Modelamiento Avanzado con Programación Entera Mixta
Parte 2/3

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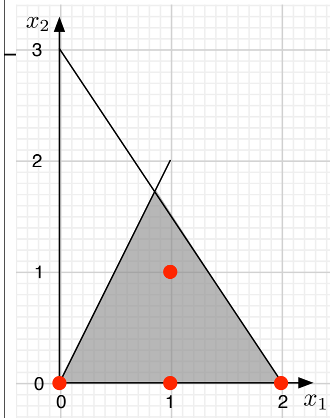
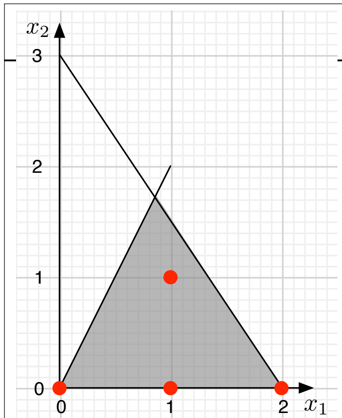
University of Pittsburgh

Universidad de Antofagasta, 2011 – Antofagasta, Chile



$$\begin{aligned} \min z := x_2 & \\ 3x_1 + 2x_2 &\le 6 \\ -2x_1 + x_2 &\le 0 \\ x_1,\, x_2 &\ge 0 \\ x_1,\, x_2 &\in \mathbb{Z} \end{aligned}$$



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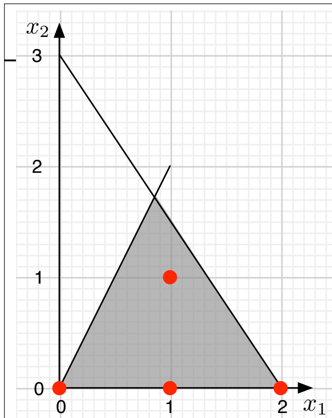

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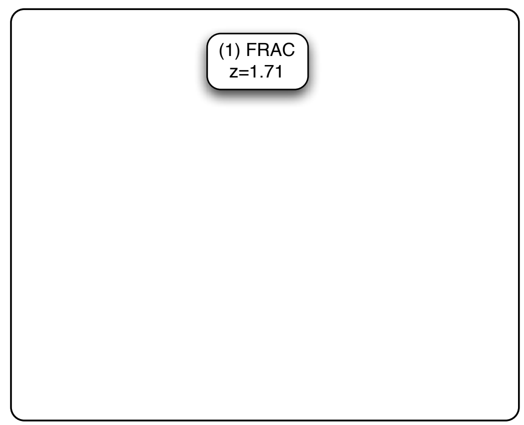

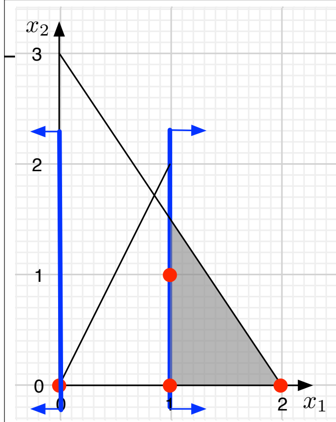

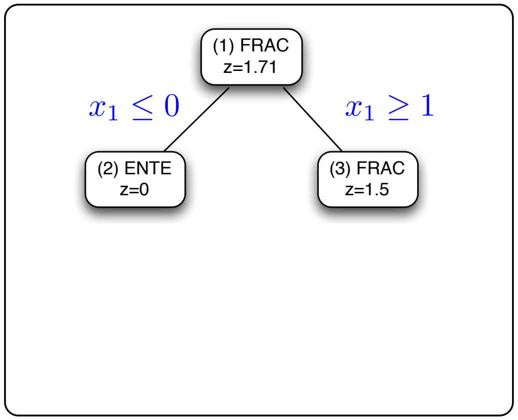

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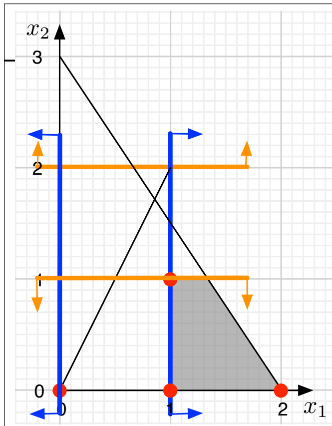

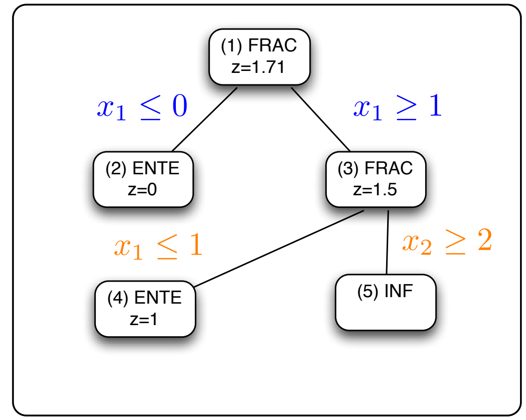

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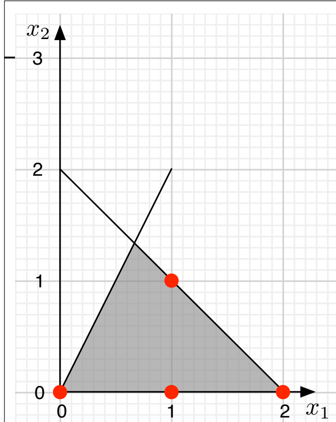

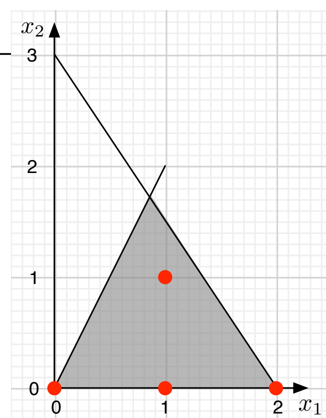

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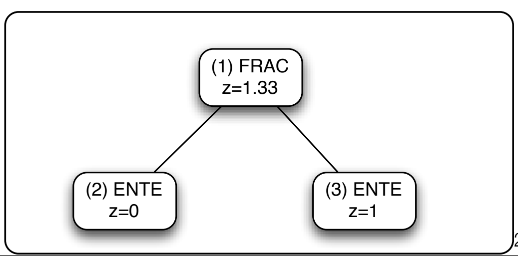


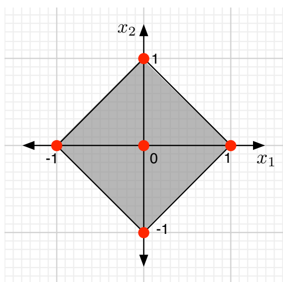
Que hace un Buen Modelo IP

- Relajación lineal fuerte
- Tamaño pequeño
- Compatibilidad con ramificación de B&B...

- Puedo tener una formulación fuerte y pequeña?
  - Si, usando el poder de las variables auxiliares (proyección)

Dos tipos de formulaciones de PE

$$S := \Big\{x \in \mathbb{Z}^n : \sum_{i=1}^{n} |x_i| \leq 1\Big\}$$



IP Models

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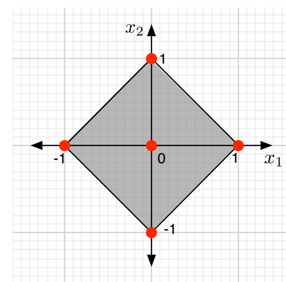

$x_1, x_2 \in \mathbb{Z}^2$

IP Models

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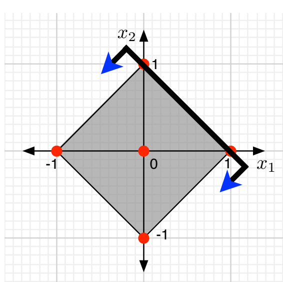

$x_1 + x_2 \leq 1$

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IP Models

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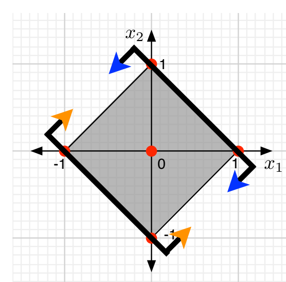

$x_1 + x_2 \leq 1$

$-x_1 - x_2 \leq 1$

$x_1, x_2 \in \mathbb{Z}^2$

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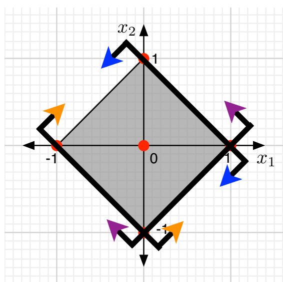

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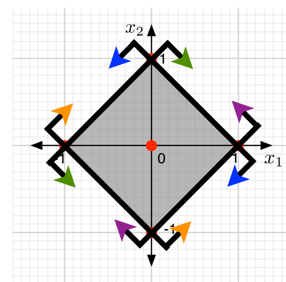

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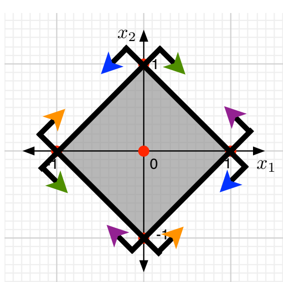
Original Space: Size= $O(2^n)$



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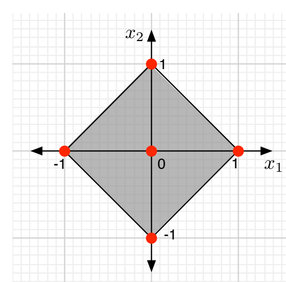

$$\begin{aligned} \sum_{i=1}^{n} y_i &\leq 1 \\ -y_i \leq x_i &\leq y_i \quad \forall i \in \{1,\ldots,n\} \\ x_i &\in \mathbb{Z} \quad \forall i \in \{1,\ldots,n\} \end{aligned}$$

Extended Formulation: Size= $O(n)$

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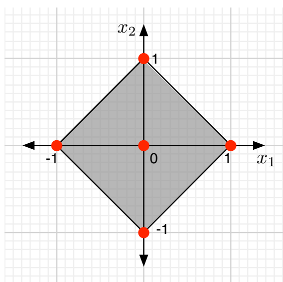

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Compact

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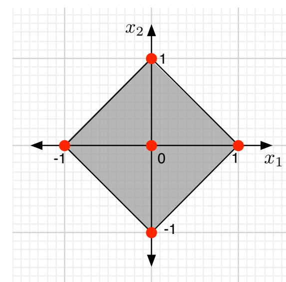

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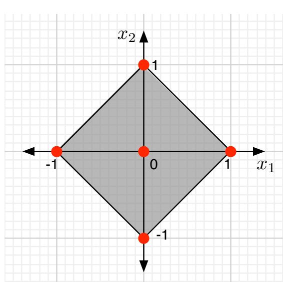

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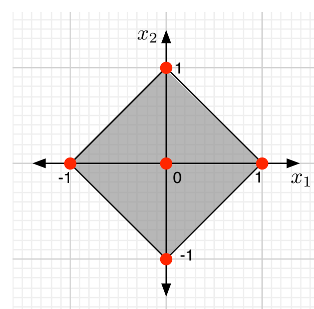
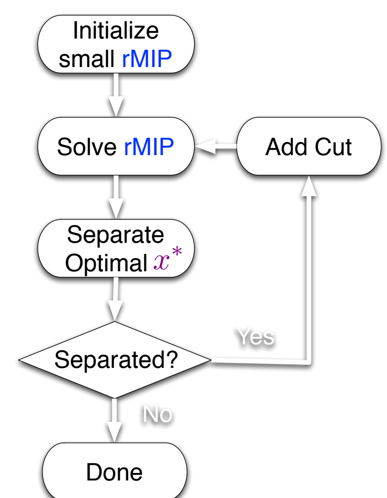
Formulaciones Grandes: Separar



$\max_{x \in \mathbb{Z}^n} \sum_{i=1}^n x_i$



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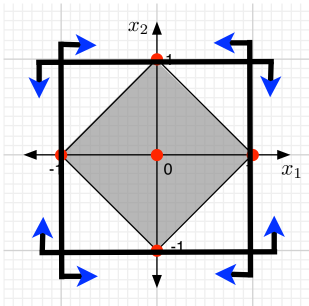

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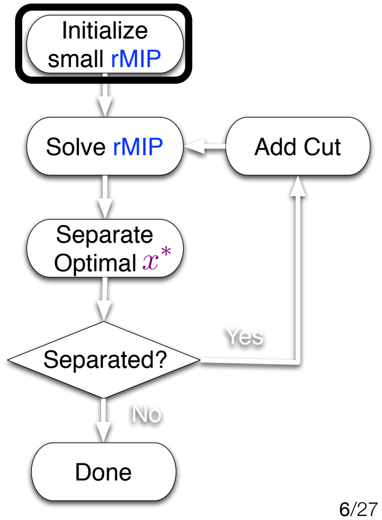

IP Models

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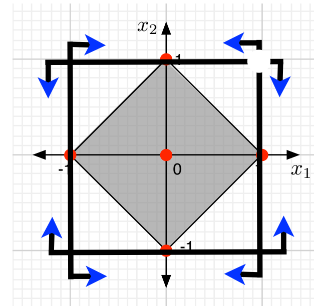

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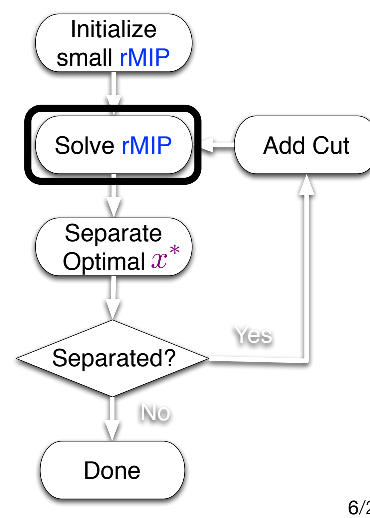

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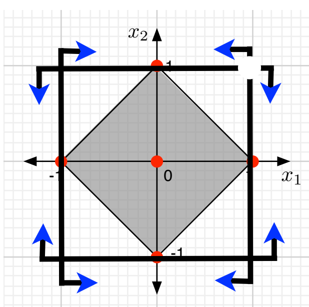

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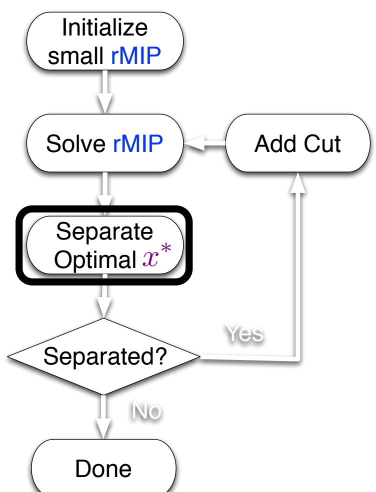

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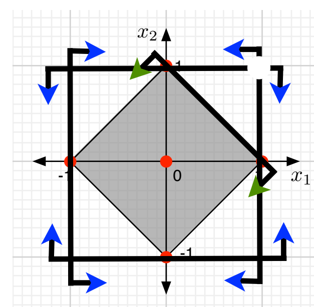

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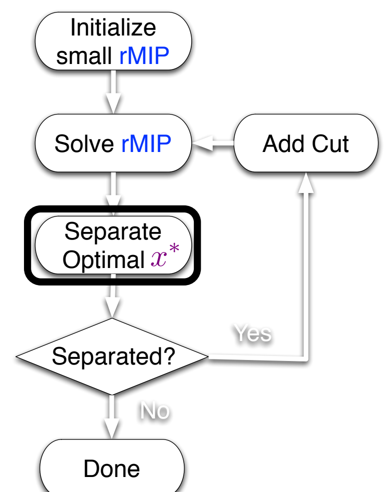

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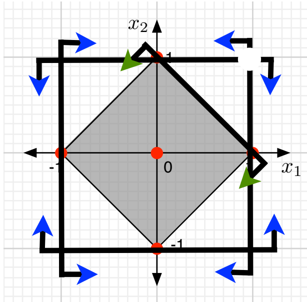

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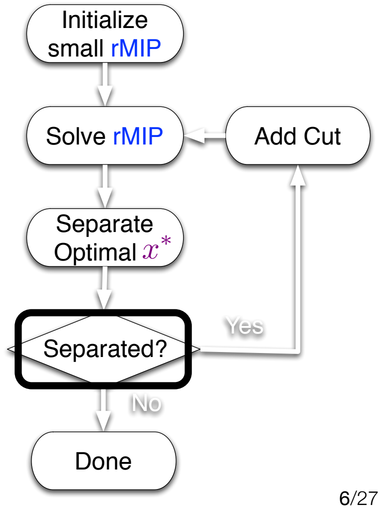

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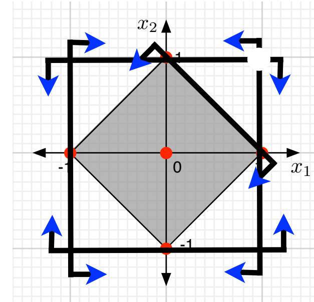

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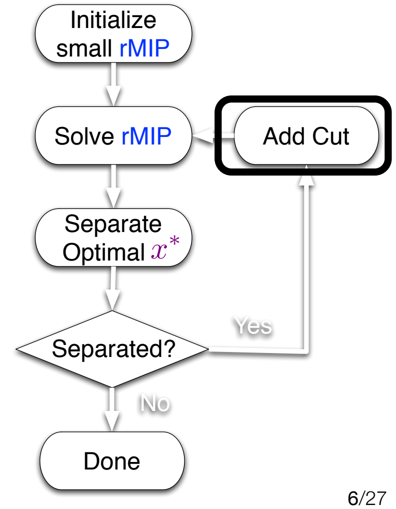

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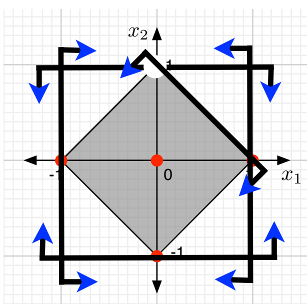

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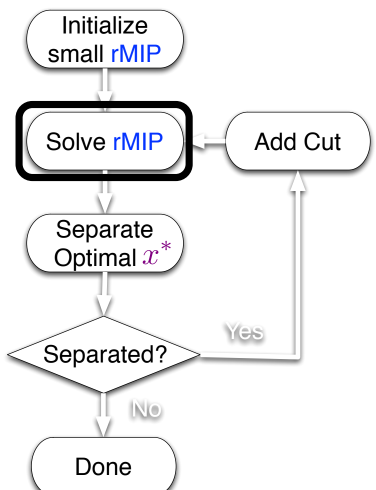

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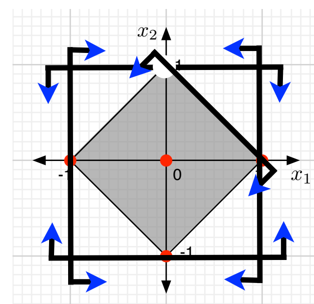

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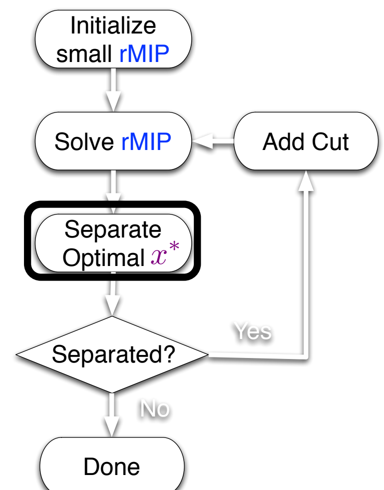

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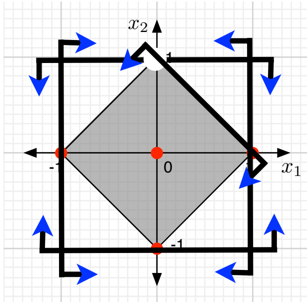

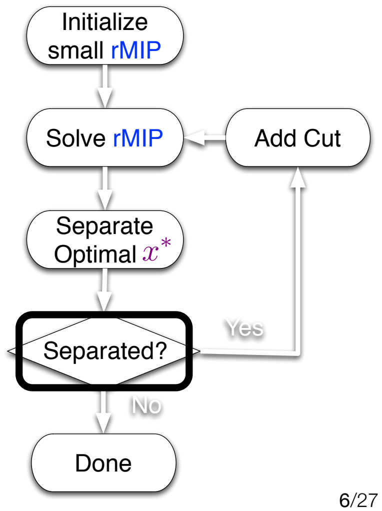

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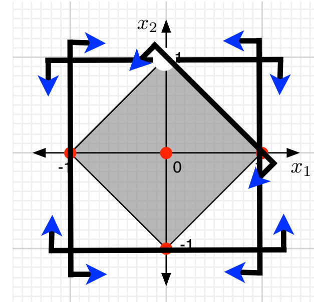

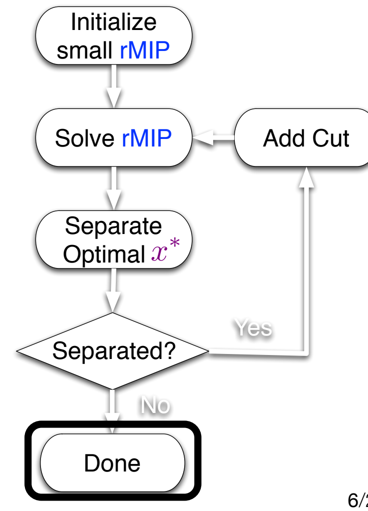

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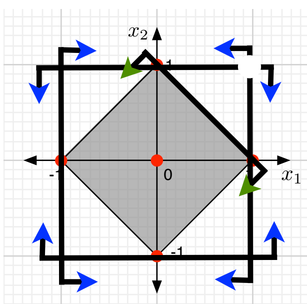
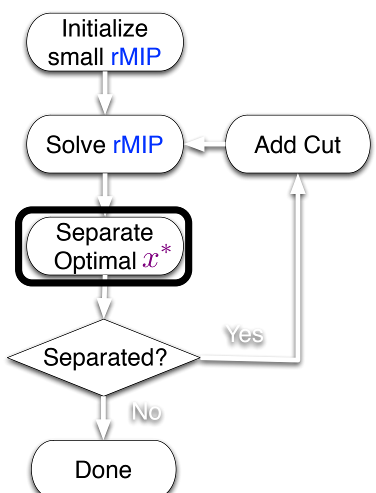
La clave es separación rápida





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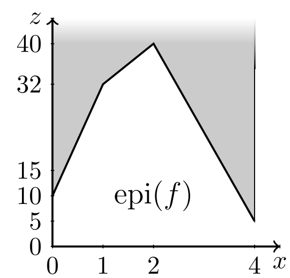
$$-1 \le x_i \le 1 \quad \forall i \in \{1,\ldots,n\}$$

$$\sum_{i=1}^{n} \mathrm{sign}(x_i^*) x_i \le 1$$

Sharp pero no localmente ideal



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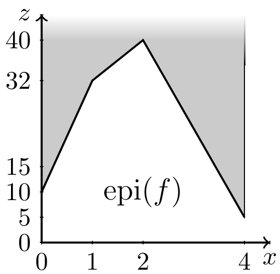

$$x = 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + 4\lambda_4$$
$$z \geq 10\lambda_1 + 32\lambda_2 + 40\lambda_3 + 5\lambda_4$$
$$\sum_{i=1}^{4} \lambda_i = 1, \quad \lambda_i \geq 0 \, \forall i \in \{1, \ldots, 4\}$$
$$\lambda_1 \leq y_1, \quad \lambda_2 \leq y_1 + y_2, \quad \lambda_3 \leq y_2 + y_3,$$
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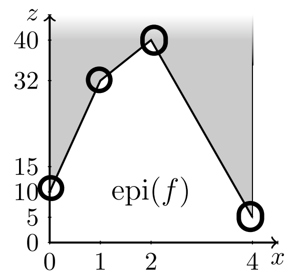
Sharp pero no localmente ideal



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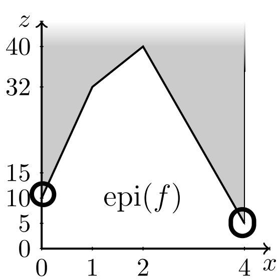

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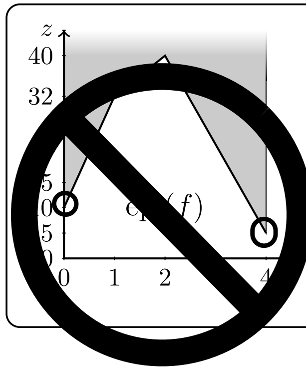

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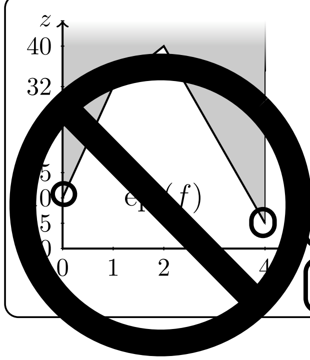

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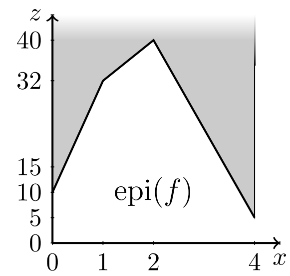

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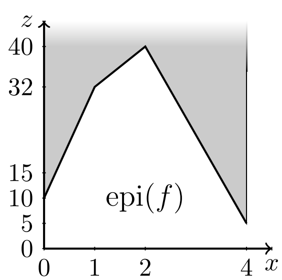

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Not Locally Ideal
\
LP has fractional extreme pt.

$$\lambda_2 = \lambda_3 = 1/2, \quad \lambda_0 = \lambda_2 = 0$$
$$y_1 = y_3 = 1/2, \quad y_2 = 0$$
$$x = 3, \quad z = 22.5$$

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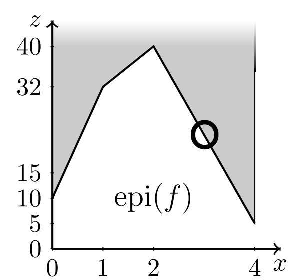

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