^2$$

$$0 \leq \lambda_1 + \lambda_5 \leq 1 - y_1$$

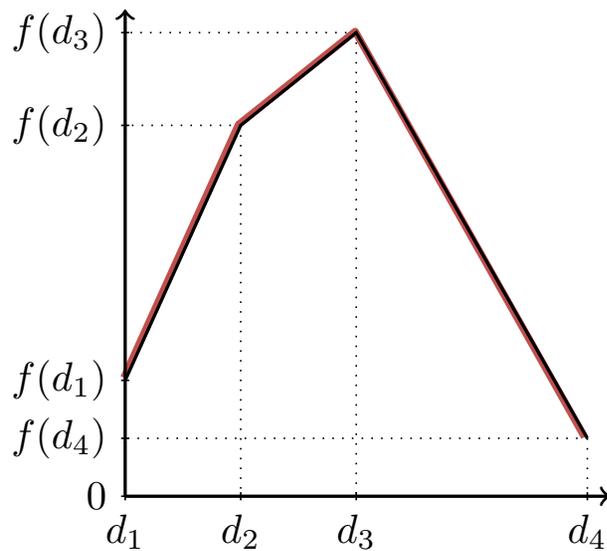
$$0 \leq \lambda_3 \leq y_1$$

$$0 \leq \lambda_4 + \lambda_5 \leq 1 - y_2$$

$$0 \leq \lambda_1 + \lambda_2 \leq y_2$$

Extended Formulation for PWL Functions

$$S = \text{gr}(f) = \bigcup_{i=1}^k \left\{ (x, z) \in \mathbb{R}^2 : \begin{array}{l} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\} \quad \text{MC Formulation:}$$



$$d_i y_i \leq x^i \leq d_{i+1} y_i \quad \forall i \in [k]$$

$$m_i x^i + c_i y_i = z^i \quad \forall i \in [k]$$

$$\sum_{i=1}^k x^i = x$$

$$\sum_{i=1}^k z^i = z$$

$$\sum_{i=1}^k y_i = 1$$

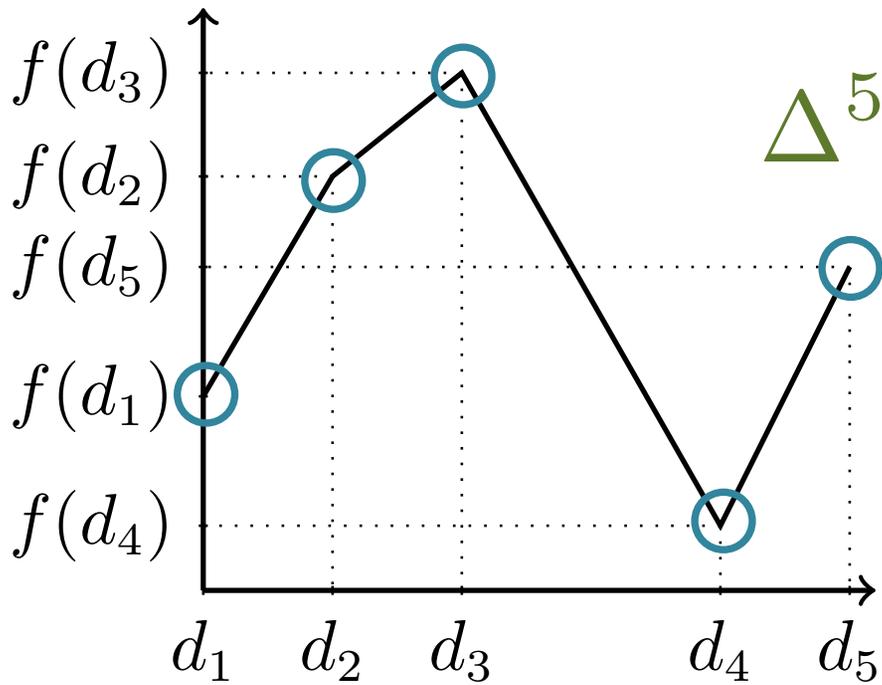
$$y \in \{0, 1\}^k$$

Abstracting Univariate Functions

$$P_i := \left\{ \lambda \in \Delta^5 : \lambda_j = 0 \quad \forall j \notin T_i \right\}$$

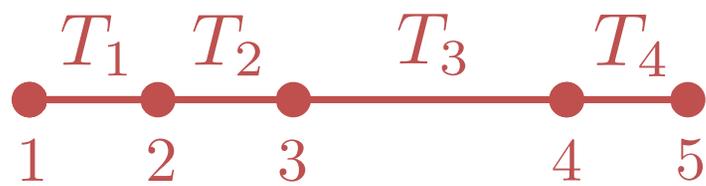
$$f(x) = \sum_{j=1}^5 \binom{x}{d_j} \frac{f(d_j)}{f(d_j)} \lambda_j$$

$$T_i := \{i, i+1\} \quad i \in \{1, \dots, 4\}$$



Δ^5

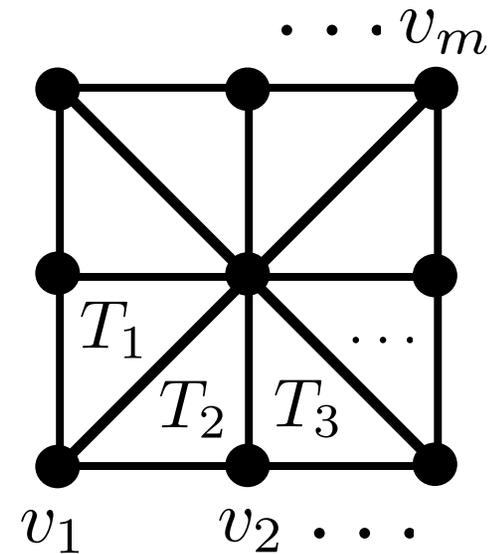
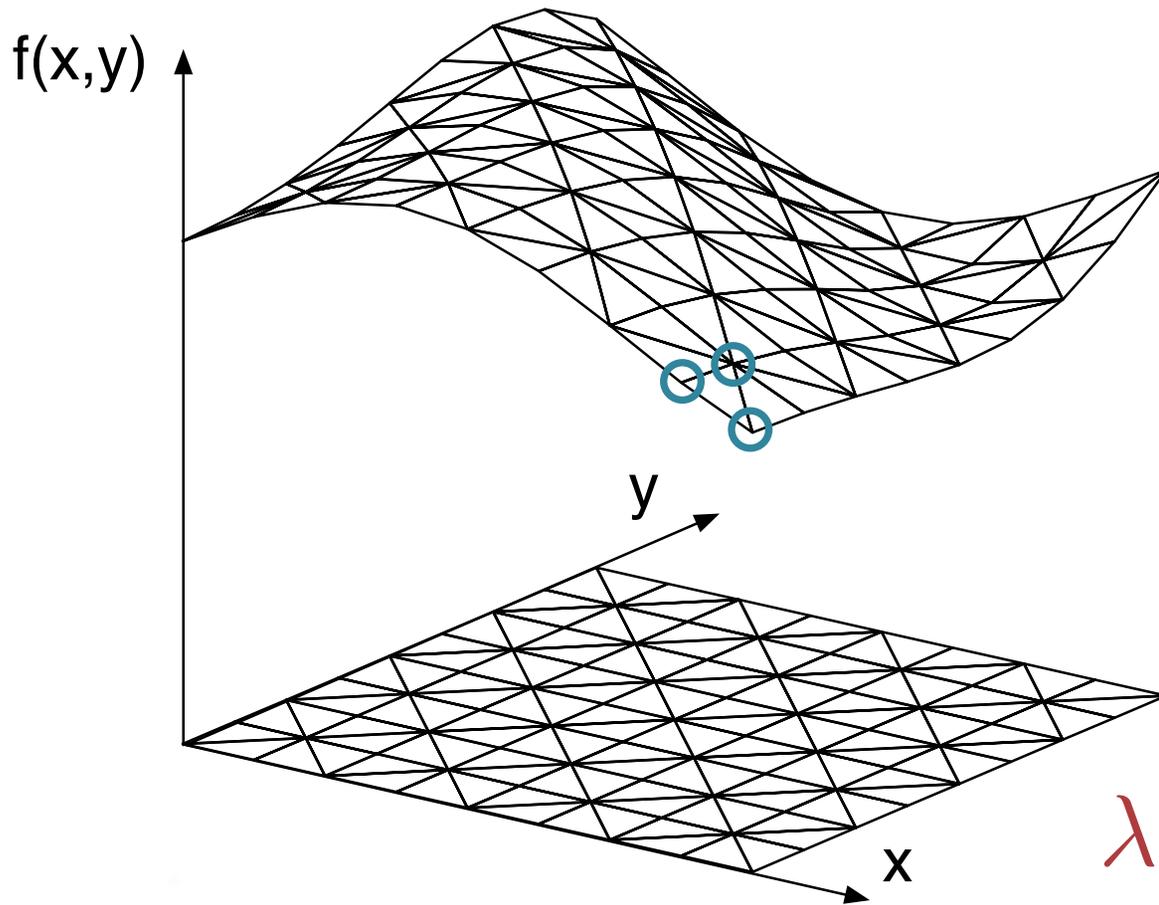
$$1 = \sum_{j=1}^5 \lambda_j, \quad \lambda_j \geq 0$$



$$\lambda \in \bigcup_{i=1}^4 P_i \subseteq \Delta^5$$

Abstraction Works for Multivariate Functions

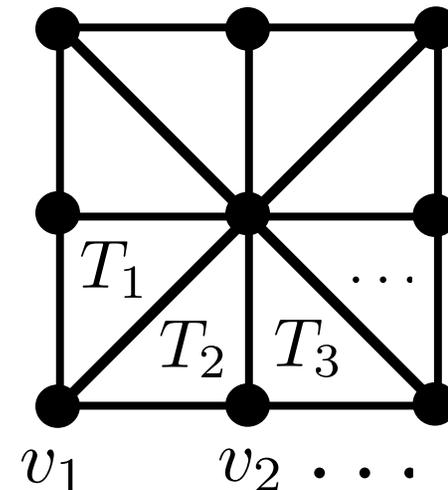
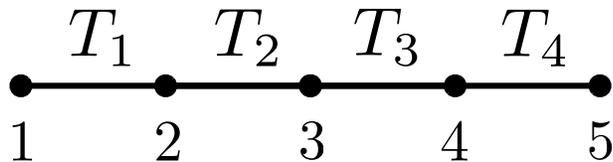
$$P_i := \{\lambda \in \Delta^m : \lambda_j = 0 \quad \forall v_j \notin T_i\}$$



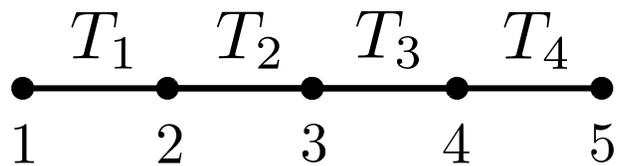
$$\lambda \in \bigcup_{i=1}^n P_i \subseteq \Delta^m$$

Complete Abstraction

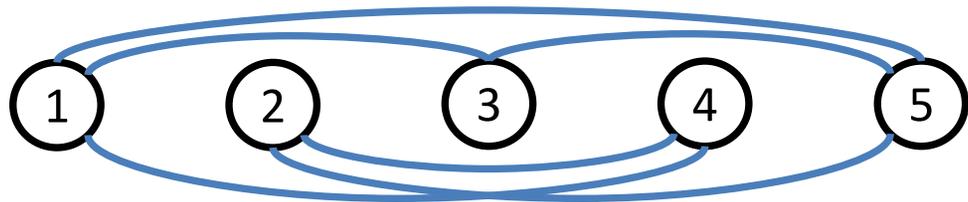
- $\Delta^V := \left\{ \lambda \in \mathbb{R}_+^V : \sum_{v \in V} \lambda_v = 1 \right\}$,
- $P_i = \left\{ \lambda \in \Delta^V : \lambda_v = 0 \quad \forall v \notin T_i \right\}$
- $\lambda \in \bigcup_{i=1}^n P_i$
- $T_i = \text{cliques of a graph}$



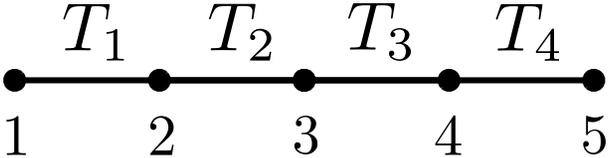
From Cliques to (Complement) Conflict Graph



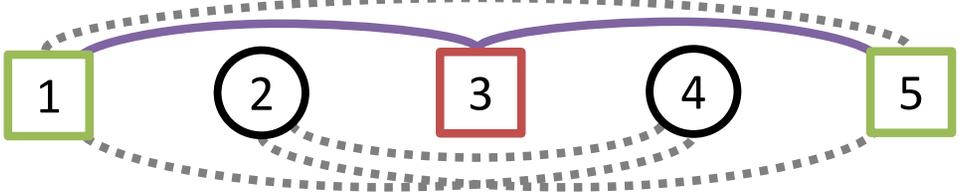
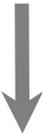
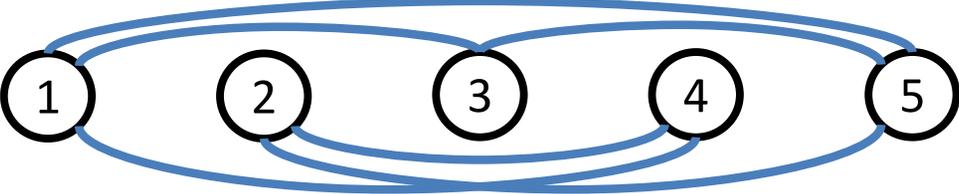
SOS2



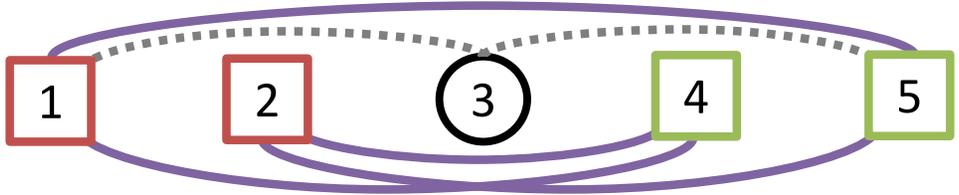
From Conflict Graph to Bi-clique Cover



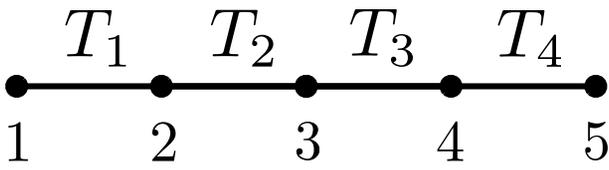
SOS2



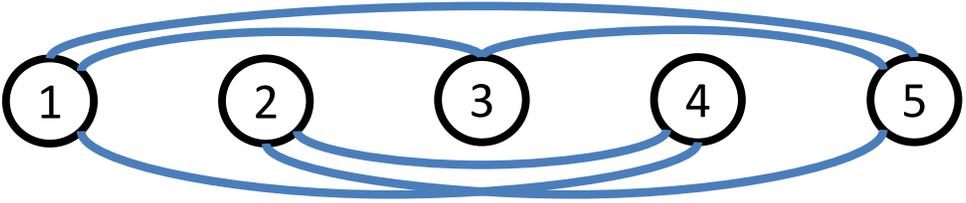
+



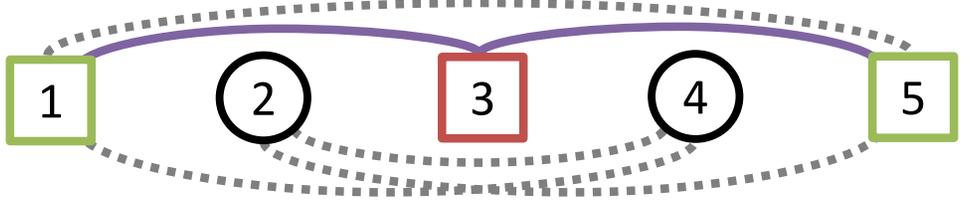
From Bi-clique Cover to Formulation



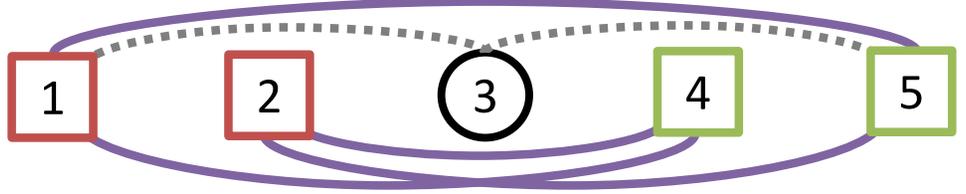
SOS2



$$\begin{aligned}
 0 &\leq \lambda_1 + \lambda_5 \leq 1 - y_1 \\
 0 &\leq \lambda_3 \leq y_1 \\
 0 &\leq \lambda_4 + \lambda_5 \leq 1 - y_2 \\
 0 &\leq \lambda_1 + \lambda_2 \leq y_2
 \end{aligned}$$



+



Ideal Formulation from Bi-clique Cover

- Conflict Graph $G = (V, E)$

$$E = \{(u, v) : u, v \in V, u \neq v, \nexists i \text{ s.t. } u, v \in T_i\}$$

- Bi-clique cover $\{(A^j, B^j)\}_{j=1}^t$, $A^j, B^j \subseteq V$

$$\forall \{u, v\} \in E \quad \exists j \text{ s.t. } u \in A^j \wedge v \in B^j$$

- Formulation

$$\sum_{v \in A^j} \lambda_v \leq 1 - y_j \quad \forall j \in [t]$$

$$\sum_{v \in B^j} \lambda_v \leq y_j \quad \forall j \in [t]$$

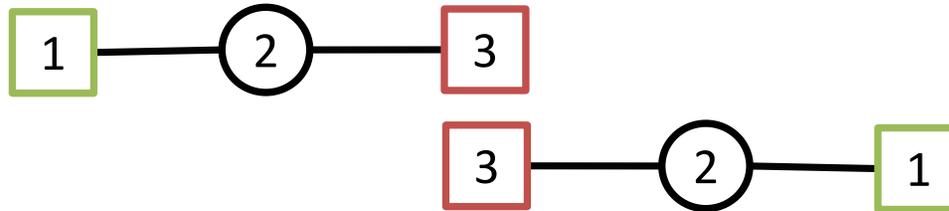
$$y \in \{0, 1\}^t$$

Recursive Construction of Cover for SOS2, Step 1

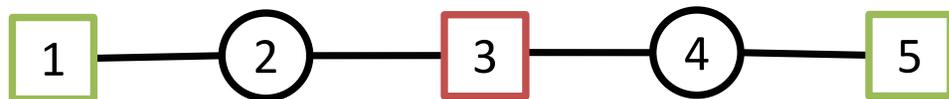
Base case $n=2^1$:



Step 1 recursion :



Reflect Graph / Cover



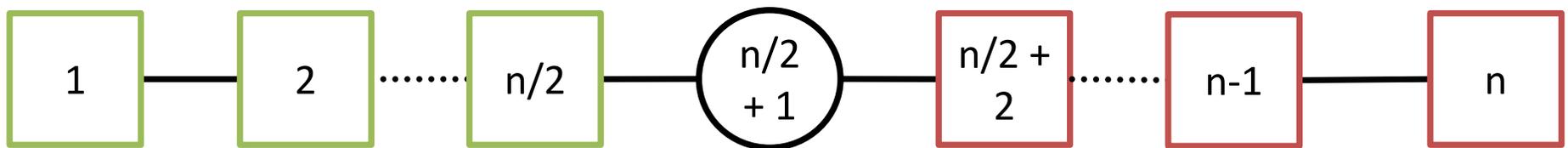
Stick Graph / Cover

Repeat for all bi-cliques from 2^{k-1}
to cover all edges within first and
last half of conflict graph

Recursive Construction of Cover for SOS2, Step 2

Only edges missing are those between first and last half of conflict graph

Step 2 : Add one more bi-clique

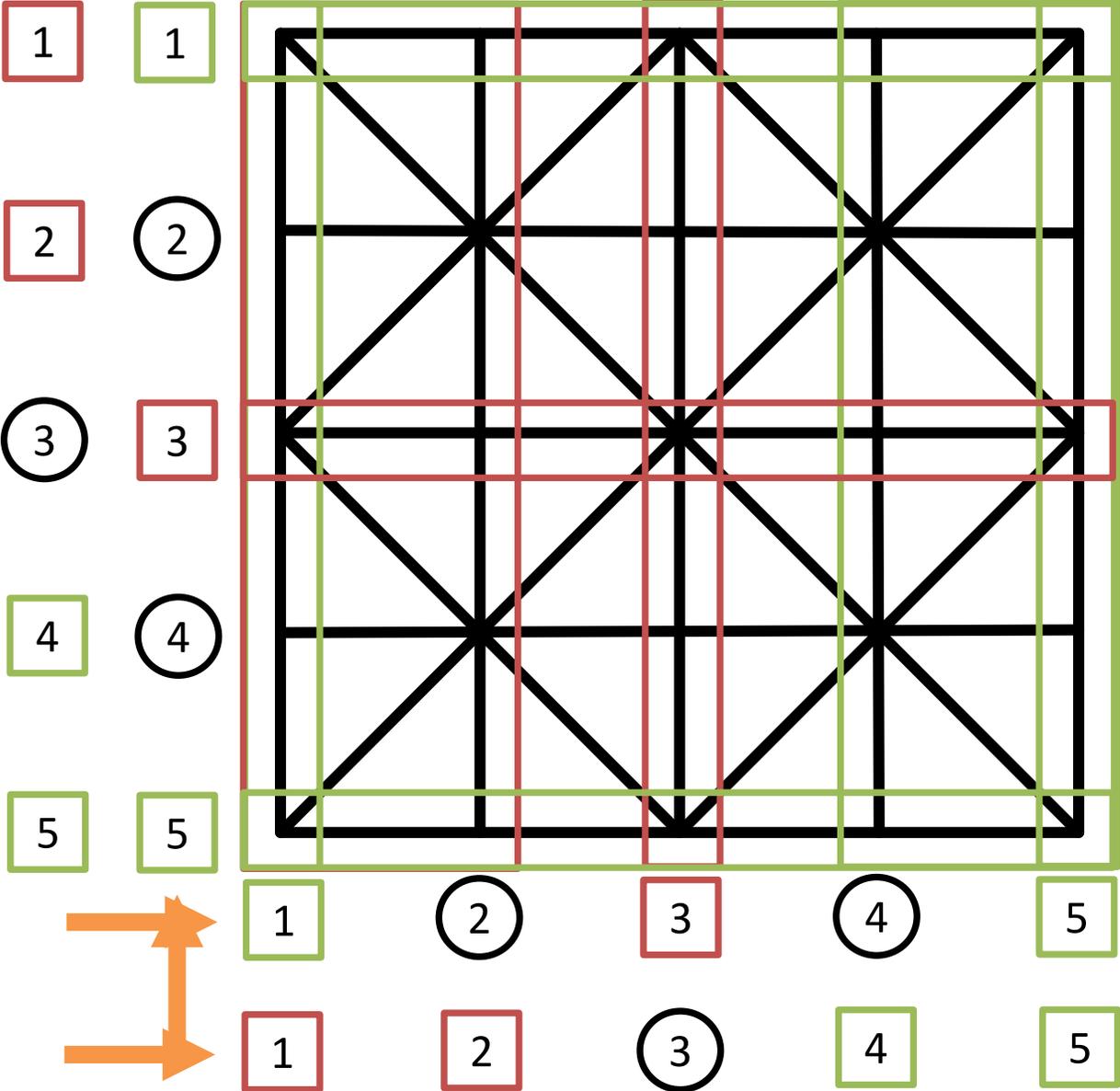


Cover has $\log_2 n$ bi-cliques.

For non-power of two just delete extra nodes.

Grid Triangulations: Step 1 = SOS2 for Inter-Box

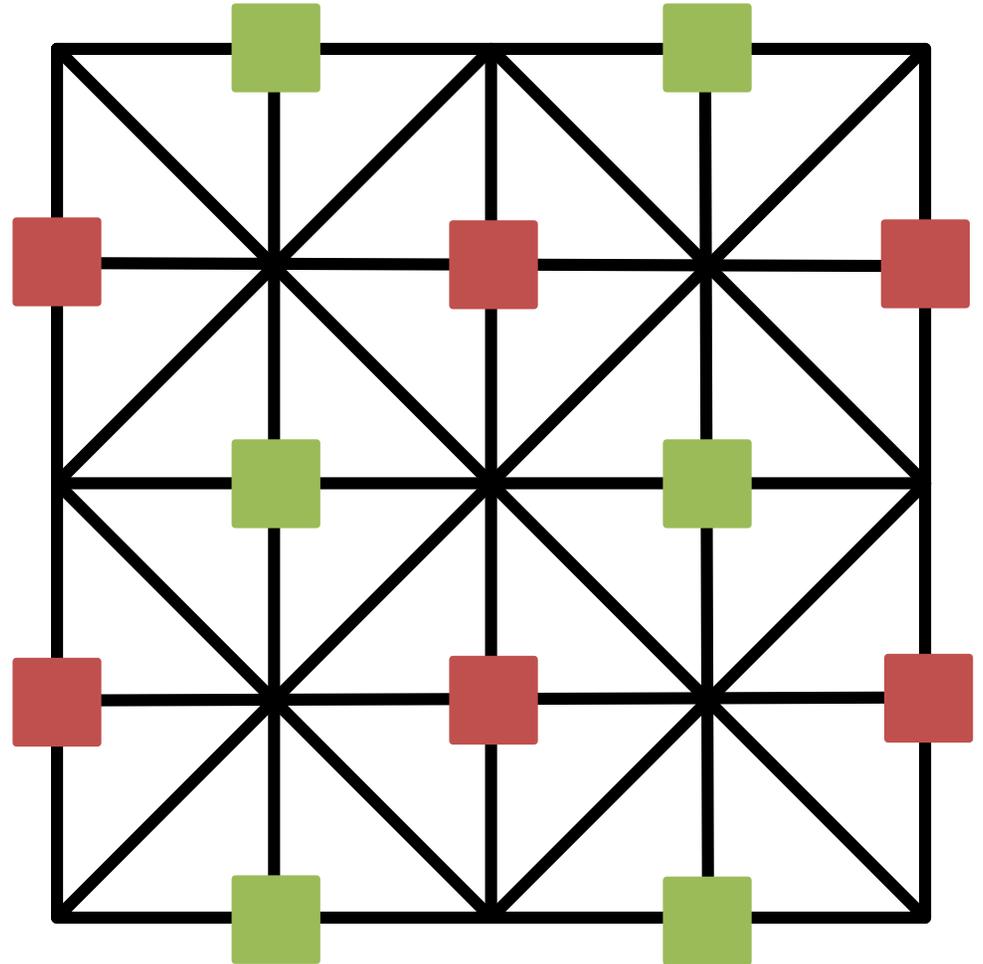
Covers all arcs between boxes



Grid Triangulations: Step 2 = Ad-hoc Intra-Box

Covers all arcs
within boxes

Sometimes 1
additional cover



Grid Triangulations: Step 2 = Ad-hoc Intra-Box

Sometimes **2**
additional covers

Sometimes more, but
always less than **9**

Simple rules to get
(near) optimal in Fall '16

