Modelamiento Avanzado con Programación Entera Mixta
Parte 3/3

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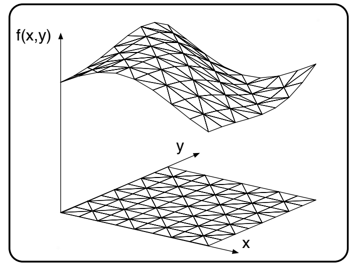
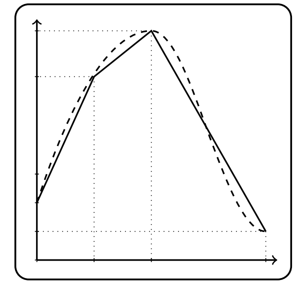
University of Pittsburgh

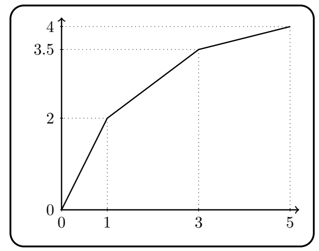
Universidad de Antofagasta, 2011 – Antofagasta, Chile

Funciones Lineales por Trazos (FLT)



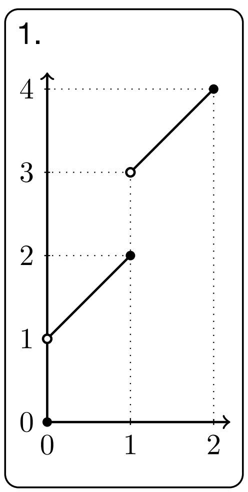
Aproximación

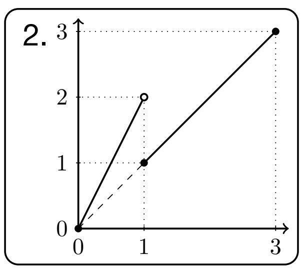


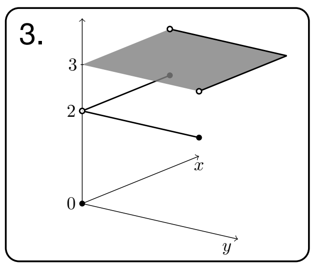


Economías de escala

Costos Fijos y Descuentos







1.Costos Fijos.
2.Descuentos (e.g. Remates).
3.Descuentos en costos Fijos

Modelar Funciones = Epigrafo

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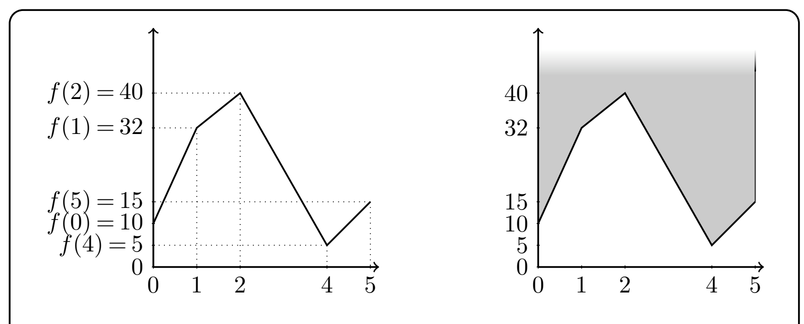
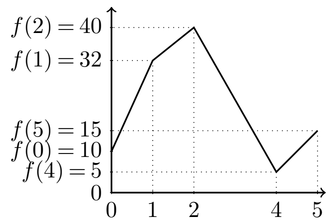


(a) f. (b) epi(f).

- Ejemplo:

Funciones Lineales por Trazos



$$f(x) := \begin{cases} 22x+10 & x \in [0,1] \\ 8x+24 & x \in [1,2] \\ -17.5x+75 & x \in [2,4] \\ 10x-35 & x \in [4,5] \end{cases}$$

DEFINITION 1. Piecewise Linear $f : D \subset \mathbb{R}^n \to \mathbb{R}$:

$$f(x) := \Big\{ m_P x + c_P \quad x \in P \quad \forall P \in \mathcal{P}.$$

for finite family of polytopes $\mathcal{P}$ such that $D = \bigcup_{P \in \mathcal{P}} P$

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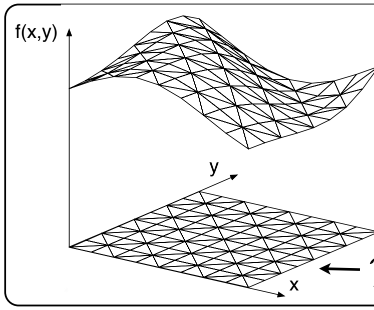

$$f(x,y) := \begin{cases} 0.48x+0.03y+6 & (x,y) \in P_1 \\ \vdots & \\ -0.4x-0.04y+8.45 & (x,y) \in P_{128} \end{cases}$$

$P_1 := \{(x,y) \in \mathbb{R} : y \geq 0,\, x \leq 1, y-x \leq 0\}$

$\vdots$

$P_{128} := \{(x,y) \in \mathbb{R} : y \geq 0,\, x \geq 7, x+y \leq 8\}$

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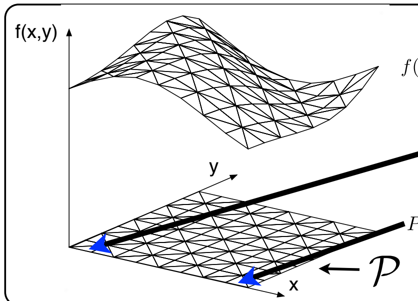

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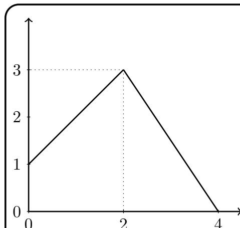
Formulacion Tradicional 1 variable



$$f(x) := \begin{cases} x+1 & x \in [0,2] \leftarrow P_1 \\ 6-3/2x & x \in [2,4] \leftarrow P_2 \end{cases}$$

$V(P) =$ vertices of P.

$\mathcal{V}(\mathcal{P}) := V(P_1) \cup V(P_2) = \{0,2,4\}.$

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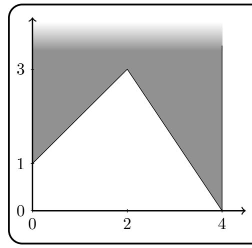

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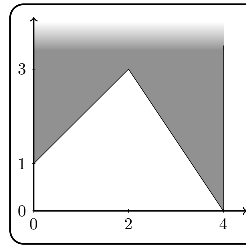

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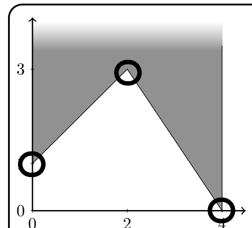

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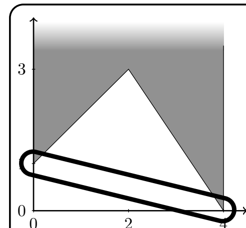

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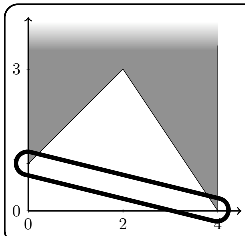

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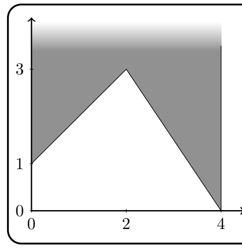

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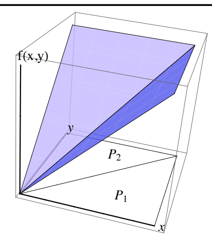
Formulacion Tradicional 2 variables



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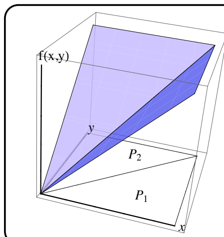

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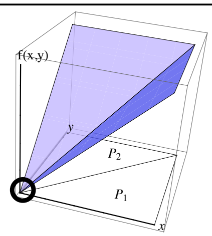

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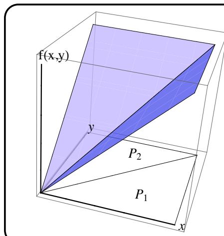

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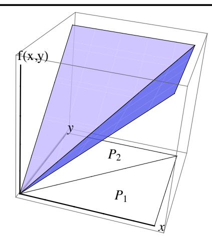

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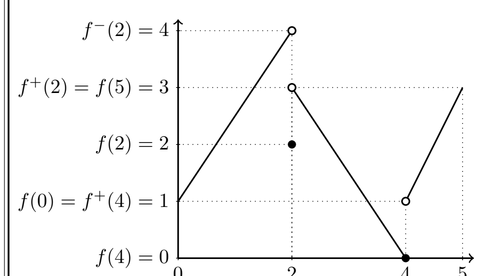
Polytopes that have (0,0) as a vertex.

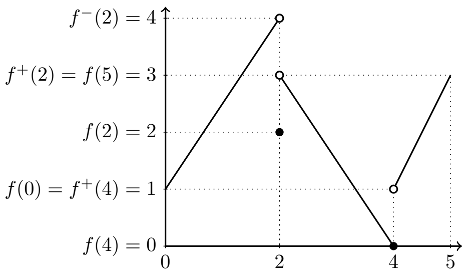
Semi-continuidad Inferior



$$f^-(d) = \lim_{\substack{x \to d \\ x \le d}} f(x)$$

$$f^+(d) = \lim_{\substack{x \to d \\ x \ge d}} f(x).$$

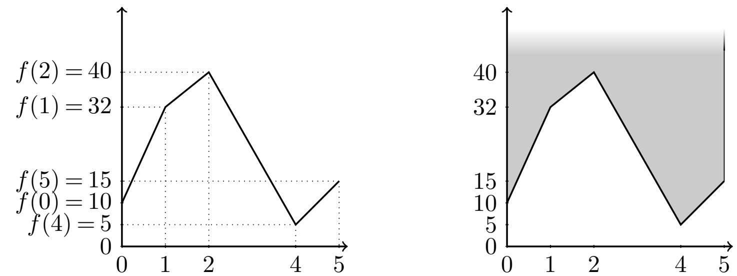
Semi-continuidad Inferior



- Semi-continuidad Inferior:

Semi-continua = epigrafo cerrado

-



(a) f. (b) epi(f).

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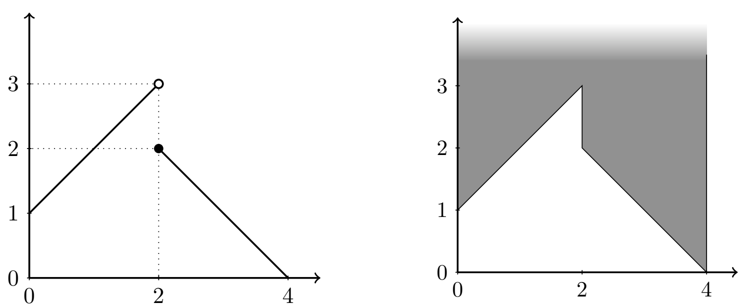

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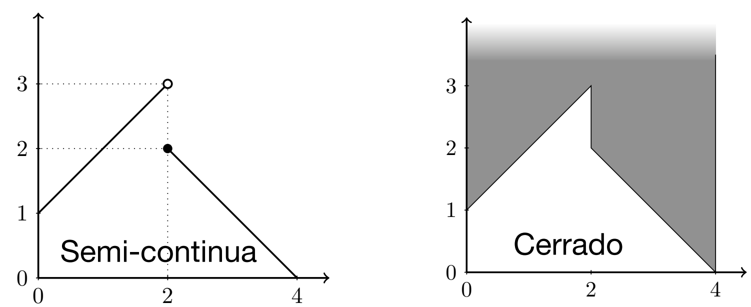

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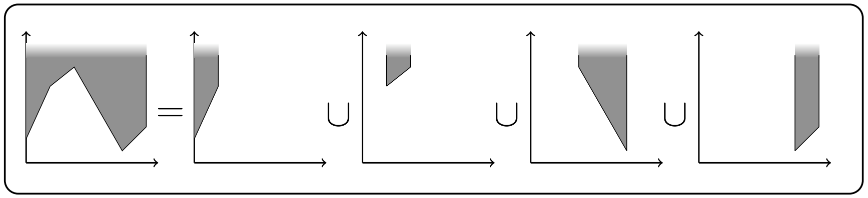
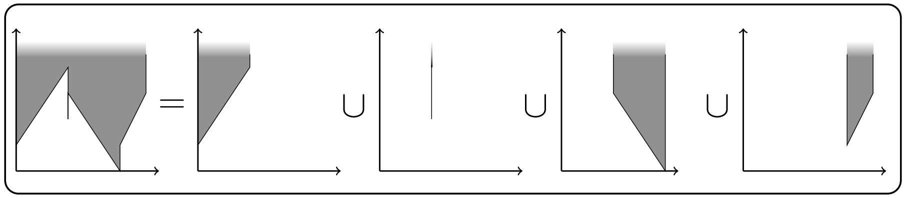
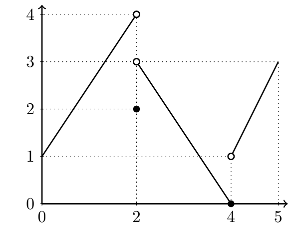
Epigrafo de FLT es unión the PL's





Funciones LT semi-continuas



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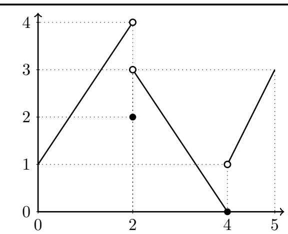
Finite family of copolytopes

$$P = \{x \in \mathbb{R}^n \,:\, a_i x \le b_i \,\forall i \in \{1,\dots,p\}, \; a_i x < b_i \,\forall i \in \{p,\dots,m\}\}$$

Funciones LT semi-continuas



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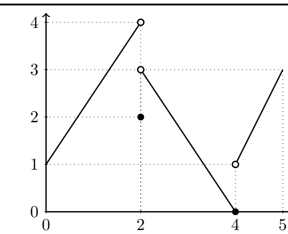

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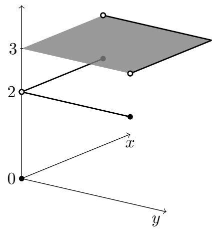

$$f(x,y) := \begin{cases} 3 & (x,y) \in (0,1]^2 \\ 2 & (x,y) \in \{(x,y) \in \mathbb{R}^2 \,:\, x = 0,\, y > 0\} \\ 2 & (x,y) \in \{(x,y) \in \mathbb{R}^2 \,:\, y = 0,\, x > 0\} \\ 0 & (x,y) \in \{(0,0)\}. \end{cases}$$

$$f(x) := \begin{cases} m_P x + c_P & x \in P \quad \forall P \in \mathcal{P} \end{cases}$$

Finite family of copolytopes

$$P = \{x \in \mathbb{R}^n \,:\, a_i x \le b_i \,\forall i \in \{1,\dots,p\},\; a_i x < b_i \,\forall i \in \{p,\dots,m\}\}$$

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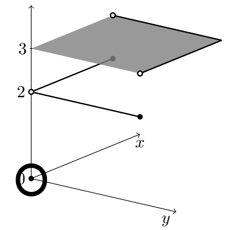

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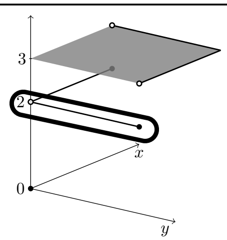

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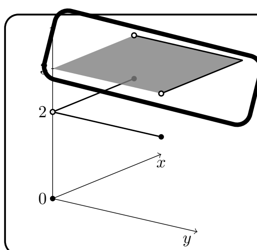

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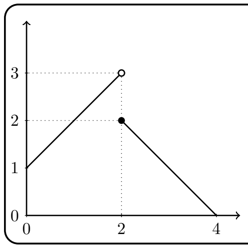
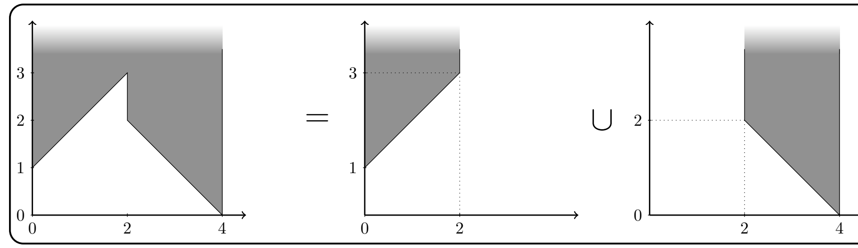
Modelo para FLT semi-continuas



(DCC)

$$f(x) := \begin{cases} x+1 & x \in [0,2) \\ 4-x & x \in [2,4] \end{cases}$$

Modelo para FLT semi-continuas



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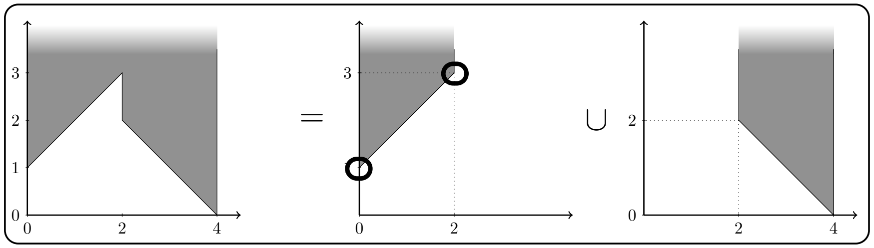


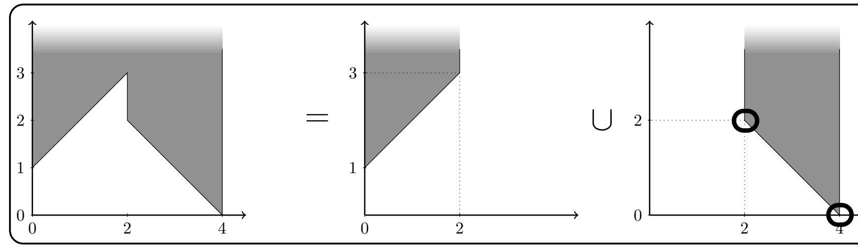
(DCC)

P_1 ↓

$$f(x) := \begin{cases} x+1 & x \in [0,2) \\ 4-x & x \in [2,4] \end{cases}$$

$$x = 0\lambda_{P_1,0} + 2\lambda_{P_1,2}$$
$$z \geq 1\lambda_{P_1,0} + 3\lambda_{P_1,2}$$
$$1 = \lambda_{P_1,0} + \lambda_{P_1,2}, \quad \lambda_{P_1,0}, \lambda_{P_1,2} \geq 0$$

Modelo para FLT semi-continuas



(DCC)

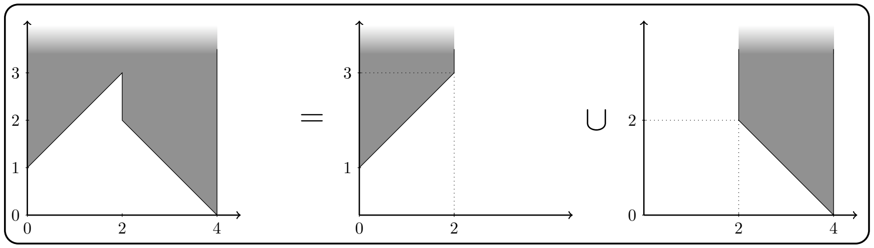
$$x = 2\lambda_{P_2,2} + 4\lambda_{P_2,4}$$
$$z \geq 2\lambda_{P_2,2} + 0\lambda_{P_2,4}$$

$$f(x) := \begin{cases} x+1 & x \in [0,2) \\ 4-x & x \in [2,4] \end{cases}$$

↑ P_2

$$1 = \lambda_{P_2,2} + \lambda_{P_2,4}, \quad \lambda_{P_2,2}, \lambda_{P_2,4} \geq 0$$

Modelo para FLT semi-continuas



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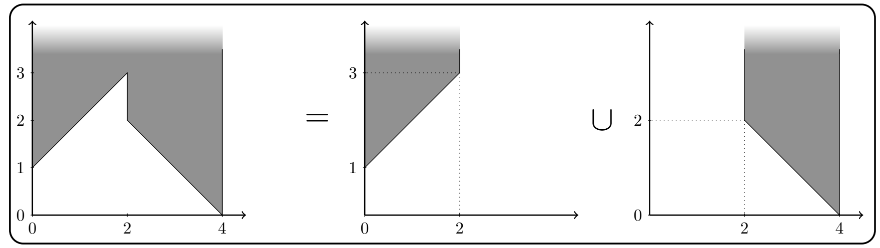
$$f(x) := \begin{cases} x+1 & x \in [0,2) \\ 4-x & x \in [2,4] \end{cases}$$

$$\begin{aligned} x &= 0\lambda_{P_1,0} + 2\lambda_{P_1,2} + 2\lambda_{P_2,2} + 4\lambda_{P_2,4} \\ z &\geq 1\lambda_{P_1,0} + 3\lambda_{P_1,2} + 2\lambda_{P_2,2} + 0\lambda_{P_2,4} \\ 1 &= \lambda_{P_1,0} + \lambda_{P_1,2}, \quad \lambda_{P_1,0}, \lambda_{P_1,2} \geq 0 \\ 1 &= \lambda_{P_2,2} + \lambda_{P_2,4}, \quad \lambda_{P_2,2}, \lambda_{P_2,4} \geq 0 \end{aligned}$$

Modelo para FLT semi-continuas



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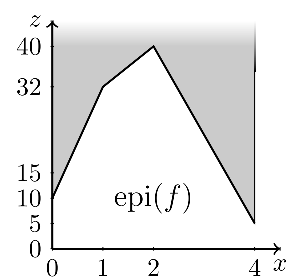
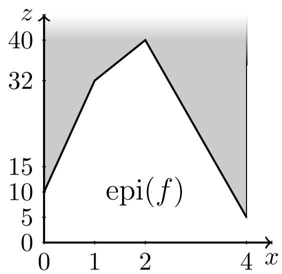
$$\begin{aligned} x &= 0\lambda_{P_1,0} + 2\lambda_{P_1,2} + 2\lambda_{P_2,2} + 4\lambda_{P_2,4} \\ z &\geq 1\lambda_{P_1,0} + 3\lambda_{P_1,2} + 2\lambda_{P_2,2} + 0\lambda_{P_2,4} \\ y_{P_1} &= \lambda_{P_1,0} + \lambda_{P_1,2}, \quad \lambda_{P_1,0}, \lambda_{P_1,2} \geq 0 \\ y_{P_2} &= \lambda_{P_2,2} + \lambda_{P_2,4}, \quad \lambda_{P_2,2}, \lambda_{P_2,4} \geq 0 \\ 1 &= y_{P_1} + y_{P_2}, \quad y_{P_1}, y_{P_2} \in \{0,1\} \end{aligned}$$

Intermedio

Formulación Tradicional de FLT



Formulación Tradicional de FLT



$$x = 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + 4\lambda_4$$
$$z \geq 10\lambda_1 + 32\lambda_2 + 40\lambda_3 + 5\lambda_4$$
$$\sum_{i=1}^{4} \lambda_i = 1, \quad \lambda_i \geq 0 \, \forall i \in \{1, \ldots, 4\}$$
$$\lambda_1 \leq y_1, \quad \lambda_2 \leq y_1 + y_2, \quad \lambda_3 \leq y_2 + y_3,$$
$$\lambda_4 \leq y_3, \quad \sum_{i=1}^{3} y_i = 1, \quad y \in \{0,1\}^3$$

Disjunciones Especiales

$$x = 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + 4\lambda_4$$
$$z \geq 10\lambda_1 + 32\lambda_2 + 40\lambda_3 + 5\lambda_4$$
$$\lambda \in [0,1]^4 \; : \; \sum_{i=1}^{4} \lambda_i = 1$$
$$\lambda_1 \leq y_1, \quad \lambda_2 \leq y_1 + y_2, \quad \lambda_3 \leq y_2 + y_3,$$
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$$x = 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + 4\lambda_4$$
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$$\left\{\lambda \in [0,1]^4 \; : \; \sum_{i=1}^{4} \lambda_i = 1\right\} =: \Delta^4$$
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Disjunciones Especiales

$x = 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + 4\lambda_4$

$z \geq 10\lambda_1 + 32\lambda_2 + 40\lambda_3 + 5\lambda_4$

$$\left\{\lambda \in [0,1]^4 \,:\, \sum_{i=1}^{4} \lambda_i = 1\right\} =: \Delta^4$$

$$\lambda_1 \leq y_1, \quad \lambda_2 \leq y_1 + y_2, \quad \lambda_3 \leq y_2 + y_3,$$

$$\lambda_4 \leq y_3, \quad \sum_{i=1}^{3} y_i = 1, \quad y \in \{0,1\}^3$$

$$y_1 = 1 \Rightarrow \lambda \in P_1 := \left\{\lambda \in \Delta^4 \,:\, \lambda_3, \lambda_4 \leq 0\right\}$$

$$y_2 = 1 \Rightarrow \lambda \in P_2 := \left\{\lambda \in \Delta^4 \,:\, \lambda_1, \lambda_4 \leq 0\right\}$$

$$y_3 = 1 \Rightarrow \lambda \in P_3 := \left\{\lambda \in \Delta^4 \,:\, \lambda_1, \lambda_2 \leq 0\right\}$$

Disjunciones Especiales

$x = 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + 4\lambda_4$

$z \geq 10\lambda_1 + 32\lambda_2 + 40\lambda_3 + 5\lambda_4$

$$\left\{\lambda \in [0,1]^4 \,:\, \sum_{i=1}^{4} \lambda_i = 1\right\} =: \Delta^4$$

$$\lambda_1 \leq y_1, \quad \lambda_2 \leq y_1 + y_2, \quad \lambda_3 \leq y_2 + y_3,$$

$$\lambda_4 \leq y_3, \quad \sum_{i=1}^{3} y_i = 1, \quad y \in \{0,1\}^3$$

$$\Bigg\} \; \lambda \in \bigcup_{i=1}^{4} P_i$$

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Disjunciones Especiales

$x = 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + 4\lambda_4$

$z \geq 10\lambda_1 + 32\lambda_2 + 40\lambda_3 + 5\lambda_4$

SOS2 Constraints

$$\left\{\lambda \in [0,1]^4 \,:\, \sum_{i=1}^{4} \lambda_i = 1\right\} =: \Delta^4$$

$$\lambda_1 \leq y_1, \quad \lambda_2 \leq y_1 + y_2, \quad \lambda_3 \leq y_2 + y_3,$$

$$\lambda_4 \leq y_3, \quad \sum_{i=1}^{3} y_i = 1, \quad y \in \{0,1\}^3$$

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$$y_3 = 1 \Rightarrow \lambda \in P_3 := \left\{\lambda \in \Delta^4 \,:\, \lambda_1, \lambda_2 \leq 0\right\}$$

Disjunciones Especiales (DE)

$$\lambda \in \bigcup_{i=1}^{m} P(F_i) \qquad P(F_i) := \{\lambda \in \Delta^n \,:\, \lambda_j \leq 0 \,\forall j \in F_i\}$$

$$\Delta^n := \left\{\lambda \in [0,1]^n \,:\, \sum_{i=1}^{n} \lambda_i = 1\right\}$$

- SOS1: .
- SOS2: .
- Funciones Lineales por trazos.

Formulación Estándar

$$x \in \bigcup_{i=1}^{m} P_i \subset \mathbb{R}^n$$

$$P_i := \{x \in \mathbb{R} \;:\; A^i x \le b^i\}$$

$$A^i \in \mathbb{R}^{r \times n},\; b^i \in \mathbb{R}^{r_i}$$

$\longrightarrow$

$$A^i x^i \le b^i y_i \quad \forall i$$

$$\sum_{i=1}^{m} x^i = x$$

$$\sum_{i=1}^{m} y_i = 1$$

$$y \in \{0,1\}^M$$

- Sharp y localmente Ideal.
- variables y restricciones.

Formulación Estándar para DE

$$\lambda \in \bigcup_{i=1}^{m} P(F_i) \qquad P(F_i) := \{\lambda \in \Delta^n \;:\; \lambda_j \le 0 \,\forall j \in F_i\}$$

$$\Delta^n := \left\{\lambda \in [0,1]^n \;:\; \sum_{i=1}^{n} \lambda_i = 1\right\}$$

$$\sum_{j=1}^{n} \lambda_j^i = y_i \qquad \sum_{i=1}^{m} \lambda^i = \lambda$$

$$0 \le \lambda_j^i \le y_i \quad \forall j \notin F_i \qquad \sum_{i=1}^{m} y_i = 1$$

$$0 \le \lambda_j^i \le 0 \quad \forall j \in F_i \qquad y \in \{0,1\}^m$$

- Sharp y localmente Ideal.

Eliminar copias de en formulacion

$$\sum_{j=1}^{n} \lambda_j^i = y_i \qquad \sum_{i=1}^{m} \lambda^i = \lambda$$

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$$\sum_{j=1}^{n} \lambda_j = 1, \quad \lambda \ge 0,$$

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- Formulación Tradicional para FLT
- Sharp pero no localmente ideal.

Formulacion Logaritmica para SOS1

$$\sum_{j=0}^{3} \lambda_j = 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3 \ge 0, \text{ at most 1 } \lambda_j \text{ is nonzero.}$$
Allowed sets: $S_0 = \{0\}$, $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$.

Formulacion Logaritmica para SOS1

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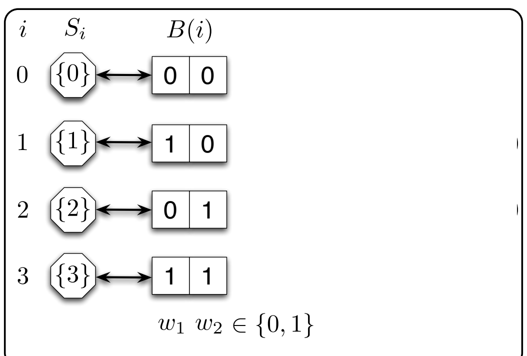

- Injective function:
- Variables:
- Idea:

Formulacion Logaritmica para SOS1

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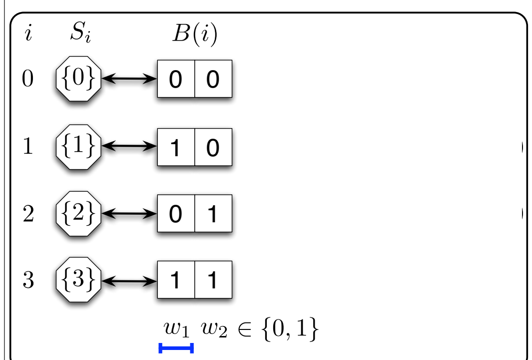

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Formulacion Logaritmica para SOS1

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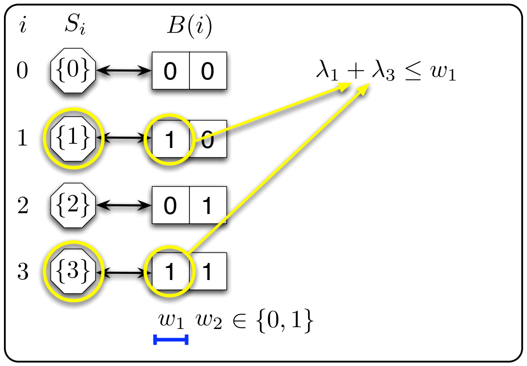

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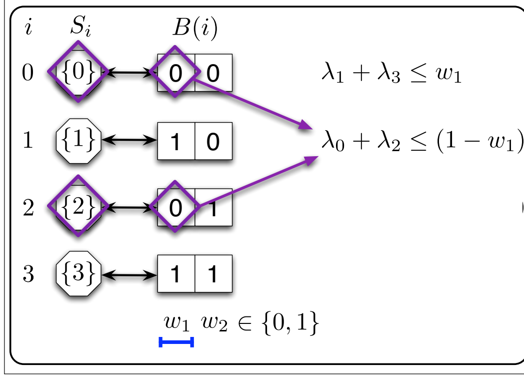

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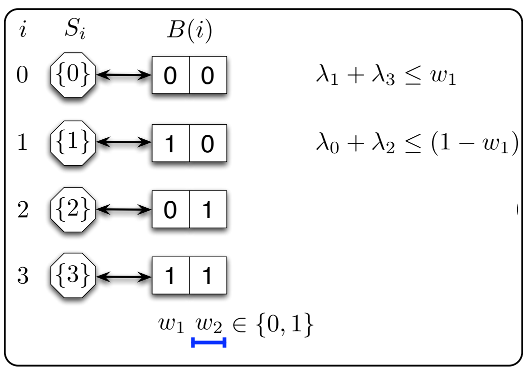

- Injective function:
- Variables:
- Idea:

Formulacion Logaritmica para SOS1

$\sum_{j=0}^{3} \lambda_j = 1, \quad \lambda_0, \lambda_1, \lambda_2, \lambda_3 \geq 0,$ at most 1 λ_j is nonzero.
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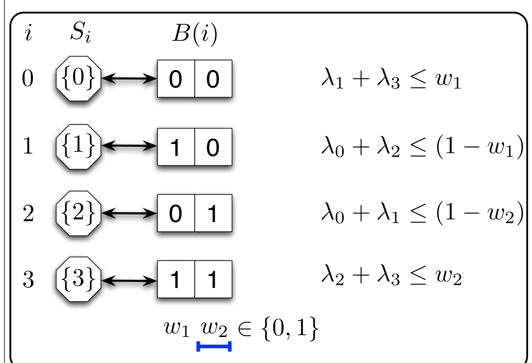

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Formulacion Logaritmica para SOS1

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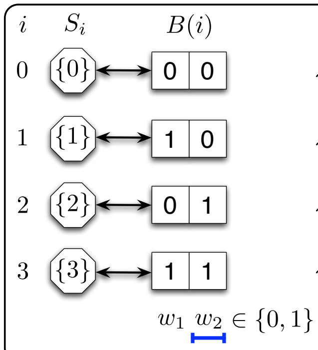

$\lambda_1 + \lambda_3 \leq w_1$

$\lambda_0 + \lambda_2 \leq (1 - w_1)$

$\lambda_0 + \lambda_1 \leq (1 - w_2)$

$\lambda_2 + \lambda_3 \leq w_2$

- In general:

Formulacion Logaritmica para SOS2

$\sum_{j=0}^{4} \lambda_j = 1, \quad \lambda_0, \ldots, \lambda_4 \geq 0,$ only 2 adjacent λ_j's ar nonzero.
Allowed sets: $S_i = \{i, i+1\} \quad$ for $i \in \{0, \ldots, 3\}$.

Formulacion Logaritmica para SOS2

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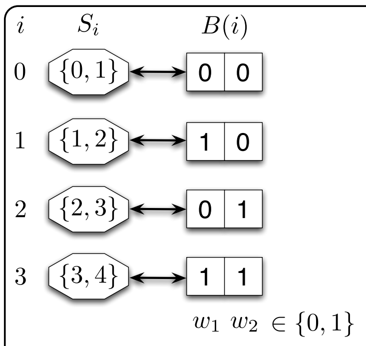

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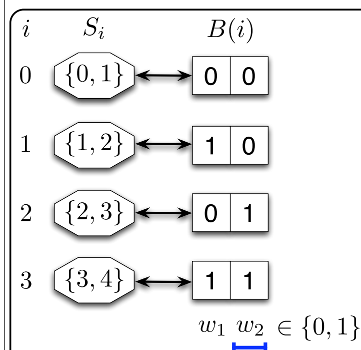

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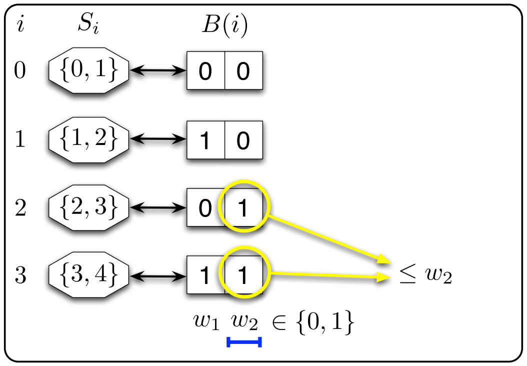

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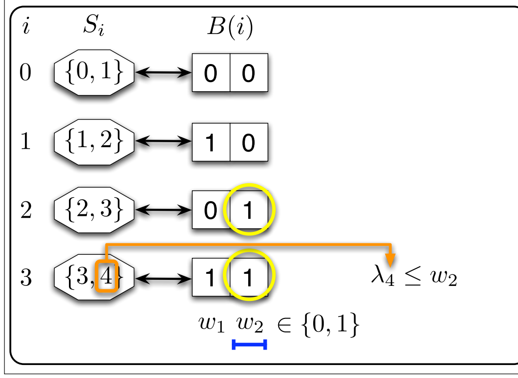

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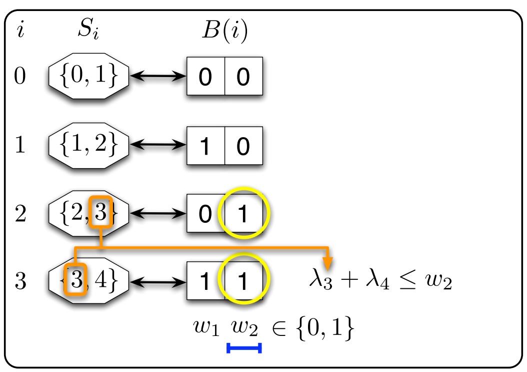

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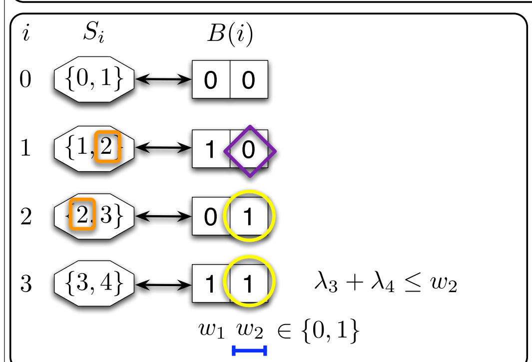

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i	S_i	$B(i)$	
0	$\{0,1\}$	0 0	$\lambda_0 \leq w_1$
1	$\{1,2\}$	1 0	$\lambda_4 \leq (1 - w_1)$
2	$\{2,3\}$	0 1	$\lambda_0 + \lambda_1 \leq (1 - w_2)$
3	$\{3,4\}$	1 1	$\lambda_3 + \lambda_4 \leq w_2$

$w_1 \ w_2 \in \{0,1\}$

- Injective function:
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Formulacion Logaritmica para SOS2

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- Where is ?!

Formulacion Logaritmica para SOS2

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Formulacion Logaritmica para SOS2

$$\sum_{j=0}^{4} \lambda_j = 1, \quad \lambda_0, \ldots, \lambda_4 \geq 0, \text{ only 2 adjacent } \lambda_j\text{'s ar nonzero.}$$

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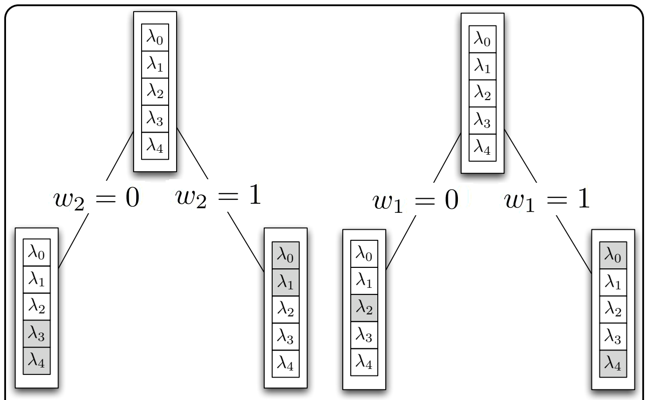
$w_1 \; w_2 \in \{0,1\}$

- Where is ?!
- In general:
- Gray Code.

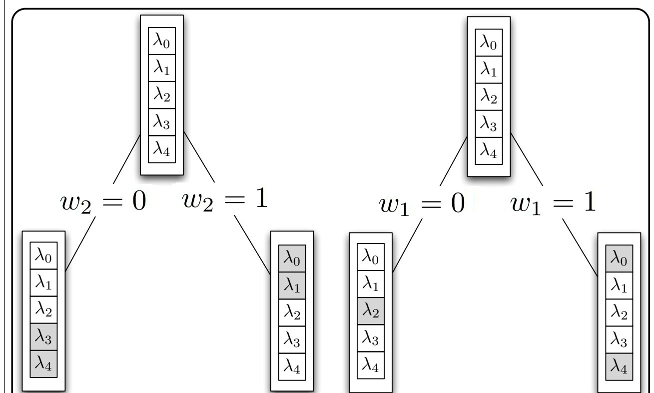
Ramificacion Independiente



$\lambda_3 + \lambda_4 \leq w_2$ $\qquad$ $\lambda_2 \leq w_1$

$\lambda_0 + \lambda_1 \leq (1 - w_2)$ $\qquad$ $\lambda_0 + \lambda_4 \leq (1 - w_1)$

Ramificacion Independiente



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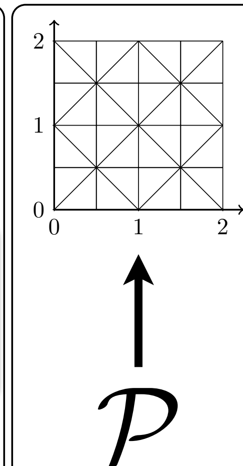
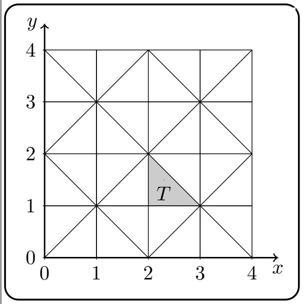


Ramificacion Independiente para FLT

- Seleccionar triángulo prohibiendo vértices.
- 2 etapas:
  - Seleccionar cuadrado con SOS2 por variable.
  - Seleccionar 1 triángulo de cada cuadtado.



Logarithmic Formulations

Ramificacion Independiente para FLT

- Seleccionar triángulo prohibiendo vértices.
- 2 etapas:
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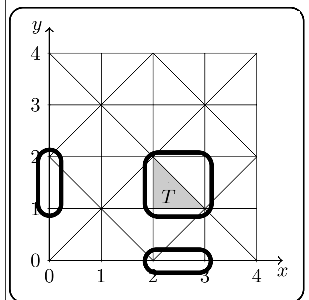

Logarithmic Formulations

Ramificacion Independiente para FLT

- Seleccionar triángulo prohibiendo vértices.
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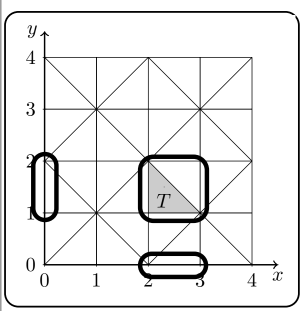

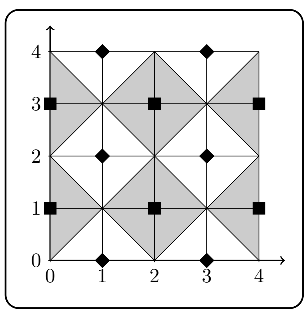

$$\bar{L} = \{(r,s) \in J : r \text{ even and } s \text{ odd}\} = \{\text{square vertices}\}$$

$$\bar{R} = \{(r,s) \in J : r \text{ odd and } s \text{ even}\} = \{\text{diamond vertices}\}$$